Module 00: Preliminaries

Econometrics II

Max Heinze (mheinze@wu.ac.at)
Department of Economics, WU Vienna

Sannah Tijani (stijani@wu.ac.at)
Department of Economics, WU Vienna

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Introduction

Installing R

Introduction to R with RStudio

Matrices and Vectors

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You also have access to **Econometrics I slides** via Canvas. **Review these** if you do not feel confident about your knowledge of what we learned in Econometrics I.

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Having statistical software (e.g., R) installed is a prerequisite for the course.

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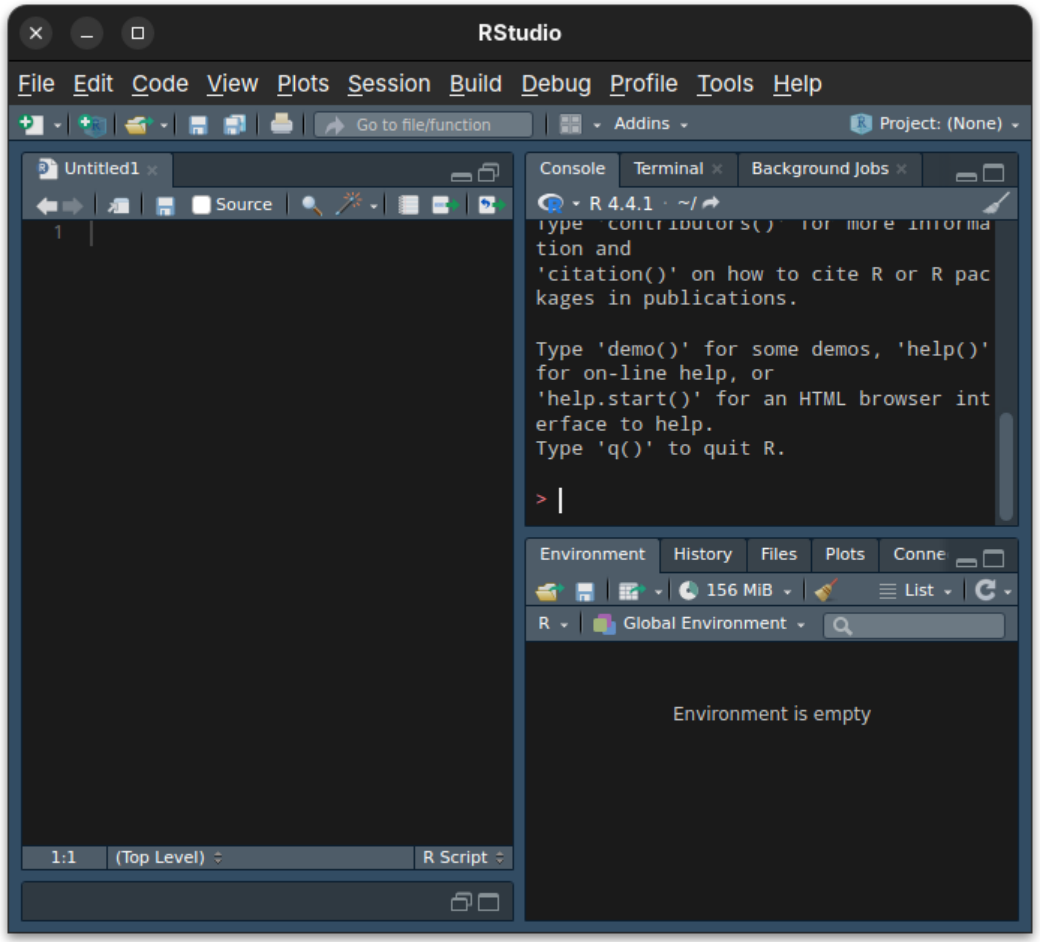

The default layout is slightly different, and the default theme is light. Both can be changed in the settings.

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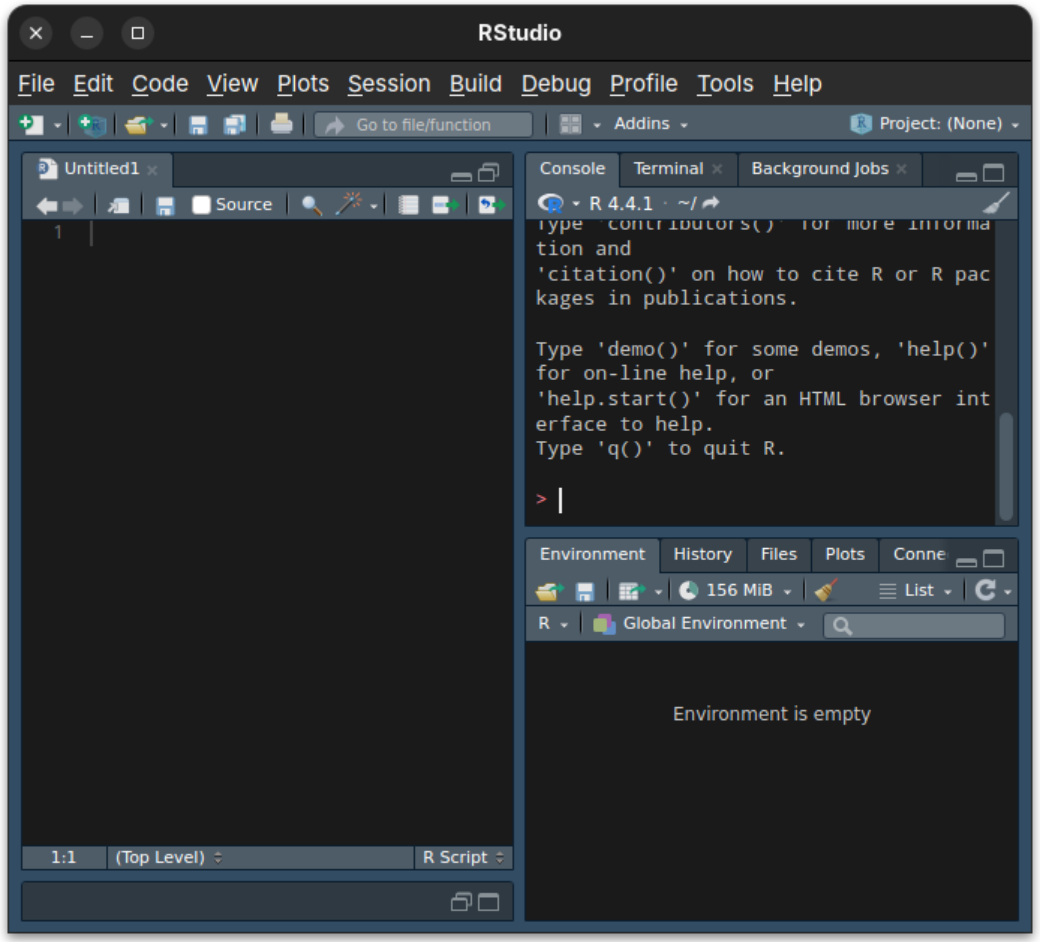

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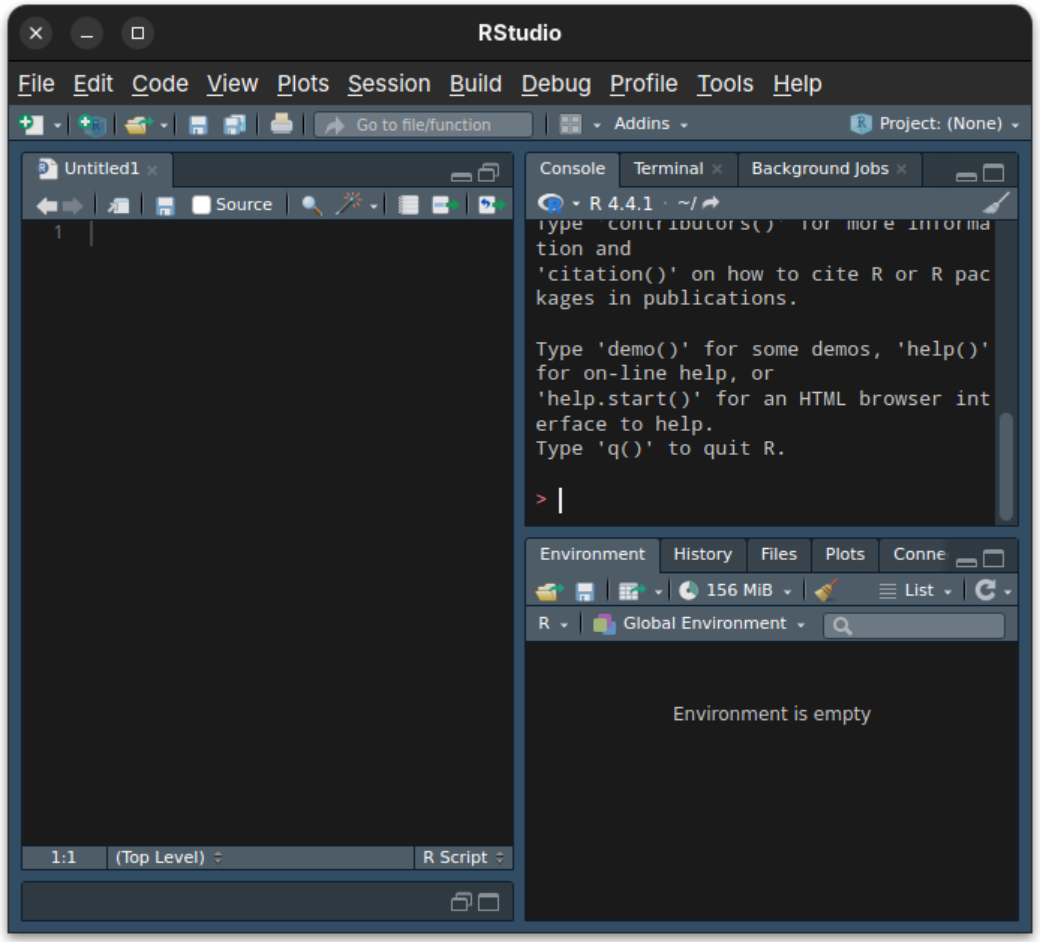

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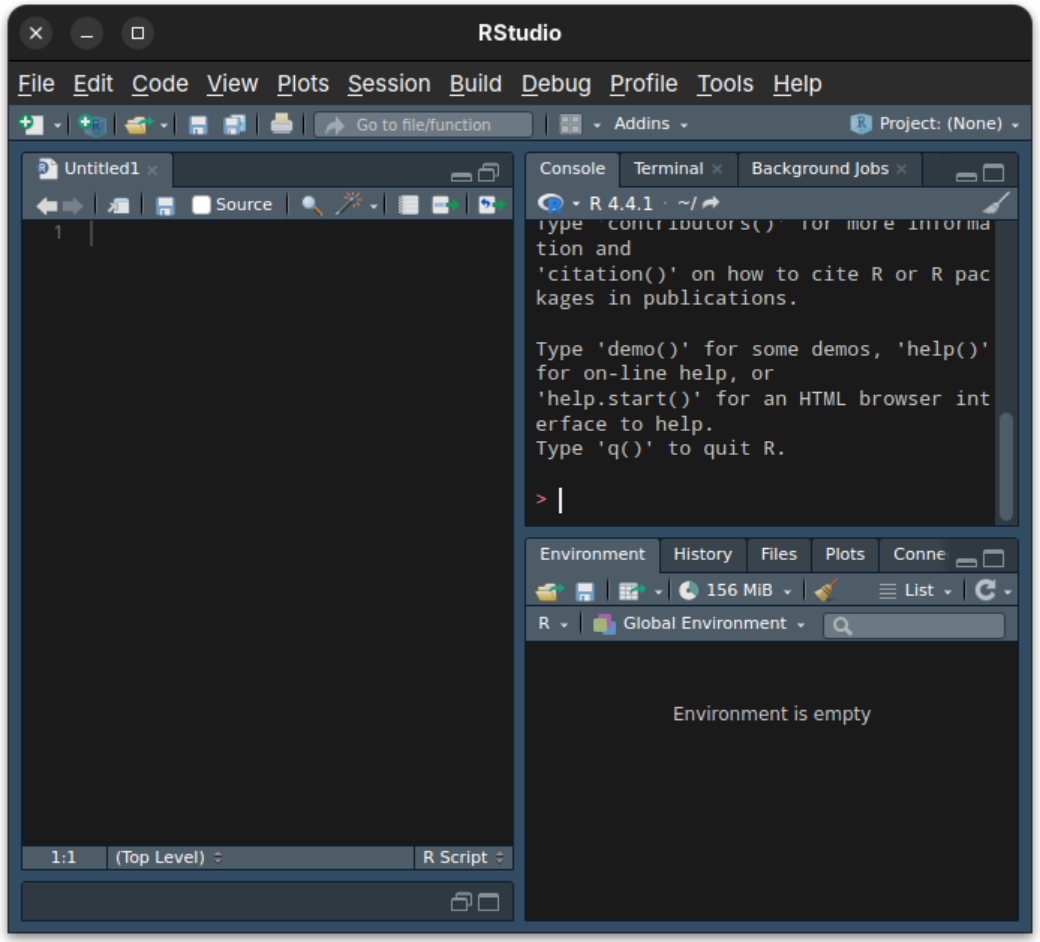

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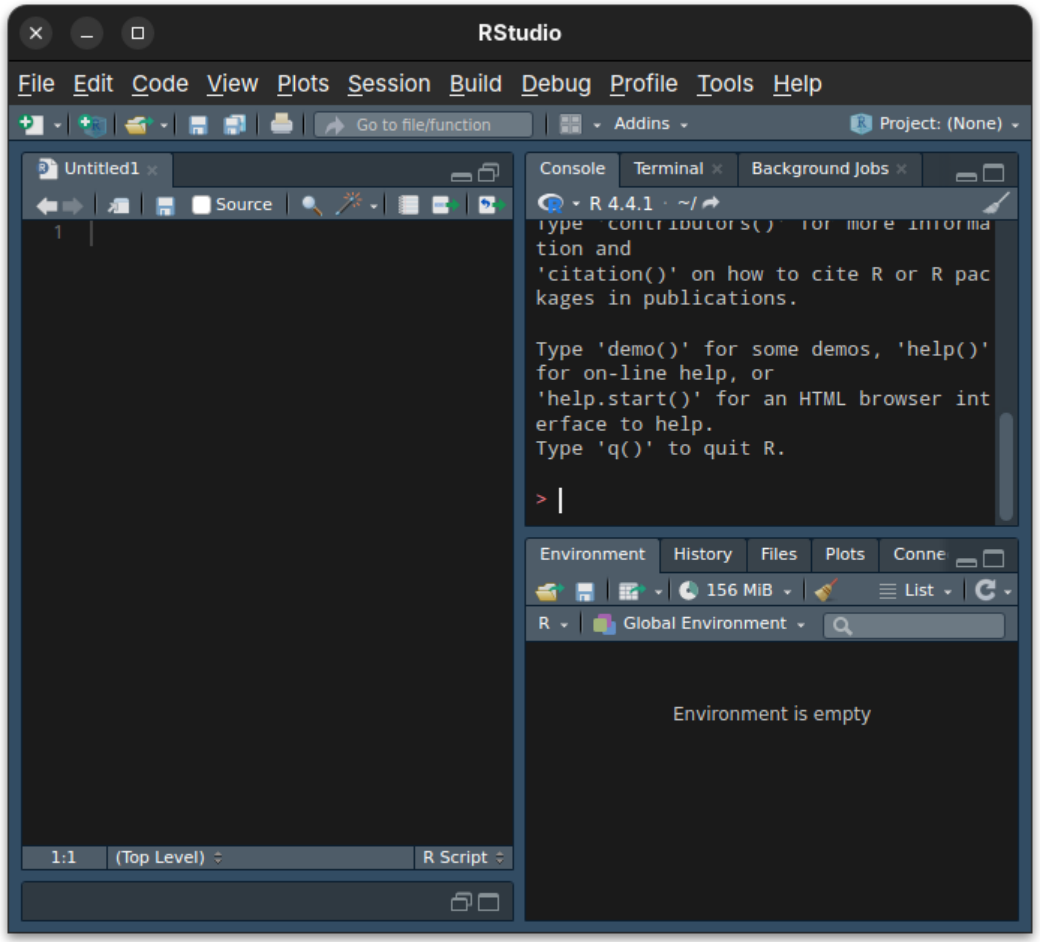

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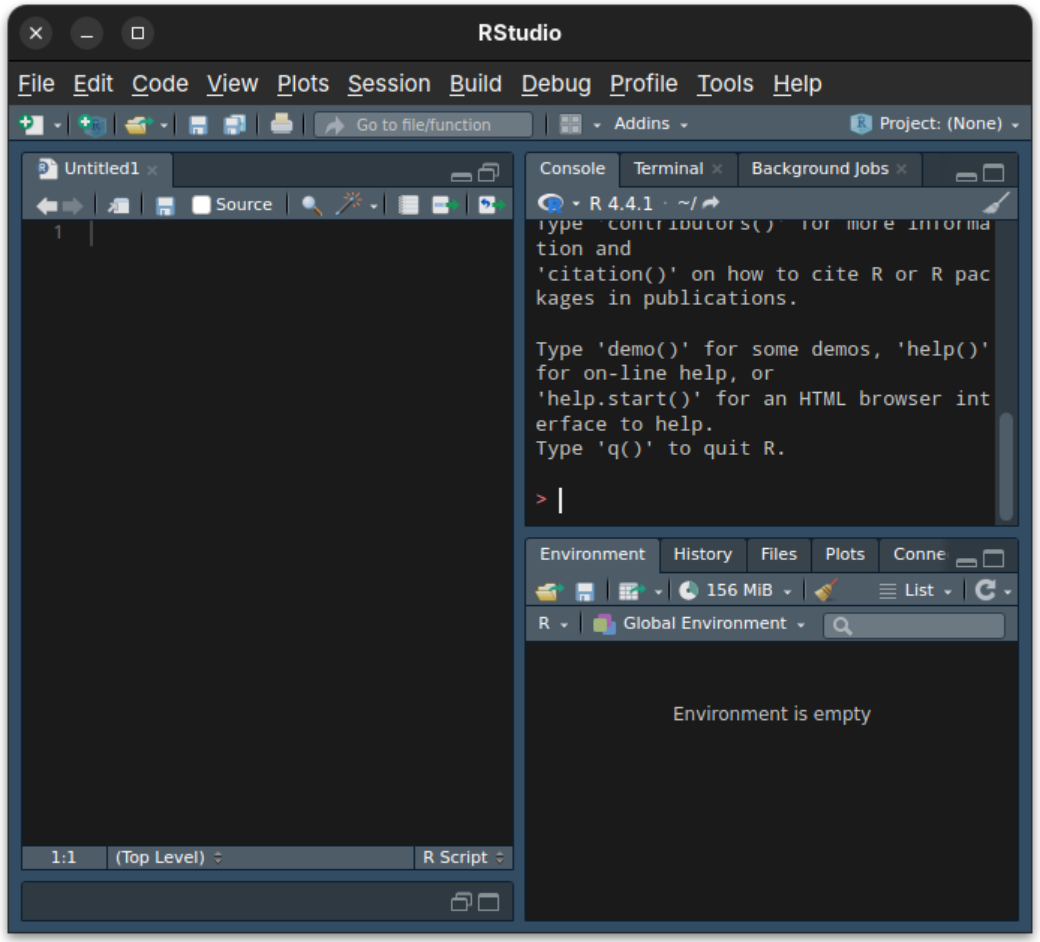

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- In the large pane on the left, we can open a **script**.

Script vs. Console

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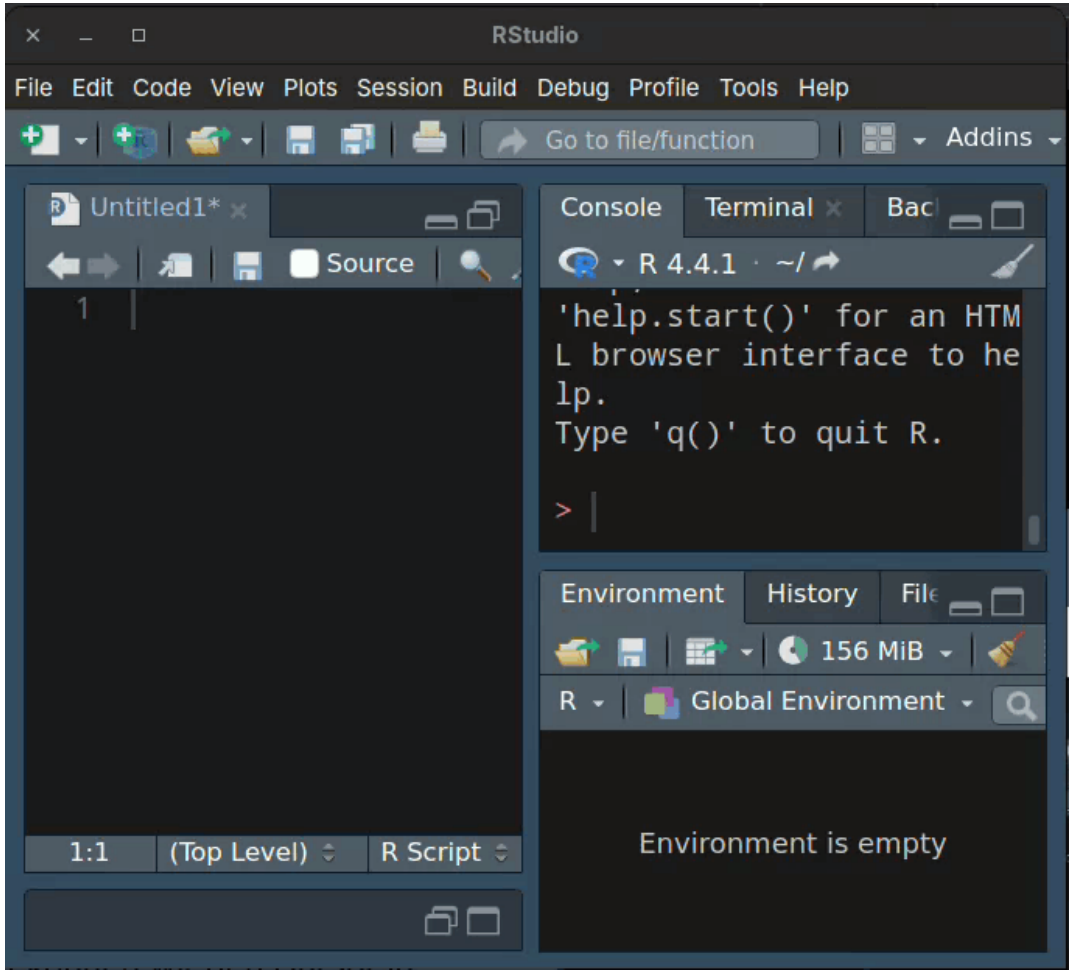

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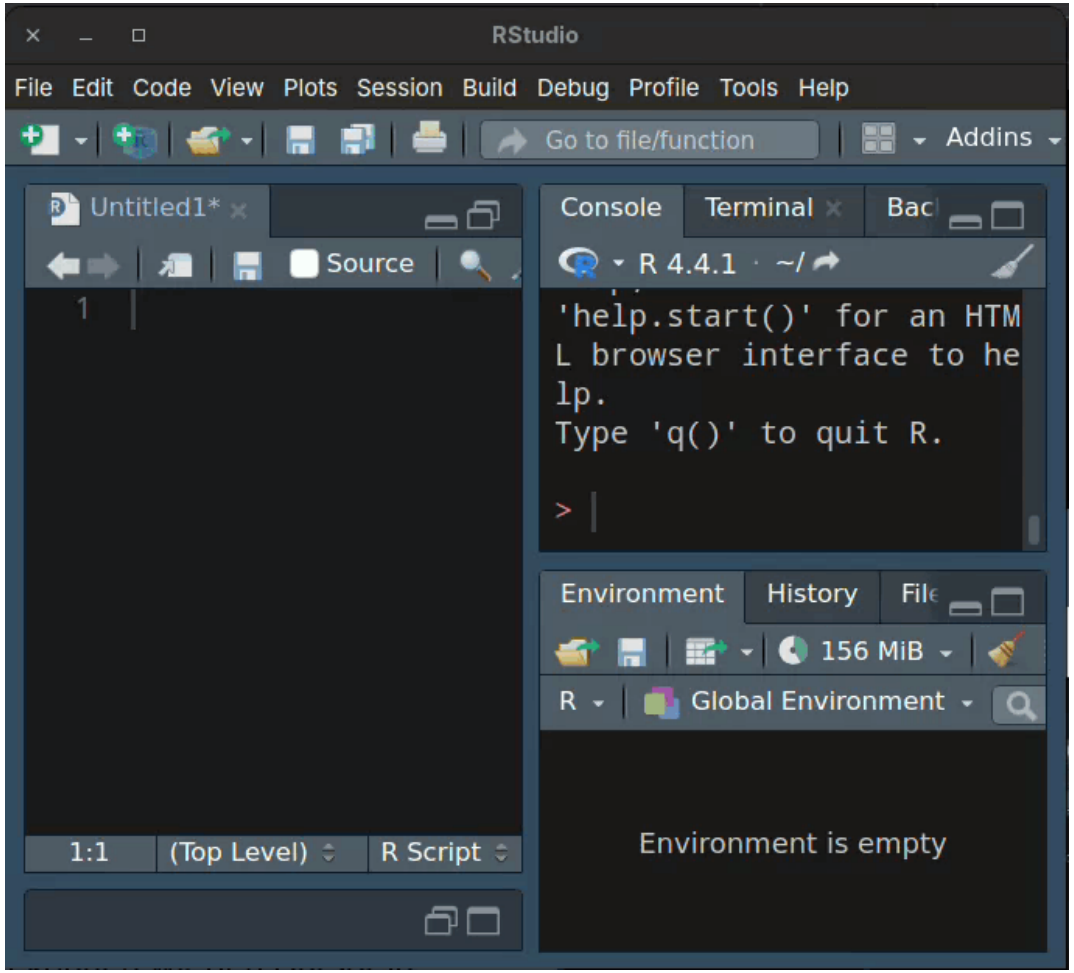

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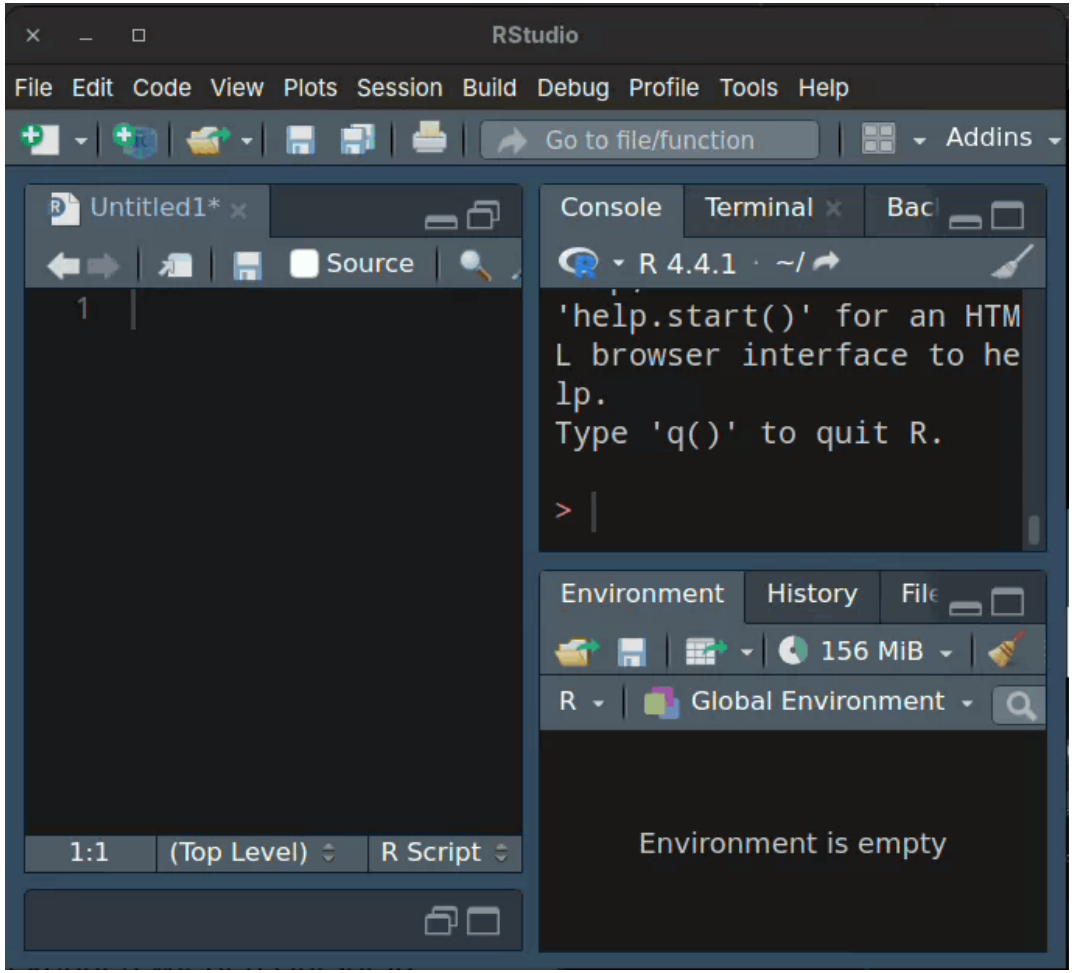

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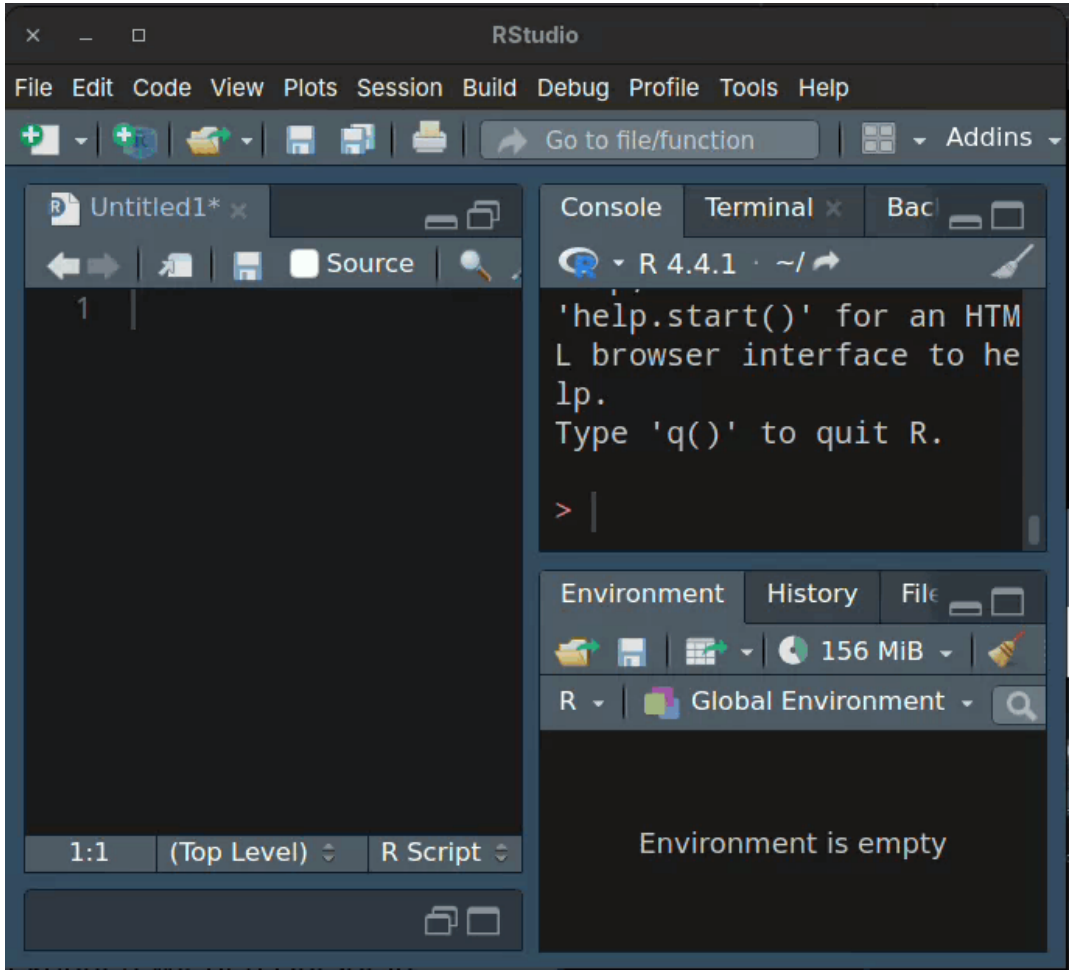

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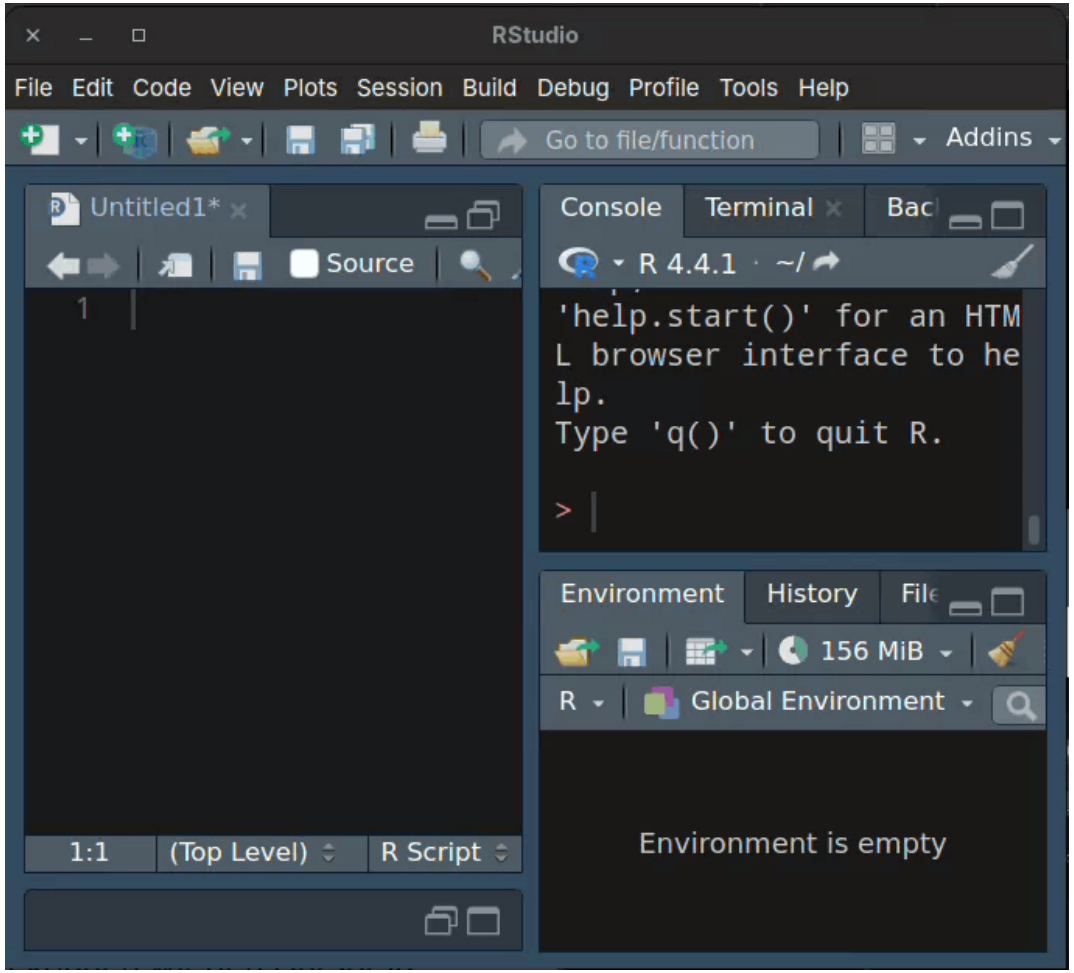
- Alternatively, we can highlight multiple lines and execute them, or execute the entire script at once.



Basics in R

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Base R comes with many useful **functions**, but sometimes we will need functions for specific econometric purposes that are not included in Base R. These are often included in packages created by developers and then (ideally) published in the Comprehensive R Archive Network (CRAN). We can install these additional packages as follows:

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The above functions **install**, **load**, and **update** packages (Note: `install.packages()` requires quotation marks). We can access a function's documentation with a `?`:

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Functions can be recognized by the parentheses. They may or may not contain arguments.

Variables and Vectors

Variables and Vectors

With `<-` (or `=`), we can **assign** a value to a variable name.

This can either be a scalar, a vector, or something else (more on that later).

We can print a variable with `print()`. It is also sufficient to just write the variable name (without `print()`).

The code on the right is **interactive** and can be modified and executed.

R Code | Start Over | Run Code

```r
# Lines starting with # are comments
# and are ignored.

my_scalar <- 12

my_vector <- c(5, 2, 13, 15)

my_character_vector <- c("Vienna", "Linz", "Graz")

# Works without print()
print(my_vector)

print(my_character_vector)

```

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1 + 3

# We can also save the result:
my_sum <- 4 + 12

c(5, 2) + c(7, 3)

2^2
sqrt(2) # Square root
log(100) # Natural logarithm
log(100, base = 10)

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We can also define **matrices** and perform calculations with them.

R Code | Start Over | Run Code

```r
my_matrix <- cbind(c(1, 2, 3), c(
2 * my_matrix
```

R Code | Start Over | Run Code

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Of course, R is primarily a **statistical programming language**. So let us simulate 100 dice rolls:

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#from the values 1 to 6
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Execute this code so we can proceed.

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We can also calculate other measures:

R Code | Start Over | Run Code

```r
summary(throws)
```

Other functions include `median()`, `min()`, `max()`, `length()`, `var()`, `sd()`, `sum()`, ...

Data

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Some datasets are already available in R, which is especially convenient for practice purposes. One example is `mtcars`. We can view the first rows using `head()`.

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When reading data, e.g., as CSV, we must assign it to a name, e.g., through `my_data <- read.csv("data.csv")`. Incidentally, we can export data in a similar way: `write.csv(my_data, "my_data.csv")`.

Dataframes

Dataframes

The structure in which data is stored is called a **dataframe**. The rows of a dataframe correspond to individual observations, and the columns correspond to variables. With `View()`, we can view the dataset in a separate window. We can also find out, e.g., the number of columns and rows:

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Using square brackets, we can access specific rows and columns. `mtcars[1,]` is the first row of `mtcars`, `mtcars[,1]` is the first column. We can also access individual variables using the following notation: `mtcars$mpg`.

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What happens when we execute this code?

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We can also use `TRUE` and `FALSE` to filter values:

R Code | Start Over | Run Code

```r
test_vector <- c(1, 2, 3)     # Define vector
test_vector > 1               # Check: What is >1?
test_vector[test_vector > 1] # Filter: Only elements >1 (so only TRUE)
```

Further Operations

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We can also combine multiple functions:

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```
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2 # How must this code be changed to calculate
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Or draw graphs:

R Code | Start Over | Run Code

```
1 hist(mtcars$mpg, breaks = 20)  # Histogram wi
2 # Try creating a boxplot with boxplot()!
```

Scatterplots

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Often, we want to draw a **scatterplot** to show the relationship between two variables.

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1 plot(mtcars$hp, mtcars$mpg)
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How strongly are `mpg` and `hp` correlated?

R Code | Start Over | Run Code

```
1 cor(mtcars$hp, mtcars$mpg)
```

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We denote matrices with bold uppercase letters ($\boldsymbol{X}$) and (column) vectors with bold lowercase letters ($\boldsymbol{x}$):

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & \cdots & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

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Bold is not strictly necessary. If we want to clarify in handwriting that something is a matrix or vector, we can instead underline them: $\underline{X}$ or $\underline{x}$.

Dimensions and Transposition

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On the previous slide, $\boldsymbol{X}$ had **dimensions** $n \times k$, and $\boldsymbol{x}$ had dimension n. $\boldsymbol{x}$ was written as a column vector, but we can also write $\boldsymbol{x}$ as a row vector. To do this, we must **transpose** the vector: Simply put, rows become columns, and columns become rows. We denote the transposition with a small apostrophe (or optionally a superscript T):

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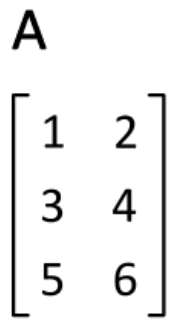
We can also **transpose matrices**. A matrix that previously had dimensions $n \times k$ will have dimensions $k \times n$ after transposition.

If we transpose a transposed matrix, we get:

$$(\boldsymbol{X}')' = \boldsymbol{X}$$

Furthermore, $(\boldsymbol{XZ})' = \boldsymbol{Z}'\boldsymbol{X}'$.



Animation from Wikipedia on matrix transposition.

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- An $n \times k$ matrix whose all elements are 0 is called a **zero matrix** and is denoted by $\mathbf{0}_{n \times k}$.

Rank

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The **rank** of a matrix is defined as the *dimension of the vector space spanned by the columns of a matrix* and is denoted by $\text{rank}(\boldsymbol{X})$. Simply put, the rank of a matrix corresponds to the **number of its linearly independent columns**. A column is linearly independent of the others if it cannot be expressed as a linear combination of them (i.e., as a sum of multiples of the other columns). Consider the following matrix:

Rank

The **rank** of a matrix is defined as the *dimension of the vector space spanned by the columns of a matrix* and is denoted by $\text{rank}(\boldsymbol{X})$. Simply put, the rank of a matrix corresponds to the **number of its linearly independent columns**. A column is linearly independent of the others if it cannot be expressed as a linear combination of them (i.e., as a sum of multiples of the other columns). Consider the following matrix:

$$\boldsymbol{X} = \begin{pmatrix} 12 & 2 & 10 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \\ 8 & 6 & 2 \end{pmatrix}$$

This matrix has rank 2. It has three columns, but the third column is a linear combination of the first two: $x_{i3} = x_{i1} + (-1) \cdot x_{i2}$.

If a matrix has the maximum possible rank for its dimensions, it is said to have **full rank**.

Matrix Addition and Scalar Multiplication

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Matrix addition occurs element by element:

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} \end{pmatrix} + \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1k} \\ z_{21} & z_{22} & \cdots & z_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nk} \end{pmatrix} = \begin{pmatrix} x_{11} + z_{11} & x_{12} + z_{12} & \cdots & x_{1k} + z_{1k} \\ x_{21} + z_{21} & x_{22} + z_{22} & \cdots & x_{2k} + z_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} + z_{n1} & x_{n2} + z_{n2} & \cdots & x_{nk} + z_{nk} \end{pmatrix}$$

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Multiplication of a matrix by a scalar also occurs element by element:

$$\alpha \cdot \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} \end{pmatrix} = \begin{pmatrix} \alpha x_{11} & \alpha x_{12} & \cdots & \alpha x_{1k} \\ \alpha x_{21} & \alpha x_{22} & \cdots & \alpha x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha x_{n1} & \alpha x_{n2} & \cdots & \alpha x_{nk} \end{pmatrix}$$

Multiplication of Two Matrices

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The **multiplication of two matrices** is somewhat more complex. Let $\boldsymbol{X}$ be a 2×3 matrix and $\boldsymbol{Z}$ be a 3×2 matrix. Then we can multiply the matrices as follows:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \cdot \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{pmatrix} = \begin{pmatrix} x_{11}z_{11} + x_{12}z_{21} & x_{11}z_{12} + x_{12}z_{22} & x_{11}z_{13} + x_{12}z_{23} \\ x_{21}z_{11} + x_{22}z_{21} & x_{21}z_{12} + x_{22}z_{22} & x_{21}z_{13} + x_{22}z_{23} \\ x_{31}z_{11} + x_{32}z_{21} & x_{31}z_{12} + x_{32}z_{22} & x_{31}z_{13} + x_{32}z_{23} \end{pmatrix}$$

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The following table helps visualize the process:

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It is easy to see that matrices can **only** be multiplied **if the number of columns in the left matrix** equals the number of **rows in the right matrix**.

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Installing R

Introduction to R with RStudio

Matrices and Vectors

Random Variables

Analysis of a Random Variable

Analysis of Two Random Variables

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We denote all possible outcomes with the corresponding lowercase letter:

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Discrete Random Variables

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A **discrete random variable** is a random variable that can have only a finite or countably infinite number of possible outcomes. If the variable is called X, we denote the outcomes as x_i and the corresponding probabilities as p_i. Note that the sum of all probabilities $\sum_i p_i$ must equal 1.

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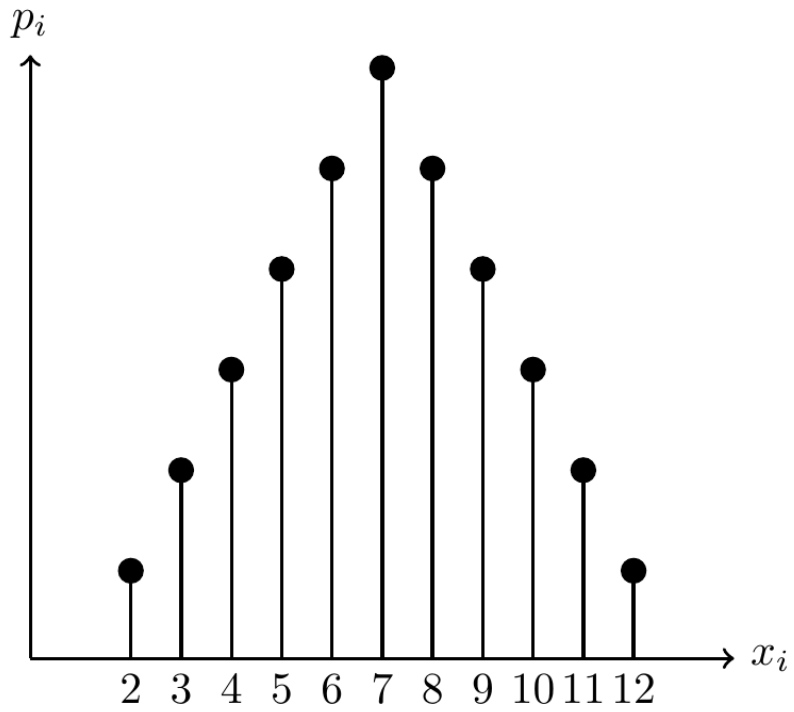
An example of a discrete random variable would be the roll of two dice. The possible outcomes are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and the corresponding probabilities are $\{\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}\}$. To the right, the **probability mass function** (PMF) is illustrated:



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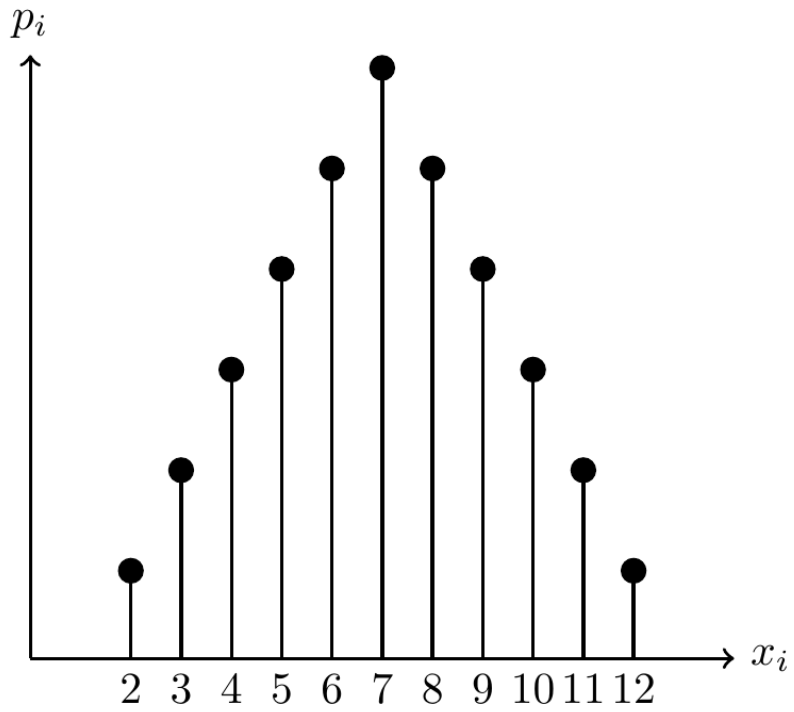

A **Bernoulli variable** is a discrete random variable that can only take on two outcomes, such as a coin toss.

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What we can do, however, is draw a **probability density function** (PDF). It tells us the probability that the outcome falls within a certain interval. The total area under the PDF is equal to 1.

Continuous Random Variables (2)

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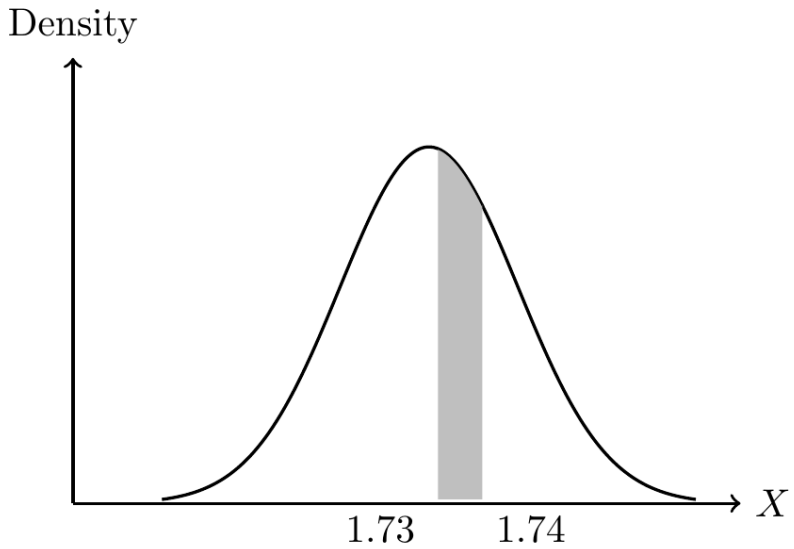
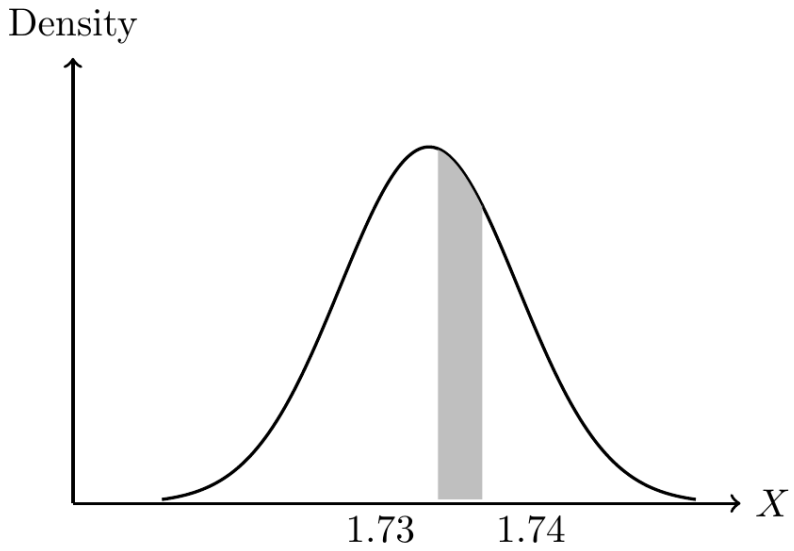
An example of such a variable would be the **height of a person**. It would make no sense to ask for the probability that a person is exactly 1.734681092536 meters tall. This probability is zero. But we can look at the PDF and determine how likely it is that the person's height lies between 1.73 and 1.74 meters:



Continuous Random Variables (2)

An example of such a variable would be the **height of a person**. It would make no sense to ask for the probability that a person is exactly 1.734681092536 meters tall. This probability is zero. But we can look at the PDF and determine how likely it is that the person's height lies between 1.73 and 1.74 meters:



Whether an infinite set of numbers is **countable** or **uncountable** can be intuitively answered. All **natural** numbers $\mathbb{N}$ are countably infinite. We can clearly specify a way to count them (start at 0, then 1, then 2, then 3, ...), we just don't know where and when the path *ends*. After all, it is still infinitely long. All **real** numbers $\mathbb{R}$, however, are uncountably infinite. We cannot define a unique way to include all numbers. Suppose we start at 0, then 0.001 what about all the numbers in between? And all the numbers between those numbers? There is no way to count them all.

Cumulative Distribution Function

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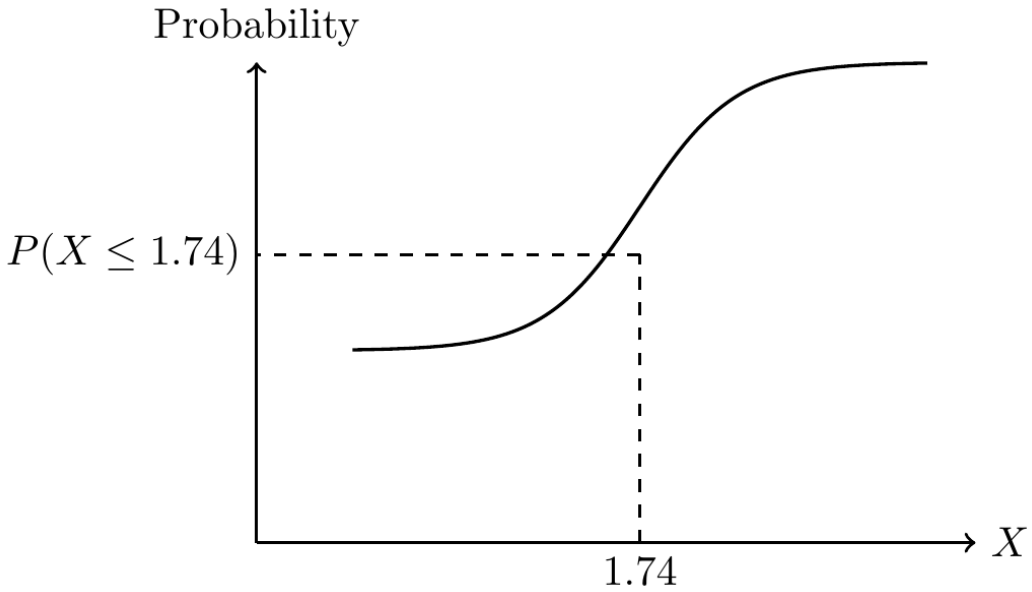
In addition to the **probability density function**, we can draw the **cumulative distribution function** (CDF). It represents the probability that the outcome is less than or equal to a certain value. The function is strictly monotonically increasing:



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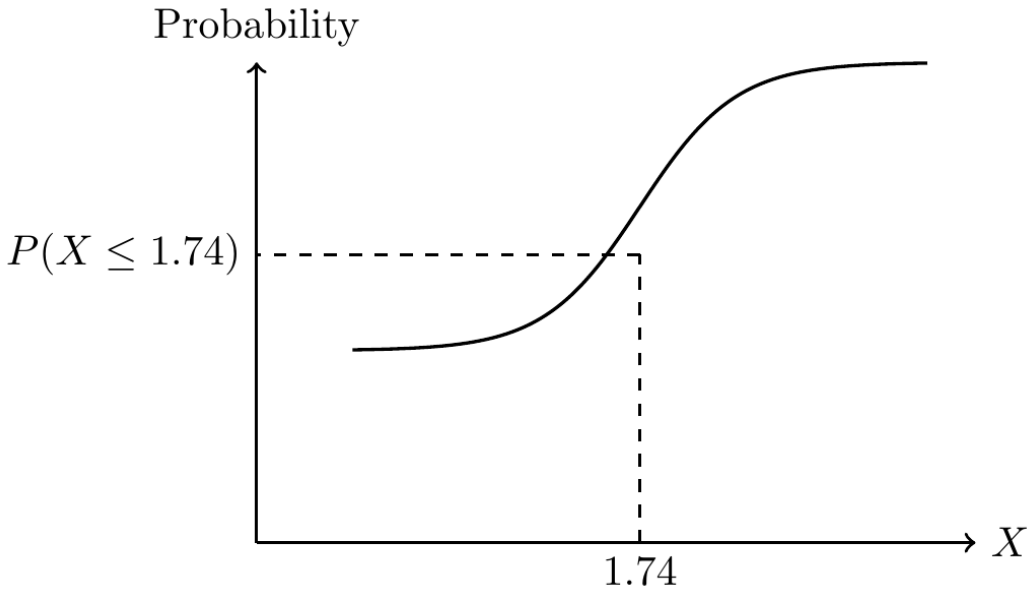
The dashed line shows how to read the plot: The value of the density function at $X = 1.74$ represents the probability that a randomly selected person is shorter than or exactly 1.74 meters tall.



Introduction to R with RStudio

Matrices and Vectors

Random Variables

Analysis of a Random Variable

Analysis of Two Random Variables

Expected Value (1)

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Outcome	Probability
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

Expected Value (2)

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The **expected value** (or **expectation**) is a concept that allows us to simply analyze what value we can *expect* when rolling the die. We calculate it as the weighted arithmetic mean of the outcomes, with their respective probabilities as weights. We denote the expected value with a capital E:

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The expected value of a fair die is 3.5. If we draw more and more outcomes from this distribution, i.e., roll the die many times, the average of all rolls will approach the expected value more and more. As long as we work with discrete variables, all of this is relatively simple to interpret. For continuous variables, it becomes more challenging, but the general intuition remains the same.

Rules for Expected Values

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  $\mathrm{E}\left(\sum_{i=1}^{n} c_i X_i\right) = \sum_{i=1}^{n} c_i \mathrm{E}(X_i)$
- For two *independent* random variables X, Y: $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$

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Often, the expected value alone is not sufficient to analyze a distribution. Imagine you own a company that manufactures screws. You have two machines producing them. You advertise that your screws are all 35 millimeters long, but in reality, the length of the screws is randomly distributed: The expected value of the screw length is 35 mm for both machines. However, machine A mostly produces screws very close to the desired length, while machine B sometimes produces screws as short as 33 mm or as long as 37 mm. What is the difference between these two machines with identical expected values?

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$$\mathrm{Var}(X) \equiv \mathrm{E}\left((X-\mu)^2\right),$$

where $\mu = \mathrm{E}(X)$.

Variance (2)

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It is evident that the variance of any constant is zero. Additionally, the following rule applies for a random variable X and constants a, b:

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X) + \mathrm{Var}(b) = a^2\mathrm{Var}(X)$$

The standard deviation, denoted as $\mathrm{sd}(X)$, is simply the square root of the variance.

Analysis of Two Random Variables

Joint Probability Distribution

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This function simply specifies the probability for each combination of X and Y. If X and Y are independent, then:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y),$$

where $f(x)$ and $f(y)$ are the probability functions for X and Y, respectively. Two random variables are **independent** if the outcome of X does not affect the probabilities of the possible outcomes of Y.

Conditional Distribution

Conditional Distribution

Another important concept is the **conditional distribution**. The conditional probability density function describes how the outcome of X affects that of Y:

$$f_{Y|X}(y|x) = P(Y = y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)}, \text{ for all } f_X(x) > 0$$

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If X and Y are independent, the outcome of X does not affect Y, and thus $f_{Y|X}(y|x) = f_Y(y)$.

Covariance

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Covariance is similar to a "two-variable version" of variance. It allows us to analyze two distributions together. It is defined as follows and denoted as $\operatorname{Cov}(X, Y)$:

$$\operatorname{Cov}(X, Y) \equiv \operatorname{E}\left((X - \mu_X)(Y - \mu_Y)\right),$$

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The sign of the covariance can be intuitively interpreted. If the covariance is positive, we expect Y to be above its mean when X is as well. If the covariance is negative, we expect Y to be below its mean when X is above its mean. Simply put, a positive covariance indicates that two variables are positively associated, and vice versa. A covariance of 0 means that there is no relationship. If X and Y are independent, the covariance is always 0.

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An association in this sense does not necessarily imply causation, but more on that in the course :)

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For constants a, b, c, d:

$$\operatorname{Cov}(aX + b, cY + d) = a \cdot c \cdot \operatorname{Cov}(X, Y)$$

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- The **law of iterated expectations**: $\mathrm{E}\left(\mathrm{E}(Y|X)\right) = \mathrm{E}(Y)$