Advanced Macroeconomics II – Assignment 1

Max Heinze (h11742049@wu.ac.at)   Tim Koenders (h12215486@wu.ac.at)

April 25, 2023

Contents

1 Business Cycle Stylized Facts **2**

2 A Real Business Cycle Model With Energy Price Shocks **7**

2.a Forming the System of Equations 7

2.b Interpretation 9

3 Understanding Impulse Responses and Model Simulation **11**

3.a Computing the Steady State Values 11

3.b Solving the Model in Dynare 11

3.c Impulse Responses I 13

3.d Impulse Responses II 15

3.e Log-Linearization 16

3.f Solving the Linear System of Equations 21

The executable code that was used in compiling the assignment is available on GitHub at https://github.com/maxmheinze/macro2.

We hereby declare that the answers to the given assignment are entirely our own, resulting from our own work effort only. Our team members contributed to the answers of the assignment in the following proportions:

Max Heinze:	50%
Tim Koenders:	50%

Signatures:

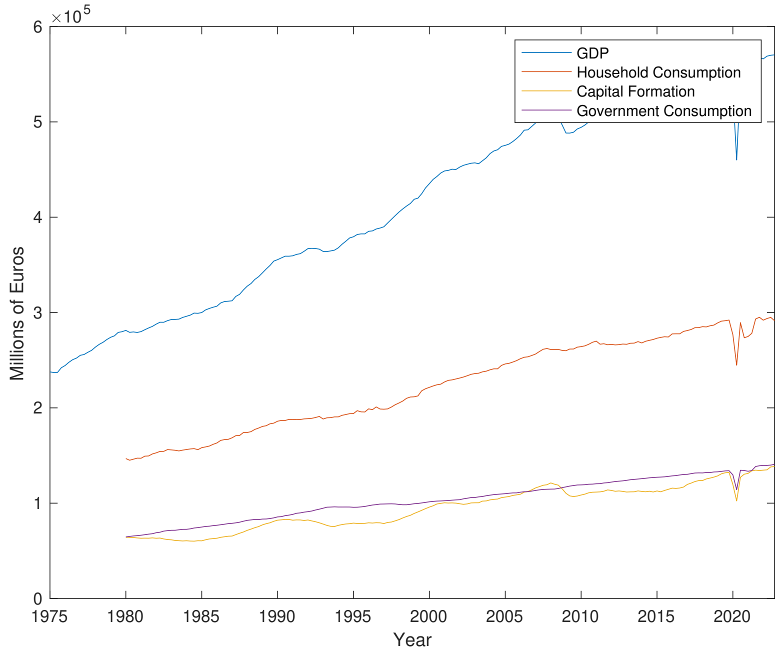
Max Heinze   Tim Koenders

1 Business Cycle Stylized Facts

Question 1

We chose to analyze data for France. The Matlab code that was used to generate the answers to this part of the exercise, as well as all other code for this assignment, is both attached to the submission and can be accessed on GitHub. The raw data is also attached.

The following are the untransformed time series:

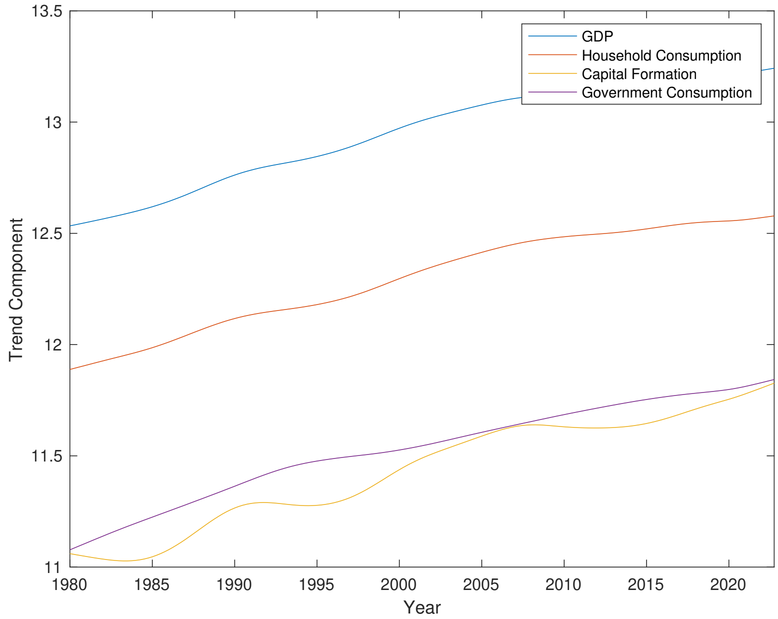


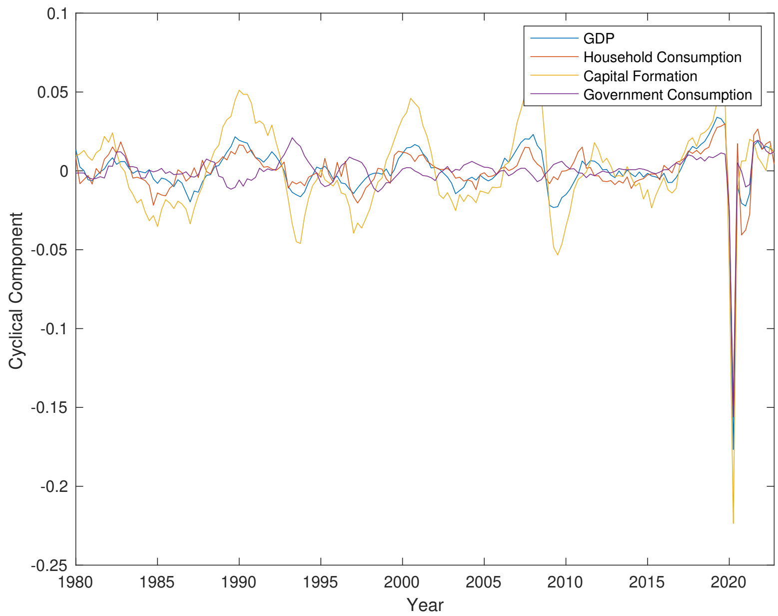
The sample means are:

$$
\begin{aligned}
C/Y &: 0.5203, \\
I/Y &: 0.2201, \\
G/Y &: 0.2418.
\end{aligned}
$$

The time series plot for GDP, household consumption, government expenditure and investment shows the strong correlation between those time series. Consumption, investment, and government expenditure make up for about 52, 22 and 24 percent of GDP, respectively.

Next, we obtain the stationary (cyclical) component using the HP filter and plot the first the trend and then the cyclical components:





The trend component of GDP, consumption, investment, and government expenditure is obtained using the HP filter with a smoothing parameter of 1600. The trend component is much smoother than the original time series and captures the long-term trends in the data. The filtered cyclical component is obtained by subtracting the trend from the original series. It represents the short-term variations in the series that are not explained by the trend. Positive values of the cyclical component indicate that the series is above its trend, while negative values indicate that the series is below its trend. From the curves, it appears that investments is the most volatile time series among the four.

```
Standard deviations of cyclical components:
Y_cycle: 0.0177
C_cycle: 0.0158
I_cycle: 0.0302
G_cycle: 0.0137
```

```
Relative Standard Deviations:
Y_cycle: 100.0000%
C_cycle: 88.8684%
I_cycle: 170.5497%
G_cycle: 77.2159%


Contemporaneous Output Correlations of Cyclical Components
--------------------------------------------------------
     Y        C        I        G
1.0000   0.8959   0.8824   0.7034
0.8959   1.0000   0.7487   0.7244
0.8824   0.7487   1.0000   0.4494
0.7034   0.7244   0.4494   1.0000


  Std_Y      Std_C       Std_I       Std_G     RSD_Y    RSD_C    RSD_I    RSD_G

--------    -------    --------    --------    -----    ------   ------   ------
0.017723    0.01575    0.030227    0.013685     100     88.868   170.55   77.216

Corr_Y_Y    Corr_Y_C    Corr_Y_I    Corr_Y_G    Corr_C_Y    Corr_C_C    Corr_C_I

--------    --------    --------    --------    --------    --------    --------
    1        0.89589     0.88243     0.70337     0.89589        1        0.74868

Corr_C_G    Corr_I_Y    Corr_I_C    Corr_I_I    Corr_I_G    Corr_G_Y    Corr_G_C

--------    --------    --------    --------    --------    --------    --------
  0.7244     0.88243     0.74868        1        0.44941     0.70337      0.7244

Corr_G_I    Corr_G_G

--------    --------
 0.44941        1
```

The first section of the table presents the standard deviations of the cyclical components of the four variables. It shows that investment (I) has the largest standard deviation (0.0302), followed by output (Y) and consumption (C), while government spending (G) has the smallest standard deviation (0.0137). The second section shows the relative standard deviations of the cyclical components (RSDs) of the four variables. The RSD is a measure of the relative volatility of a variable's cyclical component compared to the cyclical component of output. Therefore, in the given data, the RSD of the cyclical component of consumption (C_cycle) is 88.868%, indicating that its volatility is lower than that of output's cyclical component. Similarly, the RSD of the cyclical component of investment (I_cycle) is 170.55%, indicating that its volatility is higher than that of output's cyclical component.

We then split the data into two parts, Subsample 1 until the end of 2007 and Subsample 2 thereafter.

First, we report stylized facts for the first subsample:

```
Standard deviations of cyclical components for subsample 1:
Y_cycle: 0.0094
C_cycle: 0.0086
I_cycle: 0.0248
G_cycle: 0.0062


Relative Standard Deviations for subsample 1:
Y_cycle: 100.0000%
C_cycle: 91.4456%
I_cycle: 264.5138%
G_cycle: 65.8220%
```

```
Contemporaneous Output Correlations of Cyclical Components for subsample 1
-----------------------------------------------------------------------
      Y         C         I         G
 1.0000    0.7462    0.8996   -0.3419
 0.7462    1.0000    0.7859   -0.2044
 0.8996    0.7859    1.0000   -0.3618
-0.3419   -0.2044   -0.3618    1.0000


  Std_Y         Std_C         Std_I        Std_G       RSD_Y     RSD_C     RSD_I     RSD_G

---------     ---------     --------     ---------     -----     ------    ------    ------
0.0093653     0.0085641     0.024772     0.0061644      100      91.446    264.51    65.822

Corr_Y_Y    Corr_Y_C    Corr_Y_I    Corr_Y_G    Corr_C_Y    Corr_C_C    Corr_C_I

--------    --------    --------    --------    --------    --------    --------
    1        0.74623     0.89957    -0.34192     0.74623        1        0.78586

Corr_C_G    Corr_I_Y    Corr_I_C    Corr_I_I    Corr_I_G    Corr_G_Y    Corr_G_C

--------    --------    --------    --------    --------    --------    --------
 -0.20442    0.89957     0.78586        1       -0.36177    -0.34192    -0.20442

Corr_G_I    Corr_G_G

--------    --------
-0.36177        1
```

The results from subsample 1 again show that investments is the most volatile time series among the four. Furthermore, the correlation between output, consumption and investment is positive indicating the series move procyclical while government consumption appears to move countercyclical with output.

Second, we report stylized facts for the second subsample:

```
Standard deviations of cyclical components for subsample 2:
Y_cycle: 0.0273
C_cycle: 0.0241
I_cycle: 0.0386
G_cycle: 0.0217


Relative Standard Deviations for subsample 2:
Y_cycle: 100.0000%
C_cycle: 88.3432%
I_cycle: 141.5776%
G_cycle: 79.5763%


Contemporaneous Output Correlations of Cyclical Components for subsample 2
-----------------------------------------------------------------------
     Y         C         I         G
1.0000    0.9305    0.9263    0.8974
0.9305    1.0000    0.7723    0.9037
0.9263    0.7723    1.0000    0.7664
0.8974    0.9037    0.7664    1.0000
```

Std_Y	Std_C	Std_I	Std_G	RSD_Y	RSD_C	RSD_I	RSD_G
0.027282	0.024101	0.038625	0.02171	100	88.343	141.58	79.576

Corr_Y_Y	Corr_Y_C	Corr_Y_I	Corr_Y_G	Corr_C_Y	Corr_C_C	Corr_C_I
1	0.93053	0.92634	0.89742	0.93053	1	0.77229

Corr_C_G	Corr_I_Y	Corr_I_C	Corr_I_I	Corr_I_G	Corr_G_Y	Corr_G_C
0.90366	0.92634	0.77229	1	0.76641	0.89742	0.90366

Corr_G_I	Corr_G_G
0.76641	1

The results from subsample 2 again show that investments is the most volatile time series among the four. Similarly to subsample 1, the correlation between output, consumption and investment is positive, indicating the series move procyclical. However, government consumption now also moves procyclical with output.

2 A Real Business Cycle Model With Energy Price Shocks

2.a Forming the System of Equations

The benevolent social planner aims to maximize the following lifetime discounted expected utility function:

$$U = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \theta \log(1 - N_t), \tag{2.1}$$

subject to the following resource constraint:

$$C_t + K_{t+1} + P_t EN_t \leq A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t. \tag{2.2}$$

That is, the Lagrangian corresponding to this maximization problem equals the following:

$$\mathcal{L} = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} + \theta \log(1 - N_t) + \lambda_t \left(A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t - C_t - K_{t+1} - P_t EN_t \right) \right) \tag{2.3}$$

We can rewrite this Lagrangian the following way:

$$\mathcal{L} = \mathrm{E}_t \left(\begin{array}{l} \beta^t \left(\frac{C_t^{1-\sigma}-1}{1-\sigma} + \theta \log(1-N_t) + \right. \\ \qquad \left. \lambda_t \left(A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t - C_t - K_{t+1} - P_t EN_t \right) \right) + \\ \beta^{t+1} \left(\frac{C_{t+1}^{1-\sigma}-1}{1-\sigma} + \theta \log(1-N_{t+1}) + \right. \\ \qquad \left. \lambda_{t+1} \left(A_{t+1} K_{t+1}^{\alpha} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)K_{t+1} - C_{t+1} - K_{t+2} - P_{t+1} EN_{t+1} \right) \right) + \\ \beta^{t+2} \left(\frac{C_{t+2}^{1-\sigma}-1}{1-\sigma} + \theta \log(1-N_{t+2}) + \right. \\ \qquad \left. \lambda_{t+2} \left(A_{t+2} K_{t+2}^{\alpha} N_{t+2}^{\gamma} EN_{t+2}^{1-\alpha-\gamma} + (1-\delta)K_{t+2} - C_{t+2} - K_{t+3} - P_{t+2} EN_{t+2} \right) \right) + \\ \qquad \qquad \vdots \end{array} \right) \tag{2.4}$$

We get the following FOCs:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_t} &= E_t\left(\beta^t \frac{1}{C_t^\sigma}\right) - \mathrm{E}_t\left(\beta^t \lambda_t\right) &(2.5)\\
&\Rightarrow C_t^{-\sigma} - \lambda_t \overset{!}{=} 0 &(2.6)\\
\frac{\partial \mathcal{L}}{\partial K_{t+1}} &= \mathrm{E}_t\left(-\beta^t \lambda_t\right) + \mathrm{E}_t\left(\beta^{t+1}\lambda_{t+1} A_{t+1}\alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma}\right) + \mathrm{E}_t\left(\beta^{t+1}\lambda_{t+1}(1-\delta)\right) &(2.7)\\
&\Rightarrow -\lambda_t + \beta \mathrm{E}_t\left(\lambda_{t+1}\left(A_{t+1}\alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)\right)\right) \overset{!}{=} 0 &(2.8)\\
\frac{\partial \mathcal{L}}{\partial N_t} &= \mathrm{E}_t\left(-\beta^t \theta \frac{1}{1-N_t}\right) + \mathrm{E}_t\left(\beta^t \lambda_t A_t K_t^\alpha \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma}\right) &(2.9)\\
&\Rightarrow -\theta \frac{1}{1-N_t} + \lambda_t A_t K_t^\alpha \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma} \overset{!}{=} 0 &(2.10)\\
\frac{\partial \mathcal{L}}{\partial EN_t} &= \mathrm{E}_t\left(\beta^t \lambda_t A_t K_t^\alpha N_t^\gamma (1-\alpha-\gamma) EN_t^{-\alpha-\gamma}\right) - \mathrm{E}_t\left(\beta^t P_t\right) &(2.11)\\
&\Rightarrow \lambda_t A_t K_t^\alpha N_t^\gamma (1-\alpha-\gamma) EN_t^{-\alpha-\gamma} - P_t \overset{!}{=} 0 &(2.12)\\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= \mathrm{E}_t\left(\beta^t A_t K_t^\alpha N_t^\gamma EN_t^{1-\alpha-\gamma} + (1-\delta)K_t - C_t - K_{t+1} - P_t EN_t\right) &(2.13)\\
&\Rightarrow A_t K_t^\alpha N_t^\gamma EN_t^{1-\alpha-\gamma} + (1-\delta)K_t - C_t - K_{t+1} - P_t EN_t \overset{!}{=} 0 &(2.14)
\end{aligned}
$$

From (2.6), we get:

$$C_t^{-\sigma} = \lambda_t. \tag{2.15}$$

From (2.8), we get:

$$\beta \mathrm{E}_t\left(\lambda_{t+1}\left(A_{t+1}\alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)\right)\right) = \lambda_t. \tag{2.16}$$

From (2.10), we get:

$$\lambda_t A_t K_t^\alpha \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma} = \theta \frac{1}{1-N_t}. \tag{2.17}$$

From (2.12), we get:

$$\lambda_t A_t K_t^\alpha N_t^\gamma (1-\alpha-\gamma) EN_t^{-\alpha-\gamma} = P_t. \tag{2.18}$$

And from (2.14), we get:

$$A_t K_t^\alpha N_t^\gamma EN_t^{1-\alpha-\gamma} + (1-\delta)K_t = C_t + K_{t+1} + P_t EN_t. \tag{2.19}$$

In addition, we have exogenously:

$$
\begin{aligned}
\log A_{t+1} &= \rho_A \log A_t + \varepsilon_{A,t+1} &(2.20)\\
\log P_{t+1} &= \rho_P \log P_t + \varepsilon_{P,t+1} &(2.21)
\end{aligned}
$$

Substituting out for the Lagrange multiplier λ_t, we arrive at a system of 8 equations forming a nonlinear expectational system of difference equations in Y_t, C_t, I_t, K_t, N_t, EN_t, A_t:

By combining Equations 2.15 and 2.16, we get

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(C_{t+1}^{-\sigma} \left(A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} E N_{t+1}^{1-\alpha-\gamma} + (1-\delta) \right) \right). \tag{2.22}$$

By combining Equations 2.15 and 2.17, we get

$$C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma-1} E N_t^{1-\alpha-\gamma} = \theta \frac{1}{1-N_t}. \tag{2.23}$$

Let us specificy the production function as

$$Y_t = A_t K_t^{\alpha} N_t^{\gamma} (EN)_t^{1-\alpha-\gamma}. \tag{2.24}$$

By combining Equations 2.15, 2.18 and 2.24, we get:

$$(1-\alpha-\gamma) Y_t = P_t (EN)_t. \tag{2.25}$$

We can define the capital law of motion as

$$K_{t+1} = (1-\delta) K_t + I_t. \tag{2.26}$$

Finally, we combine Equations 2.19 and 2.26 and define the output identity as,

$$Y_t = C_t + I_t + P_t (EN)_t \tag{2.27}$$

Together with the two autoregressive processes, we have thus defined a system of of 8 equations forming a nonlinear expectational system of difference equations in Y_t, C_t, I_t, K_t, N_t, EN_t, A_t, P_t:

$$\begin{aligned}
C_t^{-\sigma} &= \beta \mathrm{E}_t \left(C_{t+1}^{-\sigma} \left(A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} E N_{t+1}^{1-\alpha-\gamma} + (1-\delta) \right) \right) && (2.28) \\
C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma-1} E N_t^{1-\alpha-\gamma} &= \theta \frac{1}{1-N_t} && (2.29) \\
Y_t &= A_t K_t^{\alpha} N_t^{\gamma} (EN)^{1-\alpha-\gamma} && (2.30) \\
(1-\alpha-\gamma) Y_t &= P_t (EN)_t && (2.31) \\
K_{t+1} &= (1-\delta) K_t + I_t && (2.32) \\
Y_t &= C_t + I_t + P_t (EN)_t && (2.33) \\
\log A_{t+1} &= \rho_A \log A_t + \varepsilon_{A,t+1} && (2.34) \\
\log P_{t+1} &= \rho_P \log P_t + \varepsilon_{P,t+1} && (2.35)
\end{aligned}$$

2.b Interpretation

All the first order and equilibrium conditions have econonomic interpretations.

Equation 2.15 relates the Lagrange multiplier to the additional (marginal) utility from using an additional unit of utility. This describes the intratemporal optimality.

Equation 2.16 relatesthe Lagrange multiplier to the marginal utility of consuming an additional unit of a resource, which becomes available by relaxing the resource constraint by one unit. It suggests that the social planner can choose to consume the extra unit or increase tomorrow's capital stock. In the optimum, both options should provide the same expected and discounted additional utility. The RHS of the equation represents the value of consuming the additional unit today, and the LHS can be interpreted as the return to increasing the capital stock, which includes the additional production and the remaining investment after depreciation.

Equation 2.17 represents that the optimal level of labor input is where its marginal benefit is equal to its marginal cost. Optimality requires that the marginal disutility of working one more unit equals to marginal benefit of supplying one more unit work to the labour market.

Equation 2.18 represents that the optimal level of energy input is where its marginal benefit is equal to its marginal cost. Hence, the condition states that at the optimum, the energy price equals the marginal product of energy.

Equation 2.19 represents the resource constraint which states that the sum of consumption of resources, saved capital, and the cost of using energy cannot exceed the sum of total output the undepreciated capital available at time t.

Equation 2.28 represents intertemporal optimality condition, which states that the marginal utility of consumption in period t equals the expected discounted marginal utility of consumption in period t+1. The right-hand side of the equation represents the expected utility from consuming in period t+1, which depends on the expected level of consumption, the capital stock, and the energy input. The left-hand side of the equation represents the marginal utility of consumption in period t, which depends on the consumption level and the intertemporal discount factor.

Equation 2.29 refers to the labor supply equation, which states that the marginal disutility of labor equals the marginal benefit of labor. The left-hand side represents the marginal disutility of labor, which depends on the consumption level, the capital stock, and the energy input. The right-hand side represents the marginal benefit of labor, which depends on the wage rate, the productivity of labor, and the inverse of the labor supply elasticity.

Equation 2.30 refers to the production function, which shows the relationship between output and the inputs of capital, labor, and energy.

Equation 2.31 states that the marginal product of energy is equal to the energy price. Hence, in the optimum, the price for one unit of energy is equal to the return to increasing the energy stock.

Equation 2.32 is the capital law of motion which describes how the total amount of capital in an economy changes over time. It states that the amount of capital available in the next period is equal to the sum of two terms: the capital remaining from the current period after accounting for depreciation, and the investment made in the current period.

Equation 2.33 represents the output identity related output to the sum of consumption, investment and the cost of energy inputs.

Equation 2.33 and 2.34 are stochastic processes that describe the evolution of the log of total factor productivity (A) and the log of the energy price (P), respectively.

3 Understanding Impulse Responses and Model Simulation

3.a Computing the Steady State Values

Computing the steady state values for the given parameters, $\bar{\beta} = 0.99$, $\bar{\sigma} = 1$, $\bar{\theta} = 3.48$, $\bar{\alpha} = 0.3$, $\bar{\gamma} = 0.65$, $\bar{\delta} = 0.0024$, $\bar{A} = 1$ and $\bar{P} = 1$.

First we compute $\bar{K}/\bar{Y}, \bar{EN}/\bar{Y}, \bar{C}/\bar{Y}, \bar{N}$ to get $\bar{Y}$. Then we can get $\bar{K}$, $\bar{EN}$, $\bar{C}$ and $\bar{I}$ while $\bar{A}$=$\bar{P}$=1 is given.

$$\frac{\bar{K}}{\bar{Y}} = \frac{\alpha}{\left(\frac{1}{\beta} - (1-\delta)\right)} = \frac{0.3}{\left(\frac{1}{0.99} - (1-0.025)\right)} \approx 8.547$$

$$\frac{\bar{EN}}{\bar{Y}} = 1 - \alpha - \gamma = 1 - 0.3 - 0.65 = 0.05$$

$$\frac{\bar{C}}{\bar{Y}} = 1 - \delta\frac{\bar{K}}{\bar{Y}} - \frac{\bar{EN}}{\bar{Y}} = 1 - 0.025 \times 8.547 - 0.05 \approx 0.736$$

$$\bar{N} = \frac{1}{1 + \frac{\bar{C}}{\bar{Y}}\frac{\theta}{\gamma}} = \frac{1}{1 + 0.736\left(\frac{3.48}{0.65}\right)} \approx 0.202$$

$$\bar{Y} = \left(\frac{\bar{K}}{\bar{Y}}\right)^{\frac{\alpha}{\gamma}} \bar{N} \left(\frac{\bar{EN}}{\bar{Y}}\right)^{\frac{1-\alpha-\gamma}{\gamma}} = 8.547^{\frac{0.3}{0.65}} \times 0.202 \times 0.05^{\frac{0.05}{0.65}} \approx 0.432$$

$$\bar{K} = \frac{\bar{K}}{\bar{Y}}\bar{Y} = 8.514 \times 0.433 \approx 3.697$$

$$\bar{EN} = \frac{\bar{EN}}{\bar{Y}}\bar{Y} = 0.05 \times 0.433 \approx 0.022$$

$$\bar{C} = \frac{\bar{C}}{\bar{Y}}\bar{Y} = 0.736 \times 0.433 \approx 0.319$$

Then from Equation 2.33,

$$\bar{I} = \bar{Y} - \bar{C} - \bar{P}\bar{EN} = 0.433 - 0.319 - 0.022 \times 1 \approx 0.0924$$

3.b Solving the Model in Dynare

Running our code in Dynare, we get the following result. Please refer to the attachments to this submission or to GitHub for the code. The policy functions can be seen in the table titled `POLICY AND TRANSITION FUNCTIONS`.

```
    STEADY-STATE RESULTS:

y       0
c       0
```

```
inve   0
k      0
n      0
en     0
a      0
p      0

EIGENVALUES:
         Modulus             Real        Imaginary

             0.5              0.5                0
           0.941            0.941                0
            0.95             0.95                0
           1.073            1.073                0
       3.151e+16        3.151e+16                0
       3.769e+16        3.769e+16                0
       2.597e+20        2.597e+20                0


There are 4 eigenvalue(s) larger than 1 in modulus
for 4 forward-looking variable(s)

The rank condition is verified.


MODEL SUMMARY

  Number of variables:         8
  Number of stochastic shocks: 2
  Number of state variables:   3
  Number of jumpers:           4
  Number of static variables:  2


MATRIX OF COVARIANCE OF EXOGENOUS SHOCKS
Variables          ea        ep
ea           0.000000  0.000000
ep           0.000000  0.010000

POLICY AND TRANSITION FUNCTIONS
                               y             c          inve             k
k(-1)                   0.086822      0.506355     -1.358942      0.941026
p(-1)                  -0.052990     -0.004116     -0.221416     -0.005535
a(-1)                   1.650130      0.458910      5.755226      0.143881
ea                      1.736979      0.483063      6.058133      0.151453
ep                     -0.105979     -0.008231     -0.442831     -0.011071

                               n            en             a             p
k(-1)                  -0.334645      0.086822             0             0
p(-1)                  -0.038985     -0.552990             0      0.500000
a(-1)                   0.950190      1.650130      0.950000             0
ea                      1.000200      1.736979      1.000000             0
ep                     -0.077970     -1.105979             0      1.000000


MOMENTS OF SIMULATED VARIABLES
VARIABLE             MEAN      STD. DEV.        VARIANCE        SKEWNESS        KURTOSIS
y               -0.000115       0.012691        0.000161       -0.027683        0.147622
```

c	-0.000098	0.003975	0.000016	-0.240614	0.101260
inve	-0.000174	0.050940	0.002595	-0.044038	0.153587
k	-0.000177	0.007171	0.000051	-0.247445	0.100874
n	-0.000014	0.008960	0.000080	-0.052229	0.153557
en	-0.001053	0.130905	0.017136	-0.028979	0.149346
a	0.000000	0.000000	0.000000	NaN	NaN
p	0.000938	0.118227	0.013978	0.029175	0.149515

CORRELATION OF SIMULATED VARIABLES

VARIABLE	y	c	inve	k	n	en	a	p
y	1.0000	0.5024	0.9726	0.4480	0.9520	0.9991	NaN	-0.9989
c	0.5024	1.0000	0.2876	0.9981	0.2138	0.4647	NaN	-0.4606
inve	0.9726	0.2876	1.0000	0.2278	0.9971	0.9817	NaN	-0.9826
k	0.4480	0.9981	0.2278	1.0000	0.1529	0.4091	NaN	-0.4048
n	0.9520	0.2138	0.9971	0.1529	1.0000	0.9643	NaN	-0.9655
en	0.9991	0.4647	0.9817	0.4091	0.9643	1.0000	NaN	-1.0000
a	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
p	-0.9989	-0.4606	-0.9826	-0.4048	-0.9655	-1.0000	NaN	1.0000

AUTOCORRELATION OF SIMULATED VARIABLES

VARIABLE	1	2	3	4	5
y	0.5592	0.2995	0.1435	0.0714	0.0786
c	0.9764	0.9410	0.9013	0.8622	0.8263
inve	0.5025	0.2124	0.0404	-0.0362	-0.0231
k	0.9837	0.9522	0.9145	0.8760	0.8393
n	0.4948	0.2006	0.0265	-0.0508	-0.0369
en	0.5453	0.2783	0.1183	0.0451	0.0538
a	NaN	NaN	NaN	NaN	NaN
p	0.5439	0.2761	0.1158	0.0425	0.0513

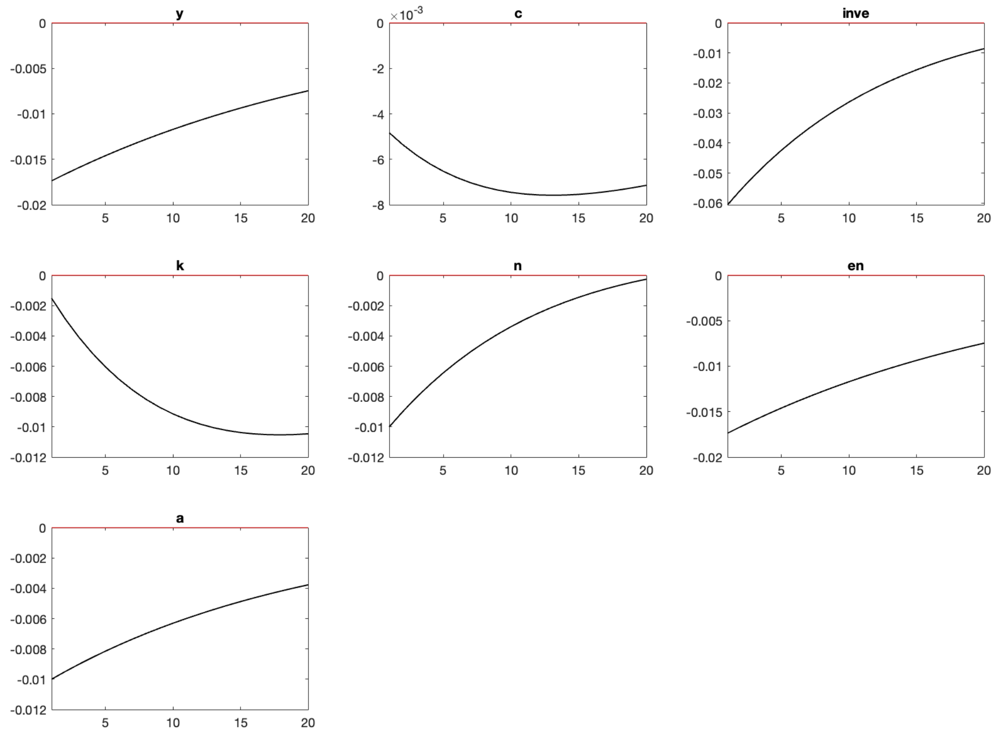
VARIANCE DECOMPOSITION SIMULATING ONE SHOCK AT A TIME (in percent)

	ea	ep	Tot. lin. contr.
y	0.00	100.05	100.05
c	0.00	100.05	100.05
inve	0.00	100.05	100.05
k	0.00	100.05	100.05
n	0.00	100.05	100.05
en	0.00	100.05	100.05
a	NaN	NaN	NaN
p	0.00	100.05	100.05

Note: numbers do not add up to 100 due to non-zero correlation of simulated shocks in small samples

3.c Impulse Responses I

We get the following impulse response functions:



Impulse Response Function of Y. At the point of impact, a 1 percent decrease in A leads to a more than proportional decrease in Y. If K is the only factor of production and predetermined, a 1 percent decrease in A directly translates to a 1 percent decrease in Y. However, when N and EN, two non-predetermined inputs, are included, Y decreases more than proportionally as the productivity of N and EN is reduced.

Impulse Response Function of C. At the point of impact, consumption (C) decreases due to the decrease in Y. Shortly after the shock, the social planner chooses to reduce consumption even further to maintain a constant level of investment and satisfy the resource constraint. This leads to a further decrease in consumption beyond the initial impact, as the planner saves more to rebuild the capital stock and increase productivity. Eventually, as the economy builds up its capital stock, the productivity gains from the increase in capital offset the negative productivity shock caused by the decrease in A, allowing consumption to return to its pre-shock level.

Impulse Response Function of K. The IRF of capital (K) shows that the effect on the capital stock at the point of impact is negligible since capital is predetermined. Over time, however, the capital stock starts to decline as agents choose to dissave and allocate some of the burden of the lower productivity to the future. The reduction in the capital stock continues until about period 15, after which it starts to flatten out. This adjustment process takes time because agents know that the decrease in productivity is persistent, and they have an incentive to reduce the capital stock so that more units of capital can be used in production during periods of relatively high productivity. Therefore, the IRF of K illustrates that the decrease in productivity has a significant and persistent negative effect on capital accumulation, which is an important factor in determining the long-term growth potential of the economy.

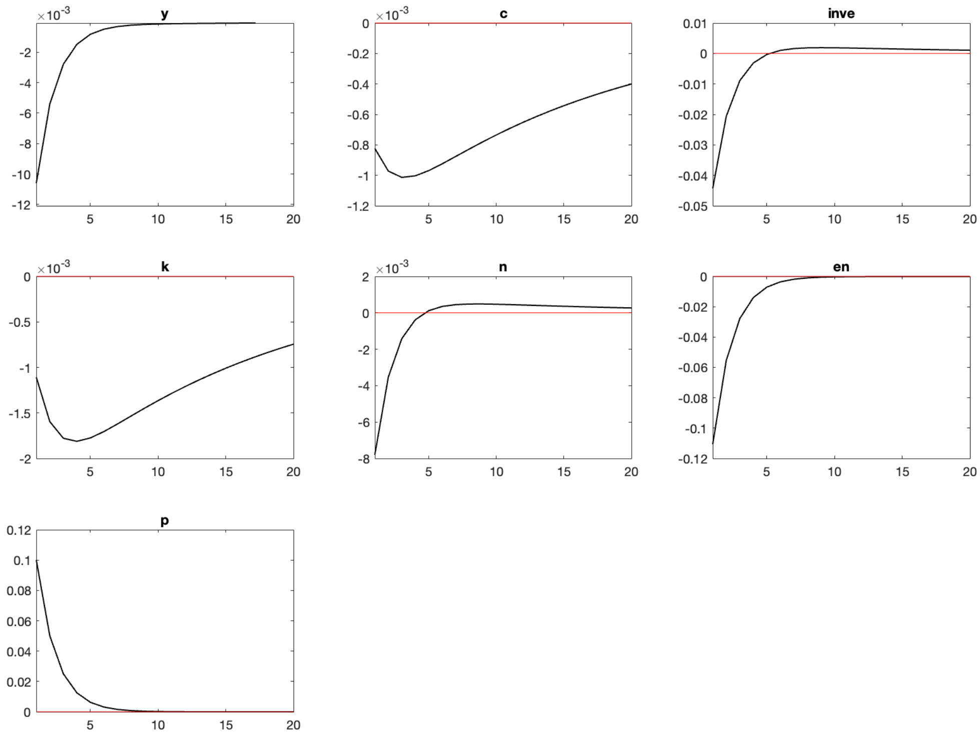
Impulse Response Function of I. At the point of impact, investments (Inve) decrease as part of Y is invested. The fact that Inve decreases by a factor of 6 shows the high volatility of investment with respect to output. The decrease in Inve is only roughly equal to the initial decrease in A after 20 periods.

Impulse Response Function of N. At the point of impact, N is reduced by a factor similar to that of A. However, N gradually returns to its pre-shock level. When A decreases, the resource constraint becomes tighter, and agents need to adjust their labor supply decisions. In particular, a decrease in A reduces the marginal product of labor and thus the real wage. Hence, people reduce their labor supply. As A increases, the real wage increases as well, and people start to increase their labor supply again to the pre-shock level.

Impulse Response Function of EN. At the point of impact, EN is reduced by a factor of about 1.5 to the initial decrease in A. A shock to variable A affects the production function by reducing the amount of capital and labor available for production. This, in turn, leads to a reduction in energy demand as energy is one of the inputs used in production. The 1.5 factor suggests that the response of energy demand to changes in A is relatively strong, indicating that energy consumption is responsive to changes in the driving force of the economy. However, the slow adjustment of EN to its pre-shock level suggests that the energy sector is less flexible in responding to shocks compared to the labor market, which adjusts more quickly.

3.d Impulse Responses II

We get the following impulse response functions:



Impulse Response Function of Y. The IRF of Y indicates that the initial impact of the shock on output is relatively small, resulting in only a 0.1 reduction. This implies that the economy is not heavily reliant on energy for production, as the energy coefficient in the production function is only 0.05.

Impulse Response Function of C. The IRF of C displays a sharp reduction in consumption at the impact, with a factor of 8, and a further decrease to a factor of 10. This indicates that the shock has a significant negative effect on consumption, as more available resources must be spent on energy instead of consumption. This can be directly seen from the resource constraint where an increase in P tightens the budget constraint, thereby reducing the availability of resources for consumption.

Impulse Response Function of I. The IRF of Inv exhibits a pattern similar to that of Y, with a temporary reduction in investment at the impact that quickly returns to its pre-shock level. This implies that the shock does not have a long-term negative effect on capital accumulation.

Impulse Response Function of K. The response of capital to the energy price shock is akin to that of consumption, with a relatively large reduction at the impact and slow recovery over time, particularly after the most significant decline in period 4. The adjustment process for capital accumulation is more sluggish than that for output or investment. The slow recovery of capital over time suggests that the negative effects of the energy price shock are persistent and may affect the long-term growth potential of the economy.

Impulse Response Function of N. The response of labor to an energy price shock is relatively quick, with N significantly reduced at the impact but returning to the pre-shock level quickly. This indicates that the labor market is relatively flexible, and workers can adjust their labor supply in response to changes in energy prices.

Impulse Response Function of EN. The response of energy demand to an energy price shock is relatively small, with only a factor of 1 reduction at the impact and a quick return to the pre-shock level. This may seem counter-intuitive, given that the elasticity of energy demand to energy prices is generally high, at least compared to cross-input price elasticities, as energy prices play a significant role in determining the cost of production and consumption. However, it may be that energy is often considered a necessity or a critical input in production processes, so consumers and producers are not very responsive to changes in energy prices in the short run.

3.e Log-Linearization

Consider again the system of 8 equations from Exercise 2:

$$
\begin{aligned}
C_t^{-\sigma} &= \beta \mathrm{E}_t \left(C_{t+1}^{-\sigma} \left(A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta) \right) \right) && (3.1)\\
C_t^{-\sigma} A_t K_t^{\alpha} \gamma N_t^{\gamma-1} EN_t^{1-\alpha-\gamma} &= \theta \frac{1}{1-N_t} && (3.2)\\
Y_t &= A_t K_t^{\alpha} N_t^{\gamma} (EN)^{1-\alpha-\gamma} && (3.3)\\
(1-\alpha-\gamma) Y_t &= P_t (EN)_t && (3.4)\\
K_{t+1} &= (1-\delta) K_t + I_t && (3.5)\\
Y_t &= C_t + I_t + P_t (EN)_t && (3.6)\\
\log A_{t+1} &= \rho_A \log A_t + \varepsilon_{A,t+1} && (3.7)\\
\log P_{t+1} &= \rho_P \log P_t + \varepsilon_{P,t+1} && (3.8)
\end{aligned}
$$

Log-linearization of Equation 3.1:

$$
C_t^{-\sigma} = \beta \mathrm{E}_t \left(C_{t+1}^{-\sigma} \left(A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta) \right) \right) \qquad (3.1)
$$

We can rewrite this equation in terms of the “hat” variables, i.e. percentage deviations from the steady state:

$$
\begin{aligned}
(\bar{C}e^{\hat{C}_t})^{-\sigma} &= \beta \mathrm{E}_t \left((\bar{C}e^{\hat{C}_{t+1}})^{-\sigma} \left(\bar{A}e^{\hat{A}_{t+1}} \alpha (\bar{K}e^{\hat{K}_{t+1}})^{\alpha-1} (\bar{N}e^{\hat{N}_{t+1}})^{\gamma} (\bar{EN}e^{\hat{EN}_{t+1}})^{1-\alpha-\gamma} + 1 - \delta \right) \right) \\
\bar{C}^{-\sigma} e^{-\sigma \hat{C}_t} &= \beta \mathrm{E}_t \left(\bar{C}^{-\sigma} e^{-\sigma \hat{C}_{t+1}} \left(\bar{A}e^{\hat{A}_{t+1}} \alpha \bar{K}^{\alpha-1} e^{(\alpha-1)\hat{K}_{t+1}}) \bar{N}^{\gamma} e^{\gamma \hat{N}_{t+1}} \bar{EN}^{1-\alpha-\gamma} e^{(1-\alpha-\gamma)\hat{EN}_{t+1}} + 1 - \delta \right) \right)
\end{aligned}
$$

The left-hand side of this is

$$
\bar{C}^{-\sigma} e^{-\sigma \hat{C}_t},
$$

where we can take the first-order taylor approximation given by $f(u) \simeq f(\bar{u}) + f_u(\bar{u})(u - \bar{u})$ (note that the steady state of the hat variables is 0):

$$
\text{LHS} \simeq \bar{C}^{-\sigma} - \sigma \bar{C}^{-\sigma} \hat{C}_t.
$$

The right-hand side is a little more complicated. Here, we take the first-order taylor approximation given by $f(u, v, w, x, y) \simeq f_u(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y})(u - \bar{u}) + f_v(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y})(v - \bar{v}) + f_w(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y})(w - \bar{w}) + f_x(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y})(x - \bar{x}) + f_y(\bar{u}, \bar{v}, \bar{w}, \bar{x}, \bar{y})(y - \bar{y})$:

$$
\begin{aligned}
\text{RHS} \simeq\ & \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} + 1 - \delta \right) \\
&+ \beta(-\sigma) \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} + 1 - \delta \right) E_t(\hat{C}_{t+1}) \\
&+ \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} \right) E_t(\hat{A}_{t+1}) \\
&+ \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha (\alpha - 1) \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} \right) E_t(\hat{K}_{t+1}) \\
&+ \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \gamma \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} \right) E_t(\hat{N}_{t+1}) \\
&+ \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} (1 - \alpha - \gamma) \bar{EN}^{1-\alpha-\gamma} \right) E_t(\hat{EN}_{t+1})
\end{aligned}
$$

Equating both sides, we get

$$
\begin{aligned}
\bar{C}^{-\sigma} - \sigma \bar{C}^{-\sigma} \hat{C}_t =\ & \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} + 1 - \delta \right) \\
&+ \beta(-\sigma) \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} + 1 - \delta \right) E_t(\hat{C}_{t+1}) \\
&+ \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} \right) E_t(\hat{A}_{t+1}) \\
&+ \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha (\alpha - 1) \bar{K}^{\alpha-1} \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} \right) E_t(\hat{K}_{t+1}) \\
&+ \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \gamma \bar{N}^{\gamma} \bar{EN}^{1-\alpha-\gamma} \right) E_t(\hat{N}_{t+1}) \\
&+ \beta \bar{C}^{-\sigma} \left(\bar{A} \alpha \bar{K}^{\alpha-1} \bar{N}^{\gamma} (1 - \alpha - \gamma) \bar{EN}^{1-\alpha-\gamma} \right) E_t(\hat{EN}_{t+1}).
\end{aligned}
$$

We can now eliminate the steady state from the left-hand side and the steady state from the right-hand side, arriving at

$$
\begin{aligned}
-\sigma\hat{C}_t =\ & (-\sigma)E_t(\hat{C}_{t+1}) \\
& + \beta\bar{C}^{-\sigma}\left(\bar{A}\alpha\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\right)E_t(\hat{A}_{t+1}) \\
& + \beta\bar{C}^{-\sigma}\left(\bar{A}\alpha(\alpha-1)\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\right)E_t(\hat{K}_{t+1}) \\
& + \beta\bar{C}^{-\sigma}\left(\bar{A}\alpha\bar{K}^{\alpha-1}\gamma\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma}\right)E_t(\hat{N}_{t+1}) \\
& + \beta\bar{C}^{-\sigma}\left(\bar{A}\alpha\bar{K}^{\alpha-1}\bar{N}^{\gamma}(1-\alpha-\gamma)\bar{EN}^{1-\alpha-\gamma}\right)E_t(\hat{EN}_{t+1}).
\end{aligned}
$$

Finally, we can use the fact that $\bar{A}\alpha\bar{K}^{\alpha-1}\bar{N}^{\gamma}\bar{EN}^{1-\alpha-\gamma} + (1-\delta) = 1-\beta(1-\delta)$, yielding

$$
-\sigma\hat{C}_t = E_t\left(-\sigma\hat{C}_{t+1} + (1-\beta(1-\delta))\left(\hat{A}_{t+1} + (\alpha-1)\hat{K}_{t+1} + \gamma\hat{N}_{t+1} + (1-\alpha-\gamma)\hat{EN}_{t+1}\right)\right). \tag{3.9}
$$

Log-linearization of Equation 3.2:

$$
C_t^{-\sigma}A_tK_t^{\alpha}\gamma N_t^{\gamma-1}EN_t^{1-\alpha-\gamma} = \theta\frac{1}{1-N_t} \tag{3.2}
$$

We can simplify this expression:

$$
\begin{aligned}
C_t^{-\sigma}A_tK_t^{\alpha}\gamma N_t^{\gamma-1}EN_t^{1-\alpha-\gamma} &= \theta\frac{1}{1-N_t} \\
C_t^{-\sigma}\gamma Y_tN_t^{-1} &= \theta\frac{1}{1-N_t} \\
(1-N_t)C_t^{-\sigma}\gamma Y_tN_t^{-1} &= \theta \\
C_t^{-\sigma}\gamma Y_tN_t^{-1} - C_t^{-\sigma}\gamma Y_t &= \theta \\
C_t^{-\sigma}Y_tN_t^{-1} - C_t^{-\sigma}Y_t &= \frac{\theta}{\gamma} \\
C_t^{-\sigma}Y_t - C_t^{-\sigma}N_tY_t &= \frac{\theta}{\gamma}N_t \\
C_t^{-\sigma}Y_t &= \frac{\theta}{\gamma}N_t + C_t^{-\sigma}N_tY_t \\
Y_t &= \frac{\theta}{\gamma}N_tC_t^{\sigma} + N_tY_t \\
Y_t &= \frac{\theta}{\gamma}N_tC_t + N_tY_t
\end{aligned}
$$

Rewriting the expression in hat variables, we get:

$$
\hat{Y}e^{\hat{Y}_t} = \frac{\theta}{\gamma}\hat{N}e^{\hat{N}_t}\hat{C}e^{\hat{C}_t} + \hat{N}e^{\hat{N}_t}\hat{Y}e^{\hat{Y}_t}
$$

The left-hand side becomes:

$$
\hat{Y}e^{\hat{Y}_t} \simeq \bar{Y}\hat{Y}_t
$$

The right-hand side becomes:

$$\frac{\theta}{\gamma}\hat{N}e^{\hat{N}_t}\hat{C}e^{\hat{C}_t} + \hat{N}e^{\hat{N}_t}\hat{Y}e^{\hat{Y}_t} \simeq \frac{\theta}{\gamma}\bar{N}\bar{C}\hat{N}_t + \frac{\theta}{\gamma}\bar{N}\bar{C}\hat{C}_t + \bar{N}\bar{Y}\hat{N}_t + \bar{N}\bar{Y}\hat{Y}_t$$

Equating left-hand side and right-hand side:

$$\bar{Y}\hat{Y}_t = \frac{\theta}{\gamma}\bar{N}\bar{C}\hat{N}_t + \frac{\theta}{\gamma}\bar{N}\bar{C}\hat{C}_t + \bar{N}\bar{Y}\hat{N}_t + \bar{N}\bar{Y}\hat{Y}_t \tag{3.10}$$

Log-linearization of Equation 3.3:

$$Y_t = A_t K_t^{\alpha} N_t^{\gamma}(EN)^{1-\alpha-\gamma} \tag{3.3}$$

Rewriting the expression in hat variables, we get

$$\bar{Y}e^{\hat{Y}_t} = \bar{A}e^{\hat{A}_t}\bar{K}^{\alpha}e^{\alpha\hat{K}_t}\bar{N}^{\gamma}e^{\gamma\hat{N}_t}\bar{EN}^{(1-\alpha-\gamma)}e^{(1-\alpha-\gamma)\hat{EN}_t}$$

The left-hand side becomes:

$$\bar{Y}e^{\hat{Y}_t} \simeq \bar{Y} + \bar{Y}\hat{Y}_t.$$

The right-hand side becomes:

$$\begin{aligned}\bar{A}e^{\hat{A}_t}\bar{K}^{\alpha}e^{\alpha\hat{K}_t}\bar{N}^{\gamma}e^{\gamma\hat{N}_t}\bar{EN}^{(1-\alpha-\gamma)}e^{(1-\alpha-\gamma)\hat{EN}_t} \simeq\ & \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)} \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\hat{A}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\alpha\hat{K}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\gamma\hat{N}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}(1-\alpha-\gamma)\hat{EN}_t.\end{aligned}$$

Equating both sides,

$$\begin{aligned}\bar{Y} + \bar{Y}\hat{Y}_t = & \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)} \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\hat{A}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\alpha\hat{K}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\gamma\hat{N}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}(1-\alpha-\gamma)\hat{EN}_t.\end{aligned}$$

Subtracting the steady state from both sides,

$$\begin{aligned}\bar{Y}\hat{Y}_t = & \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\hat{A}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\alpha\hat{K}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}\gamma\hat{N}_t \\ & + \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)}(1-\alpha-\gamma)\hat{EN}_t.\end{aligned}$$

Since $\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\bar{EN}^{(1-\alpha-\gamma)} = \bar{Y}$, this can be simplified to:

$$\begin{aligned} \bar{Y}\hat{Y}_t &= \bar{Y}(\hat{A}_t + \alpha\hat{K}_t + \gamma\hat{N}_t + (1-\alpha-\gamma)\hat{EN}_t) \\ \hat{Y}_t &= \hat{A}_t + \alpha\hat{K}_t + \gamma\hat{N}_t + (1-\alpha-\gamma)\hat{EN}_t \end{aligned} \tag{3.11}$$

Log-linearization of Equation 3.4:

$$(1-\alpha-\gamma)Y_t = P_t(EN)_t \tag{3.4}$$

Rewriting the expression in hat variables, we get

$$(1-\alpha-\gamma)\bar{Y}e^{\hat{Y}_t} = \bar{P}e^{\hat{P}_t}\bar{EN}e^{\hat{EN}_t}$$

The left-hand side becomes:

$$(1-\alpha-\gamma)\bar{Y}e^{\hat{Y}_t} \simeq (1-\alpha-\gamma)\bar{Y} + (1-\alpha-\gamma)\bar{Y}\hat{Y}_t$$

The right-hand side becomes:

$$\bar{P}e^{\hat{P}_t}\bar{EN}e^{\hat{EN}_t} \simeq \bar{P}\bar{EN} + \bar{P}\bar{EN}\hat{P}_t + \bar{P}\bar{EN}\hat{EN}_t$$

Equating both sides:

$$(1-\alpha-\gamma)\bar{Y} + (1-\alpha-\gamma)\bar{Y}\hat{Y}_t = \bar{P}\bar{EN} + \bar{P}\bar{EN}\hat{P}_t + \bar{P}\bar{EN}\hat{EN}_t$$

Eliminating the steady state:

$$(1-\alpha-\gamma)\bar{Y}\hat{Y}_t = \bar{P}\bar{EN}\hat{P}_t + \bar{P}\bar{EN}\hat{EN}_t \tag{3.12}$$

Log-linearization of Equation 3.5:

$$K_{t+1} = (1-\delta)K_t + I_t \tag{3.5}$$

Rewriting the expression in hat variables, we get:

$$\bar{K}e^{\hat{K}_{t+1}} = (1-\delta)\bar{K}e^{\hat{K}_t} + \bar{I}e^{\hat{I}_t}$$

The left-hand side becomes:

$$\bar{K}e^{\hat{K}_{t+1}} \simeq \bar{K}\hat{K}_{t+1}$$

The right-hand side becomes:

$$(1-\delta)\bar{K}e^{\hat{K}_t} + \bar{I}e^{\hat{I}_t} \simeq (1-\delta)\bar{K}\hat{K}_t + \bar{I}\hat{I}_t$$

Equating both sides yields:

$$\bar{K}\hat{K}_{t+1} = (1-\delta)\bar{K}\hat{K}_t + \bar{I}\hat{I}_t \tag{3.13}$$

Log-linearization of Equation 3.6:

$$Y_t = C_t + I_t + P_t(EN)_t \tag{3.6}$$

Rewriting the expression in hat variables, we get

$$\bar{Y}e^{\hat{Y}_t} = \bar{C}e^{\hat{C}_t} + \bar{I}e^{\hat{I}_t} + (\bar{P}e^{\hat{P}_t})(\bar{EN}e^{\hat{EN}_t})$$

The left-hand side becomes:

$$\bar{Y}e^{\hat{Y}_t} \simeq \bar{Y} + \bar{Y}\hat{Y}_t.$$

The right-hand side becomes:

$$\bar{C}e^{\hat{C}_t} + \bar{I}e^{\hat{I}_t} + (\bar{P}e^{\hat{P}_t})(\bar{EN}e^{\hat{EN}_t}) \simeq \bar{C} + \bar{I} + \bar{P}\bar{EN} + \bar{C}\hat{C}_t + \bar{I}\hat{I}_t + \bar{P}\bar{EN}\hat{P}_t + \bar{P}\bar{EN}\hat{EN}_t$$

Thus, the log-linearized equation is:

$$\bar{Y} + \bar{Y}\hat{Y}_t = \bar{C} + \bar{I} + \bar{P}\bar{EN} + \bar{C}\hat{C}_t + \bar{I}\hat{I}_t + \bar{P}\bar{EN}\hat{P}_t + \bar{P}\bar{EN}\hat{EN}_t$$

Subtracting the steady state and dividing by $\bar{Y}$, we get

$$\begin{aligned}
\bar{Y}\hat{Y}_t &= \bar{C}\hat{C}_t + \bar{I}\hat{I}_t + \bar{P}\bar{EN}\hat{P}_t + \bar{P}\bar{EN}\hat{EN}_t \\
\hat{Y}_t &= \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t + \frac{\bar{P}\bar{EN}}{\bar{Y}}\hat{P}_t + \frac{\bar{P}\bar{EN}}{\bar{Y}}\hat{EN}_t \qquad (3.14)
\end{aligned}$$

Equations 3.7 and 3.8 are already linear:

$$\log A_{t+1} = \rho_A \log A_t + \varepsilon_{A,t+1} \tag{3.15}$$

$$\log P_{t+1} = \rho_P \log P_t + \varepsilon_{P,t+1} \tag{3.16}$$

3.f Solving the Linear System of Equations

From the previous subquestion, our log-linearized equations are:

$$
\begin{aligned}
-\sigma\hat{C}_t &= E_t\Big(-\sigma\hat{C}_{t+1} + (1-\beta(1-\delta)) \\
&\qquad \Big(\hat{A}_{t+1} + (\alpha-1)\hat{K}_{t+1} + \gamma\hat{N}_{t+1} + (1-\alpha-\gamma)\hat{EN}_{t+1}\Big)\Big) && (3.9)\\
\bar{Y}\hat{Y}_t &= \frac{\theta}{\gamma}\bar{N}\bar{C}\hat{N}_t + \frac{\theta}{\gamma}\bar{N}\bar{C}\hat{C}_t + \bar{N}\bar{Y}\hat{N}_t + \bar{N}\bar{Y}\hat{Y}_t && (3.10)\\
\hat{Y}_t &= \hat{A}_t + \alpha\hat{K}_t + \gamma\hat{N}_t + (1-\alpha-\gamma)\hat{EN}_t && (3.11)\\
(1-\alpha-\gamma)\bar{Y}\hat{Y}_t &= \bar{P}\bar{EN}\hat{P}_t + \bar{P}\bar{EN}\hat{EN}_t && (3.12)\\
\bar{K}\hat{K}_{t+1} &= (1-\delta)\bar{K}\hat{K}_t + \bar{I}\hat{I}_t && (3.13)\\
\hat{Y}_t &= \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t + \frac{\bar{P}\bar{EN}}{\bar{Y}}\hat{P}_t + \frac{\bar{P}\bar{EN}}{\bar{Y}}\hat{EN}_t && (3.14)\\
\log A_{t+1} &= \rho_A \log A_t + \varepsilon_{A,t+1} && (3.15)\\
\log P_{t+1} &= \rho_P \log P_t + \varepsilon_{P,t+1} && (3.16)
\end{aligned}
$$

We now need to map the equations into matrix format in order for us to be able to solve them. That is, we need to construct two matrices $\boldsymbol{A}, \boldsymbol{B}$ s.t.

$$
\boldsymbol{A}E_t\boldsymbol{z}_{t+1} = \boldsymbol{B}\boldsymbol{z}_t, \tag{3.17}
$$

where $\boldsymbol{z}_t = \begin{pmatrix} \hat{K}_t \\ \hat{A}_t \\ \hat{P}_t \\ \hat{C}_t \\ \hat{N}_t \\ \hat{Y}_t \\ \hat{I}_t \\ \hat{EN}_t \end{pmatrix}$ and $\boldsymbol{z}_{t+1} = \begin{pmatrix} \hat{K}_{t+1} \\ \hat{A}_{t+1} \\ \hat{P}_{t+1} \\ \hat{C}_{t+1} \\ \hat{N}_{t+1} \\ \hat{Y}_{t+1} \\ \hat{I}_{t+1} \\ \hat{EN}_{t+1} \end{pmatrix}$.

We therefore construct the following matrices:

$$
\boldsymbol{A} = \begin{pmatrix}
(\alpha-1)(1-\beta(1-\delta)) & (1-\beta(1-\delta)) & 0 & -\sigma & \gamma(1-\beta(1-\delta)) & 0 & 0 & (1-\gamma-\alpha)(1-\beta(1-\delta)) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{K} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\boldsymbol{B} = \begin{pmatrix}
0 & 0 & 0 & -\sigma & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\theta}{\gamma}\bar{N}\bar{C} & \frac{\theta}{\gamma}\bar{N}\bar{C} + \bar{N}\bar{Y} & -\bar{Y} + \bar{N}\bar{Y} & 0 & 0 \\
\alpha & 1 & 0 & 0 & \gamma & -1 & 0 & (1-\alpha-\gamma) \\
0 & 0 & \bar{P}\bar{EN} & 0 & 0 & -(1-\alpha-\gamma)\bar{Y} & 0 & \bar{P}\bar{EN} \\
(1-\delta)\bar{K} & 0 & 0 & 0 & 0 & 0 & \bar{I} & 0 \\
0 & 0 & \frac{\bar{P}\bar{EN}}{\bar{Y}} & \frac{\bar{C}}{\bar{Y}} & 0 & -1 & \frac{\bar{I}}{\bar{Y}} & \frac{\bar{P}\bar{EN}}{\bar{Y}} \\
0 & \rho_A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_P & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Solving the linear system of equations $\boldsymbol{A}E_t\boldsymbol{z}_{t+1} = \boldsymbol{B}\boldsymbol{z}_t$ (see the code attached to the submission) yields

$$
\boldsymbol{G} = \begin{pmatrix} \phi_{CK} & \phi_{CA} & \phi_{CP} \\ \phi_{NK} & \phi_{NA} & \phi_{NP} \\ \phi_{YK} & \phi_{YA} & \phi_{YP} \\ \phi_{IK} & \phi_{IA} & \phi_{IP} \\ \phi_{EN\,K} & \phi_{EN\,A} & \phi_{EN\,P} \end{pmatrix} = \begin{pmatrix} 0.5064 & 0.4834 & -0.0082 \\ -0.3346 & 1.0009 & -0.0780 \\ 0.0869 & 1.7376 & -0.1060 \\ -1.3592 & 6.0623 & -0.4432 \\ 0.0870 & 1.7400 & -1.1062 \end{pmatrix}
$$

and

$$
\boldsymbol{H} = \begin{pmatrix} \phi_{KK} & \phi_{KA} & \phi_{KP} \\ 0 & \rho_A & 0 \\ 0 & 0 & \rho_P \end{pmatrix} = \begin{pmatrix} 0.9410 & 0.1515 & -0.0111 \\ 0 & 0.95 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}.
$$

This corresponds to the results we got earlier from solving the model using Dynare in Exercise 3.b.