Wave equation 1D

Wave equation (1D):

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

0.0.1 Weak Form

$$\begin{aligned}
w(x)\frac{\partial^2 u}{\partial x^2} &= w(x)\frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2} & (2)\\
\frac{\partial}{\partial x}\left(w(x)\frac{\partial u}{\partial x}\right) - \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} &= w(x)\frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2} & (3)\\
\int\limits_{\Omega}\frac{\partial}{\partial x}\left(w(x)\frac{\partial u}{\partial x}\right)dx - \int\limits_{\Omega}\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dx &= \int\limits_{\Omega} w(x)\frac{1}{c^2(x)}\frac{\partial^2 u}{\partial^2 t}dx & (4)
\end{aligned}$$

Because $w(\Gamma_{\text{Dirichlet}}) = 0$ we get $\int\limits_{\Omega}\frac{\partial}{\partial x}\left(w(x)\frac{\partial u}{\partial x}\right)dx = w(x)\underbrace{\left.\frac{\partial u}{\partial x}\right|_{\Gamma_{\text{Neumann}}}}_{q(\Gamma)}$. Rearranging yields:

$$\int\limits_{\Omega}\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dx = w(x)\underbrace{\left.\frac{\partial u}{\partial x}\right|_{\Gamma_{\text{Neumann}}}}_{q(\Gamma)} - \int\limits_{\Omega} w(x)\frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2}dx \tag{5}$$

0.0.2 Discretization

$$\begin{aligned}
u(x) &\approx U(x,t) = \Phi(x)^T\mathbf{C} & (6)\\
w(x) &\approx W(x,t) = \mathbf{W}^T\Phi(x) & (7)
\end{aligned}$$

$\ddot{\mathbf{C}}$ is a column with nodal accelerations. Differentiation yields:

$$\frac{\partial u}{\partial x} \approx \Phi_x^T \mathbf{C} \tag{8}$$

$$\frac{\partial w}{\partial x} \approx \mathbf{W}^T \Phi_x \tag{9}$$

$$\frac{\partial^2 u}{\partial t^2} \approx \Phi(x)^T \ddot{\mathbf{C}} \tag{10}$$

Substitution yields:

$$\mathbf{W}^T \int_\Omega \Phi_x \Phi_x^T d\Omega \mathbf{C} = \mathbf{W}^T \Phi(\Gamma) q(\Gamma) - \mathbf{W}^T \int_\Omega \Phi(x)\Phi(x)^T \frac{1}{c^2(x)} d\Omega \ddot{\mathbf{C}}$$

$$\underbrace{\int_\Omega \Phi_x \Phi_x^T d\Omega}_{K} \mathbf{C} = \underbrace{\underbrace{\int_\Gamma q(\Gamma)\Phi(x) d\Gamma}_{\mathbf{F}_\Gamma}}_{\mathbf{F}_{\text{tot}}} - \underbrace{\int_\Omega \Phi(x)\Phi(x)^T \frac{1}{c^2(x)} d\Omega}_{M} \ddot{\mathbf{C}} \tag{11}$$

When $\mathbf{F}_{\text{tot}} = \mathbf{0}$ we write:

$$K\mathbf{C} = -M\ddot{\mathbf{C}} \tag{12}$$

$$\ddot{\mathbf{C}} = -M^{-1}K\mathbf{C} \tag{13}$$

$$\mathbf{Y} = \begin{pmatrix} \mathbf{C} \\ \dot{\mathbf{C}} \end{pmatrix} \tag{14}$$

$$\dot{\mathbf{Y}} = \underbrace{\begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix}}_{A} \mathbf{Y} \tag{15}$$

$$\dot{\mathbf{Y}} = A\mathbf{Y} \tag{16}$$

with $\mathbf{Y}(t = 0)$ an initial condition.

FEM 3D

$$\nabla \cdot \sigma + \mathbf{f} = \rho \ddot{\mathbf{u}} \tag{17}$$

0.0.3 Weak form

$$\mathbf{w} \cdot (\nabla \cdot \sigma) + \mathbf{w} \cdot \mathbf{f} = \mathbf{w} \cdot \rho \ddot{\mathbf{u}} \tag{18}$$

$$\iiint_\Omega \mathbf{w} \cdot (\nabla \cdot \sigma) d\Omega + \iiint_\Omega \mathbf{w} \cdot \mathbf{f} d\Omega = \iiint_\Omega \mathbf{w} \cdot \rho \ddot{\mathbf{u}} d\Omega \tag{19}$$

$$\iiint_\Omega \nabla \cdot (\sigma \cdot \mathbf{w}) d\Omega - \iiint_\Omega (\nabla \mathbf{w})^T : \sigma d\Omega + \iiint_\Omega \mathbf{w} \cdot \mathbf{f} d\Omega = \iiint_\Omega \mathbf{w} \cdot \rho \ddot{\mathbf{u}} d\Omega \tag{20}$$

$$\iiint_\Omega (\nabla \mathbf{w})^T : \sigma d\Omega + \iiint_\Omega \mathbf{w} \cdot \rho \ddot{\mathbf{u}} d\Omega = \iiint_\Omega \mathbf{w} \cdot \mathbf{f} d\Omega + \iint_\Gamma \mathbf{w} \cdot \mathbf{q} d\Gamma \tag{21}$$

with $:$ the double dot-product.

0.0.4 Discretization

Discretization on element level yields:

$$\mathbf{u} \approx H\mathbf{U} \tag{22}$$

$$\mathbf{w} \approx H\mathbf{W} \tag{23}$$

with

$$H = \left(\begin{array}{ccc|ccc|ccc|ccc} \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 & 0 & \phi_4 & 0 & 0 \\ 0 & \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 & 0 & \phi_4 & 0 \\ 0 & 0 & \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 & 0 & \phi_4 \end{array} \right) \tag{24}$$

where $\phi_i = \phi_i(x,y,z)$ and

$$\mathbf{U}_e = \begin{pmatrix} U_{1x} \\ U_{1y} \\ U_{1z} \\ U_{2x} \\ U_{2y} \\ U_{2z} \\ U_{3x} \\ U_{3y} \\ U_{3z} \\ U_{4x} \\ U_{4y} \\ U_{4z} \end{pmatrix}, \quad \mathbf{W}_e = \begin{pmatrix} W_{1x} \\ W_{1y} \\ W_{1z} \\ W_{2x} \\ W_{2y} \\ W_{2z} \\ W_{3x} \\ W_{3y} \\ W_{3z} \\ W_{4x} \\ W_{4y} \\ W_{4z} \end{pmatrix} \tag{25}$$

The number of degrees of freedom per element equals 3 (3D). Substitution yields:

$$\mathbf{W}_e^T \iiint_{\Omega_e} B^T DBd\Omega \mathbf{U}_e + \mathbf{W}_e^T \iiint_{\Omega_e} \rho H^T H d\Omega \ddot{\mathbf{U}}_e = \mathbf{W}_e^T \iiint_{\Omega_e} H^T \mathbf{f} d\Omega + \mathbf{W}_e^T \iint_{\Gamma_e} H^T \mathbf{q} d\Gamma_e \tag{26}$$

$$\underbrace{\iiint_{\Omega_e} B^T DBd\Omega}_{K_e} \mathbf{U}_e + \underbrace{\iiint_{\Omega_e} \rho H^T H d\Omega}_{M_e} \ddot{\mathbf{U}}_e = \underbrace{\underbrace{\iiint_{\Omega_e} H^T \mathbf{f} d\Omega}_{\mathbf{F}_{\Omega_e}} + \underbrace{\iint_{\Gamma_e} H'^T \mathbf{q} d\Gamma_e}_{\mathbf{F}_{\Gamma_e}}}_{\mathbf{F}_{\mathrm{tot}_e}} \tag{27}$$

waarin

$$H' = \left(\begin{array}{ccc|ccc|ccc} \phi_1' & 0 & 0 & \phi_2' & 0 & 0 & \phi_3' & 0 & 0 \\ 0 & \phi_1' & 0 & 0 & \phi_2' & 0 & 0 & \phi_3' & 0 \\ 0 & 0 & \phi_1' & 0 & 0 & \phi_2' & 0 & 0 & \phi_3' \end{array}\right) \tag{28}$$

Short:

$$M_e \ddot{\mathbf{U}}_e + K_e \mathbf{U}_e = \mathbf{F}_{\mathrm{tot}_e} \tag{29}$$

0.0.5 Element matrices

$$K_e = \iiint_{\Omega_e} B^T DBd\Omega = B^T DB \cdot \mathrm{Volume}(e) \tag{30}$$

with

$$B = \left(\begin{array}{ccc|ccc|ccc|ccc}
\frac{\partial\phi_1}{\partial x} & 0 & 0 & \frac{\partial\phi_2}{\partial x} & 0 & 0 & \frac{\partial\phi_3}{\partial x} & 0 & 0 & \frac{\partial\phi_4}{\partial x} & 0 & 0 \\
0 & \frac{\partial\phi_1}{\partial y} & 0 & 0 & \frac{\partial\phi_2}{\partial y} & 0 & 0 & \frac{\partial\phi_3}{\partial y} & 0 & 0 & \frac{\partial\phi_4}{\partial y} & 0 \\
0 & 0 & \frac{\partial\phi_1}{\partial z} & 0 & 0 & \frac{\partial\phi_2}{\partial z} & 0 & 0 & \frac{\partial\phi_3}{\partial z} & 0 & 0 & \frac{\partial\phi_4}{\partial z} \\
\frac{\partial\phi_1}{\partial y} & \frac{\partial\phi_1}{\partial x} & 0 & \frac{\partial\phi_2}{\partial y} & \frac{\partial\phi_2}{\partial x} & 0 & \frac{\partial\phi_3}{\partial y} & \frac{\partial\phi_3}{\partial x} & 0 & \frac{\partial\phi_4}{\partial y} & \frac{\partial\phi_4}{\partial x} & 0 \\
0 & \frac{\partial\phi_1}{\partial z} & \frac{\partial\phi_1}{\partial y} & 0 & \frac{\partial\phi_2}{\partial z} & \frac{\partial\phi_2}{\partial y} & 0 & \frac{\partial\phi_3}{\partial z} & \frac{\partial\phi_3}{\partial y} & 0 & \frac{\partial\phi_4}{\partial z} & \frac{\partial\phi_4}{\partial y} \\
\frac{\partial\phi_1}{\partial z} & 0 & \frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_2}{\partial z} & 0 & \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_3}{\partial z} & 0 & \frac{\partial\phi_3}{\partial x} & \frac{\partial\phi_4}{\partial z} & 0 & \frac{\partial\phi_4}{\partial x}
\end{array}\right) \tag{31}$$

and

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & 0 & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix} \tag{32}$$

where E is the Young's modulus of the material and ν the Poisson-factor.

$$\begin{aligned} M_e &= \iiint\limits_{\Omega_e} \rho H^T H d\Omega \\ &\approx \rho \begin{pmatrix} \overline{\phi}_1 & 0 & 0 \\ 0 & \overline{\phi}_1 & 0 \\ 0 & 0 & \overline{\phi}_1 \\ \overline{\phi}_2 & 0 & 0 \\ 0 & \overline{\phi}_2 & 0 \\ 0 & 0 & \overline{\phi}_2 \\ \overline{\phi}_3 & 0 & 0 \\ 0 & \overline{\phi}_3 & 0 \\ 0 & 0 & \overline{\phi}_3 \\ \overline{\phi}_4 & 0 & 0 \\ 0 & \overline{\phi}_4 & 0 \\ 0 & 0 & \overline{\phi}_4 \end{pmatrix} \left(\begin{array}{ccc|ccc|ccc|ccc} \overline{\phi}_1 & 0 & 0 & \overline{\phi}_2 & 0 & 0 & \overline{\phi}_3 & 0 & 0 & \overline{\phi}_4 & 0 & 0 \\ 0 & \overline{\phi}_1 & 0 & 0 & \overline{\phi}_2 & 0 & 0 & \overline{\phi}_3 & 0 & 0 & \overline{\phi}_4 & 0 \\ 0 & 0 & \overline{\phi}_1 & 0 & 0 & \overline{\phi}_2 & 0 & 0 & \overline{\phi}_3 & 0 & 0 & \overline{\phi}_4 \end{array} \right) \cdot \text{Volume}(e) \\ &= \rho \left(\frac{1}{4}\right)^2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \cdot \text{Volume}(e) \end{aligned} \tag{33}$$

where $\overline{\phi}_i = \phi_i(\overline{x}, \overline{y}, \overline{z})$.

$$\mathbf{F}_{\Omega_e} = \iiint\limits_{\Omega_e} H^T \mathbf{f}(x,y,z) d\Omega \approx \frac{1}{4} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{f}(\overline{x}, \overline{y}, \overline{z}) \cdot \text{Volume}(e) \tag{34}$$

$$\mathbf{F}_{\Gamma_e} = \iint\limits_{\Gamma_e} H'^T \mathbf{q}(x,y,z) d\Gamma_e \approx \frac{1}{3} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{q}(\overline{x}, \overline{y}, \overline{z}) \cdot \mathrm{Area}(\Gamma_e) \qquad (35)$$