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Joint Dynamic Pricing of Multiple Perishable Products Under Consumer Choice

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In response to competitive pressures, firms are increasingly adopting revenue management opportunities afforded by advances in information and communication technologies. Motivated by these revenue management initiatives in industry, we consider a dynamic pricing problem facing a firm that sells given initial inventories of multiple substitutable and perishable products over a finite selling horizon. Because the products are substitutable, individual product demands are linked through consumer choice processes. Hence, the seller must formulate a joint dynamic pricing strategy while explicitly incorporating consumer behavior. For an integrative model of consumer choice based on linear random consumer utilities, we model this *multiproduct dynamic pricing* problem as a stochastic dynamic program and analyze its optimal prices. The consumer choice model allows us to capture the linkage between product differentiation and consumer choice, and readily specializes to the cases of vertically and horizontally differentiated assortments. When products are vertically differentiated, our results show monotonicity properties (with respect to quality, inventory, and time) of the optimal prices and reveal that the optimal price of a product depends on higher quality product inventories only through their aggregate inventory rather than individual availabilities. Furthermore, we show that the price of a product can be decomposed into the price of its adjacent lower quality product and a markup over this price, with the markup depending solely on the aggregate inventory. We exploit these properties to develop a polynomial-time, exact algorithm for determining the optimal prices and the profit. For a horizontally differentiated assortment, we show that the profit function is unimodal in prices. We also show that individual, rather than aggregate, product inventory availability drives pricing. However, we find that monotonicity properties observed in vertically differentiated assortments do not hold.

Key words: dynamic pricing; revenue management; perishable products; consumer choice; vertical and horizontal product assortments; efficient algorithm
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1. Introduction

Increasing global competition is forcing companies to rethink their existing pricing and sales strategies and explore new opportunities afforded by advances in information and communication technologies. With these technologies, firms have extensive reach to customers and to information systems throughout their own organizations, allowing them to collect market data, learn about customer behavior, understand market segments, and change prices dynamically. As a result, sellers can now continuously monitor availability and demand, and adjust prices dynamically to maximize profits. Such demand management principles offer firms an opportunity to make substantial gains—a recent McKinsey study (Marn et al. 2003) estimates that for a typical S&P 1500 company, a 1% improvement in pricing can lead to an 8% improvement in profits. Recognizing these opportunities, managers in several industries including travel, hospitality, public utility, and equipment rental (see Talluri and van Ryzin 2004b for a detailed profile of applications in these and other industries) are increasingly using revenue management tools such as dynamic pricing to maximize profits when facing uncertain demand.

In many industrial applications, firms offer their customers a line of differentiated products. Often, these products are *perishable* (e.g., room inventories for a particular day expire at the end of the day) and *substitutable* (e.g., either a standard or a deluxe room can meet consumer demand). To meet uncertain demand in these contexts, firms must adopt a proactive demand management strategy such as dynamic pricing to maximize revenues. Because individual demands for substitutable products are linked by consumer choice processes, managers must determine a *joint* pricing strategy based on consumer behavior. For instance, a hotel might have various types of rooms (e.g., standard versus deluxe rooms) that differ in the amenities and facilities available for the guests. In this case, the demand for an individual room type depends not only on the price and nonprice characteristics of that room type, but also on those of the other room types. As a result, the hotel must understand the choices that consumers make when facing such a product assortment and determine the prices for different room types jointly. Similarly, a discount airline that offers parallel flights (with different arrival and departure times within the same day for a particular origin–destination flight leg) must determine the fares for these itineraries jointly and model how consumers choose these flight itineraries. Although both examples require a joint dynamic pricing strategy, consumer behavior in each context is different and follows the nature of product differentiation. In the hotel example, because room types can be ordered based on their quality, they are *vertically* differentiated; consequently, if all the room types were priced the same, consumers would prefer deluxe rooms over standard rooms. In contrast, in the airline example, customer preferences are not uniformly ordered and the different flight itineraries are *horizontally* differentiated; that is, if all flight itineraries were priced the same, some consumers may prefer an early departure whereas others may choose a later departure.

Motivated by these revenue management applications, this paper studies a dynamic pricing problem facing a firm that sells given initial inventories of multiple substitutable and perishable products over a finite selling season. We refer to this problem as the *multiproduct dynamic pricing* (MPDP) problem. In the MPDP problem, the demand for each product depends on the price and nonprice characteristics of all products in the assortment, and the behavior of consumers, which in turn depends on the nature of product differentiation. To understand consumer utility and derive consumer choice probabilities, we use a linear random utility framework. This utility framework is integrative and is the basis of choice models for two important cases of product differentiation—*vertical* and *horizontal*. For each case, we formulate a stochastic dynamic program (DP) for the MPDP problem and analyze the structure of the optimal prices. Overall, our results offer valuable managerial insights and facilitate actionable multiproduct dynamic pricing policies that are easy to understand, compute, and implement. We describe these in detail next.

The MPDP models we study have applications in several industries. For example, the MPDP model for vertically differentiated products applies to the hospitality (e.g., pricing hotel rooms that offer different amenities), entertainment (e.g., pricing of event tickets for different seat locations), agriculture (e.g., pricing perishable agriculture goods of different grades), and information technology (e.g., pricing of advertisement slots at different positions on web pages) industries. Our analysis of this practical, relevant problem leads to several important contributions.

For the MPDP problem with vertically differentiated products, which we refer to as the V model, we provide a complete analysis of the structure of the optimal prices and the value function. We show that the optimal prices exhibit (1) *quality monotonicity* (the optimal price of a higher quality product is always higher than that of a lower quality product), (2) *inventory monotonicity* (when a product's inventory level increases, the firm must set a lower price not only for that product, but also for the other products in the assortment), and (3) *time monotonicity* (as the end of the sales horizon approaches, the firm must reduce the prices for all its products). Furthermore, we show that the optimal price of a product is governed by an *aggregation* of the inventories of *higher* quality products (rather than their individual availabilities) and the *individual* inventory of *lower* quality products. In addition, the aggregate inventory of higher quality products *alone* determines the *price difference* between two adjacent products. When the product inventories are in surplus (that is, when the total inventory of all products is greater than the maximum possible future demand), we also prove that the surplus units, starting from the lowest quality, are of no value to the firm, and hence can be removed from inventory. These results imply that the pricing policy for the V model should be driven by the *aggregate*, rather than the *individual*, inventory availability. We exploit these structural properties to develop a polynomial-time, exact algorithm that decomposes the multidimensional DP for the V model into a series of one dimensional DPs.

For the MPDP problem with horizontally differentiated products, which we refer to as the H model, we capture consumer behavior using the *multinomial logit* (MNL) model, a specialization of the linear random utility framework. Although the profit function could be non-quasiconcave in prices in this case (Hanson and Martin 1996), we show that it is unimodal to establish the sufficiency of the first-order conditions in determining optimal prices. Additionally, we show that the optimal prices depend on the individual inventory availability, instead of the aggregate inventory availability as in the V model. We also prove that the firm should charge a uniform price for all the products with inventory surplus (i.e., the inventory of each product by itself can meet the maximum possible future demand), and should charge a premium over the uniform price for any product with an inventory shortfall (i.e., the inventory of each product by itself cannot meet the maximum possible future demand). Consequently, unlike the V model, the firm extracts greater value from a product with inventory shortfall than a product with inventory surplus, regardless of their respective attribute ratings. Finally, we illustrate with a counterexample that the optimal prices do not conform to monotonicity properties that hold for the V model.

The remainder of this paper is organized as follows. Section 2 provides a review of related literature and positions our work relative to others. Section 3 introduces a linear random utility framework and presents a DP formulation for the MPDP problem. We build on this framework in §§4 and 5 to study the MPDP problems specific to vertical and horizontal product assortments. Section 6 uses numerical examples to highlight contrasts between the V and H models. Finally, §7 describes managerial insights and offers future research directions. For all the results in the paper, we provide proofs in the e-companion.[1]

2. Related Literature

As the importance of revenue management has grown in practice, so has academic research on dynamic pricing and capacity management. The papers by McGill and van Ryzin (1999), Elmaghraby and Keskinocak (2003), Bitran and Caldentey (2003), and the recent book by Talluri and van Ryzin (2004b) provide comprehensive surveys of the literature in revenue management. We focus our review on models that consider dynamic pricing over a finite selling horizon with no replenishment opportunities and classify this literature into two categories—single and multiproduct dynamic pricing models. For a single-product setting, Gallego and van Ryzin (1994) and Zhao and Zheng (2000) study dynamic pricing models with a continuous-time formulation. They show that the optimal price decreases with increasing inventory and decreasing remaining time. Focusing on contexts where continuous updating of prices might not be feasible, Feng and Gallego (1995), Bitran and Mondschein (1997), and Feng and Xiao (2000) consider models that allow for only a finite number of price changes.

In the studies that develop pricing models for multiple products, pricing decisions are linked because of joint capacity constraints and/or because of demand correlations. Given starting inventories of components, Gallego and van Ryzin (1997) model the problem of determining the price for multiple products over a finite selling horizon. Because their model is difficult to solve, they develop heuristics based on the deterministic solution to the problem and show that these are asymptotically optimal. Karaesmen and van Ryzin (2004) consider the substitutability of inventories to determine overbooking limits in a two-period model. In a case where a firm uses a single resource to produce multiple products, Maglaras and Meissner (2006) explore the relation between dynamic pricing and capacity control and show that the dynamic pricing problem in Gallego and van Ryzin (1997) and the capacity control approach (for example, Lee and Hersh 1993) can be reduced to a common formulation. Liu and Milner (2006) study the multiproduct pricing problem with a common price constraint. However, these papers do not explicitly model individual consumer choices.

Talluri and van Ryzin (2004a) and Zhang and Cooper (2005) model consumer choice behavior explicitly when considering booking limit (*capacity control*) policies for airline revenue management. Focusing on a single-leg yield management problem with exogenous fares, Talluri and van Ryzin (2004a) model how consumers choose from multiple fare products in determining the booking limits for various fare classes. Zhang and Cooper (2005) extend their model and consider capacity control for parallel flights. For an airline revenue management application that considers parallel flights and consumer choice, Zhang and Cooper (2009) model a *pricing control* problem. They acknowledge the complexity of the DP, construct heuristics, and test performance using a numerical study. Dong et al. (2009) examine both the initial inventory and subsequent dynamic pricing decisions with a multinomial logit model of consumer choice. Their work focuses on horizontally differentiated products, uses numerical experiments to demonstrate the value of dynamic pricing, and illustrates the value of their approach in determining near-optimal initial inventories. Using a modification of a budget-constrained choice model in Hauser and Urban (1986), Bitran et al. (2005) formulate a continuous-time problem to determine the optimal prices of a vertical assortment. Their model assumes that individual customers have a deterministic, common valuation of each product's quality, but vary along two characteristics—the budget for the product purchase and the valuation of the outside option. As a result, they use a bivariate random process in their model of consumer choice. Using this choice

[1] An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

model, they derive pricing policies for the deterministic equivalent of the problem and the case of unlimited inventory.

We also address a pricing control problem; however, our model differs from the existing research along the following lines. Using a linear random consumer utility model, we develop an integrative framework of dynamic pricing for various models of consumer choice and product differentiation. We consider both the cases of vertical and horizontal product differentiation and derive strong analytical results. For vertically differentiated products, unlike Bitran et al. (2005), by assuming that outside option valuations are common (a reasonable assumption given easy access to markets and information in many contexts) and by focusing on consumer sensitivity to quality, our model incorporates key factors involved in consumer choice in a manner that is conducive for analysis and parameter estimation. Hence, we are able to identify interesting properties of the optimal solution and exploit the insights to develop an efficient algorithm. For horizontally differentiated products, the non-quasiconcavity of the profit function with respect to prices has led other researchers (Dong et al. 2009, Song and Xue 2007) to examine profit as a function of market share. In contrast, we show that the profit function is unimodal with respect to price, demonstrating the sufficiency of the first-order conditions. Overall, our analysis reveals strikingly different pricing characteristics for the V and H models, underscoring the importance of studying the impact of the nature of product differentiation on the MPDP problem with appropriate consumer choice models.

3. Model Formulation

We first introduce a linear random utility framework to model consumer choice (and hence demand) in a differentiated assortment. Then, we develop a dynamic programming formulation of the MPDP problem. Finally, we comment on the structure of the MPDP profit function.

3.1. A Linear Random Utility Model for Differentiated Products

To model individual consumer choice from a differentiated assortment, we adopt an approach that is consistent with the literature on discrete choice models (described in detail in Anderson et al. 1992, Roberts and Lilien 1993). This approach treats products as bundles of attributes and captures consumer preferences for the attributes through associated random parameters. Specifically, we consider a model from the economic literature (Caplin and Nalebuff 1991, Train 2003, Hensher and Greene 2003) that assumes that a consumer's utility is linear in random parameters. In the following discussion, let ζ^k represent a random copy of random variable ζ with distribution $\Gamma(\zeta)$, i.e., ζ^k is a random draw from $\Gamma(\zeta)$. We describe the random utility u_j^k that the kth arriving customer would get from the purchase of product j with attribute rating q_j at price p_j as

$$u_j^k = \theta^k q_j - p_j + \mu \xi_j^k, \quad j = 1, 2, \ldots, n, \tag{1}$$

where $\{\theta^k\}_{k\geq 1}$ and $\{\xi_j^k\}_{k\geq 1}$ are two independent random sequences, and $\mu \geq 0$ is a scalar. In (1), all consumers have a *common* rating q_j of variant j, and θ^k represents the kth customer's *sensitivity* to q_j, with the distribution of θ^k capturing the heterogeneity of consumer sensitivities to q_j. On the other hand, the random term $\mu\xi_j^k$ captures idiosyncratic customer preferences (from factors such as aesthetics) for individual products, with μ measuring the degree of such preferences. Note that, all consumers have the same sensitivity to price in (1). We assume that a consumer is rational and purchases product j that maximizes his utility, i.e., $u_j^k = \max_i\{u_i^k, i = 1, 2, \ldots, n, n+1\}$, where $u_{n+1}^k \equiv 0$ is the value of the outside option, product $n+1$. The utility function (1) is highly flexible and embeds the two *extremes* of product differentiation—vertical and horizontal differentiation—as special cases. We introduce these cases next.

When μ is zero, (1) does not have the random term $\mu\xi_j$ and directly reflects consumers' valuation of product attributes when consumers agree on product attribute values. This choice model is a *pure characteristics demand* model (Berry and Pakes 2007) and has been widely used to describe vertical demand (e.g., Bresnahan 1987, Tirole 1988, Wauthy 1996, Bhargava and Choudhary 2008). In §4, we study the MPDP problem for vertically differentiated products using this specification of (1), with μ equal to zero and θ following a uniform distribution between 0 and 1.

When μ is positive, the random terms θ and $\mu\xi_j$ in (1) capture consumers' sensitivity to product quality and their idiosyncratic preferences for products, respectively; as such, the utility function (1) contains both vertical and horizontal elements of consumer choice. Researchers often make suitable assumptions regarding the random parameters for analytical tractability. Suppose ξ_j follows the standard Gumbel distribution, and θ follows a general continuous distribution F with support $[0, \infty]$. Then, (1) yields the *mixed logit model* (McFadden and Train 2000, Train 2003), whose choice probability for product j is given by

$$\alpha_j(p_1, p_2, \ldots, p_n) = \int_0^\infty \frac{e^{(\theta q_j - p_j)/\mu}}{1 + \sum_{k=1}^n e^{(\theta q_k - p_k)/\mu}}\, dF(\theta), \quad j = 1, 2, \ldots, n, \tag{2}$$

and the choice probability for the outside option, which has $p_{n+1} = q_{n+1} \equiv 0$, is $\alpha_{n+1}(p_1, p_2, \ldots, p_n) = 1 - \sum_{j=1}^{n} \alpha_j(p_1, p_2, \ldots, p_n)$. Because (2) does not have a closed-form expression, the mixed logit model is not conducive for analysis. To overcome this difficulty, researchers assume that θ follows a degenerate distribution whose support consists of a single value (i.e., consumers have common sensitivity to product attributes), resulting in the well-known MNL model. In the MNL model, because of the random term $\mu\xi_j$ in (1), products cannot be universally ordered by their valuations, thereby capturing the behavior of consumers facing horizontally differentiated products. In §5, we use the MNL probabilities to study the H model.

Finally, we note that Berry and Pakes (2007) show that the mixed logit model approaches the demand model for vertically differentiated products as μ approaches zero. Therefore, our vertical and horizonal consumer choice models are special cases of the mixed logit model. We also note that researchers have developed effective estimation procedures for parameters of both vertical and horizontal choice models (see Train 2003, Berry and Pakes 2007, Song 2007), allowing for their wide applicability. Assuming the availability of reliable estimates of the choice model parameters, we formulate and analyze the MPDP problem next.

3.2. Dynamic Programming Formulation

Consider the tactical pricing problem facing a firm that sells n substitutable products with indices $j = 1, 2, \ldots, n$ over a finite selling season. The firm starts the selling season with a *given* initial inventory k_j of product j and is unable to replenish these inventories during the season. Inventories of product j are perishable and have a fixed rating of q_j throughout the selling season. Note that in some applications, firms may have the ability to upgrade/downgrade a unit of inventory to a higher/lower valuation during the selling season; our model does not factor this option. We divide the selling season into T time periods such that each period has at most one customer arrival, and assume that each customer requires no more than one unit of inventory. This demand arrival model is similar to others in the revenue management literature (for instance, Gerchak et al. 1985, Talluri and van Ryzin 2004a, Zhang and Cooper 2009). Let λ_t denote the probability of a customer arrival in period t. We index the time periods in reverse chronological order; that is, $t = 0$ and $t = T$ correspond to the *end* and the *beginning* of the selling season. The firm's objective is to maximize the total revenues from the selling season by selecting an appropriate price for each product in every period; that is, in each period t, for every current inventory level $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the firm must determine the price vector $\mathbf{p}_t = (p_{1t}, p_{2t}, \ldots, p_{jt})$. Depending on the prices that the firm quotes, customers may purchase one of the products in the assortment or not purchase any product at all. Let $\alpha_j(\mathbf{p}_t)$ be the probability that a consumer chooses to buy a product j in period t, given price vector $\mathbf{p}_t$. We can express this probability by $\alpha_j(\mathbf{p}_t) = \mathbb{P}(u_j^k = \max_i\{u_i^k, i = 1, 2, \ldots, n, n+1\})$. At an inventory level $\mathbf{x}$ and time t, because some products may have zero inventory, we define a state-dependent action space

$$\mathcal{P}_{\mathbf{x}} = \{\mathbf{p}_t \geq \mathbf{0}\colon \alpha_j(\mathbf{p}_t) = 0 \text{ if } x_j = 0, j = 1, \ldots, n\}. \quad (3)$$

The exact form of $\mathcal{P}_{\mathbf{x}}$ depends on the choice probabilities, which we will specify in §§4 and 5. Given inventory vector $\mathbf{x}$, let $V_t(\mathbf{x})$ denote the optimal expected revenue from period t to the end of the season. Then, we formulate the MPDP problem as the following dynamic program:

$$V_t(\mathbf{x}) = \max_{\mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}} \left\{ \sum_{j=1}^{n} \lambda_t \alpha_j(\mathbf{p}_t)(p_{jt} + V_{t-1}(\mathbf{x} - \mathbf{e}_j)) + \lambda_t \alpha_{n+1}(\mathbf{p}_t) V_{t-1}(\mathbf{x}) + (1 - \lambda_t) V_{t-1}(\mathbf{x}) \right\}, \quad (4)$$

with boundary conditions $V_t(\mathbf{0}) = 0$ for $t = 0, 1, \ldots, T$, and $V_0(\mathbf{x}) = 0$ for all $\mathbf{x}$, where $\mathbf{e}_j$ is a vector of size n with 1 at the jth entry and zeros elsewhere. Because $\sum_{j=1}^{n+1} \alpha_j(\mathbf{p}_t) = 1$, we can rewrite the optimality equation (4) as

$$V_t(\mathbf{x}) = \max_{\mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}} \left\{ \sum_{j=1}^{n} \lambda_t \alpha_j(\mathbf{p}_t)(p_{jt} + V_{t-1}(\mathbf{x} - \mathbf{e}_j) - V_{t-1}(\mathbf{x})) \right\} + V_{t-1}(\mathbf{x}). \quad (5)$$

Let us define the difference functions of $V_t(\mathbf{x})$ with respect to t and x_j by

$$\Delta_t V_t(\mathbf{x}) = V_t(\mathbf{x}) - V_{t-1}(\mathbf{x}) \quad \text{for } t = 0, 1, \ldots, T, \quad \text{and}$$
$$\Delta_{x_j} V_t(\mathbf{x}) = V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_j) \quad \text{for } j = 1, 2, \ldots, n.$$

Here, $\Delta_t V_t(\mathbf{x})$ represents the maximum expected gain, with inventory level $\mathbf{x}$ in period t, if the firm had one additional selling period (marginal value of *time*). Similarly, $\Delta_{x_j} V_t(\mathbf{x})$ is the maximum expected gain, with inventory level $\mathbf{x}$ in period t, if the firm had one more unit of product j inventory to sell (*marginal value of inventory*). Using this notation, we define

$$G_t(\mathbf{x}, \mathbf{p}_t) = \sum_{j=1}^{n} \lambda_t \alpha_j(\mathbf{p}_t)(p_{jt} - \Delta_{x_j} V_{t-1}(\mathbf{x})), \quad \mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}, \quad (6)$$

and rewrite (5) as follows:

$$\Delta_t V_t(\mathbf{x}) = V_t(\mathbf{x}) - V_{t-1}(\mathbf{x}) = \max_{\mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}} \{G_t(\mathbf{x}, \mathbf{p}_t)\}. \quad (7)$$

Intuitively, $G_t(\mathbf{x}, \mathbf{p}_t)$ is the the expected additional gain realized in period t by selling a single unit of inventory $\mathbf{x}$ (of any product) at price $\mathbf{p}_t$. We can interpret $G_t(\mathbf{x}, \mathbf{p}_t)$ as the *marginal value of time* at inventory level $\mathbf{x}$ in period t when product prices are set at $\mathbf{p}_t$. Consequently, we determine the optimal prices so as to maximize $G_t(\mathbf{x}, \mathbf{p}_t)$. Hence, identifying the structure of $G_t(\mathbf{x}, \mathbf{p}_t)$ is the key for finding the solution to (7). We discuss the structure of $G_t(\mathbf{x}, \mathbf{p}_t)$ next.

3.3. Structure of Value Function $G_t(\mathbf{x}, \mathbf{p}_t)$

For the special case of a firm offering a single product, we can show that $G_t(x, p_t)$ is a quasiconcave function of p_t under the mixed logit choice model (see §EC.1 in the e-companion for proof and related results). However, this structural property need *not* hold in general when customers choose from multiple products. Hanson and Martin (1996) give a counterexample to illustrate the non-quasiconcavity of their profit function under the MNL model, which is a special case of the mixed logit model. We can extend this counterexample to show that our objective value function $G_t(\mathbf{x}, \mathbf{p}_t)$ under mixed logit choices is also non-quasiconcave in $\mathbf{p}_t$. Furthermore, the mixed logit choice probabilities, which are not in closed form, are not analytically tractable. Other researchers (e.g., Train 2003, Hensher and Greene 2003) have recognized this difficultly and resorted to numerical methods such as simulation and quadrature.

In this paper, we investigate the structure of $G_t(\mathbf{x}, \mathbf{p}_t)$ for two specifications of the mixed logit model corresponding to the two product differentiation schemes—vertical and horizontal differentiation. In §4, we prove that when $\mu = 0$ (*vertical differentiation*), $G_t(\mathbf{x}, \mathbf{p}_t)$ is concave in $\mathbf{p}_t$. In §5, we show that, although $G_t(\mathbf{x}, \mathbf{p}_t)$ need not be quasiconcave in $\mathbf{p}_t$ when $\mu > 0$ and θ is a constant (*horizontal differentiation*), it is *unimodal* in $\mathbf{p}_t$. Hence, the first-order conditions for $G_t(\mathbf{x}, \mathbf{p}_t)$ would yield optimal product prices in both cases. The first-order conditions, obtained by setting the partial derivatives of $G_t(\mathbf{x}, \mathbf{p}_t)$ with respect to p_{jt} to zero are given by

$$\frac{\partial G_t(\mathbf{x}, \mathbf{p}_t)}{\partial p_{jt}} = \sum_{k=1}^{n} \lambda_t \frac{\partial \alpha_k(\mathbf{p}_t)}{\partial p_{jt}} (p_{kt} - \Delta_{x_k} V_{t-1}(\mathbf{x})) + \lambda_t \alpha_j(\mathbf{p}_t) = 0, \quad j = 1, 2, \ldots, n. \tag{8}$$

The conditions in (8) form a system of equations that the optimal price vector $\mathbf{p}_t$ must satisfy. We write this relation in matrix form as

$$\mathbf{p}_t = \mathbf{h}(\mathbf{p}_t(\mathbf{x})) + \Delta_{\mathbf{x}} V_{t-1}(\mathbf{x}), \tag{9}$$

where $\mathbf{h}(\mathbf{p}_t) = (h_1(\mathbf{p}_t), \ldots, h_n(\mathbf{p}_t))$ is an n-dimensional vector given by

$$\mathbf{h}(\mathbf{p}_t) = -\boldsymbol{\alpha}(\mathbf{p}_t) \left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} \right)^{-1}, \tag{10}$$

where $\partial \boldsymbol{\alpha}(\mathbf{p}_t)/\partial \mathbf{p}_t$ is the Jacobian matrix of $\boldsymbol{\alpha}(\mathbf{p}_t) = (\alpha_1(\mathbf{p}_t), \alpha_2(\mathbf{p}_t), \ldots, \alpha_n(\mathbf{p}_t))$, with $\partial \alpha_i(\mathbf{p}_t)/\partial p_{jt}$ as its element (i, j), and $(\partial \boldsymbol{\alpha}(\mathbf{p}_t)/\partial \mathbf{p}_t)^{-1}$ is the inverse of the Jacobian matrix.

Equation (9) shows that the optimal price of product j at inventory level $\mathbf{x}$ in period t is composed of two terms—the first term $h_j(\mathbf{p}_t)$ is the *current* marginal value of the product j's inventory, and the second term $\Delta_{x_j} V_{t-1}(\mathbf{x})$ is the *future* marginal value of the product j's inventory, at inventory level $\mathbf{x}$. These two terms highlight the decision maker's need to incorporate short- and long-term considerations in determining the optimal prices.

4. Pricing of Vertically Differentiated Products

From the utility function (1) with $\mu = 0$ and θ as a uniform $[0, 1]$ random variable, we first develop a quality-based choice model for vertically differentiated products in §4.1. Then, we formulate the MPDP problem with this particular choice model in §4.2. We derive structural properties of the optimal prices and discuss managerial implications in §4.3. Finally, in §4.4, we synthesize these results to develop an effective and exact algorithm to solve this problem.

4.1. A Quality-Based Choice Model for Vertical Differentiation

Consider a consumer who must choose from n products that have different quality ratings. Let product j have a quality index q_j (common to all consumers), and assume that product quality can be ordered as $q_1 > q_2 > \cdots > q_n > 0$. We model consumer utility as a variant of the linear random utility function (1) with random coefficient θ being uniformly distributed between 0 and 1, and $\mu \xi_j$ excluded ($\mu = 0$). Accordingly, we can express a typical consumer's utility from the purchase of product j at price p_{jt} as $u_{jt} = \theta q_j - p_{jt}$. In this setup, if any two products i and j have the same price, then a consumer would prefer product i over product j when $q_i > q_j$. A consumer can also choose the *no-purchase* option, product $n + 1$, with $q_{n+1} \equiv p_{n+1,t} \equiv 0$, for all $0 \le t \le T$.

A utility maximizing customer facing price $\mathbf{p}_t$ in period t would choose product j at time t if $u_{jt} \ge \max_{k \ne j} \{u_{kt}\}$, or equivalently if $\theta q_j - p_{jt} \ge \theta q_k - p_{kt}$, $\forall k \ne j$, $k = 1, \ldots, n + 1$. This further implies that a consumer will choose product j with probability

$$\alpha_j(\mathbf{p}_t) = \begin{cases} P\left(\max_{k>1} \left\{ \dfrac{p_{1t} - p_{kt}}{q_1 - q_k} \right\} \le \theta \right), & j = 1; \\ P\left(\max_{k>j} \left\{ \dfrac{p_{jt} - p_{kt}}{q_j - q_k} \right\} \le \theta \le \min_{k<j} \left\{ \dfrac{p_{kt} - p_{jt}}{q_k - q_j} \right\} \right), & j = 2, \ldots, n; \\ P\left(\min_{k<n+1} \left\{ \dfrac{p_{kt}}{q_k} \right\} \ge \theta \right), & j = n + 1. \end{cases} \tag{11}$$

To obtain an explicit expression of these choice probabilities, first note that for product $j = n$, Equation (11) implies that it is sufficient to consider the candidate prices $p_{n-1,t}$ and p_{nt} satisfying

$$0 \le \frac{p_{nt} - p_{n+1,t}}{q_n - q_{n+1}} \le \frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n}. \tag{12}$$

In (12), the first inequality is self-explanatory; the second inequality is valid, for otherwise the choice probability in (11) for product n would be zero, the same as the null probability when the second inequality in (12) assumes an equality. Similarly, for product $j = n-1$, (11) implies that it is sufficient to consider the candidate prices $p_{n-2,t}$, $p_{n-1,t}$, and $p_{n,t}$ such that

$$\max\left\{\frac{p_{n-1,t} - p_{n,t}}{q_{n-1} - q_n}, \frac{p_{n-1,t} - p_{n+1,t}}{q_{n-1} - q_{n+1}}\right\} = \frac{p_{n-1,t} - p_{n,t}}{q_{n-1} - q_n} \le \frac{p_{n-2,t} - p_{n-1,t}}{q_{n-2} - q_{n-1}},$$

where the equality in the above expression is the result of (12). Following a similar argument, we can show inductively that for any product j, we only need to consider the prices that satisfy

$$\max_{k>j}\left\{\frac{p_{jt} - p_{k,t}}{q_j - q_k}\right\} = \frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}} \le \frac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j}, \quad \text{for } j = 2, \dots, n; \tag{13}$$
$$\max_{k>1}\left\{\frac{p_{1t} - p_{kt}}{q_1 - q_k}\right\} = \frac{p_{1t} - p_{2,t}}{q_1 - q_2} \le 1, \quad \text{for } j = 1.$$

Furthermore, to ensure $\mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}$, we need to force (13) to an equality so that $\alpha_j(\mathbf{p}_t) = 0$ whenever $x_j = 0$, $j = 1, \dots, n$. Together, our observations imply that, given $\mathbf{x}$, it is sufficient to restrict the candidate prices to the following set of *quality-aligned prices*, which we denote as $\widetilde{\mathcal{P}}_{\mathbf{x}}$:

$$\widetilde{\mathcal{P}}_{\mathbf{x}} = \left\{ \mathbf{p}_t\colon \begin{array}{l} 1 \ge \dfrac{p_{1t} - p_{2t}}{q_1 - q_2} \ge \dfrac{p_{2t} - p_{3t}}{q_2 - q_3} \\ \quad \ge \cdots \ge \dfrac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n} \ge \dfrac{p_{nt} - p_{n+1,t}}{q_n - q_{n+1}} \ge 0; \\ \dfrac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}} = \dfrac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j}, \\ \qquad \text{if } x_j = 0, j = 2, 3, \dots, n; \\ \dfrac{p_{1t} - p_{2t}}{q_1 - q_2} = 1, \quad \text{if } x_1 = 0. \end{array} \right\}, \tag{14}$$

The first constraint of (14) requires that the "price increment per unit quality," $(p_{jt} - p_{j+1,t})/(q_j - q_{j+1})$, is decreasing in quality index j. This condition is equivalent to saying that the candidate prices $\{p_{jt}\}$ must form an *increasing convex mapping* of the quality values $\{q_j\}$ over the interval $[0, 1]$. A managerial implication of this result is that the firm has greater pricing power with higher quality products. Because (14) is an increasing mapping, the firm can charge higher prices for higher quality products (*quality monotonicity*). In addition, because (14) is a convex mapping, the firm's premium for a higher quality product is nondecreasing in quality, allowing it to charge a larger price per unit quality for a higher quality product. Finally, because (11)–(14) do not depend on distributional assumptions, this observation holds for any distribution of θ.

By restricting the prices $\mathbf{p}_t$ to the set $\widetilde{\mathcal{P}}_x$, we have

$$\max_{k>j}\left\{\frac{p_{jt} - p_{kt}}{(q_j - q_k)}\right\} = \frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}} \quad \text{and}$$

$$\min_{k<j}\left\{\frac{p_{k,t} - p_{jt}}{q_k - q_j}\right\} = \frac{p_{j-1,t} - p_{j,t}}{q_{j-1} - q_j}.$$

Therefore, we can write the choice probabilities (11) as

$$\alpha_j(\mathbf{p}_t) = \begin{cases} 1 - \dfrac{p_{1t} - p_{2t}}{q_1 - q_2}, & j = 1; \\ \dfrac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j} - \dfrac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}}, & j = 2, \dots, n; \\ 1 - \displaystyle\sum_{i=1}^{n} \alpha_i(\mathbf{p}_t) = \dfrac{p_{nt}}{q_n}, & j = n+1. \end{cases} \tag{15}$$

4.2. Optimal Pricing for Vertical Differentiation

By substituting the purchase probabilities from (15) into (7), we write $G_t(\mathbf{x}, \mathbf{p}_t)$, defined in (6), as

$$\begin{aligned} G_t(\mathbf{x}, \mathbf{p}_t) = {} & \lambda_t\left(1 - \frac{p_{1t} - p_{2t}}{q_1 - q_2}\right)(p_{1t} - \Delta_{x_1} V_{t-1}(\mathbf{x})) \\ & + \sum_{k=2}^{n-1} \lambda_t\left(\frac{p_{k-1,t} - p_{kt}}{q_{k-1} - q_k} - \frac{p_{kt} - p_{k+1,t}}{q_k - q_{k+1}}\right) \\ & \cdot (p_{kt} - \Delta_{x_k} V_{t-1}(\mathbf{x})) \\ & + \lambda_t\left(\frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n} - \frac{p_{nt}}{q_n}\right) \\ & \cdot (p_{nt} - \Delta_{x_n} V_{t-1}(\mathbf{x})), \quad \mathbf{p}_t \in \widetilde{\mathcal{P}}_{\mathbf{x}}. \end{aligned} \tag{16}$$

We first assume $\mathbf{x} > \mathbf{0}$. Clearly, $G_t(\mathbf{x}, \mathbf{p}_t)$ in (16) is a quadratic function of $\mathbf{p}_t$. Theorem 1 states that $G_t(\mathbf{x}, \mathbf{p}_t)$ is also a concave function of $\mathbf{p}_t$.

Theorem 1. *In the V model, $G_t(\mathbf{x}, \mathbf{p}_t)$ is a concave function of $\mathbf{p}_t$.*

Because of Theorem 1, the first-order conditions for $G_t(\mathbf{x}, \mathbf{p}_t)$, given in (9), are sufficient to determine optimal product prices. The Jacobian matrix of probability vector $\boldsymbol{\alpha}(\mathbf{p}_t)$ for given $\mathbf{p}_t \in \widetilde{\mathscr{P}}_{\mathbf{x}}$ is

$$\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} = \begin{pmatrix} \frac{-1}{q_1-q_2} & \frac{1}{q_1-q_2} & 0 & 0 & \cdots & 0 \\ \frac{1}{q_1-q_2} & \frac{-(q_1-q_3)}{(q_1-q_2)(q_2-q_3)} & \frac{1}{q_2-q_3} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{q_{n-1}-q_n} & \frac{-q_{n-1}}{(q_{n-1}-q_n)q_n} \end{pmatrix}. \tag{17}$$

The Jacobian of $\boldsymbol{\alpha}(\mathbf{p}_t)$ in (17) is a tridiagonal symmetric matrix with negative diagonal entries and nonnegative nondiagonal entries, with each entry independent of the price vector $\mathbf{p}_t$. Because all the nontridiagonal entries are zero, an infinitesimal change in the price of product j will result in demand substitution, but only to an adjacent product. Moreover, this demand substitution occurs at a linear rate, i.e., when the price of product j increases, the likelihood that a customer will buy j decreases at a *linear* rate and the probability that the customer will buy $j-1$ or $j+1$ increases at a *linear* rate. Also, $\partial\alpha_{n+1}(\mathbf{p}_t)/\partial p_{nt} = 1/q_n$ and $\partial\alpha_{n+1}(\mathbf{p}_t)/\partial p_{jt} = 0$ for $j = 1, 2, \ldots, n-1$, implying that the total purchase probability (market share) depends *only* on p_{nt}, the price of the lowest quality product.

We can derive the inverse of the Jacobian matrix in (17) as

$$\left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t}\right)^{-1} = -\begin{pmatrix} q_1 & q_2 & \cdots & q_{n-1} & q_n \\ q_2 & q_2 & \cdots & q_{n-1} & q_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n-1} & q_{n-1} & \cdots & q_{n-1} & q_n \\ q_n & q_n & \cdots & q_n & q_n \end{pmatrix}. \tag{18}$$

Substituting (17) and (18) into

$$\mathbf{h}(\mathbf{p}_t) = -\boldsymbol{\alpha}(\mathbf{p}_t)\left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p})}{\partial \mathbf{p}_t}\right)^{-1},$$

we get

$$h_j(\mathbf{p}_t) = \left(1 - \frac{p_{1t}-p_{2t}}{q_1-q_2}, \ldots, \frac{p_{j-1,t}-p_{jt}}{q_{j-1}-q_j} - \frac{p_{j,t}-p_{j+1,t}}{q_j-q_{j+1}}, \ldots, \frac{p_{n-1,t}-p_{nt}}{q_{n-1}-q_n} - \frac{p_{nt}}{q_n}\right) \times \begin{pmatrix} q_j \\ q_j \\ \vdots \\ q_j \\ q_{j+1} \\ \vdots \\ q_n \end{pmatrix} = q_j - p_{jt}, \quad j = 1, 2, \ldots, n. \tag{19}$$

Because the optimal prices satisfy $p_{jt}(\mathbf{x}) = q_j - p_{jt}(\mathbf{x}) + \Delta_{x_j} V_{t-1}(\mathbf{x})$, from (9), we obtain

$$p_{jt}(\mathbf{x}) = \tfrac{1}{2}(q_j + \Delta_{x_j} V_{t-1}(\mathbf{x})), \quad j = 1, 2, \ldots, n. \tag{20}$$

Thus, if $x_j > 0$, the optimal price $p_{jt}(\mathbf{x})$ is the average of the quality rating of product j and the future marginal value of product j inventory. Note that $p_{jt}(\mathbf{x}) \in [q_j/2, q_j]$. Using (16), we obtain

$$\begin{aligned} V_t(\mathbf{x}) &= G_t(\mathbf{x}, \mathbf{p}_t(\mathbf{x})) + V_{t-1}(\mathbf{x}) \\ &= \lambda_t\left(q_1 - 2p_{1t}(\mathbf{x}) + \sum_{k=1}^{n-1} \frac{(p_{kt}(\mathbf{x}) - p_{k+1,t}(\mathbf{x}))^2}{q_k - q_{k+1}} + \frac{p_{nt}^2(\mathbf{x})}{q_n}\right) \\ &\quad + V_{t-1}(\mathbf{x}). \end{aligned} \tag{21}$$

Now, suppose $x_j = 0$ in $\mathbf{x}$ for some j. Then, we adopt the following two steps to determine the price for each product. In the first step, we ignore the zero-inventory products and solve the problem for the positive-inventory products alone using the procedure described in (20) and (21). In the second step, we set the prices for the zero-inventory products as follows. Recall that if $x_j = 0$, the price $p_{jt}(\mathbf{x})$ needs to satisfy

$$\frac{p_{1t}(\mathbf{x}) - p_{2t}(\mathbf{x})}{q_1 - q_2} = 1 \quad \text{for } j = 1 \quad \text{and}$$

$$\frac{p_{jt}(\mathbf{x}) - p_{j+1,t}(\mathbf{x})}{q_j - q_{j+1}} = \frac{p_{j-1,t}(\mathbf{x}) - p_{jt}(\mathbf{x})}{q_{j-1} - q_j} \quad \text{for } j = 2, \ldots, n;$$

or, equivalently,

$$p_{jt}(\mathbf{x}) = \begin{cases} p_{2t}(\mathbf{x}) + (q_1 - q_2), & j = 1; \\ \dfrac{(q_j - q_{j+1})p_{j-1,t}(\mathbf{x}) + (q_{j-1} - q_j)p_{j+1,t}(\mathbf{x})}{q_{j-1} - q_{j+1}}, & \\ \qquad j = 2, \ldots, n. \end{cases} \tag{22}$$

This provides us with a recursive relation to determine the price of a zero-inventory product from that of its adjacent products. By using the recursion (iteratively, if necessary), we can express the price of each zero-inventory product by the price(s) of its adjacent, positive-inventory product(s). Because the prices of the positive-inventory products have been determined in the first step, the prices for the zero-inventory products are determined uniquely. The value function in (21) still holds, except that the summation is over index j such that $x_j > 0$. Therefore, we can assume $\mathbf{x} > 0$ without loss of generality and do so for the remainder of this section.

4.3. Structural Properties of the V Model

In this section, we first uncover a special structure of the V model that highlights the importance of *aggregating* product inventories in descending order of product quality (Theorem 2). Then, using insights that stem from this structure, we establish the properties of the optimal prices (Theorem 3) and value (Theorem 4).

To facilitate this discussion, we refer to the aggregate inventory of the first j highest quality products, $|\mathbf{x}|_j = \sum_{i=1}^{j} x_i$, as product j's *aggregate inventory*, and say that product j has an *aggregate inventory surplus* if $|\mathbf{x}|_j \geq t$, i.e., if $|\mathbf{x}|_j$ is sufficient to satisfy the potential remaining demand. The following theorem derives core properties that state that the marginal value of product j's inventory depends on $(x_1, x_2, \ldots, x_j)$ *only* through their sum $|\mathbf{x}|_j$, and if product j's aggregate inventory is in surplus, the marginal value of any lower quality inventory is zero.

Theorem 2. *For the* V *model,*

$$\Delta_{x_j} V_t(\mathbf{x}) = \Delta_{x_j} V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j)$$

for any $i < j$. Furthermore, if $|\mathbf{x}|_j \geq t$, then $\Delta_{x_k} V_t(\mathbf{x}) = 0$ for any $k > j$.

To understand Theorem 2, we first examine the value of product j's inventory in period $t = 1$. Because the inventories have zero future marginal value, it follows from (20) that $p_{j,1}(\mathbf{x}) = q_j/2$ for all j. With these optimal prices, the choice probability for the highest quality product is $\alpha_{1,1} = 0.5$, and for any lower quality product, it is $\alpha_{j,1} = 0$, for $j > 1$. This stems from the observation in §4.2 that the optimal prices form an increasing convex mapping of the quality ratings, giving the firm greater pricing power with higher quality products. Consequently, when $t = 1$, the firm will set the prices such that the highest quality product *alone* has a positive choice probability. Effectively, the firm aggregates its inventories in descending order of product quality and extracts positive value only from a single unit of the highest quality product. Through induction, we find that this result can be extended to other periods $t > 1$; that is, if the inventories are sufficient to meet maximum potential demand t, then only the first t units of the *rank-ordered, aggregated inventory* have a positive value. This implies that values of all remaining inventories are zero, regardless of the quality levels of the first t units. We also find that, even when the inventories are scarce ($<t$), the principle of inventory aggregation by descending order of quality holds; that is, the marginal value of product j depends on $|\mathbf{x}|_j$ rather than the individual inventory levels of the first j products.

Theorem 2 has two key implications. First, by recursively applying Theorem 2, we can obtain

$$\begin{aligned}\Delta_{x_j} V_t(\mathbf{x}) &= \Delta_{x_j} V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j)\\ &= \cdots = \Delta_{x_j} V_t(0, \ldots, 0, |\mathbf{x}|_j, x_{j+1}, \ldots, x_n).\end{aligned} \quad (23)$$

Note that although $|\mathbf{x}|_j$ captures the impact of all higher quality inventories, the marginal value of product j still depends on the individual inventory levels of lower quality products. Second, from Theorem 2, for $i < i' \leq j$,

$$\begin{aligned}V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_j) &= \Delta_{x_j} V_t(\mathbf{x}) = \Delta_{x_j} V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'})\\ &= V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'}) - V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'} - \mathbf{e}_j),\end{aligned}$$

or, equivalently,

$$\begin{aligned}&V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'})\\ &\quad = V_t(\mathbf{x} - \mathbf{e}_j) - V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'} - \mathbf{e}_j).\end{aligned} \quad (24)$$

Equation (24), in essence, states that the marginal value of having one less unit of higher quality product i and one more unit of lower quality product i' is independent of $(x_{i'}, \ldots, x_n)$, the inventories of products with quality no better than i'. As we will see next, Theorem 2 and its key implications are instrumental in establishing the optimal prices for the V model. In the following discussion, we say that products i and j are *adjacent* if $x_i > 0$, $x_j > 0$, and $x_k = 0$ for $i < k < j$.

Theorem 3. *For the* V *model, the optimal prices $\mathbf{p}_t(\mathbf{x})$ have the following properties*:

(a) *If $|\mathbf{x}|_j \geq t$, then $p_{kt}(\mathbf{x}) = q_k/2$ for $k \geq j$ and $\alpha_k(\mathbf{p}_t(\mathbf{x})) = 0$ for $k > j$.*

(b) *$p_{jt}(\mathbf{x})$ depends on the inventory levels of the first j products only through their sum $|\mathbf{x}|_j$. Specifically, for $i' < i \leq j$,*

$$\begin{aligned}p_{jt}(\mathbf{x}) &= p_{jt}(\mathbf{x} - \mathbf{e}_{i'} + \mathbf{e}_i)\\ &= \cdots = p_{jt}(0, \ldots, 0, |\mathbf{x}|_j, x_{j+1}, \ldots, x_n).\end{aligned} \quad (25)$$

(c) *Let products i and j be two adjacent products. Then the price difference $p_{it}(\mathbf{x}) - p_{jt}(\mathbf{x})$ is strictly positive and* *depends only on the total inventory of the first i products $|\mathbf{x}|_i$. Specifically, for $i = 1, \ldots, n-1$, and $j > i$,*

$$\begin{aligned} p_{it}(\mathbf{x}) - p_{jt}(\mathbf{x}) &= \tfrac{1}{2}\big(q_i - q_j + V_{t-1}(0,\ldots,0,|\mathbf{x}|_i,0,\ldots,0) \\ &\qquad - V_{t-1}(0,\ldots,0,|\mathbf{x}|_i - 1,0,\ldots,0,1,0,\ldots,0)\big), \quad (26) \end{aligned}$$

where $|\mathbf{x}|_i$ and $|\mathbf{x}|_i - 1$ are in the ith position of the two vectors and 1 is in the jth position.

When the firm has an aggregate inventory surplus for some product j, then, as stated in part (a), the firm should set prices as $p_{kt}(\mathbf{x}) = q_k/2$, for $k \geq j$, and for the rest of the selling season, because $\Delta_{x_k} V_{t-1}(\mathbf{x}) = 0$ by Theorem 2. At this constant price, we have $(p_{k-1,t}(\mathbf{x}) - p_{kt}(\mathbf{x}))/(q_{k-1} - q_k) = (p_{kt}(\mathbf{x}) - p_{k+1,t}(\mathbf{x}))/(q_k - q_{k+1})$, for $k > j$, implying that the purchase probability for any lower quality product $k > j$ is zero. Consequently, inventories of lower quality products have no effect on the firm's pricing strategy for higher quality products. Parts (b) and (c) follow from the two key implications of Theorem 2 stated in (23) and (24), respectively, and emphasize that the principle of inventory aggregation by quality applies uniformly throughout the season. In particular, part (b) suggests that when setting the price $p_{jt}(\mathbf{x})$ for product j, the firm should treat the inventories $(x_1, \ldots, x_j)$ of the first j highest quality products at the *aggregate level* $|\mathbf{x}|_j$. Note that, for two adjacent products i and j, the price $p_{it}(\mathbf{x})$ of the higher quality product depends on both the aggregate inventory $|\mathbf{x}|_i$ and individual inventories $(x_j, \ldots, x_n)$ of the lower quality products. By expressing $p_{it}(\mathbf{x})$ as the sum of two components—the price of the adjacent lower quality product $p_{jt}(\mathbf{x})$ and a *markup* over this price—part (c) specifies how $p_{it}(\mathbf{x})$ depends on the *aggregate* inventory of high quality products and *individual* inventories of lower quality products. Specifically, the markup component, which depends solely on $|\mathbf{x}|_i$, captures the impact of the aggregate higher quality inventory, and the $p_{jt}(\mathbf{x})$ component, which depends on $|\mathbf{x}|_i + x_j$ and $(x_{j+1}, \ldots, x_n)$, factors both the aggregate higher quality inventory and individual lower quality inventories. We exploit these properties to develop a polynomial-time exact algorithm in §4.4.

Theorem 3 has several managerial and operational implications that highlight the importance of aggregate inventory properties in determining optimal prices. When inventory is abundant relative to demand (i.e., when demand is lower than that anticipated by strategic capacity choices or when the end of the season approaches), Theorem 3 provides the firm with a simple rule, via the notion of aggregate inventory surplus of a product, to determine which product inventories to market and which to ignore. Even when inventories are scarce, inventory aggregation by quality is a key principle. In this case, the principle suggests that in determining the price of any product j, managers can treat j as the product with the highest quality in the assortment with corresponding inventory of $|\mathbf{x}|_j$. By accounting for inventories of higher quality products $i < j$ through the aggregate inventory $|\mathbf{x}|_j$, this principle emphasizes that it is the *echelon-type inventory position* $|\mathbf{x}|_j$ of product j, rather than individual availabilities of higher quality products, that determines the price of the product. Moreover, the price of product j can be determined as a markup over the adjacent lower quality product, where the markup itself is determined *solely* by the the aggregate inventory $|\mathbf{x}|_j$ of product j. Together, these results (based on the aggregation principle) provide great insights into the manager's problem of jointly pricing of vertically differentiated product inventories over time.

Theorem 4. *For the* V *model,*
(a) *$\Delta_{x_j} V_t(\mathbf{x})$ is nondecreasing in t (or equivalently, $\Delta_t V_t(\mathbf{x})$ is nondecreasing in x_j);*
(b) *$\Delta_{x_j} V_t(\mathbf{x})$ is nonincreasing in x_i, $i \neq j$;*
(c) *$\Delta_{x_j} V_t(\mathbf{x})$ is nonincreasing in x_j;*
(d) *if $\lambda_t \geq \lambda_{t+1}$, then $\Delta_t V_t(\mathbf{x})$ is nonincreasing in t.*

Theorem 4 implies that $V_t(\mathbf{x})$ is supermodular in time t and product j's inventory x_j, submodular in inventory levels of products i and j (x_i and x_j, respectively), and concave in product j's inventory x_j. Furthermore, $V_t(\mathbf{x})$ is also concave in time t if the arrival probability λ_t is nonincreasing in t. Because the optimal price satisfies $p_{jt}(\mathbf{x}) = \frac{1}{2}(q_j + \Delta_{x_j} V_{t-1}(\mathbf{x}))$, we know that $p_{jt}(\mathbf{x})$ carries all the structural properties of $\Delta_{x_j} V_{t-1}(\mathbf{x})$. The next corollary follows directly from Theorem 4.

Corollary 1. *For the* V *model, optimal price $p_{jt}(\mathbf{x})$ is nondecreasing function in t, strictly increasing in j, nonincreasing in x_j, and nonincreasing in x_i for $i \neq j$.*

Corollary 1 generalizes the monotonicity results from a single product (see §EC.1 in the e-companion) to an assortment of vertically differentiated products. Specifically, Corollary 1 shows that the optimal prices exhibit (1) *quality monotonicity* (a higher quality product is always priced higher than a lower quality product; as mentioned earlier, this property follows from the fact that the optimal prices must form an increasing convex mapping of the quality ratings), (2) *inventory monotonicity* ($p_{jt}(\mathbf{x})$ becomes lower if the inventory of *any product* becomes higher), and (3) *time monotonicity* (the price for any product is nonincreasing when the end of the selling season approaches). As we will see in §6, these monotonicity properties do not always hold for the H model.

4.4. An Efficient and Exact Algorithm for the V Model

We can effectively translate the structural results in §4.3 into an exact, polynomial-time computational algorithm. Exploiting the structure of the optimal prices, this algorithm decomposes the multidimensional state and action spaces of the V model into a sequence of single-dimensional state and action space DPs, thereby drastically reducing the computational effort, both in memory requirements and running time. In the following discussion, we first highlight the key insights that drive our algorithm and then present a formal description of the procedure.

To describe this algorithm, we first define the following notation that helps us identify those products that have positive inventories at a given inventory level. Suppose inventory level $\mathbf{x}$ has $m(\mathbf{x}) \le n$ products with positive inventories. Let $k_r(\mathbf{x})$, $r = 1, \ldots, m(\mathbf{x})$, be the product with the rth highest quality rating among the products with nonzero inventory levels. Accordingly, $\{k_1(\mathbf{x}), \ldots, k_{m(\mathbf{x})}(\mathbf{x})\}$ represents the set of products with nonzero inventories at inventory level $\mathbf{x}$. For convenience, hereafter, we express $k_r(\mathbf{x})$ simply as k_r, and $m(\mathbf{x})$ as m.

Our algorithm recursively uses the price difference function (26), stated in Theorem 3(c), to obtain optimal prices $p_{k_r,t}(\mathbf{x})$, starting from the price of the lowest quality product, $p_{k_m,t}(\mathbf{x})$. To use (26), our first step is to solve a sequence of single-dimensional DP problems,

$$V_t^i(x) := V_t(0, \ldots, 0, x, 0, \ldots, 0), \quad \text{for } i = 1, \ldots, n,$$

where x appears in the ith position, and

$$V_t^{i,j}(x-1,1) := V_t(0,\ldots,0,x-1,0,\ldots,0,1,0,\ldots,0), \quad \text{for } i=1,\ldots,n-1, j>i,$$

where $x-1$ appears in the ith position and 1 in the jth position.

In the second step, we determine the optimal price for each product, computing the prices for positive inventory products first and then using (22) for the remaining zero-inventory products. Recall from Theorem 3(b) that the price of product k_m depends only on its aggregate inventory $|\mathbf{x}|_{k_m}$. Accordingly, the optimal price $p_{k_m,t}(\mathbf{x})$ of the lowest quality product k_m with positive inventory can be obtained as $p_{k_m,t}(\mathbf{x}) = \frac{1}{2}(q_{k_m} + V_{t-1}^{k_m}(|\mathbf{x}|_{k_m}) - V_{t-1}^{k_m}(|\mathbf{x}|_{k_m} - 1))$. Next, our algorithm progresses from the lowest quality product k_m to the adjacent higher quality product k_{m-1} using (26) to obtain the price $p_{k_{m-1},t}(\mathbf{x})$ as

$$p_{k_{m-1},t}(\mathbf{x}) = p_{k_m,t}(\mathbf{x}) + \tfrac{1}{2}\big(q_{k_{m-1}} - q_{k_m} + V_{t-1}^{k_{m-1}}(|\mathbf{x}|_{k_{m-1}}) - V_{t-1}^{k_{m-1},k_m}(|\mathbf{x}|_{k_{m-1}}, 1)\big).$$

Because we have already established the price $p_{k_m,t}(\mathbf{x})$ and computed the optimal values $V_{t-1}^{k_{m-1},k_m}(x-1,1)$ and $V_{t-1}^{k_{m-1}}(x)$, we can readily compute $p_{k_{m-1},t}(\mathbf{x})$. Recursively, for any two adjacent products i and j, starting with the optimal single-dimensional DP solutions $V_{t-1}^{i,j}(x-1,1)$ and $V_{t-1}^{i}(x)$ and the computed price $p_{jt}(\mathbf{x})$, we can determine price $p_{it}(\mathbf{x})$ using the corresponding price equation.

We can also derive the value function $V_t(\mathbf{x})$ directly from the single-dimensional DPs $V_{t-1}^{i,j}(x-1,1)$ and $V_{t-1}^{i}(x)$. From (24), for the two highest quality adjacent products k_1 and k_2,

$$\begin{aligned} V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_{k_1} + \mathbf{e}_{k_2}) &= V_t(0,\ldots,0,x_{k_1},0,\ldots,0) \\ &\quad - V_t(0,\ldots,0,x_{k_1}-1,0,\ldots,0,1,0,\ldots,0) \\ &= V_t^{k_1}(x_{k_1}) - V_t^{k_1,k_2}(x_{k_1}-1,1). \end{aligned}$$

Using this relationship recursively, we obtain

$$\begin{aligned} V_t(\mathbf{x}) &= V_t(\mathbf{x}-\mathbf{e}_{k_1}+\mathbf{e}_{k_2}) + V_t^{k_1}(x_{k_1}) - V_t^{k_1,k_2}(x_{k_1}-1,1) \\ &= V_t(\mathbf{x}-2\mathbf{e}_{k_1}+2\mathbf{e}_{k_2}) + V_t^{k_1}(x_{k_1}-1) \\ &\quad - V_t^{k_1,k_2}(x_{k_1}-2,1) + V_t^{k_1}(x_{k_1}) - V_t^{k_1,k_2}(x_{k_1}-1,1) \\ &= V_t(0,\ldots,0,|\mathbf{x}|_{k_2},x_{k_2+1},\ldots,x_n) \\ &\quad + \sum_{x=1}^{|\mathbf{x}|_{k_1}} V_t^{k_1}(x) - \sum_{x=0}^{|\mathbf{x}|_{k_1}-1} V_t^{k_1,k_2}(x,1) \\ &= V_t(0,0,|\mathbf{x}|_{k_3},x_{k_3+1},\ldots,x_n) \\ &\quad + \sum_{x=1}^{|\mathbf{x}|_{k_2}} V_t^{k_2}(x) - \sum_{x=0}^{|\mathbf{x}|_{k_2}-1} V_t^{k_2,k_3}(x,1) \\ &\quad + \sum_{x=1}^{|\mathbf{x}|_{k_1}} V_t^{k_1}(x) - \sum_{x=0}^{|\mathbf{x}|_{k_1}-1} V_t^{k_1,k_2}(x,1) \\ &= \cdots \\ &= \sum_{r=1}^{m}\sum_{x=1}^{|\mathbf{x}|_{k_r}} V_t^{k_r}(x) - \sum_{j=2}^{m}\sum_{x=0}^{|\mathbf{x}|_{k_{r-1}}-1} V_t^{k_{r-1},k_r}(x-1,1), \\ &\qquad \mathbf{x} \le \boldsymbol{\kappa}, 1 \le t \le T. \end{aligned} \tag{27}$$

We formally describe the steps in our exact algorithm next.

Algorithm 1 (An exact algorithm for Model V)

Step 1. **Determine the value functions** $V_t^i(x)$ **and** $V_t^{i,j}(x,1)$: For all t, compute $V_t^i(x)$ for $i = 1, \ldots, n$, and $V_t^{i,j}(x,1)$ for all $i = 1, \ldots, n-1$ and $j > i$.

Step 2. **Determine the optimal prices** $p_{jt}(\mathbf{x})$, $j = 1, \ldots, n$:

Step 2a. For all t, set price $p_{k_m,t}(\mathbf{x})$ for the lowest quality product with positive inventory as

$$p_{k_m,t}(\mathbf{x}) = \tfrac{1}{2}\big(q_{k_m} + V_{t-1}^{k_m}(|\mathbf{x}|_{k_m}) - V_{t-1}^{k_m}(|\mathbf{x}|_{k_m} - 1)\big). \tag{28}$$

Step 2*b*. For all t, starting with $r = m$ and $2 \le r \le m$, recursively set the price $p_{k_{r-1},t}(\mathbf{x})$ of the adjacent lower quality product with positive inventory as

$$p_{k_{r-1},t}(\mathbf{x}) = p_{k_r,t}(\mathbf{x}) + \tfrac{1}{2}\big(q_{k_{r-1}} - q_{k_r} + V_{t-1}^{k_{r-1}}(|\mathbf{x}|_{k_{r-1}}) - V_{t-1}^{k_{r-1},k_r}(|\mathbf{x}|_{k_{r-1}} - 1, 1)\big). \quad (29)$$

Step 2*c*. For all t and any j with $x_j = 0$, set $p_{jt}(\mathbf{x})$ as

$$p_{jt}(\mathbf{x}) = \begin{cases} p_{2t}(\mathbf{x}) + (q_1 - q_2), & j = 1; \\ \dfrac{(q_j - q_{j+1})p_{j-1,t}(\mathbf{x}) + (q_{j-1} - q_j)p_{j+1,t}(\mathbf{x})}{q_{j-1} - q_{j+1}}, & \\ \qquad j = 2, \ldots, n. & \end{cases} \quad (30)$$

Step 3. **Compute the value function** $V_t(\mathbf{x})$: For all t, $1 \le r < m$, and $0 \le \mathbf{x} \le \boldsymbol{\kappa}$, set

$$V_t(\mathbf{x}) = \sum_{r=1}^{m} \sum_{x=1}^{|\mathbf{x}|_{k_r}} V_t^{k_r}(x) - \sum_{r=2}^{m} \sum_{x=1}^{|\mathbf{x}|_{k_{r-1}}-1} V_t^{k_{r-1},k_r}(x, 1). \quad (31)$$

We close this section with a brief discussion of the complexity of our algorithm versus that of the standard backward induction algorithm. Suppose $\kappa = \max_j\{\kappa_j\}$ is the maximum of the initial product inventory levels. Then, solving for the value function $V_t(\mathbf{x})$ for $1 \le t \le T$ using backward induction would need an *exponential* number of state evaluations, resulting in a running time of $O(\kappa^n T)$. In contrast, the computational complexity of our algorithm depends on that of the single-dimensional DPs V_t^i for all $i = 1, \ldots, n$ and $V_t^{i,j}$ for all $i < j$ and $j = 2, \ldots, n$. Because there are $(n^2 + n)/2$ such DPs, each requiring a running time of $O(n\kappa T)$, the total running time is $O(n^3 \kappa T)$. Therefore, Algorithm 1 is a *polynomial-time* exact algorithm. We can also show that the savings in memory requirements are similar to the savings in the running time. Clearly, our exact algorithm drastically reduces the computational complexity of the backward induction algorithm and can be implemented for large-sized problems.

5. Pricing of Horizontally Differentiated Products

Consider a firm that offers n horizontally differentiated products. Product j has an attribute rating q_j that is common to all the consumers. Building on (1), we express the utility function of a typical consumer from the purchase of product j by the well-known MNL discrete choice model (see McFadden 1980, Anderson et al. 1992, Anderson and De Palma 1992), $u_j = \theta q_j - p_j + \mu \xi_j$, where θ is a constant, denoting the mean sensitivity of consumers to value q_j, random variable ξ_j follows the standard Gumbel distribution with zero mean and unit variance, and μ is a positive parameter. Suppose the utility of an outside option is normalized to zero. In this case, the mixed logit model in §3.1 reduces to the MNL model, and the choice probability (2) becomes

$$\alpha_j(\mathbf{p}_t) = \frac{e^{(\theta q_j - p_{jt})/\mu}}{1 + \sum_{k=1}^{n} e^{(\theta q_j - p_{kt})/\mu}}, \quad j = 1, 2, \ldots, n. \quad (32)$$

The MNL model has been extensively used in econometrics to describe consumer choice (Berkovec 1985, Train 1986), in marketing for pricing and production decisions (Ben-Akiva and Lerman 1985), and in the operations management area (Zhang and Cooper 2009, Dong et al. 2009).

We can formulate the H model by substituting the probabilities from (32) in the optimality Equation (7). Then $G_t(\mathbf{x}, \mathbf{p}_t)$, from (6), can be expressed as

$$G_t(\mathbf{x}, \mathbf{p}_t) = \sum_{j=1}^{n} \frac{\lambda_t e^{(\theta q_j - p_{jt})/\mu}}{1 + \sum_{k=1}^{n} e^{(\theta q_k - p_{kt})/\mu}} (p_{jt} - \Delta_{x_j} V_{t-1}(\mathbf{x})), \quad \mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}. \quad (33)$$

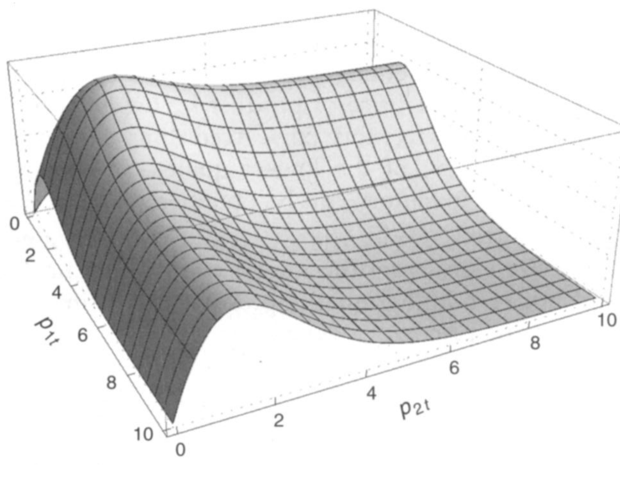
To ensure $\mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}$, that is, $\alpha_j(\mathbf{p}) = 0$ when $x_j = 0$, we will set $p_{jt} = \infty$ if $x_j = 0$.

Although $G_t(\mathbf{x}, \mathbf{p}_t)$ in (33) is not a quasiconcave function of $\mathbf{p}_t$ (see Hanson and Martin 1996 for a counterexample with a profit function similar to $G_t(\mathbf{x}, \mathbf{p}_t)$), we can show that the first-order conditions in (9) are sufficient to determine optimal product prices because $G_t(\mathbf{x}, \mathbf{p}_t)$ is unimodal in $\mathbf{p}_t$ (as illustrated in Figure 1). We establish this result in the following theorem.

THEOREM 5. *For the* H *model*, $G_t(\mathbf{x}, \mathbf{p}_t)$ *is a unimodal function of* $\mathbf{p}_t$.

Recognizing the non-quasiconcavity of the MNL profit function in prices, Hanson and Martin (1996) develop a global optimization approach that determines optimal prices by solving a related logit profit

Figure 1 **Structure of $G_t(\mathbf{x}, \mathbf{p}_t)$ for Two Products with $q_1 = 2$, $q_2 = 1$, and $\mu = 1$**



function. Other researchers (Dong et al. 2009, Song and Xue 2007) have taken a different approach to establish the structure of the MNL profit function. They express the MNL profit as a function of its choice probabilities (rather than prices) and prove that it is concave in these choice probabilities. In contrast, our approach *directly* establishes the necessity and sufficiency of the first-order conditions with respect to price by proving the unimodality of the MNL profit function.

Using Theorem 5, from the first-order conditions for $G_t(\mathbf{x}, \mathbf{p}_t)$, we can derive the optimal price for product j in period t as

$$p_{jt}(\mathbf{x}) = \mu\left(1+\sum_{k=1}^{n} e^{(\theta q_k - p_{kt}(\mathbf{x}))/\mu}\right) + \Delta_{x_j} V_{t-1}(\mathbf{x}) \quad \text{for } j=1,2,\ldots,n. \tag{34}$$

We refer the reader to §EC.6 in the e-companion for the details of this derivation. When the firm has surplus inventory of product j ($x_j \geq t$), the future value of a surplus unit is zero, that is, $\Delta_{x_j} V_{t-1}(\mathbf{x}) = 0$, if $x_j - t \geq 0$. Consequently, from (34), all products that have surplus inventories should be priced the same. However, such a uniform pricing scheme is not optimal for products that have inventory shortfalls ($x_j < t$). From (34), we see that any product with an inventory shortfall, regardless of its attribute rating q_j, commands a *higher price* than the uniform price set for the products with inventory surplus. In other words, inventory shortfall of a product translates into a premium charged over the uniform price of products with surplus inventories in the assortment, regardless of their respective attribute ratings. This result helps us gain a better understanding of what drives price differences among products in an assortment. In practice, we observe that variants in a product category are offered at different prices. The static, single-period pricing models in the literature have failed to explain these differences, concluding instead that the products within a category should be offered at a similar price, despite differences in attribute ratings. Our result shows that the key driver of price differentiation in a horizontal assortment is individual inventory availability, rather than the attribute rating, of the product. We formally state this result next.

Theorem 6. *For the* H *model,*

(a) *prices of all products with surplus inventories are the same, i.e.,* $p_{jt}(\mathbf{x}) = p_{kt}(\mathbf{x})$ *if* $x_j, x_k \geq t$;

(b) *the price of a product with inventory shortfall is always higher than that of a product with inventory surplus, i.e.,* $p_{it}(\mathbf{x}) > p_{jt}(\mathbf{x})$ *if* $x_i < t$ *and* $x_j \geq t$.

The properties stated above are in sharp contrast to the pricing structure in the V model, where, because of the universal ordering of product quality, a product with a higher rating is always priced higher than a product with a lower rating, regardless of their respective inventories. Moreover, the notion of *inventory surplus* in the H model must be defined at each *individual* inventory level; in contrast, because of the universal ordering of quality in the V model, the inventory surplus is defined at the *aggregate* level. Consequently, the optimal prices in the H model are driven by *individual* inventory levels of products, rather than the *aggregate* inventory level as in the V model.

Note that the results stated in Theorem 6 do not hold in general for the mixed logit model, where θ follows a general distribution. Berry and Pakes (2007) show that the mixed logit model converges to the pure characteristic demand model as μ approaches zero. This implies that the MPDP model with linear random utility and uniformly distributed θ converges to the V model as $\mu \to 0$. From our result in the V model, we know that differentiated pricing $q_j/2$ (as opposed to uniform pricing) is optimal for the product with inventory surplus.

6. Numerical Results

6.1. Behavior of Optimal Prices of the V Model

First, we illustrate the three monotonicity properties—inventory, time, and quality—of the optimal prices as discussed in Corollary 1. We then examine the behavior of the price difference between adjacent products $p_{jt}(\mathbf{x}) - p_{j+1,t}(\mathbf{x})$ and demonstrate that it depends only on $|\mathbf{x}|_j$.

We select the parameters in the numerical example as follows. We consider a firm offering customers ($\lambda_t = \lambda = 0.8$) three *vertically differentiated* products, with corresponding quality ratings of $q_1 = 10$, $q_2 = 6$, and $q_3 = 2$. Figure 2 shows the three optimal prices as a function of the inventory level of product 1 in period $t = 40$, with the inventory levels of products 2 and 3 fixed as $x_2 = x_3 = 5$. Figure 3 depicts the optimal prices as a function of the remaining time, at the fixed inventory levels $x_1 = x_2 = x_3 = 5$. Figures 2 and 3 illustrate the quality, time and inventory monotonicity properties that we established in Corollary 1. Furthermore, as x_1 increases (Figure 2) or t decreases (Figure 3), the price of product j converges to $q_j/2$, $j = 1, 2, 3$, as established in Theorem 3(a).

Next we use the same example to illustrate Theorem 3(c), which states that the price difference between any two *adjacent* products i and j with positive inventories depends only on $|\mathbf{x}|_i$. In our example, it means that the difference $p_{1t}(\mathbf{x}) - p_{2t}(\mathbf{x})$ depends only on x_1, and $p_{2t}(\mathbf{x}) - p_{3t}(\mathbf{x})$ depends only on $x_1 + x_2$. Figures 4 and 5 illustrate that for a fixed x_1, $p_{1t}(\mathbf{x}) -$

Figure 2 **Prices for Model V as a Function of x_1 When $x_2 = x_3 = 5$ and $t = 40$**

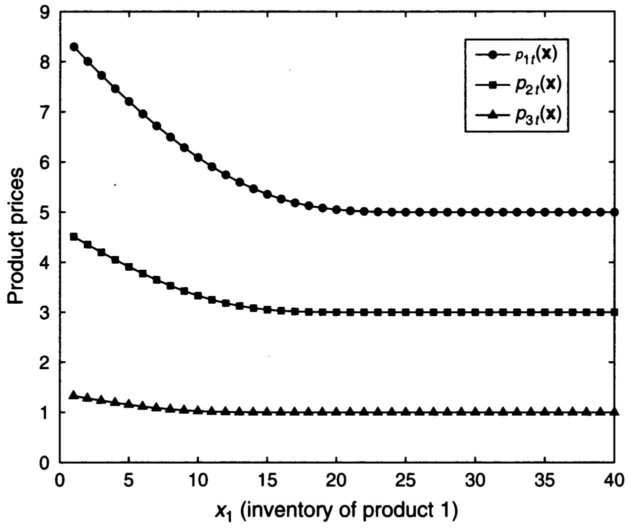
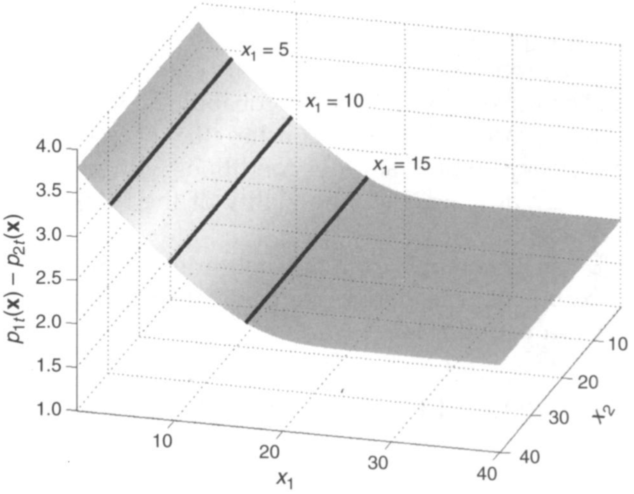


Figure 4 **Price Difference of Products 1 and 2 as a Function of x_1 and x_2 When $x_3 = 5$ and $t = 40$**



$p_{2t}(\mathbf{x})$ is a constant, and for a fixed $x_1 + x_2$, $p_{2t}(\mathbf{x}) - p_{3t}(\mathbf{x})$ is a constant.

We also conducted numerical tests for different distributions of θ, including triangular and beta distributions. Results (reported in §EC.7 in the e-companion) showed that the inventory aggregation and monotonicity (quality, time, and inventory) properties of the V model still hold, demonstrating the robustness of our results for the V model.

6.2. Behavior of Optimal Prices of the H Model

Suppose that the firm in the previous example offers a horizontally differentiated assortment. We retain the attribute values q_j that we specified earlier (noting that the attribute may refer to popularity rather than quality) to ensure an appropriate comparison with the model V example. We let $\theta = 0.5$, corresponding to the mean customer sensitivity to the attribute value. In this example, we vary the factor μ, which reflects the degree of horizontal differentiation among products, to understand its impact. Figures 6 and 7, the counterparts of Figures 2 and 3 for the V model, respectively, illustrate how the optimal prices vary with inventory, attribute value, and time. These figures demonstrate that the optimal prices for the H model do not necessarily possess attribute rating monotonicity (e.g., in Figure 6, for $\mu = 1.5$, $p_{2t}(\mathbf{x})$ starts to dominate $p_{1t}(\mathbf{x})$ when x_1 becomes larger), inventory monotonicity (e.g., in Figure 6, $p_{2t}(\mathbf{x})$ first decreases and then increases in x_1), and time monotonicity (e.g., in Figure 7, the prices for both products 2 and 3 first decrease and then increase in t). Our computational experience suggests that the nonmonotonic behavior of prices with respect to time and inventory is due to the different effects that variable changes have on

Figure 3 **Prices for Model V as a Function of t When $x_1 = x_2 = x_3 = 5$**

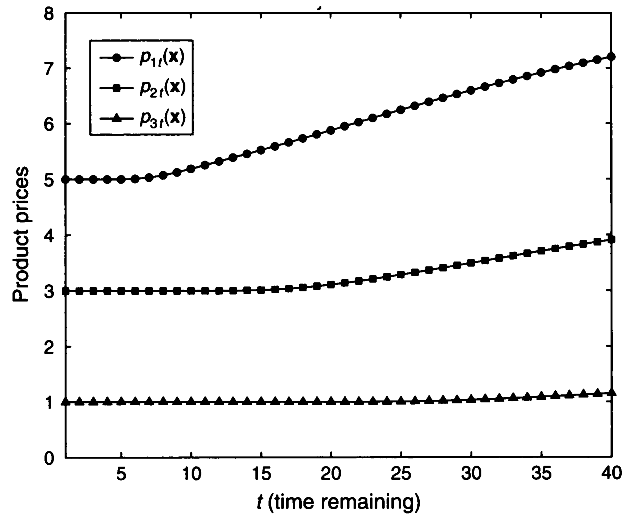


Figure 5 **Price Difference of Products 2 and 3 as a Function of x_1 and x_2 When $x_3 = 5$ and $t = 40$**

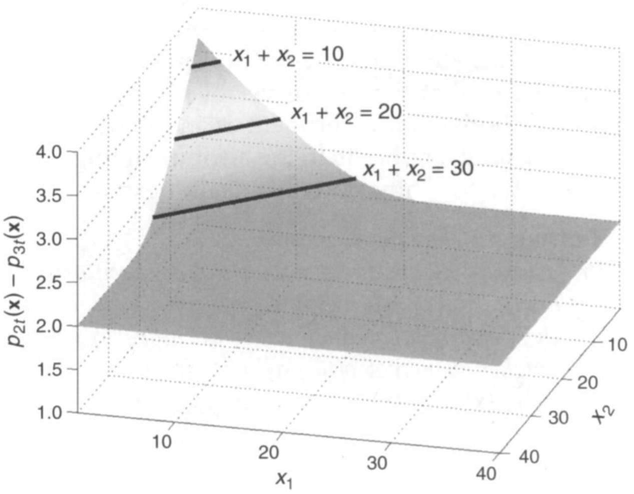


Figure 6 **Prices for Model H as a Function of x_1 When $x_2 = x_3 = 5$, $t = 40$, $\mu = 1.0$ or 1.5**

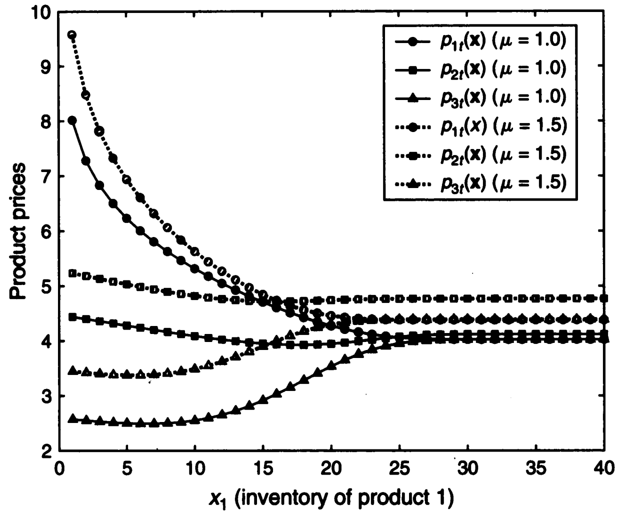
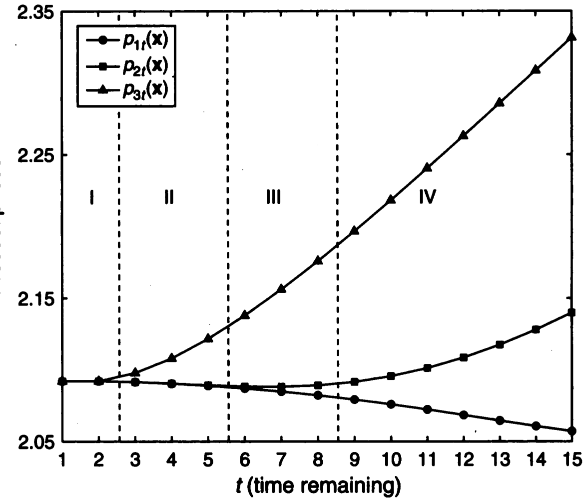


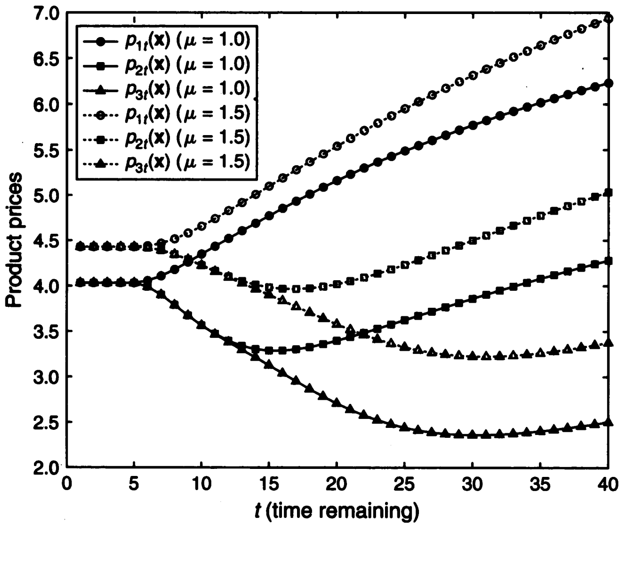
Figure 8 **Prices for Model H as a Function of t When $x_1 = 8$, $x_2 = 5$, $x_3 = 2$, $\mu = 1$**



the two components of the optimal price—the current value $h_j(\mathbf{p}_t)$ and the future value $\Delta_{x_j} V_{t-1}(\mathbf{x})$. Specifically, when time or inventory varies, one of the two terms may increase, whereas the other decreases. As a result, the sum of the two terms can have nonmonotone behavior depending on the relative magnitude of each term. In understanding the nonmonotonic behavior with respect to attribute rating, first observe that the current value $h_j(\mathbf{p}_t)$ is uniform across products, and the differences in prices arise from the future marginal value $\Delta_{x_j} V_{t-1}(\mathbf{x})$ of product inventories. Now, suppose the higher quality product has a surplus and the lower quality product a shortfall. Then the future marginal value of the higher quality product is zero, whereas the lower quality product has a positive future marginal value. Such scenarios can lead to nonmonotonic behavior in attribute rating.

Furthermore, the optimal prices of all three products, at each setting of μ, converge to the same value, meaning that the *uniform pricing* policy starts to take effect when product inventory is abundant relative to demand. Figures 6 and 7 show sharp contrasts with Figures 2 and 3, in which the pattern of *price differentiation* persists regardless of the inventory levels and time remaining, and the optimal prices always mirror their attribute values at all times. In both Figures 6 and 7, we observe that a higher value of μ, which signifies a greater degree of horizontal differentiation, allows the firm to charge higher prices. Higher values of μ result in a larger dispersion in consumer valuations, leading to an increased willingness to pay by consumers.

Figure 7 **Prices for Model H as a Function of t When $x_1 = x_2 = x_3 = 5$, $\mu = 1.0$ or 1.5**



Next, we illustrate Theorem 6 using another three-product H model with $q_1 = 3$, $q_2 = 2$, $q_3 = 1$, $\mu = 1$, $\theta = 0.5$, and $\lambda_t = 0.8$. Figure 8 depicts the change in product prices over time when $x_1 = 8$, $x_2 = 5$, and $x_3 = 2$. Note that, in Region I ($t = 1, 2$), because all three products have surplus inventories, $p_{1t}(\mathbf{x}) = p_{2t}(\mathbf{x}) = p_{3t}(\mathbf{x})$. In Region II ($t = 3, 4, 5$), products 1 and 2 have surplus inventories, whereas product 3 has an inventory shortfall. Hence, $p_{1t}(\mathbf{x}) = p_{2t}(\mathbf{x}) < p_{3t}(\mathbf{x})$, although product 3 has the lowest attribute rating. In Region III ($t = 6, 7, 8$), product 1 has surplus and products 2 and 3 have shortfalls, and we have $p_{1t}(\mathbf{x}) < p_{2t}(\mathbf{x})$ and $p_{1t}(\mathbf{x}) < p_{3t}(\mathbf{x})$. Finally, in Region IV ($t > 9$), all three products have shortfall inventories, and hence the firm charges different prices.

7. Conclusions

Motivated by applications in industry, we study the dynamic pricing problem of a firm that sells given initial inventories of multiple perishable products over a finite selling season. Using an integrative linear random utility framework, we present a detailed analysis of the structural properties of the MPDP models for both vertically and horizontally differentiated products.

The results in this paper have the following important managerial implications.

(1) Vertically and horizontally differentiated products have fundamentally different pricing policy structures. These differences suggest that managers need to select a consumer choice model that is compatible with the specific nature of product differentiation in their applications. The profitability of a firm may be significantly compromised if an inappropriate consumer choice model is used in making pricing decisions.

(2) When products can be universally ordered based on their attributes, managers can charge premiums for products ranked higher by consumers, for products with scarce inventories, and for all products as the end of the season approaches. The optimal price of a product depends on higher quality product inventories only through the aggregate inventory level rather than their individual availabilities. Additionally, this aggregate inventory solely determines the product's markup over an adjacent lower quality product in the assortment. Our exact algorithm allows managers to readily apply the insights and analysis in this paper to practical-sized problems.

(3) If consumer preferences for products are dispersed, the optimal pricing policy is driven by individual product availability. Regardless of product attribute ratings, managers should select a uniform price for products with surplus inventories and charge premiums for those with inventory shortfalls. In general, higher attribute value, lower inventory availability, and longer time to sell do not necessarily permit the firm to charge a higher price.

By establishing the importance of incorporating product differentiation in dynamic pricing decisions, our work explores an interesting and novel dimension of the revenue management problem and offers opportunities for interesting future work. For instance, developing models to incorporate firm's initial inventory choices, optimal assortment decisions, alternative distributions of customer sensitivities, multiproduct budget and resource constraints, and strategic consumers in this framework would greatly facilitate the deployment of these models in practice. Exploring industry-specific considerations such as multiday stays (hospitality) and group purchases (travel) would also greatly enrich the model.

8. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

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References

Anderson, S. P., A. De Palma. 1992. The logit as a model of product differentiation. *Oxford Econom. Papers, New Ser.* **44**(1) 51–67.

Anderson, S. P., A. De Palma, J. F. Thisse. 1992. *Discrete Choice Theory of Product Differentiation*. MIT Press, Cambridge, MA.

Ben-Akiva, M. E., S. R. Lerman. 1985. *Discrete Choice Analysis: Theory and Application to Travel Demand*. MIT Press, Cambridge, MA.

Berkovec, J. 1985. New car sales and used car stocks: A model of the automobile market. *RAND J. Econom.* **16**(2) 195–214.

Berry, S., A. Pakes. 2007. The pure characteristics demand model. *Internat. Econom. Rev.* **48**(4) 1193–1225.

Bhargava, H. K., V. Choudhary. 2008. When is versioning optimal for information goods? *Management Sci.* **54**(5) 1029–1035.

Bitran, G., R. Caldentey. 2003. An overview of pricing models for revenue management. *Manufacturing Service Oper. Management* **5**(3) 203–229.

Bitran, G., S. V. Mondschein. 1997. Periodic pricing of seasonal products in retailing. *Management Sci.* **43**(1) 64–79.

Bitran, G., R. Caldentey, R. Vial. 2005. Pricing policies for a family of products under demand correlation. Working paper, New York University, New York.

Bresnahan, T. F. 1987. Competition and collusion in the American automobile industry: The 1955 price war. *J. Indust. Econom.* **35**(4) 457–482.

Caplin, A., B. Nalebuff. 1991. Aggregation and imperfect competition: On the existence of equilibrium. *Econometrica* **59**(1) 25–59.

Dong, L., P. Kouvelis, Z. Tian. 2009. Dynamic pricing and inventory control of substitute products. *Manufacturing Service Oper. Management* **11**(2) 317–339.

Elmaghraby, W., P. Keskinocak. 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Sci.* **49**(10) 1287–1309.

Feng, Y., G. Gallego. 1995. Optimal starting times for end-of-season sales and optimal starting times for promotional fares. *Management Sci.* **41**(9) 1371–1391.

Feng, Y., B. Xiao. 2000. A continuous-time yield management model with multiple prices and reversible price changes. *Management Sci.* **46**(5) 644–657.

Gallego, G., G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* **40**(8) 999–1020.

Gallego, G., G. van Ryzin. 1997. A multiproduct dynamic pricing problem and its applications to network yield management. *Oper. Res.* **45**(1) 24–41.

Gerchak, Y. M., M. Parlar, T. K. M. Yee. 1985. Optimal rationing policies and production quantities for products with several demand classes. *Canadian J. Admin. Sci.* **2**(1) 161–176.

Hanson, W., K. Martin. 1996. Optimizing multinomial logit profit functions. *Management Sci.* **42**(7) 992–1003.

Hauser, J. R., G. L. Urban. 1986. The value priority hypotheses for consumer budget plans. *J. Consumer Res.* **12**(4) 446–462.

Hensher, D. A., W. H. Greene. 2003. The mixed logit model: The state of practice. *Transportation* **30**(2) 133–176.

Karaesmen, I., G. van Ryzin. 2004. Overbooking with substitutable inventory classes. *Oper. Res.* **52**(1) 83–104.

Lee, T. C., M. Hersh. 1993. A model for dynamic airline seat inventory control with multiple seat bookings. *Transportation Sci.* **27**(3) 252–265.

Liu, B., J. Milner. 2006. Multiple-item dynamic pricing under a common pricing constraint. Working paper, University of Toronto, Toronto.

Maglaras, C., J. Meissner. 2006. Dynamic pricing strategies for multiproduct revenue management problems. *Manufacturing Service Oper. Management* **8**(2) 136–148.

Marn, M. V., E. V. Roegner, C. C. Zawada. 2003. The power of pricing. *McKinsey Quart.* (February 2003) 26–39.

McFadden, D. 1980. Econometric models for probabilistic choice among products. *J. Bus.* **53**(3) 513–529.

McFadden, D., K. E. Train. 2000. Mixed MNL models for discrete response. *J. Appl. Econometrics* **15**(5) 447–470.

McGill, J. I., G. J. van Ryzin. 1999. Revenue management: Research overview and prospects. *Transportation Sci.* **33**(2) 233–256.

Roberts, J. H., G. L. Lilien. 1993. Explanatory and predictive models of consumer behavior. J. Eliashberg, G. L. Lilien, eds. *Handbooks in Operations Research and Management Science*, Vol. 5, Chap. 2. Elsevier Science, Amsterdam, 27–82.

Song, J. R., Z. Xue. 2007. Demand management and inventory control for substitutable products. Working paper, Duke University, Durham, NC.

Song, M. 2007. Measuring consumer welfare in the CPU market: Anapplication of the pure-characteristics demand model. *RAND J. Econom.* **38**(2) 429–446.

Talluri, K., G. van Ryzin. 2004a. Revenue management under a general discrete choice model of consumer behavior. *Management Sci.* **50**(1) 15–33.

Talluri, K., G. J. van Ryzin. 2004b. *The Theory and Practice of Revenue Management*. Kluwer Academic Publishers, Boston.

Tirole, J. 1988. *The Theory of Industrial Organization*. MIT Press, Cambridge, MA.

Train, K. 1986. *Qualitative Choice Analysis: Theory, Econometrics, and an Application to Automobile Demand*. MIT Press, Cambridge, MA.

Train, K. E. 2003. *Discrete Choice Methods with Simulation*. Cambridge University Press, Cambridge, UK.

Wauthy, X. 1996. Quality choice in models of vertical differentiation. *J. Indust. Econom.* **44**(3) 345–353.

Zhang, D., W. L. Cooper. 2005. Revenue management for parallel flights with customer-choice behavior. *Oper. Res.* **53**(3) 415–431.

Zhang, D., W. L. Cooper. 2009. Pricing substitutable flights in airline revenue management. *Eur. J. Oper. Res.* **197**(3) 848–861.

Zhao, W., Y.-S. Zheng. 2000. Optimal dynamic pricing for perishable assets with nonhomogeneous demand. *Management Sci.* **46**(3) 375–388.