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Suppose  $t_1$  and  $t_2$  are two points in the domain of the curve with  $t_2 \geq t_1$ . We want expression for the distance between the points  $(x(t_2), y(t_2))$  and  $(x(t_1), y(t_1))$ .

That is we want to evaluate

$$\Delta s = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2}$$

From the mean value theorem we have that there exists a  $t_x \in [t_1, t_2]$  such that

$$\begin{aligned}\Delta x &= x(t_2) - x(t_1) \\ &= \frac{dx(t_x)}{dt}(t_2 - t_1) \\ &= \frac{dx(t_x)}{dt}\Delta t\end{aligned}$$

where

$$\frac{dx(t_x)}{dt} = \left. \frac{dx}{dt} \right|_{t=t_x}$$

and  $\Delta t = t_2 - t_1$ .

Similarly for  $y$  there exists a value  $t_y \in [t_1, t_2]$  such that

$$\begin{aligned}\Delta y &= y(t_2) - y(t_1) \\ &= \frac{dy(t_y)}{dt}(t_2 - t_1) \\ &= \frac{dy(t_y)}{dt}\Delta t\end{aligned}$$

Thus we have

$$\Delta s = \sqrt{\left(\frac{dx(t_x)}{dt}\right)^2 + \left(\frac{dy(t_y)}{dt}\right)^2} \Delta t$$

But to get the Riemann sum we need the differential to be evaluated at the points  $(x(t_1), y(t_1))$ .

$$\begin{aligned}\Delta s &= \sqrt{\left(\frac{dx(t_x)}{dt}\right)^2 + \left(\frac{dy(t_y)}{dt}\right)^2} \Delta t \\ &= \sqrt{\left(\frac{dx(t_1)}{dt}\right)^2 + \left(\frac{dy(t_1)}{dt}\right)^2} (t_2 - t_1) + \epsilon(t_1, t_2)\end{aligned}$$

where

$$\epsilon(t_1, t_2) = \sqrt{\left(\frac{dx(t_x)}{dt}\right)^2 + \left(\frac{dy(t_y)}{dt}\right)^2} - \sqrt{\left(\frac{dx(t_1)}{dt}\right)^2 + \left(\frac{dy(t_1)}{dt}\right)^2}$$

Now we have

$$\begin{aligned}
|\epsilon(t_1, t_2)| &= \left| \sqrt{\left(\frac{dx(t_x)}{dt}\right)^2 + \left(\frac{dy(t_y)}{dt}\right)^2} - \sqrt{\left(\frac{dx(t_1)}{dt}\right)^2 + \left(\frac{dy(t_1)}{dt}\right)^2} \right| \\
&\leq \sqrt{\left(\frac{dx(t_x)}{dt} - \frac{dx(t_1)}{dt}\right)^2 + \left(\frac{dy(t_y)}{dt} - \frac{dy(t_1)}{dt}\right)^2} \\
&\leq \left| \frac{dx(t_x)}{dt} - \frac{dx(t_1)}{dt} \right| + \left| \frac{dy(t_y)}{dt} - \frac{dy(t_1)}{dt} \right|
\end{aligned}$$

where the first inequality comes from the reverse triangle inequality  $||a| - |b|| \leq |a - b|$  and the second comes from the regular triangle inequality  $|a + b| \leq |a| + |b|$ .

Finally the first derivatives of  $x$  and  $y$  are continuous we can make  $\epsilon(t_1, t_2)$  arbitrary small as we let  $\Delta t$  become small.