1 Converting x and y Coordinates into Fresnel Integrals

First, we need to convert the Euler version into the Fresnel version. We start with the x coordinate first.

The Euler version with constant of proportionality equal to α is given by

$$x(s) = \int_0^s \cos \frac{\alpha u^2}{2} du$$

and we have that the standard Fresnel version is

$$C(z) = \int_0^z \cos \frac{\pi t^2}{2} dt$$

Apply the following transformation $t = \frac{\sqrt{\alpha}}{\sqrt{\pi}} u$ we have $du = \frac{\sqrt{\pi}}{\sqrt{\alpha}} dt$ and then

$$\begin{aligned} x(s) &= \int_0^s \cos \frac{\alpha u^2}{2} du \\ &= \frac{\sqrt{\pi}}{\sqrt{\alpha}} \int_0^{\frac{\sqrt{\alpha}}{\sqrt{\pi}} s} \cos \frac{\pi t^2}{2} dt \\ &= \frac{\sqrt{\pi}}{\sqrt{\alpha}} C\left(\frac{\sqrt{\alpha}}{\sqrt{\pi}} s\right) \end{aligned}$$

The same holds the second integral for the y coordinate.

$$y(s) = \frac{\sqrt{\pi}}{\sqrt{\alpha}} S\left(\frac{\sqrt{\alpha}}{\sqrt{\pi}} s\right)$$

where

$$S(z) = \int_0^z \sin \frac{\pi t^2}{2} dt$$

2 Calculating Angle

Second we need to calculate the angle at the tangency point. In particular we cant to know when we have turned through $\frac{\pi}{2}$ the first time.

Be we know from the definition of the Euler spiral that

$$\begin{aligned} \frac{dy}{dx} &= \tan\left(\frac{\alpha s^2}{2}\right) \\ &= \tan\left(\frac{\pi}{2}\right) \end{aligned}$$

which implies s^* the arc length when the angle turned is $\frac{\pi}{2}$ must satisfy

$$s^* = \frac{\sqrt{\pi}}{\sqrt{\alpha}}$$

Substituting into the equations above we get

$$\begin{aligned} x(s^*) &= \frac{\sqrt{\pi}}{\sqrt{\alpha}} C(1) \\ y(s^*) &= \frac{\sqrt{\pi}}{\sqrt{\alpha}} S(1) \end{aligned}$$

3 Matching The Radius

We want to make sure that the radius matches the height when at the apex i.e. when we have turned through $\frac{\pi}{2}$.

$$\begin{aligned} r &= y(s^*) \\ &= \frac{\sqrt{\pi}}{\sqrt{\alpha}} S\left(1\right) \end{aligned}$$

which gives

$$\alpha = \frac{\pi^2}{r^2} S(1)^2$$