# The Euler Spiral

## History, Discovery and Derivation

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Arcana, May 2022

# Outline

- History of the Euler Spiral
- Definitions: Curves, Lengths and Curvature
- Relating the Curve Length to its Curvature
- Calculating Fresnel Integrals and the Euler Spiral
- Plotting the Euler Spiral and other Fun Curves
- Another (Re)discovery: Designing Railways and Roads

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# History: Bernoulli's Cantilever Problem

- In 1694 James Bernoulli first studied the Cantilever Problem.
- What shape does a thin horizontal beam of negligible mass fixed at one end with weight on the other have?
- He called it an *elastica*.
- He also posed the converse problem: What shape must a pre-curved beam have in order to be horizontal and straight when a weight is added to the non fixed end?
- He claimed $a^2 = sR$ where $a$ is constant, $R$ is the radius of curvature and $s$ is arc length (more on this later).

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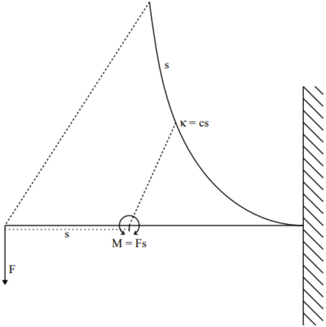
# Euler's Solution

- In 1744 Euler gave the first solution and derived his curve.
- When the curve is horizontal the moment $M$ at any point is equal to the Force $F$ times the distance $s$ from the Force.
- From elastic theory the curvature, $\kappa$ at that point is proportional to the moment.
- Assuming the curve doesn't stretch the distance from the force is proportional to arc length.
- Thus curvature is proportional to the arc length - the Euler Spiral!

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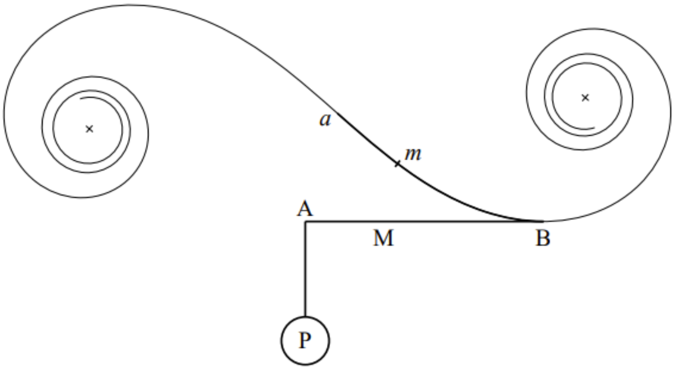
# Euler's Original Spiral

Figure: Reconstruction of Euler's original drawing with spiral superimposed

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# What is a Curve?

Let's define some of the concepts and derive the maths behind the Euler Spiral.

**Definition (A Parametric Curve)**

A parametric curve is a smooth function that has the form $x = g(t)$ and $y = h(t)$ defined on an open interval $(a, b)$. The set of points traced out by the curve is called the trace.

Note that everything we look at here is in 2 dimensions $\mathbb{R}^2$.

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# How Long is a Curve?

**Definition (Arc Length)**

The length of a curve $s$ is given by $s = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2} dt$

- The velocity at time $t$ is $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ and hence the speed at time $t$ is $\sqrt{x'(t)^2 + y'(t)^2}$. Distance travelled (arc length) is the integral of speed with respect to time.
- Approximate the trace by line segments. The total length of line segments converges to the arc length as the segments get smaller and smaller.

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# Some Simple Examples

- Circle: We can define a circle with radius $r$ as $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$. The arc length is

$$s = \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\, dt = [r]_0^{2\pi} = 2\pi r$$

- Parabola: We can define a parabola as $x(t) = t$ and $y(t) = t^2$. The arc length between 0 and 1 is

$$\begin{aligned} s &= \int_0^1 \sqrt{1+4t^2}\, dt \\ &= \left[\frac{1}{2} t\sqrt{1+4t^2} + \frac{1}{4}\ln\left(2t + \sqrt{1+4t^2}\right)\right]_0^1 \\ &\approx 1.48 \end{aligned}$$

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# How Curved is a Curve?

The curvature at a point on the curve is the reciprocal of the radius of the circle that approximates the curve.

**Definition (Curvature)**

The curvature of curve is $\kappa$ given by $\kappa = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$

Note: Curvature has a signed in 2 dimensions $\mathbb{R}^2$. Positive curvature corresponds "bending to the left" while negative curvature "bends to the right".

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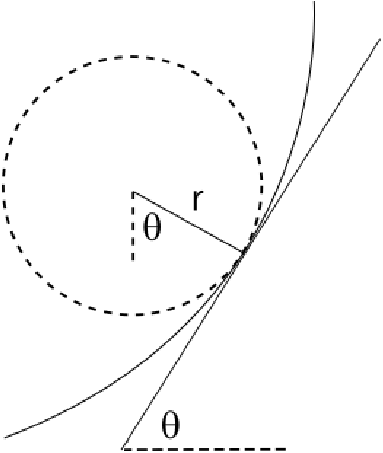
# Explaining Definition for Curvature




1. For any circle with radius $r$ we have $s = r\theta \implies$ for "kissing" circle $\frac{ds}{dt} = r\frac{d\theta}{dt}$
2. $\frac{ds}{dt} = \sqrt{x'^2 + y'^2}$.
3. $\tan\theta = \frac{y'(t)}{x'(t)} \implies$ $\sec^2\theta\frac{d\theta}{dt} = \frac{x'y''-y'x''}{x'^2}$.
4. $\frac{1}{\cos^2\theta} = \frac{x'^2+y'^2}{x'^2} \implies$ $\frac{d\theta}{dt} = \frac{x'y''-y'x''}{x'^2+y'^2}$
5. Combing gives us $r = \frac{ds/dt}{d\theta/dt} = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$
6. Finally $\kappa = \frac{1}{r}$

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# Some Simple Examples

- Circle: Recall $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$

$$\kappa = \frac{r^2\sin^2 t + r^2\cos^2 t}{\left(r^2\sin^2 t + r^2\cos^2 t\right)^{\frac{3}{2}}} = \frac{1}{r}$$

- Parabola: Recall $x(t) = t$ and $y(t) = t^2$.

$$\kappa = \frac{2}{(1+4t^2)^{\frac{3}{2}}}$$

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# Relating Curve Length to Curvature

Let's try the following parametrisation for $x$ and $y$

$$\begin{aligned} x(t) &= \int_0^t \cos f(u) du \\ y(t) &= \int_0^t \sin f(u) du \end{aligned}$$

This gives us

$$\begin{aligned} x' = x'(t) = \cos f(t) \quad &\text{and} \quad x'' = -f'(t) \sin f(t) \\ y' = y'(t) = \sin f(t) \quad &\text{and} \quad y'' = f'(t) \cos f(t) \end{aligned}$$

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# Relating Curve Length to Curvature

This gives us the following for slope, arc length and curvature

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sin f(t)}{\cos f(t)} = \tan f(t) \\
s &= \int_0^t \sqrt{\cos^2 f(u) + \sin^2 f(u)}du = t \\
\kappa &= \frac{f'(t)\cos^2 f(t) + f'(t)\sin^2 f(t)}{\cos^2 f(t) + \sin^2 f(t)} = f'(t)
\end{aligned}$$

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# Define the Curve by Curve Length and Curvature

We can replace the "time" variable $t$ by the arc length $s$. And the curvature at point $t$ is $f'(t)$. Which means

$$f(t) = \int \kappa(t)dt$$

Thus the equations for the curve become

$$\begin{aligned} x = x(s) &= \int_0^s \cos\left(\int_0^u \kappa(t)dt\right) du \\ y = y(s) &= \int_0^s \sin\left(\int_0^u \kappa(t)dt\right) du \end{aligned}$$

Hence the curve is defined by arc length and curvature alone.

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# A Very Simple Example

We can make the curvature $\kappa$ constant and equal to 1. Then $\int_0^u \kappa(t)dt = u$ and

$$\begin{aligned} x = x(s) &= \int_0^s \cos u du = \sin s \\ y = y(s) &= \int_0^s \sin u du = -\cos s + 1 \end{aligned}$$

which is the parametric curve for a circle with centre $(0, 1)$ and radius 1.

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# The Euler Spiral

Recall: Euler defined his curve as one where the curvature is proportional to arc length.

$$\boxed{\kappa(s) = s}$$

Then $\int_0^u \kappa(t)dt = \frac{u^2}{2}$ and

$$\begin{aligned} x = x(s) &= \int_0^s \cos \frac{u^2}{2} du \\ y = y(s) &= \int_0^s \sin \frac{u^2}{2} du \end{aligned}$$

But since these integrals can't be solved analytically how were they calculated?

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# Solving the Integrals: Euler

- In 1744 Euler derived these integrals.

$$\int ds \ \text{fin.} \ \frac{ss}{2aa} \ \& \int ds \ \text{cof.} \ \frac{ss}{2aa}$$

- He derived a series expansion which is still a viable method for small $s$.

$$x = \frac{s^3}{1.3b^2} - \frac{s^7}{1.2.3.7b^6} + \frac{s^{11}}{1.2.3.4.5.11b^{10}} - \frac{s^{15}}{1.2....7.15b^{14}} + \&c.$$

$$y = s - \frac{s^5}{1.2.5b^4} + \frac{s^9}{1.2.3.4.9b^8} - \frac{s^{13}}{1.2.3...6.13b^{12}} + \&c.$$

- In 1781 he proved the integrals for limits between 0 and $\infty$ are equal to $\frac{a\sqrt{\pi}}{2}$

# Solving the Integrals: Fresnel and Cornu

- In 1818 Augustin Fresnel rediscovered these integrals when he investigated the diffraction of light through a slit. He showed that the intensity (under some assumptions) was

$$\left(\int_0^s \cos\left(\pi t^2/2\right)\,dt\right)^2 + \left(\int_0^s \sin\left(\pi t^2/2\right)\,dt\right)^2$$

- Up to a factor of $\pi$ the integrals are the same as the ones Euler derived.
- These integrals are now called the *Fresnel Integrals*.
- Fresnel calculated them for values of $s$ between 0.1 and 5.1.
- In 1874 Alfred Cornu calculated values and plotted the Euler spiral accurately. Hence the Euler spiral is also known as a Cornu spiral.

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# Solving the Integrals: Today with Python

```python
def x_func(upper, lower, func):
  def integrand(t):
    return np.cos(func(t))
  result, _ = integrate.quad(integrand, upper, lower)
  return result
```

```python
def y_func(upper, lower, func):
  def integrand(t):
    return np.sin(func(t))
  result, _ = integrate.quad(integrand, upper, lower)
  return result
```

Euler Spiral

```python
[7]  k = s
     spy.integrate(k, s)
```

$$\frac{s^2}{2}$$

```python
[8]  def func_euler(x):
         return 0.5 * np.power(x, 2)
```

```python
[9]  arc_length, x_coord, y_coord = x_y_coordinates(x_func, y_func, func_euler, s_steps)
```

Nowadays we use computers and numerical methods to evaluate these integrals.

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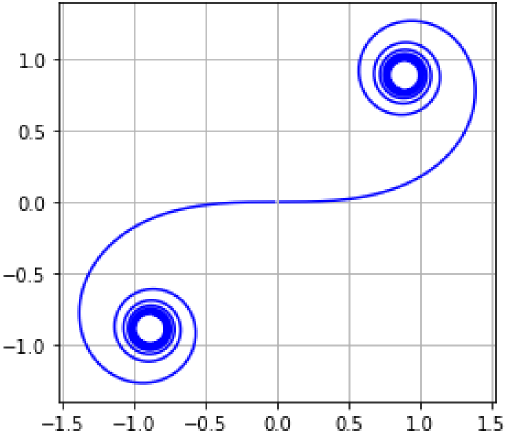
# The Euler Spiral

Figure: The Euler Spiral aka Cornu Curve

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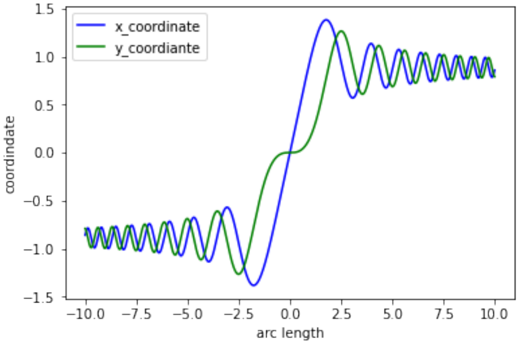
# The Euler Spiral - $x$ $y$ Coordinates

Figure: Fresnel Integrals with arguments $\frac{u^2}{2}$




As expected these converge to $\pm\frac{\sqrt{\pi}}{2} \approx 0.8862$.

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# Other Fun Curves: Even Powers of $s$

Figure: $\kappa(s) = s^2$

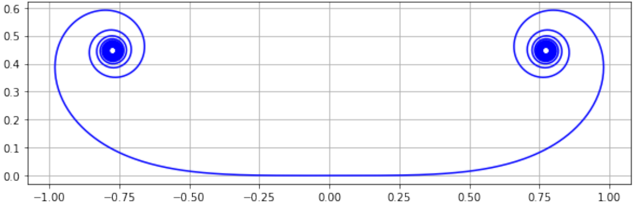
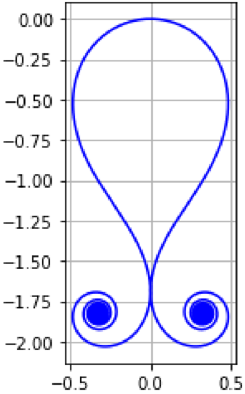


Figure: $\kappa(s) = s^2 - 2.19$

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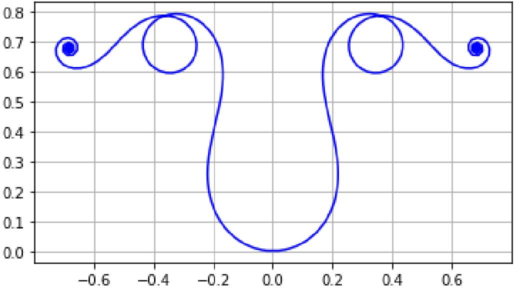
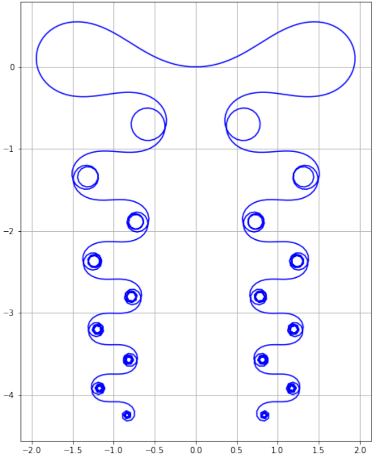
# More Fun Curves: Polynomials

Figure: $\kappa(s) = 5s^4 - 18s^2 + 5$



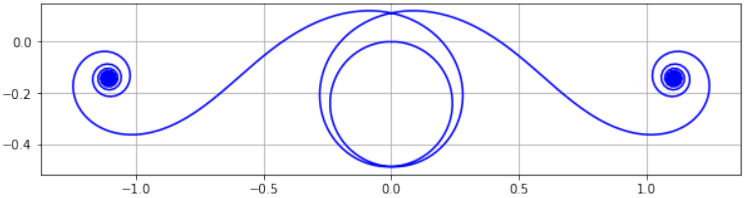
# More Fun Curves: Trigonometric Functions

Figure: $\kappa(s) = \cos(s) - s\sin(s)$



# More Fun Curves: Hyperbolic Functions

Figure: $\kappa(s) = \sinh(s) - 5.19$

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# Another (Re)discovery of the Euler Spiral

- As trains became faster in the 19th century, the Euler spiral was rediscovered by railway designers.
- Railway engineers discovered that a track shape with gradually varying curvature provided a smooth riding experience.
- In 1899 Railway engineer Arthur Talbot solved the problem mathematically and derived the same integrals as Fresnel.
- His solution was “a curve whose degree-of-curve increases directly along the curve.”
- He derived a series expansion for the integrals almost identical to Euler’s 1744 series.

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# Designing Roads and Railways

- Transition curves are used to link straight sections of motorways or railways.
- They are designed to give passengers a smooth ride with no sudden changes in acceleration.

Figure: Cloverleaf Motorway Interchange

# Why Transition Curves are Euler Spirals

- The acceleration along the transition curve is given by

$$a = s''(t)\vec{T} + \kappa s'(t)^2\vec{N}$$

  Where $\vec{T}$ is the unit tangent vector and $\vec{N}$ is the unit normal vector.
- If the car/train is going round the curve at constant speed $s'(t) = \mathit{constant}$ and $s''(t) = 0$.
- The acceleration at constant speed only depends on the curvature $\kappa$ and speed $s'(t)$ in the direction of the normal vector.

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# Case 1: Semicircular Curves

A closed track is made up of four segments.

1. A straight track of length 1km
2. A semicircular track of length 1km. This has radius $1,000/\pi m \approx 318.31m$
3. A straight track of length 1km
4. A semicircular track of length 1km.

Let the vehicle go around the track at a constant speed of $60km/h = 16\frac{2}{3}m/s$.

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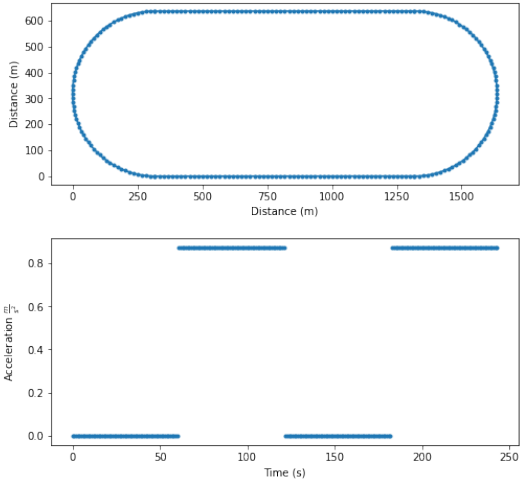
# Case 1: Position and Acceleration

Figure: Position and Acceleration




Note: The acceleration is a step function.
As a passenger you would feed the full centrifugal force pushing you outward the moment you entered the curve.

# Case 2: Euler Spiral Transition Curves

- A closed track with same width and height.
- Replace semicircles with two parts of an Euler Spiral.
- Recall that the curvature is proportional to arc length. $\kappa = \alpha s$ for some $\alpha$.
- The width and height of an Euler Spiral that turns through $\pi/2$ is given by

$$\begin{aligned} width &= \sqrt{\pi/\alpha}\,C(1) \\ height &= \sqrt{\pi/\alpha}\,S(1) \end{aligned}$$

where $S(u)$ and $C(u)$ are the standard Fresnel integrals.

$$\alpha = \frac{\pi S(1)^2}{r^2} \approx 5.95 \times 10^{-6}$$

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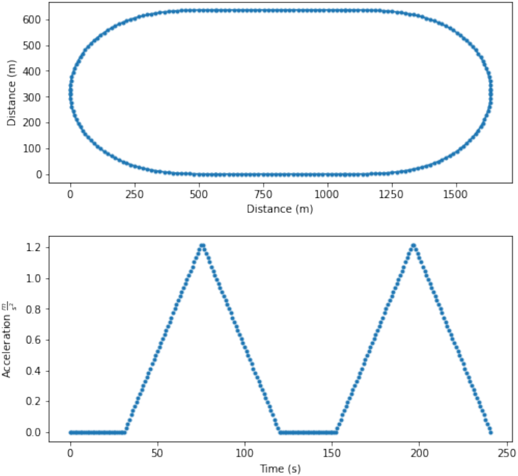
# Case 2: Euler Spiral Transition Curves

Figure: Position and Acceleration




Note: The acceleration increases linearly as we move through the curve. The maximum acceleration however at the apex is greater than with the semicircular track. But the ride is more comfortable now!

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# Summary

- The Euler Spiral has the property that the curvature is proportional to the arc length.
- It was first discovered by James Bernoulli in 1694 investigating the cantilever problem.
- Euler derived Integrals for the x,y coordinates in 1744 which later became know as Fresnel integrals.
- Rediscovered in the 19th century for designing railway and features of the spiral appear in modern day motorways.

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