Cornu Spirals

From Euler to Ferrari

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Outline

- Definitions: Curves, Lengths and Curvature
- Relating the Curve Length to its Curvature
- The Euler Spiral
- Other Fun Curves
- "Curvature determines the Curve"
- Application 1: Designing Roads and Railways
- Application 2: The Racing Line

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What is a Curve?

Before we begin note that everything we look at here is in 2 dimensions $\mathbb{R}^2$

Definition (A Parametric Curve)

A parametric curve is a twice differentiable function that has the form $x = g(t)$ and $y = h(t)$ defined on an open interval (a, b). The set of points traced out by the curve is called the trace.

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How Long is a Curve?

Definition (Arc Length)

The length of a curve s is given by $s = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2} dt$

Intuition: The velocity at time t is $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ and hence the speed at time t is $\sqrt{x'(t)^2 + y'(t)^2}$. Distance travelled (arc length) is the integral of speed with respect to time.

Alternatively approximate the trace by line segments. The length of line segments converges to the arc length as the segments get smaller and smaller.

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Some Simple Examples

- Circle: We can define a circle with radius r as $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$. The arc length is

$$s = \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}dt = [r]_0^{2\pi} = 2\pi r$$

- Parabola: We can define a parabola as $x(t) = t$ and $y(t) = t^2$. The Arc Length between 0 and 1 is

$$\begin{aligned} s &= \int_0^1 \sqrt{1+4t^2}dt \\ &= \left[\frac{1}{2}t\sqrt{1+4t^2} + \frac{1}{4}\ln\left(2t + \sqrt{1+4t^2}\right)\right]_0^1 \\ &\approx 1.48 \end{aligned}$$

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How Curved is a Curve?

The curvature at a point on the curve is the reciprocal of the radius of the circle that approximates the curve.

Definition (Curvature)

The curvature of curve is κ given by $\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$

Note: Curvature is signed in two dimensions. Positive curvature corresponds "bending to the left" while negative curvature "bends to the right".

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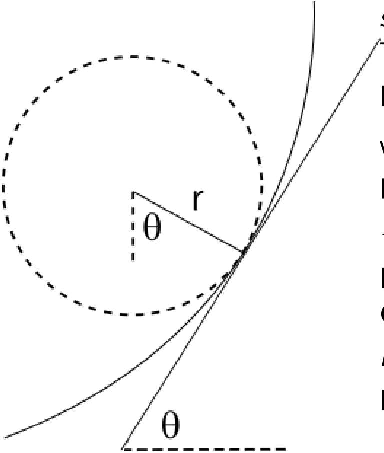
Explaining Definition for Curvature



For any circle with radius r we have $s = r\theta$.
Therefore for "kissing" circle $\frac{ds}{dt} = r\frac{d\theta}{dt}$
But $\tan\theta = \frac{y'(t)}{x'(t)}$ and taking derivatives w.r.t. t we have $\sec^2\theta \frac{d\theta}{dt} = \frac{x'y''-y'x''}{x'^2}$.
Further $\frac{1}{\cos^2\theta} = \frac{x'^2+y'^2}{x'^2}$ therefore $\frac{d\theta}{dt} = \frac{x'y''-y'x''}{x'^2+y'^2}$
From above $\frac{ds}{dt} = \sqrt{x'^2+y'^2}$.
Combing gives us
$r = \frac{ds/dt}{d\theta/dt} = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$
Finally note $\kappa = \frac{1}{r}$

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Some Simple Examples

- Circle: Recall $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$

$$\kappa = \frac{r^2\sin^2 t + r^2\cos^2 t}{\left(r^2\sin^2 t + r^2\cos^2 t\right)^{\frac{3}{2}}} = \frac{1}{r}$$

- Parabola: Recall $x(t) = t$ and $y(t) = t^2$.

$$\kappa = \frac{2}{(1+4t^2)^{\frac{3}{2}}}$$

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Relating Curve Length to Curvature

Let's try the following parametrisation for x and y

$$\begin{aligned} x(t) &= \int_0^t \cos f(u)du \\ y(t) &= \int_0^t \sin f(u)du \end{aligned}$$

This gives us

$$\begin{aligned} x' = x'(t) = \cos f(t) \quad &\text{and} \quad x'' = -f'(t)\sin f(t) \\ y' = y'(t) = \sin f(t) \quad &\text{and} \quad y'' = f'(t)\cos f(t) \end{aligned}$$

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Relating Curve Length to Curvature

This gives us the following for slope, arc length and curvature

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\sin f(t)}{\cos f(t)} = \tan f(t) \\
s &= \int_0^t \sqrt{\cos^2 f(u) + \sin^2 f(u)}du = t \\
\kappa &= \frac{f'(t)\cos^2 f(t) + f'(t)\sin^2 f(t)}{\cos^2 f(t) + \sin^2 f(t)} = f'(t)
\end{aligned}
$$

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Define the Curve by Curve Length and Curvature

We can get replace "time" variable t by the curve length s. And the curvature at point t is $f'(t)$. Which means

$$f(t) = \int \kappa(t) dt$$

Thus the equations for the curve become

$$\begin{aligned} x = x(s) &= \int_0^s \cos\left(\int_0^u \kappa(t) dt\right) du \\ y = y(s) &= \int_0^s \sin\left(\int_0^u \kappa(t) dt\right) du \end{aligned}$$

Hence the curve is defined by curve length and curvature alone.

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A Very Simple Example

We can make the curvature κ constant and equal to 1. Then $\int_0^u \kappa(t)dt = u$ and

$$\begin{aligned} x = x(s) &= \int_0^s \cos u du = \sin s \\ y = y(s) &= \int_0^s \sin u du = -\cos s + 1 \end{aligned}$$

which is the parametric curve for a circle with centre $(0, 1)$ and radius 1.

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The Euler Spiral

A more interesting example is the curvature equal to the arc length.

$$\boxed{\kappa(s) = s}$$

Then $\int_0^u \kappa(t)dt = \frac{u^2}{2}$ and

$$\begin{aligned} x = x(s) &= \int_0^s \cos\frac{u^2}{2}du \\ y = y(s) &= \int_0^s \sin\frac{u^2}{2}du \end{aligned}$$

These integrals are called "Fresnel Integrals".
They cannot be solved analytically. But what do they look like numerically?

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Making a Plotter in Python

```
def x_func(upper, lower, func):
  def integrand(t):
    return np.cos(func(t))
  result, _ = integrate.quad(integrand, upper, lower)
  return result
```

```
def y_func(upper, lower, func):
  def integrand(t):
    return np.sin(func(t))
  result, _ = integrate.quad(integrand, upper, lower)
  return result
```

Euler Spiral

```
[7]  k = s
     spy.integrate(k, s)
```

$\frac{s^2}{2}$

```
[8]  def func_euler(x):
         return 0.5 * np.power(x, 2)
```

```
[9]  arc_length, x_coord, y_coord = x_y_coordinates(x_func, y_func, func_euler, s_steps)
```

First write functions to calculate x and y coordinates More words here

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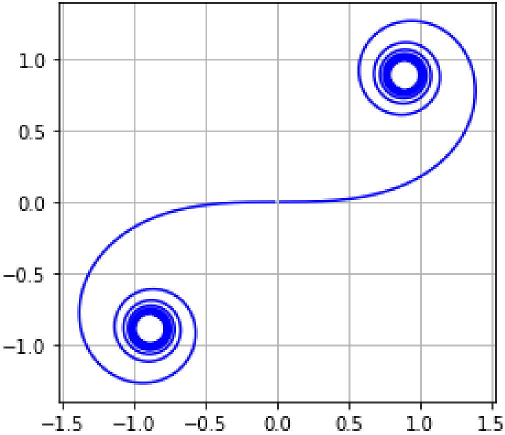
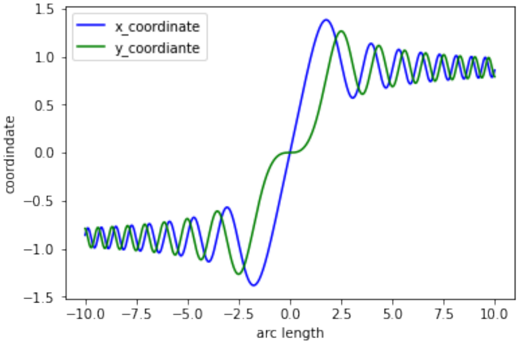
The Euler Spiral

Figure: The Euler Spiral aka Cornu Curve



The Euler Spiral - x y Coordinates

Figure: Fresnel Integrals with arguments $\frac{u^2}{2}$



These converge to $\pm\frac{\sqrt{\pi}}{2} \approx 0.8862$.

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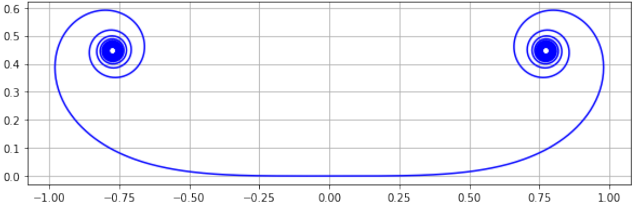
Other Fun Curves: Even Powers of s

Figure: $\kappa(s) = s^2$



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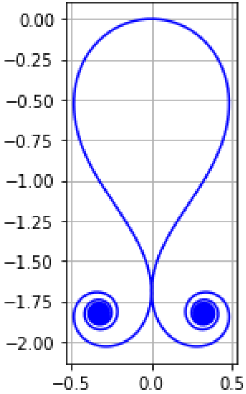
More Fun Curves: Mix in a Bit of a Circle

Figure: $\kappa(s) = s^2 - 2.19$



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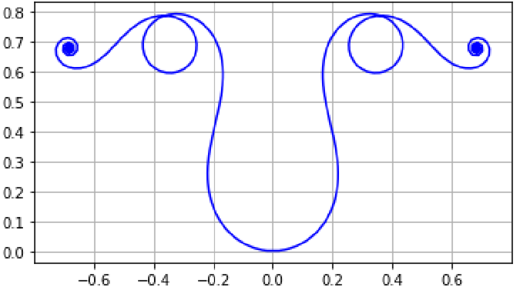
More Fun Curves: Polynomials

Figure: $\kappa(s) = 5s^4 - 18s^2 + 5$



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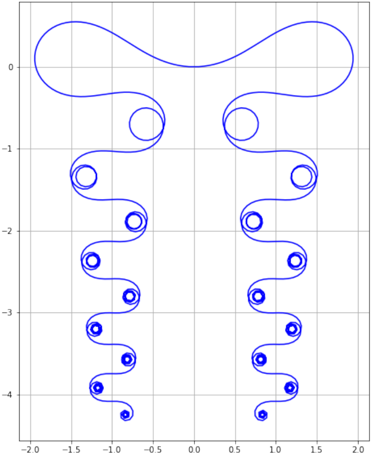
More Fun Curves: Trigonometric Functions

Figure: $\kappa(s) = \cos(s) - s\sin(s)$



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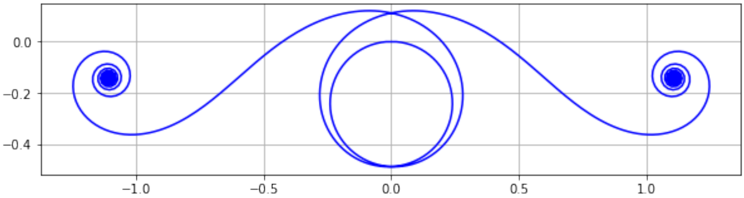
More Fun Curves: Hyperbolic Functions

Figure: $\kappa(s) = \sinh(s) - 5.19$



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"Curvature determines the Curve"

Diffraction and Fresnel Integrals

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Designing Roads and Railways

1. Transition curves are used to link straight sections of motorways or railways.
2. They are designed to give passengers a smooth ride.
3. In particular so sudden changes in acceleration.

Figure: Cloverleaf Motorway Interchange

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Why Transition Curves are Euler Spirals

- The acceleration along the transition is given by

$$a = s''(t)\vec{T} + \kappa s'(t)^2\vec{N}$$

  Where $\vec{T}$ is the unit tangent vector and $\vec{N}$ is the unit normal vector.
- If the car/train is going round the curve at constant speed $s'(t) = \textit{constant}$ and $s''(t) = 0$.
- The acceleration at constant speed only depends on the curvature κ and speed $s'(t)$ in the direction of the normal vector.

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The Racing Line