The Euler Spiral

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Outline

- Definitions and Derivations: Parametric Curves, Lengths and Curvature
- Relating the Curve Length to its Curvature
- Calculating Fresnel Integrals and the Euler Spiral
- Plotting the Euler Spiral and other Fun Curves
- Application: Designing Railways and Roads

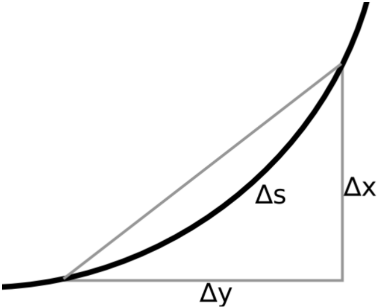
Parametric Curves

All curves in this talk are in $\mathbb{R}^2$.

Definition (A Parametric Curve)

A parametric curve is a smooth function that has the form $x = g(t)$ and $y = h(t)$ defined on an open interval (a, b). The set of points traced out by the curve is called the trace.

How Long is a Curve?



- Use Pythagoras to approximate Δs
- $\Delta s \approx \sqrt{\Delta y^2 + \Delta x^2}$

Derivation Arc Length Integral

- Write the whole Curve Length ,S, as a Riemann Sum and make it into Integral
- $\mathrm{S} \approx \sum_{i=1}^{n} \sqrt{\Delta y_i^2 + \Delta x_i^2}$
- Factor out Δx_i^2
- $\mathrm{S} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\Delta y_i^2 / \Delta x_i^2 + 1} \Delta x_i$
- $\mathrm{S} = \int_a^b \sqrt{(dy/dx)^2 + 1} dx$

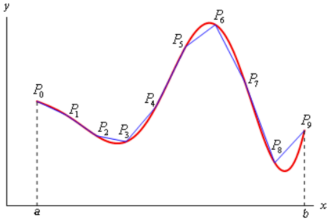
Derivation of Arc Length Integral

- $S = \int_a^b \sqrt{(dy/dx)^2 + 1} dx$
- $\frac{dx/dt}{dx/dt} = 1$ so
- $S = \int_a^b \sqrt{(\frac{dy/dt}{dx/dt})^2 + (\frac{dx/dt}{dx/dt})^2} dx$
- Factor out $(\frac{dx/dt}{dx/dt})^2$ giving us:
- $S = \int_a^b \sqrt{(dy/dt)^2 + (dx/dt)^2} dt$

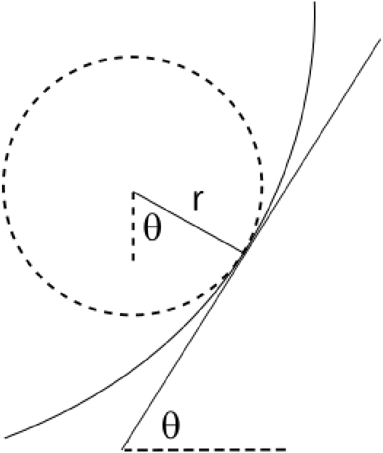
Arc Length Integral

Theorem (Arc Length)

The arc length between two points the curve for $T = a$ and $T = b$ is given by $S = \int_a^b \sqrt{(dy/dt)^2 + (dx/dt)^2}dt$



Deriving Formula for Curvature



- Firstly Curvature is 1/radius of Osculating Circle.
- Or the rate of Change of the angle the tangent makes with the x axis with respect to arc length.
- $ds = rd\theta \implies 1/r = d\theta/ds$
- $\frac{ds}{dt} = \sqrt{x'^2 + y'^2}$ from arc length.

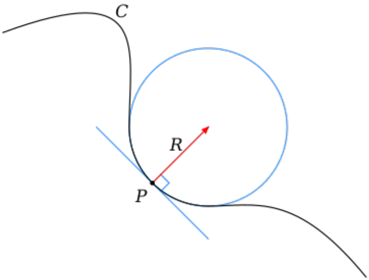
Deriving Formula for Curvature

- $\tan\theta = \frac{y'(t)}{x'(t)} \implies \sec^2\theta \frac{d\theta}{dt} = \frac{x'y''-y'x''}{x'^2}$.
- $sec^2\theta = tan^2\theta + 1 = \frac{x'^2+y'^2}{x'^2} \implies \frac{d\theta}{dt} = \frac{x'y''-y'x''}{x'^2+y'^2}$
- Combining gives us $1/r = \frac{d\theta/dt}{ds/dt} = d\theta/ds = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$
- Curvature, $\kappa = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$

How Curved is a Curve?

Theorem (Curvature)

The curvature of curve is κ which is the reciprocal of the radius of the osculating circle given by $\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$



Circle

- Circle: The parametric equations for a circle with radius r as $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$. The arc length is

$$s = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt = [r]_0^{2\pi} = 2\pi r$$

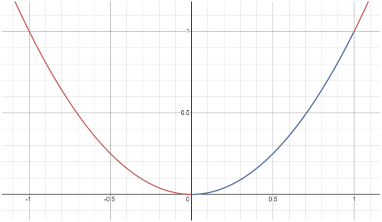
- Using the same Parametrisations we can work out the Curvature

$$\kappa = \frac{r^2\sin^2 t + r^2\cos^2 t}{\left(r^2\sin^2 t + r^2\cos^2 t\right)^{\frac{3}{2}}} = \frac{1}{r}$$

Parabola Curve Length

- Parabola: We can define a parabola as $x(t) = t$ and $y(t) = t^2$. The arc length between 0 and 1 is

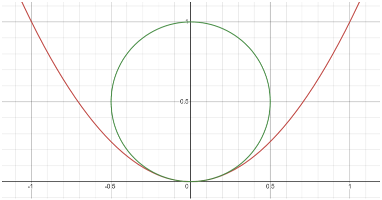
$$\begin{aligned} s &= \int_0^1 \sqrt{1+4t^2}dt \\ &= \left[\frac{1}{2}t\sqrt{1+4t^2} + \frac{1}{4}\ln\left(2t+\sqrt{1+4t^2}\right)\right]_0^1 \\ &\approx 1.48 \end{aligned}$$



Parabola Curvature

- Parabola: Recall $x(t) = t$ and $y(t) = t^2$.

$$\kappa = \frac{2}{(1+4t^2)^{\frac{3}{2}}}$$



Relating Curve Length to Curvature

Let's try the following parametrisation for x and y

$$\begin{aligned} x(t) &= \int_0^t \cos f(u) du \\ y(t) &= \int_0^t \sin f(u) du \end{aligned}$$

This gives us

$$\begin{aligned} x' = x'(t) = \cos f(t) \quad &\text{and} \quad x'' = -f'(t) \sin f(t) \\ y' = y'(t) = \sin f(t) \quad &\text{and} \quad y'' = f'(t) \cos f(t) \end{aligned}$$

Relating Curve Length to Curvature

This gives us the following for slope, arc length and curvature

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\sin f(t)}{\cos f(t)} = \tan f(t) \\
s &= \int_0^t \sqrt{\cos^2 f(u) + \sin^2 f(u)}du = t \\
\kappa &= \frac{f'(t)\cos^2 f(t) + f'(t)\sin^2 f(t)}{\cos^2 f(t) + \sin^2 f(t)} = f'(t)
\end{aligned}
$$

Define the Curve by Curve Length and Curvature

We can replace the variable t by the arc length s.
And the curvature at point t is $f'(t)$. Which means

$$f(t) = \int \kappa(t)dt$$

Thus the equations for the curve become

$$\begin{aligned} x = x(s) &= \int_0^s \cos\left(\int_0^u \kappa(t)dt\right) du \\ y = y(s) &= \int_0^s \sin\left(\int_0^u \kappa(t)dt\right) du \end{aligned}$$

Hence the curve is defined by arc length and curvature alone.

A Very Simple Example

We can make the curvature κ constant and equal to 1. Then $\int_0^u \kappa(t)dt = u$ and

$$\begin{aligned} x = x(s) &= \int_0^s \cos u du = \sin s \\ y = y(s) &= \int_0^s \sin u du = -\cos s + 1 \end{aligned}$$

which is the parametric curve for a circle with centre $(0,1)$ and radius 1.

Recall: Euler defined his curve as one where the curvature is proportional to arc length.

$$\boxed{\kappa(s) = s}$$

Then $\int_0^u \kappa(t)dt = \frac{u^2}{2}$ and

$$\begin{aligned} x = x(s) &= \int_0^s \cos \frac{u^2}{2} du \\ y = y(s) &= \int_0^s \sin \frac{u^2}{2} du \end{aligned}$$

But since these integrals can't be solved analytically how were they calculated?

Solving the Integrals: Euler

- In 1744 Euler derived these integrals.

$$\int ds \text{ fin. } \frac{ss}{2aa} \ \& \int ds \text{ cof. } \frac{ss}{2aa}$$

- He derived a series expansion which is still a viable method for small s.

$$x = \frac{s^3}{1.3b^2} - \frac{s^7}{1.2.3.7b^6} + \frac{s^{11}}{1.2.3.4.5.11b^{10}} - \frac{s^{15}}{1.2....7.15b^{14}} + \&c.$$

$$y = s - \frac{s^5}{1.2.5\,b^4} + \frac{s^9}{1.2.3.4.9b^8} - \frac{s^{13}}{1.2.3...6.13b^{12}} + \&c.$$

- In 1781 he proved the integrals for limits between 0 and ∞ are equal to $\frac{a\sqrt{\pi}}{2}$

Solving the Integrals: Fresnel and Cornu

- In 1818 Augustin Fresnel rediscovered these integrals when he investigated the diffraction of light through a slit. He showed that the intensity (under some assumptions) was

$$\left(\int_0^s \cos\left(\pi t^2/2\right)\,dt\right)^2 + \left(\int_0^s \sin\left(\pi t^2/2\right)\,dt\right)^2$$

- Up to a factor of π the integrals are the same as the ones Euler derived.
- These integrals are now called the *Fresnel Integrals*.
- Fresnel calculated them for values of s between 0.1 and 5.1.

Solving the Integrals: Today with Python

```python
def x_func(upper, lower, func):
  def integrand(t):
    return np.cos(func(t))
  result, _ = integrate.quad(integrand, upper, lower)
  return result
```

```python
def y_func(upper, lower, func):
  def integrand(t):
    return np.sin(func(t))
  result, _ = integrate.quad(integrand, upper, lower)
  return result
```

Euler Spiral

```python
[7]  k = s
     spy.integrate(k, s)
```

$$\frac{s^2}{2}$$

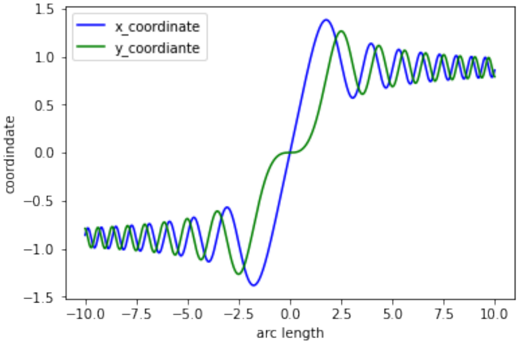
```python
[8]  def func_euler(x):
         return 0.5 * np.power(x, 2)
```

```python
[9]  arc_length, x_coord, y_coord = x_y_coordinates(x_func, y_func, func_euler, s_steps)
```

Nowadays we use computers and numerical methods to evaluate these integrals.

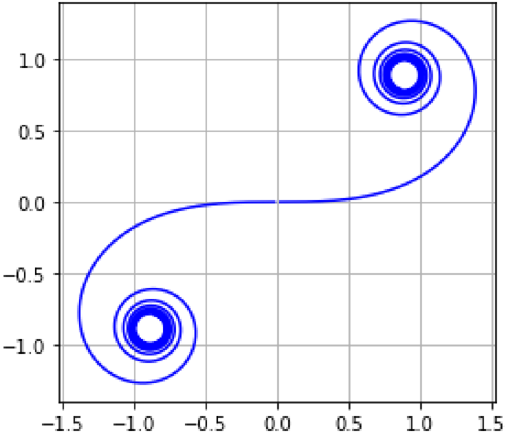
The Euler Spiral - x y Coordinates

Figure: Fresnel Integrals with arguments $\frac{u^2}{2}$



As expected these converge to $\pm\frac{\sqrt{\pi}}{2} \approx 0.8862$.

Figure: The Euler Spiral aka Cornu Curve



Eulers First Drawing

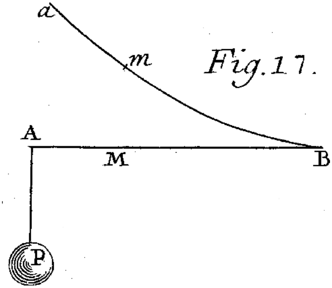


Figure: Euler's First Drawing from a 1744 Publication. P refers to a weight.

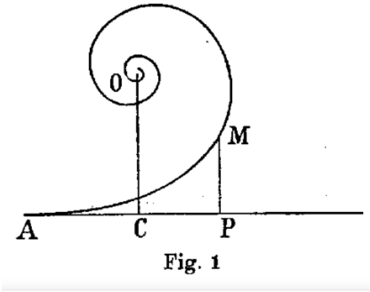
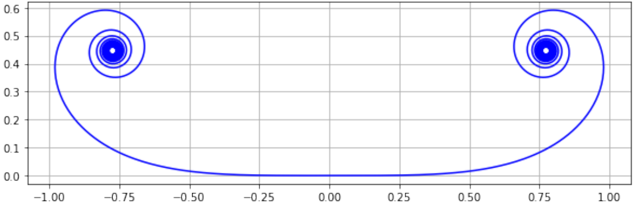


Figure: Euler's drawing of full Spiral in 1781 after solving integrals to ∞

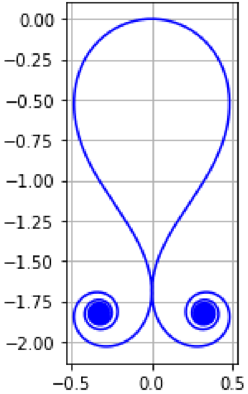
Other Fun Curves: Even Powers of s

Figure: $\kappa(s) = s^2$



More Fun Curves: Mix in a Bit of a Circle

Figure: $\kappa(s) = s^2 - 2.19$



More Fun Curves: Polynomials

Figure: $\kappa(s) = 5s^4 - 18s^2 + 5$

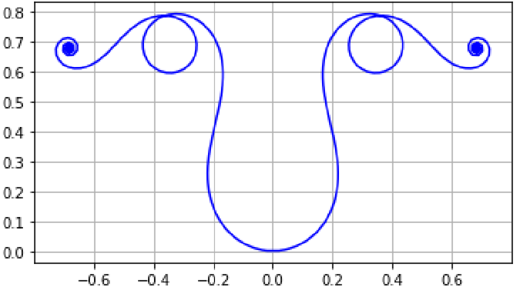
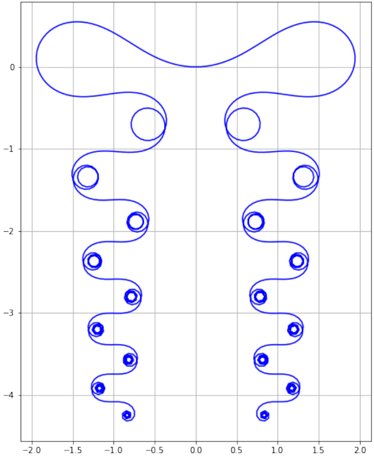
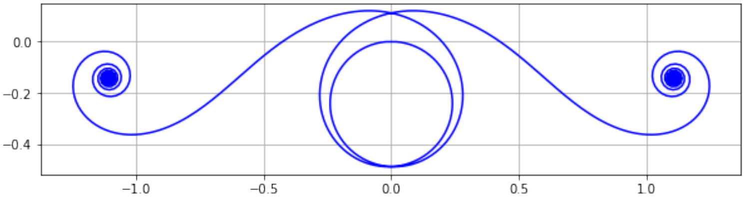


Figure: $\kappa(s) = \cos(s) - s\sin(s)$



More Fun Curves: Hyperbolic Functions

Figure: $\kappa(s) = \sinh(s) - 5.19$



A major use of the Euler Spiral

- As trains became faster in the 19th century, the Euler spiral was rediscovered by railway designers.
- Railway engineers discovered that a track shape with gradually varying curvature provided a smooth riding experience.
- In 1899 railway engineer Arthur Talbot solved the problem mathematically and derived the same integrals as Fresnel.
- His solution was “a curve whose degree-of-curve increases directly along the curve.”
- He derived a series expansion for the integrals almost identical to Euler’s 1744 series.

Designing Roads and Railways

- Transition curves are used to link straight sections of motorways or railways.
- They are designed to give passengers a smooth ride with no sudden changes in acceleration.

Figure: Cloverleaf Motorway Interchange

Why Transition Curves are Euler Spirals

- The acceleration along the transition curve is given by

$$a = s''(t)\vec{T} + \kappa s'(t)^2\vec{N}$$

  Where $\vec{T}$ is the unit tangent vector and $\vec{N}$ is the unit normal vector.
- If the car/train is going round the curve at constant speed $s'(t) = constant$ and $s''(t) = 0$.
- The acceleration at constant speed only depends on the curvature κ and speed $s'(t)$ in the direction of the normal vector.

Case 1: Semicircular Transition Curves

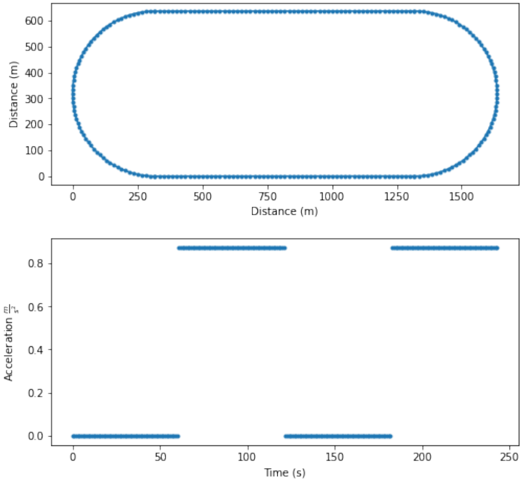
A closed track is made up of four segments.

1. A straight track of length 1km
2. A semicircular track of length 1km. This has radius $1,000/\pi m \approx 318.31m$
3. A straight track of length 1km
4. A semicircular track of length 1km.

Let the vehicle go around the track at a constant speed of $60km/h = 16\frac{2}{3}m/s$.

Case 1: Position and Acceleration

Figure: Position and Acceleration



Note: The acceleration is a step function.
As a passenger you would feed the full centrifugal force ($F = mv^2/r$) pushing you outward the moment you entered the curve.

Case 2: Euler Spiral Transition Curves

- A closed track with same width and height.
- Replace semicircles with two parts of an Euler Spiral.
- Recall that the curvature is proportional to arc length. $\kappa = \alpha s$ for some α.
- The width and height of an Euler Spiral that turns through $\pi/2$ is given by

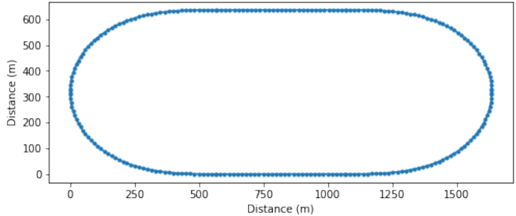
$$\begin{aligned} width &= \sqrt{\pi/\alpha}C(1) \\ height &= \sqrt{\pi/\alpha}S(1) \end{aligned}$$

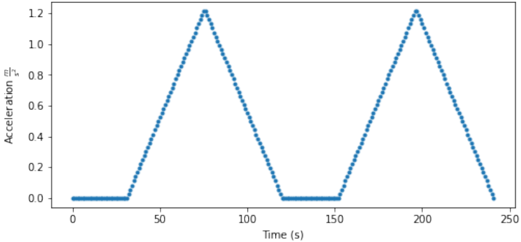
where $S(u)$ and $C(u)$ are the standard Fresnel integrals.

$$\alpha = \frac{\pi S(1)^2}{r^2} \approx 5.95 \times 10^{-6}$$

Case 2: Euler Spiral Transition Curves

Figure: Position and Acceleration





Note: The acceleration increases linearly as we move through the curve. The maximum acceleration however at the apex is greater than with the semicircular track. But the ride is more comfortable now!

Thanks for listening!