The Euler Spiral

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Outline

- Definitions and Derivations: parametric curves, lengths and curvature
- Relating curvature to the curve length
- Calculating arc length integrals
- Plotting the Euler spiral and other fun curves
- Application: designing railways and roads

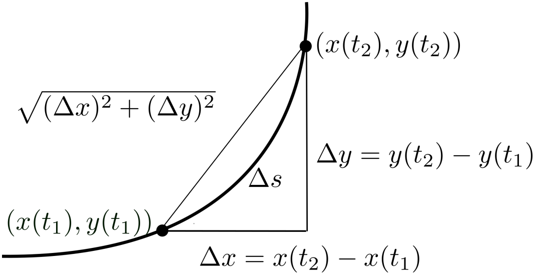
Parametric curves

All curves in this talk are in $\mathbb{R}^2$.

Definition (Parametric curve)

A parametric curve is a smooth function that is defined on an open interval (a, b) and takes values in $\mathbb{R}^2$ of the form $(x(t), y(t))$. The set of points traced out by the curve is called the **trace**.

Length of a curve



- When $\Delta t = t_2 - t_1$ is small
- $\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Length of a curve segment

- From the *mean value theorem* there exits a $t_x \in [t_1, t_2]$ such that

$$\Delta x = x'(t_x)\Delta t$$

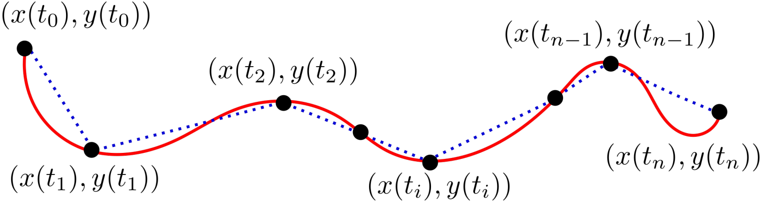
  where $x'(t_x) = \frac{dx}{dt}\Big|_{t=t_x}$
- Similarly there exists $t_y \in [t_1, t_2]$ such that

$$\Delta y = y'(t_y)\Delta t$$

  where $y'(t_y) = \frac{dy}{dt}\Big|_{t=t_y}$
- $$\Delta s \approx \sqrt{x'(t_x)^2 + y'(t_y)^2}\Delta t$$

Derivation arc length integral

- Partition the domain (a, b) into n small intervals: Let $t_0, t_1, \ldots t_n$ be such that $t_0 = a$ and $t_n = b$ and $t_{i-1} < t_i$ for all $i = 1, \ldots n$.
- The whole curve length, S, is sum of curve length segments between each $(x(t_{i-1}), y(t_{i-1}))$ and $(x(t_i), y(t_i))$.



Derivation arc length integral

- Approximate, S, with a Riemann sum

$$S \approx \sum_{i=1}^{n} \sqrt{x'(t_{x_i})^2 + y'(t_{y_i})^2}\Delta t_i$$

where $\Delta t_i = t_i - t_{i-1}$ $t_{x_i} \in [t_{i-1}, t_i]$ and $t_{y_i} \in [t_{i-1}, t_i]$ for all i.

- Let $n \to \infty$ such that $\Delta t_i \to 0$ for all i then

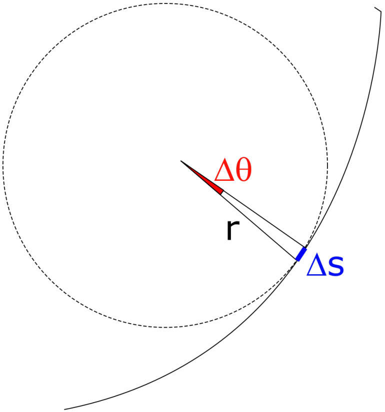
$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{x'(t_{x_i})^2 + y'(t_{y_i})^2}\Delta t_i \\ &= \int_{t=a}^{t=b} \sqrt{x'(t)^2 + y'(t)^2}dt \end{aligned}$$

Arc length integral

Theorem (Arc Length)

The arc length between two points, $t = a$ and $t = b$, on the curve is given by $S = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt$

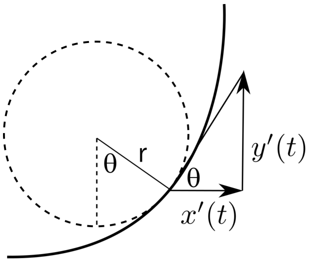
Deriving formula for curvature



- Firstly curvature is 1/radius of the osculating circle.
- Or the rate of change of the angle the tangent makes with the x axis with respect to arc length.
- $ds = r d\theta \implies \frac{1}{r} = \frac{d\theta}{ds}$
- From arc length

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} \qquad (1)$$

Deriving formula for curvature



- $\tan\theta = \frac{y'(t)}{x'(t)}$
- Differentiate w.r.t. t gives $\sec^2\theta\frac{d\theta}{dt} = \frac{x'(t)y''(t)-y'(t)x''(t)}{x'(t)^2}$ where $x''(t) = \frac{d^2x}{dt^2}$ and $y''(t) = \frac{d^2y}{dt^2}$
- $sec^2\theta = tan^2\theta + 1 = \frac{x'(t)^2+y'(t)^2}{x'(t)^2}$
- Therefore

$$\frac{d\theta}{dt} = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^2 + y'(t)^2} \qquad (2)$$

Deriving formula for curvature

- Combining equations 1 and 2 gives us

$$\begin{aligned}\frac{1}{r} &= \frac{d\theta}{ds} \\ &= \frac{d\theta/dt}{ds/dt} \\ &= \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}\end{aligned}$$

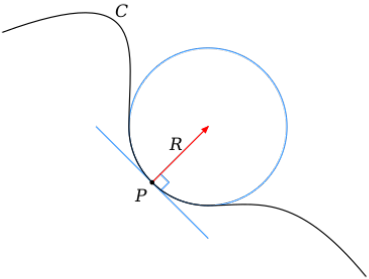
- Curvature, $\kappa = \frac{1}{r}$

How curved is a curve?

Theorem (Curvature)

The curvature of curve is κ which is the reciprocal of the radius of the osculating circle given by

$$\kappa = \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$



Curvature of a circle

- Circle: The parametric equations for a circle with radius r are $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$.
- The arc length is

$$s = \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\, dt = [r]_0^{2\pi} = 2\pi r$$

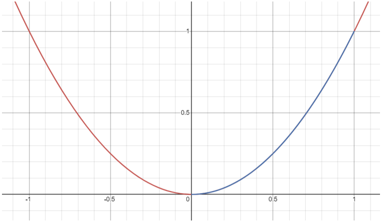
- Using the same parametrisations we can work out the curvature

$$\kappa = \frac{r^2 \sin^2 t + r^2 \cos^2 t}{\left(r^2 \sin^2 t + r^2 \cos^2 t\right)^{\frac{3}{2}}} = \frac{1}{r}$$

Parabola curve length

- Parabola: For parabola $y = x^2$ the parametric equations are $x(t) = t$ and $y(t) = t^2$. The arc length between 0 and 1 is

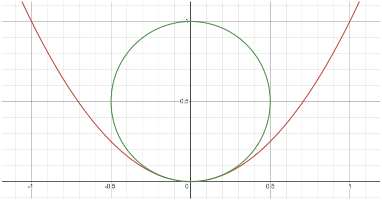
$$\begin{aligned} s &= \int_0^1 \sqrt{1+4t^2}\,dt \\ &= \left[\frac{1}{2}t\sqrt{1+4t^2} + \frac{1}{4}\ln\left(2t+\sqrt{1+4t^2}\right)\right]_0^1 \\ &\approx 1.48 \end{aligned}$$



Parabola curvature

- Parabola: Recall $x(t) = t$ and $y(t) = t^2$.

$$\kappa = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}}$$



Relating curve length to curvature

Let's try the following parametrisation for x and y

$$\begin{aligned} x(t) &= \int_0^t \cos f(u) du \\ y(t) &= \int_0^t \sin f(u) du \end{aligned}$$

This gives us

$$\begin{aligned} x' = x'(t) = \cos f(t) &\quad \text{and} \quad x'' = -f'(t)\sin f(t) \\ y' = y'(t) = \sin f(t) &\quad \text{and} \quad y'' = f'(t)\cos f(t) \end{aligned}$$

Relating curve length to curvature

This gives us the following for slope, arc length and curvature

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sin f(t)}{\cos f(t)} = \tan f(t) \\
s &= \int_0^t \sqrt{\cos^2 f(u) + \sin^2 f(u)} du = t \\
\kappa &= \frac{f'(t)\cos^2 f(t) + f'(t)\sin^2 f(t)}{\cos^2 f(t) + \sin^2 f(t)} = f'(t)
\end{aligned}$$

Define the curve by curve length and curvature

We can replace the variable t by the arc length s.
And the curvature at point t is $f'(t)$. Which means

$$f(u) = \int_0^u \kappa(t)dt$$

Thus the equations for the curve become

$$\begin{aligned} x = x(s) &= \int_0^s \cos\left(\int_0^u \kappa(t)dt\right) du \\ y = y(s) &= \int_0^s \sin\left(\int_0^u \kappa(t)dt\right) du \end{aligned}$$

Hence the curve is defined by arc length and curvature alone.

A very simple example

We can make the curvature κ constant and equal to 1. Then $\int_0^u \kappa(t)dt = u$ and

$$\begin{aligned} x = x(s) &= \int_0^s \cos u du = \sin s \\ y = y(s) &= \int_0^s \sin u du = -\cos s + 1 \end{aligned}$$

which is the parametric curve for a circle with centre $(0, 1)$ and radius 1.

The Euler spiral

Recall: Euler defined his curve as one where the curvature is proportional to arc length.

$$\boxed{\kappa(s) = s}$$

Then $\int_0^u \kappa(t)dt = \frac{u^2}{2}$ and

$$\begin{aligned} x = x(s) &= \int_0^s \cos \frac{u^2}{2} du \\ y = y(s) &= \int_0^s \sin \frac{u^2}{2} du \end{aligned}$$

But since these integrals can't be solved analytically how were they calculated?

Solving the integrals: Euler

- In 1744 Euler derived these integrals.

$$\int ds \text{ fin. } \frac{ss}{2aa} \;\&\; \int ds \text{ cof. } \frac{ss}{2aa}$$

- He derived a series expansion which is still a viable method for small s.

$$x = \frac{s^3}{1.3b^2} - \frac{s^7}{1.2.3.7b^6} + \frac{s^{11}}{1.2.3.4.5.11b^{10}} - \frac{s^{15}}{1.2....7.15b^{14}} + \&c.$$

$$y = s - \frac{s^5}{1.2.5\,b^4} + \frac{s^9}{1.2.3.4.9b^8} - \frac{s^{13}}{1.2.3...6.13b^{12}} + \&c.$$

- In 1781 he proved the integrals for limits between 0 and ∞ are equal to $\frac{a\sqrt{\pi}}{2}$

Solving the integrals: Fresnel and Cornu

- Fresnel rediscovered these integrals when investigating the diffraction of light through a slit. He showed that the light intensity (under some assumptions) was

$$\left(\int_0^s \cos\left(\pi t^2/2\right)\, dt\right)^2 + \left(\int_0^s \sin\left(\pi t^2/2\right)\, dt\right)^2$$

- Up to a factor of π the integrals are the same as the ones Euler derived.
- These integrals are now called the *Fresnel integrals*.
- Fresnel calculated them for values of s between 0.1 and 5.1.

Solving the integrals: using Python

```
def x_func(upper, lower, func):
  def integrand(t):
    return np.cos(func(t))
  result, _ = integrate.quad(integrand, upper, lower)
  return result
```

```
def y_func(upper, lower, func):
  def integrand(t):
    return np.sin(func(t))
  result, _ = integrate.quad(integrand, upper, lower)
  return result
```

Euler Spiral

```
[7]  k = s
     spy.integrate(k, s)
```

$\frac{s^2}{2}$

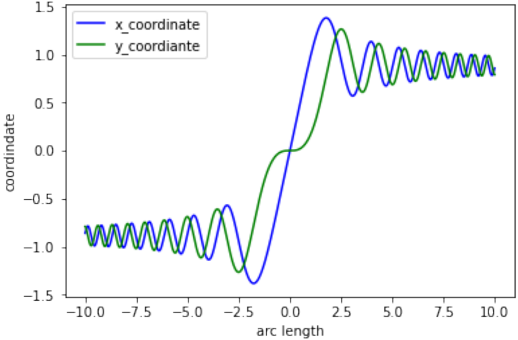
```
[8]  def func_euler(x):
         return 0.5 * np.power(x, 2)
```

```
[9]  arc_length, x_coord, y_coord = x_y_coordinates(x_func, y_func, func_euler, s_steps)
```

Nowadays computers and numerical methods are used to evaluate these integrals.

The Euler spiral - x y Coordinates

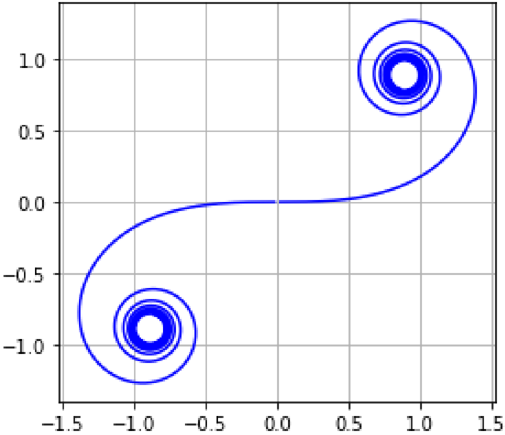
Figure: Fresnel integrals with arguments $\frac{u^2}{2}$



As expected these converge to $\pm\frac{\sqrt{\pi}}{2} \approx 0.8862$.

The Euler Spiral

Figure: The Euler spiral aka Cornu curve



Eulers first drawing

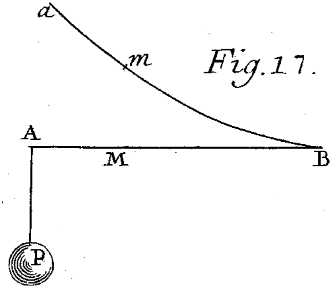


Figure: Euler's first drawing from a 1744 publication. P refers to a weight.

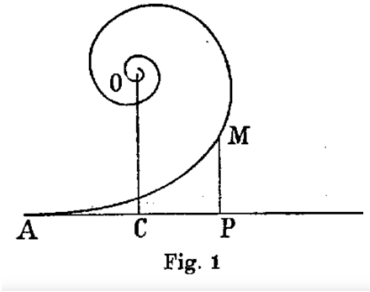
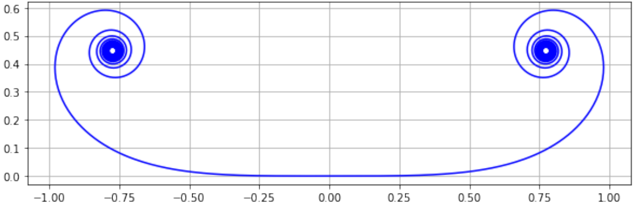


Figure: Euler's drawing of full spiral in 1781 after solving integrals to ∞

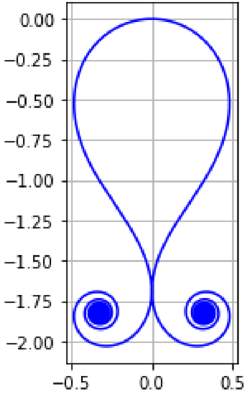
Other fun curves: even powers of s

Figure: $\kappa(s) = s^2$



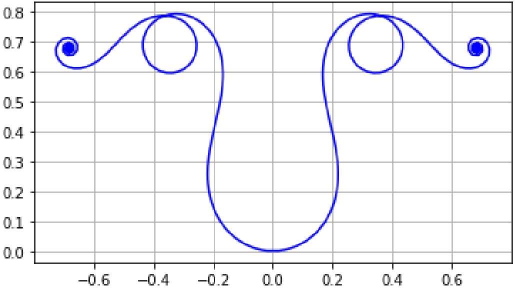
More fun curves: mix in a bit of a circle

Figure: $\kappa(s) = s^2 - 2.19$



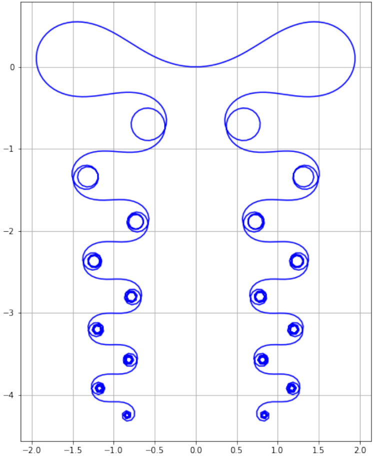
More fun curves: polynomials

Figure: $\kappa(s) = 5s^4 - 18s^2 + 5$



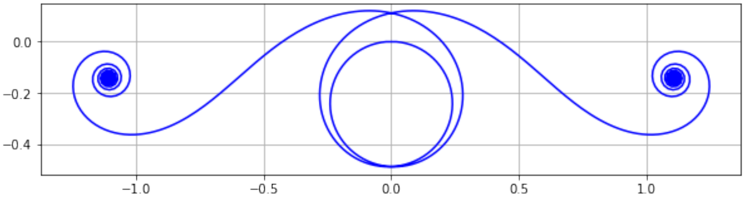
More fun curves: trigonometric functions

Figure: $\kappa(s) = \cos(s) - s\sin(s)$



More fun curves: hyperbolic functions

Figure: $\kappa(s) = \sinh(s) - 5.19$



A major use of the Euler Spiral

- In the 19th century, the Euler spiral was rediscovered by railway designers.
- They realised that a track shape with gradually varying curvature provided a smoother riding experience.
- In 1899 Arthur Talbot solved the design problem mathematically deriving the same integrals as Fresnel.
- His series expansion for the integrals was almost identical to Euler's 1744 series.

Designing roads and railways

- Transition curves are used to link straight sections of motorways or railways.
- They are designed to give passengers a smooth ride with no sudden changes in acceleration.

Figure: Cloverleaf motorway interchange

Why transition curves are Euler spirals

- The acceleration along the transition curve is given by

$$a = s''(t)\vec{T} + \kappa s'(t)^2\vec{N}$$

  Where $\vec{T}$ is the unit tangent vector and $\vec{N}$ is the unit normal vector, $s(t)$ is the curve length, $s'(t) = \frac{ds}{dt}$ and $s''(t) = \frac{d^2s}{dt^2}$.
- If the car/train is going round the curve at constant speed $s'(t) = constant$ and $s''(t) = 0$.
- The acceleration at constant speed only depends on the curvature κ and speed $s'(t)$ in the direction of the normal vector.

Case 1: Semicircular transition curves

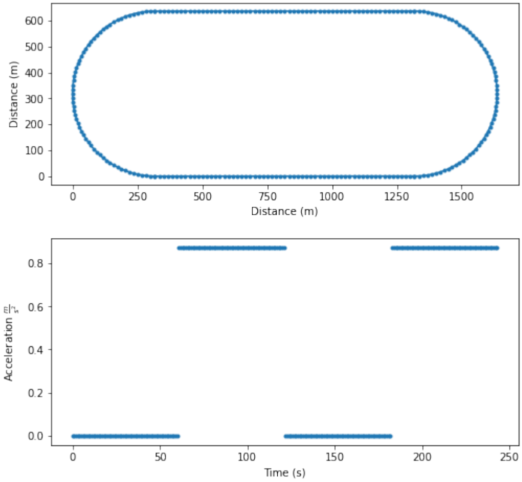
A closed track is made up of four segments.

1. A straight track of length 1km
2. A semicircular track of length 1km. This has radius $1,000/\pi m \approx 318.31m$
3. A straight track of length 1km
4. A semicircular track of length 1km.

Let the vehicle go around the track at a constant speed of $60km/h = 16\frac{2}{3}m/s$.

Case 1: Position and acceleration

Figure: Position and acceleration



Note: Acceleration is a step function.
As a passenger you would feed the full centrifugal force ($F = mv^2/r$) pushing you outward the moment you entered the curve.

Case 2: Euler spiral transition curves

- A closed track with same width and height.
- Replace semicircles with two parts of an Euler Spiral.
- Recall that the curvature is proportional to arc length. $\kappa = \alpha s$ for some α.
- The width and height of an Euler Spiral that turns through $\pi/2$ is given by

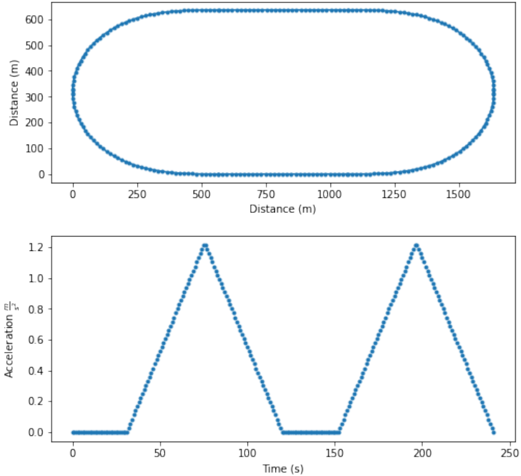
$$\begin{aligned} width &= \sqrt{\pi/\alpha}C(1) \\ height &= \sqrt{\pi/\alpha}S(1) \end{aligned}$$

where $S(u)$ and $C(u)$ are the standard Fresnel integrals.

$$\alpha = \frac{\pi S(1)^2}{r^2} \approx 5.95 \times 10^{-6}$$

Case 2: Euler spiral transition curves

Figure: Position and acceleration



Note: Acceleration increases linearly as we move through the curve.
Howvever the maximum acceleration at the apex is greater than with the semicircular track but the ride is now more comfortable.

Thanks for listening!