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Suppose t_1 and t_2 are two points in the domain of the curve with $t_2 \geq t_1$. We want expression for the distance between the points $(x(t_2), y(t_2))$ and $(x(t_1), y(t_1))$.

That is we want to evaluate

$$\Delta s = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2}$$

From the mean value theorem we have that there exits a $t_x \in [t_1, t_2]$ such that

$$\begin{aligned}
\Delta x &= x(t_2) - x(t_1) \\
&= \frac{dx(t_x)}{dt}(t_2 - t_1) \\
&= \frac{dx(t_x)}{dt}\Delta t
\end{aligned}$$

where

$$\frac{dx(t_x)}{dt} = \left.\frac{dx}{dt}\right|_{t=t_x}$$

and $\Delta t = t_2 - t_1$.

Similarly for y there exists a value $t_y \in [t_1, t_2]$ such that

$$\begin{aligned}
\Delta y &= y(t_2) - y(t_1) \\
&= \frac{dy(t_y)}{dt}(t_2 - t_1) \\
&= \frac{dy(t_y)}{dt}\Delta t
\end{aligned}$$

Thus we have

$$\Delta s = \sqrt{\left(\frac{dx(t_x)}{dt}\right)^2 + \left(\frac{dy(t_y)}{dt}\right)^2}\Delta t$$

But to get the Riemann sum we need the differential to be evaluated at the points $(x(t_1), y(t_1))$.

$$\begin{aligned}
\Delta s &= \sqrt{\left(\frac{dx(t_x)}{dt}\right)^2 + \left(\frac{dy(t_y)}{dt}\right)^2}\Delta t \\
&= \sqrt{\left(\frac{dx(t_1)}{dt}\right)^2 + \left(\frac{dy(t_1)}{dt}\right)^2}(t_2 - t_1) + \epsilon(t_1, t_2)
\end{aligned}$$

where

$$\epsilon(t_1, t_2) = \sqrt{\left(\frac{dx(t_x)}{dt}\right)^2 + \left(\frac{dy(t_y)}{dt}\right)^2} - \sqrt{\left(\frac{dx(t_1)}{dt}\right)^2 + \left(\frac{dy(t_1)}{dt}\right)^2}$$

Now we have

$$\begin{aligned}
|\epsilon(t_1,t_2)| &= \left|\sqrt{\left(\frac{dx(t_x)}{dt}\right)^2+\left(\frac{dy(t_y)}{dt}\right)^2}-\sqrt{\left(\frac{dx(t_1)}{dt}\right)^2+\left(\frac{dy(t_1)}{dt}\right)^2}\right| \\
&\leq \sqrt{\left(\frac{dx(t_x)}{dt}-\frac{dx(t_1)}{dt}\right)^2+\left(\frac{dy(t_y)}{dt}-\frac{dy(t_1)}{dt}\right)^2} \\
&\leq \left|\frac{dx(t_x)}{dt}-\frac{dx(t_1)}{dt}\right|+\left|\frac{dy(t_y)}{dt}-\frac{dy(t_1)}{dt}\right|
\end{aligned}$$

where the first inequality comes from the reverse triangle inequality $||a|-|b||\leq|a-b|$ and the second comes from the regular triangle inequality $|a+b|\leq|a|+|b|$.

Finally the first derivatives of x and y are continuous we can make $\epsilon(t_1,t_2)$ arbitrary small as we let Δt become small.