Probabilities

$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$ # probability of B, given that A is true

if $P(B|A) = P(B) \;||\; P(A|B) = P(A) \;||\; P(A \cap B) = P(A)P(B)$, then A and B are independent

Bayes' rule:

$P(A) = P(A \cap B) + P(A \cap B') = P(B)P(A|B) + P(B')P(A|B')$

$P(B_r|A) = \frac{P(B_r|\cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$

Random variables

$f(x) = P(X = x)\ (dis\ only) = F'(x) = F(x) - F(x - 1)$ # probability math/density function

$F(x) = P(X <= x) = \sum_{-\infty}^{\infty} f(x) = \sum_{x}\sum_{y} f(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)dxdy$ # cumulative probability/density fn

$F(x) = x \quad for\ linear\ distribution\ [0, 1]$

$P(a < x < b) = F(b) - F(a) = \sum_{a<x<b} f(x) = \int_{a}^{b} f(x)dx$

$P(x > a) = 1 - F(x)$

$g(x) = \int f(x, y)dy = \int + \int$ (h(y) dx) - marginal distribution (**over range of y)(remember piecewise)**

$P(0 < x < 1) = \int_{0}^{1} g(h)dx$ # alternative way

$\int e^{ax}dx = \frac{e^{ax}}{a}$

$\int \frac{1}{x^n}dx = \frac{-1}{(n-1)x^{n-1}}$

Mean and Variance

$\mu = E(X) = \sum_{x} xf(x) = \int_{-\infty}^{\infty} xf(x)dx$

$E(Y) = \int\int yf(x, y)dydx$ **(remember reflection)**

$E(g(X)) = \sum_{x} g(x)f(x) = \int_{-\infty}^{\infty} g(x)f(x)dx$

$E(g(X, Y)) = \sum_{x}\sum_{y} g(x, y)f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy$

$Var(X) > 0; = \sigma^2 = \sigma_x^2 = E((x - \mu)^2) = E(X^2) - E(X)^2 = \sum_{x}(x - \mu)^2 f(x) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$

$Standard\ deviation = \sigma = \sqrt{Var(x)}$

$Cov(X,Y) = \sigma_{XY} = E((X - \sigma_X)(Y - \sigma_Y)) = E(XY) - E(X)E(Y) = \int\int(x - \sigma_X)(y - \sigma_Y)f(x,y)dxdy$

$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(y)}}$ $\# \ correlation \ coefficient \ in \ [-1, 1]$

$if \ X = Y, \ Cov(X,Y) = Var(x) \ and \ \rho_{XX} = \frac{\sigma_{XX}}{\sigma_X\sigma_X} = \frac{Var(x)}{Var(x)} = 1$

$if \ X = -Y, \ \rho_{X(-X)} = \frac{\sigma_{X(-X)}}{\sigma_X\sigma_{-X}} = \frac{-Var(x)}{Var(x)} = -1$

$E(aX + bY + C) = aE(X) + bE(Y) + c$ ← **use this to simplify**

$Var(aX + bY + C) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$ ← **use this to simplify**

$Cov(a_1X_1 + a_2X_2, Y) = a_1Cov(X_1, Y) + a_2Cov(X_2, Y)$

$Cov(X, bY_1 + bY,) = b_1Cov(X, Y_1) + b_2Cov(X, Y_2)$

$Cov(a_1X_1 + a_2X_2, bY_1 + bY) = \sum_{i=1}^{2}\sum_{j=1}^{2} a_ib_jCov(X_i, Y_j)$

If X and Y are independent: $E(XY) = E(X)E(Y) \Rightarrow Cov(X,Y) = 0 \ and \ \rho_{XY} = 0$

$Var(aX + bY + C) = a^2Var(X) + b^2Var(Y)$

$Cov(X,Y) = Cov(Y,X)$

$if \ f(y) = 1 \ in \ 7 \le y \le 8, \ then \ let \ X = Y - 7 \ and \ Var(X) = Var(y) \ and \ Var(e^Y) = Var(e^{x+7})$

$\int udv = uv - \int vdu$

Uniform Distribution

$f(x; A, B) = \frac{1}{B-A} \ if \ A \le x \le B, \ else \ 0$

$E(x) = \frac{A+B}{2}$

$Var(x) = \frac{(B-A)^2}{12}$

Normal Distribution

$n(x, \mu, \sigma^2) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$ # density function

$n(x) = n(x, 0, 1) = \frac{e^{-\frac{x}{2}}}{\sqrt{2\pi}}$ # standard normal distribution

if X is st. norm. var., $\sigma X + \mu$ is a norm. var., with var σ^2

Symmetric about $x = \mu$ with max height of $\frac{1}{\sigma\sqrt{2\pi}}$

Inflection points at $x = \mu \pm \sigma$

$X \sim N(\mu, \sigma^2)$ # rand var X distributed with mean and variance

$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dx$ (sum from $-\infty \ to \ x$)

$\Phi(0) = \frac{1}{2}$ (midpoint) $\Phi(-\infty) = 0$ $\Phi(\infty) = 1$ $\Phi(x) + \Phi(-x) = 1$

$if \ X \sim N(0,1)$, then $P(a \le X \le b) = \Phi(b) - \Phi(a)$ $P(X \le b) = \Phi(b)$ $P(X \ge a) = 1 - \Phi(a)$

$if\ X \sim N(\mu, \sigma^2)$, then $P(a \le X \le b) = \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})$ $\quad P(X \le b) = \Phi(\frac{b-\mu}{\sigma})$ $\quad P(X \ge a) = 1 - \Phi(\frac{a-\mu}{\sigma})$

$middle$ 75%: $P(\frac{\mu-\alpha-\mu}{\sigma} \le X \le \frac{\mu+\alpha-\mu}{\sigma})$

Chi-Squared Distribution

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{x} dx,\ \ for\ \alpha > 1$

$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

$\Gamma(n) = (n - 1)!\ \ \ for\ int\ n > 0$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$f(x; v) = \frac{x^{\frac{v}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}\ \ for\ x > 0,\ else\ 0$

$\mu = v$

$\sigma^2 = 2v$

$E(S^2) = \sigma^2$ # S^2 is same as for samples

$E(x^2) = Var(x_i) + E(x_i)^2 = \sigma^2 + \mu^2$

$E(x_i x_j) = E(x_i)E(x_j) = \mu^2$

$\frac{(n-1)S^2}{\sigma^2} \sim X^2$ n-1 degrees of freedom ← **use this ($x^2\ test$) to verify claims** (compare result with table)

Samples

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ # mean

$\hat{X} = Y_{\frac{n+1}{2}}\ if\ n\ is\ odd\ else\ \frac{1}{2}(Y_{\frac{n}{2}} + Y_{\frac{n}{2}+1})$ # median

Mode: most common value(s)

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n(n-1)}(n\sum_{i=1}^{n} X_i^2 - (\sum_{i=1}^{n} X_i)^2)$ # variance

$S = \sqrt{S^2}$ # standard deviation

$R = max\{X_1, \dots, X_n\} - min\{X_1, \dots, X_n\}$ # range

Central Limit Theorem

$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{(X_1 + \dots + X_n) - n\mu}{\sigma\sqrt{n}} \approx Z \sim N(0, 1)\ \ for\ n \ge 30\ (lim)$ ← **prefer this formula**

$\bar{X} \approx \mu + \frac{\sigma Z}{\sqrt{n}}$

$X_i = (int)(bool)heads$

$P(X_i = 1) = \frac{1}{2}$

$\mu = E(x_1) = P(x_i = 1) * 1 + P(x_i = 0) * 0 = \frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}$

$\sigma^2 = E((x_i - \mu)^2) = P(x_i = 1)(1 - P(x_i = 1))^2 + P(x_i = 0)(0 - P(x_i = 0))^2 = \frac{1}{2}(1 - \frac{1}{2})^2 + \frac{1}{2}(0 - \frac{1}{2})^2 =$

$= \frac{1}{4} = p(1 - p) = E(x_i^2) - E(x_i)^2 = \mu - \mu^2$ ← **take a square root of the result**

$x_1 + ... + x_n \leq 40 \Leftrightarrow \bar{x} \leq 0.4$ # # of heads ≤ 40

$P(\bar{x} \leq 0.4) = P(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq \frac{0.4-\mu}{\frac{\sigma}{\sqrt{n}}}) = P(Z \leq -2) = 0.0228$

EXAM

$\mu = E(X) = \sum_x xf(x) = \int_{-\infty}^{\infty} xf(x)dx$

$Standard\ deviation = \sigma = \sqrt{Var(x)}$

$E(g(X, Y)) = \sum_x\sum_y g(x, y)f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy$ **(remember reflection) (TRIPLE CHECK!!!)**

$Var(X) > 0; = \sigma^2 = E((x - \mu)^2) = E(X^2) - E(X)^2 = \sum_x (x - \mu)^2 f(x) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$

$Cov(X, Y) = \sigma_{XY} = E((X - \sigma_X)(Y - \sigma_Y)) = E(XY) - E(X)E(Y) = \int\int(x - \sigma_X)(y - \sigma_Y)f(x, y)dxdy$

$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(y)}}$ # $correlation\ coefficient\ in\ [-1, 1]$

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If X and Y are independent: $E(XY) = E(X)E(Y) \Rightarrow Cov(X, Y) = 0\ and\ \rho_{XY} = 0\ and\ var\ is\ simpler$

$if\ f(y) = 1\ in\ 7 \leq y \leq 8,\ then\ let\ X = Y - 7\ and\ Var(X) = Var(y)\ and\ Var(e^Y) = Var(e^{x+7})$

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$if\ X \sim N(\mu, \sigma^2)$, then $P(a \leq X \leq b) = \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})$ $P(X \leq b) = \Phi(\frac{b-\mu}{\sigma})$ $P(X \geq a) = 1 - \Phi(\frac{a-\mu}{\sigma})$

$middle$ 75%: $P(\frac{\mu-\alpha-\mu}{\sigma} \leq X \leq \frac{\mu+\alpha-\mu}{\sigma}) = \frac{3}{4}$ (solve for a)

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^x dx,\ \ for\ \alpha > 1 \quad \Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \quad \Gamma(n) = (n - 1)! \quad for\ int\ n > 0 \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$

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$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n(n-1)}(n\sum_{i=1}^{n} X_i^2 - (\sum_{i=1}^{n} X_i)^2)$ # variance. **remember sqrt:** $S = \sqrt{S^2}$ # standard deviation

$Z = \frac{X_1 + ... + X_n - n\mu}{\sqrt{n}\sigma} \qquad X_i = (int)(bool)heads$

$P(X_i = 1) = \frac{1}{2}$

$\mu = E(x_1) = P(x_i = 1) * 1 + P(x_i = 0) * 0 = \frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}$ (or 1*⅔ + -2*⅓=0 and square 1,-2 for var)

$\sigma^2 = E((x_i - \mu)^2) = P(x_i = 1)(1 - P(x_i = 1))^2 + P(x_i = 0)(0 - P(x_i = 0))^2 = \frac{1}{2}(1 - \frac{1}{2})^2 + \frac{1}{2}(0 - \frac{1}{2})^2 =$

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