Midterm exam

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1 Rules

The rules of the oral midterm are as follows.

- First, the student will choose at random one of the four tickets presented below.
- The student has at most 30 minutes to prepare without using any additional materials.
- The student has 15 minutes to present the proof(s) of statements that are written on the ticket chosen. During this discussion the student:
  - must be able to explain every part of the proof(s) of the statement(s) of the ticket;
  - must be able to define every concept that is used in the proof of the ticket;
  - must be able to give simple examples about every concept that is used in proof of the ticket;

2 Tickets

Ticket 1

(a) From formulas $\mathcal{F}_1, \ldots, \mathcal{F}_n$ we can conclude the formula $\mathcal{G}$ if and only if the formula $\mathcal{F}_1 \wedge \cdots \wedge \mathcal{F}_n \Rightarrow \mathcal{G}$ is a tautology (Theorem 4.7).

(b) Prove that for every set A, B and C the following equality holds

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

(Theorem 6.33).

Ticket 2 Let a be an integer and b be a natural number. Then there exist unique integers q and r, such that

$$a = bq + r \quad \textit{and} \quad 0 \leq r < b.$$

(Theorem 7.5)

Ticket 3

(a) Let a be an integerer. Prove that $a \mid 1$ if and only if $a \in \{-1, 1\}$. (Proposition 7.3)

(b) Prove that the set of prime numbers is infinite. (Theorem 7.8)

Ticket 4

(a) Prove that $\sqrt{2}$ is irrational (Proposition 8.9).

(b) Prove that there exist irrational numbers x and y such that x^y is a rational number. (Proposition 8.20)

Ticket 1

(a) From formulas $\mathcal{F}_1, \ldots, \mathcal{F}_n$ we can conclude the formula $\mathcal{G}$ if and only if the formula $\mathcal{F}_1 \wedge \cdots \wedge \mathcal{F}_n \Rightarrow \mathcal{G}$ is a tautology.

(b) Prove that for every set A, B and C the following equality holds

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Ticket 2 Let a be an integer and b be a natural number. Then there exist unique integers q and r, such that

$$a = bq + r \quad \textit{and} \quad 0 \leq r < b.$$

$(2n+1)(2n+1) =$
$= \underbrace{\underbrace{4n^2 + 4 \cdot n}_{\text{even}} + 1}_{\text{odd}}$

Ticket 3

(a) Let a be an integerer. Prove that $a \mid 1$ if and only if $a \in \{-1, 1\}$.

(b) Prove that the set of prime numbers is infinite.

Ticket 4

(a) Prove that $\sqrt{2}$ is irrational.

(b) Prove that there exist irrational numbers x and y such that x^y is a rational number.

$\sqrt{2}$ is irrational

let's assume $x = \sqrt{2}$ $y = \sqrt{2}$

a) if $\sqrt{2}^{\sqrt{2}}$ is rational then proved otherwise

assume $X = \sqrt{2}^{\sqrt{2}}$ and

$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$ rational

Ticket 1

a) From formulas $F_1, \ldots, F_n$ we can conclude the formula G iff the formula $F_1 \wedge \ldots \wedge F_n \Rightarrow G$ is tautology

Proof

We need to show that:

1) if $F_1, \ldots, F_n \models G$ then $F_1 \wedge \ldots \wedge F_n \Rightarrow G$ is tautology

2) vice versa

We address each direction of $\Leftrightarrow$ separatly

Proof for (1):

if $F_1, \ldots, F_2 \models G$ then in all these valuations where $F_1, \ldots F_n$ are true G is true. Therefore, formula $F_1 \wedge \ldots \wedge F_n \Rightarrow G$ is true. In valuations where some of formulas are false, the formula $F_1 \wedge \ldots \wedge F_n \Rightarrow G$ is true because assumption of implication is false

Therefore $F_1 \wedge \ldots \wedge F_n \Rightarrow G$ is tautology

Proof for (2)

if $F_1 \wedge \ldots \wedge F_2 \Rightarrow G$ is tautology then by def of tautology for any interpretation of variables formula is true. It means no interpretations where for all $F_1, \ldots F_n$ true and G false therefore by def. $F_1, \ldots, F_n \models G$ (G concludes on logical equivalence of $F_1 \ldots F_n$)

b) Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Let's take an arbitrary $(x,y) \in ((A \times B) \cap (A \times C))$

By def of intersection

$\equiv (x,y) \in (A \times B) \wedge (x,y) \in (A \times C) \overset{\text{By def of cart. product}}{\equiv}$

By commutative law

$\equiv x \in A \wedge y \in B \wedge x \in A \wedge y \in C \equiv$

By def. of Intersection

$\equiv \underbrace{x \in A \wedge x \in A}_{x \in A} \wedge y \in B \wedge y \in C \equiv$

By def. Cart. prod.

$\equiv x \in A \wedge y \in (B \cap C) \equiv (x,y) \in A \times (B \cap C)$

Ticket 2

$a \in \mathbb{Z}$ $b \in \mathbb{N}$, then exist unique

$q \in \mathbb{Z}$ and $r \in \mathbb{Z}$,
quotient reminder

$$a = bq + r \quad \text{and} \quad 0 \le r < b$$

Division algorithm page 306 Gary Chartran

First we show that exists $q \in \mathbb{Z}$ and $r \in \mathbb{Z}$, $a = bq + r$ $0 \le r < b$.
We will verify uniqueness later.
Consider set

$$S = \{a - bx : x \in \mathbb{Z} \text{ and } a - bx \ge 0\}$$

S is not empty if by checking that $x = 0$ therefore $a \in S$

A non-empty set S of real numbers is said to be well-ordered if every non empty subset of S has a least element

Alternative way of proving non empty.

Set S is well ordered which means it has a least element r such that $r \ge 0$.

Theorem
For each integer m, the set

$$S = \{i \in \mathbb{Z} : i \ge m)$$

is well-ordered

Since $r \in S$ then exists q such that $r = a - bq$ and therefore

$a = bq + r$ with $r \geq 0$

Next we need to show that $r < b$

Assume, to the contrary, that $r \geq b$ then

$t = r - b \geq 0 \Rightarrow t < r$

$r = t + b$

$a = bq + t + b$

$a = b(1+q) + t$

$t = a - b(1+q) \in S$ this contradicts that r is the smallest element in the set S

therefor $r < b$ as decired

Next show uniqueness of r and q

assume exists $r_1 \neq r$ and $q_1 \neq q$,

$a = bq_1 + r_1$

Ticket 3 a)

Let a be an integer. Prove $a \mid 1$ iff $a \in \{1, -1\}$

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Since $a \mid 1$ then $\exists\, b \in \mathbb{Z}$, $ab = 1$

b) prove by contradiction

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finite number of primes

$P = \{p_1, p_2, \dots, p_n\}$

if we multiply those n primes

$p = p_1 \cdot p_2 \cdot \dots \cdot p_n$

let $m = p + 1$

if we rewrite as

$1 = m \cdot 1 + r \cdot (-1)$ - linear combination

$\gcd(m, r) = 1$

Theorem

let $a, b \in \mathbb{Z}$ not both equal to zero. Then $\gcd(a,b) = 1$ iff there exists integers s and t such that $1 = as + bt$

Corollary

Every integer exceeding 1 has a prime factor

Since every prime divides p.
no prime divides n, which
contradicts corollary $\square$

Ticket 4

p 80 konspekt

a) Prove that $\sqrt{2}$ is irrational

Let's prove by contradiction.

Irrational number can not be written in the form a/b

Assume $\sqrt{2}$ is rational number therefore exist integers r and $s \neq 0$

$\sqrt{2} = \frac{r}{s}$ without loss in generality we assume r/s is irreducible fraction. By taking a square of this

$2 = \frac{r^2}{s^2} \Rightarrow r^2 = 2s^2$. Therefore r^2 is even number and thus r is even.

Exists $k \in \mathbb{Z}$, $2k = r$

$4k^2 = 2s^2$

$s^2 = 2k^2 \Rightarrow s^2$ - is even and s is even

but it contradicts assumption that r/s is irreducible fraction $\Rightarrow \sqrt{2}$ is irrational b)

two cases