Diagonalising a matrix

$$S^{-1} A S = \Lambda$$

matrix of evectors

! not allways possible

Suppose n independent evectors of A

Put them in columns of S

$$AS = A \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 x_1, & \dots, & \lambda_n x_n \\ | & & | \end{bmatrix}$$

$$= S \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} = S\Lambda$$

diagonal eigenvalue matrix

$$A = S \Lambda S^{-1}$$

new factorization replacement for LU or QR

Independent x from different λ:
Eigenvectors $x_1 \ldots x_j$ that correspond to distinct (all different) evalues are lin. indep.. An n by n matrix that has n dif-t evalues (no repeated λ's) must be diagonilizable

Let's consider A^2

if

$$Ax = \lambda x$$

$$A^2 x = \lambda^2 x$$

$$A = S \Lambda S^{-1}$$

$$A^2 = S \Lambda \underbrace{S^{-1} S} \Lambda S^{-1} = S \Lambda^2 S^{-1}$$

$$A^k = S \Lambda^k S^{-1}$$

Theorem

$A^k \to 0$ as $k \to +\infty$

if all $|\lambda_i| < 1$

A is sure to have n indep evectors (and be diagonizable) if all the λ's are different (no repeated λ's)

Repeated eigenvectors may or may not have n indep evectors

Example: upper triangular

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \lambda_{1,2} = 2$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

An example whe m. has not enough evectors.

Equation $u_{k+1} = A u_k$ starting with given u_0

$$u_1 = A u_0 \qquad u_2 = A u_1 = A^2 u_0$$

$$\boxed{u_k = A^k u_0}$$

To really solve: write

linear combinatio of eigenvectors

$$u_0 = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

$$A u_0 = c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n$$

$$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + \dots + c_n \lambda_n^{100} x_n$$

$$A^{100} u_0 = \Lambda^{100} S c$$

Example: Fibonacci

$$F_{100} \approx C_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$$

f. number growing about this factor

$$\left.\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} \end{aligned}\right]$$

trick

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}_{u_k}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1{,}618$$

$$\lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -.618$$

$$X_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 X_1 + c_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

When things
evolving in
time
findin e vectors
and values

Recitation

Find formula for C^k where

$$C = \begin{bmatrix} 2b-a & a-b \\ 2b-2a & 2a-b \end{bmatrix}$$

$$C^k = S\Lambda^k S^{-1}$$

$$|C-\lambda I| = \begin{vmatrix} 2b-a-\lambda & a-b \\ 2b-2a & 2a-b-\lambda \end{vmatrix} =$$

$$= (2b-a-\lambda)(2a-b-\lambda) - (2b-2a)(a-b) =$$

$$= 4ab - 2b^2 - 2b\lambda - 2a^2 + ab + a\lambda - 2a\lambda + \lambda b + \lambda^2$$

$$+ 2(a-b)^2 = \lambda^2 \cancel{-2b^2} \cancel{-2a^2} + 5ab - b\lambda - a\lambda$$

$$\cancel{+2a^2} - 4ab \cancel{+2b^2} = \lambda^2 - \lambda(a+b) + ab$$

$$D = (a+b)^2 - 4ab = (a-b)^2$$

$$\lambda_1 = \frac{(a+b)+(a-b)}{2} = a \qquad \lambda_2 = b$$

$$\begin{bmatrix} 2b-2a & a-b \\ 2b-2a & a-b \end{bmatrix} \qquad \begin{bmatrix} b-a & a-b \\ 2b-2a & 2a-2b \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \qquad S^{-1} = \frac{C^T}{|S|} = \frac{1}{-1/2}\begin{bmatrix} 1 & -1 \\ -1 & 1/2 \end{bmatrix}^T =$$

$$= -2 \begin{bmatrix} 1 & -1 \\ -1 & 1/2 \end{bmatrix}$$

$$C^k = \underset{S}{\begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix}} \underset{\Lambda}{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k} \underset{S^{-1}}{\begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}}$$

C^{100} when $a = b = 1$

$$C^{100} = \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{100} \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix} = I$$