Determinants

Properties of Det-s:

① $\det I = 1$

② Exchange rows: reverse sign of determinant $\Rightarrow$ $\det P = 1$ even permutations, -1 odd

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

③a $$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

③b $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Linear Each Row

④ 2 equal rows lead to $\det = 0$

Exchange 2 rows $\rightarrow$ Same matrix

(5) Subract $\ell \times$ row i from row k
Determinant doesn't change

$$\begin{vmatrix} a & b \\ c-\ell a & d-\ell b \end{vmatrix} \overset{(3b)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -\ell a & -\ell d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \underbrace{\cancel{\ell \begin{vmatrix} a & b \\ a & b \end{vmatrix}}}_{0}$$

by property 4

(6) Row of zeros $\to$ $\det A = 0$

(7) $U = \begin{bmatrix} d_1 & * & & \\ 0 & d_2 & & \\ 0 & 0 & \ddots & \\ 0 & \cdots & 0 & d_n \end{bmatrix} = (d_1)(d_2)\cdots(d_n)$

product of pivots

Rule 5 to make diagonal m.
3a to factor
& identity m.

(8) $\det A = 0$ when A is singular
$\downarrow$
row of zeros

when A is invertable $\det \neq 0$
$A \to U \to D \to d_1 \cdot d_2 \cdots d_n$

(9) $\det AB = \det(A)\det(B)$

$$\det A^{-1} = \frac{1}{\det A}$$

$$A^{-1}A = I$$

$$\det(A^{-1}A) = \det(I)$$

$$\det A^2 = (\det A)^2$$

$\det 2A = 2^n \det A$ (from 3a)

(10) $\det A^T = \det A$

Proof:

$$|A^T| = |A|$$

$$\downarrow$$

$$|U^T L^T| = |LU|$$

$$|U^T|\,|L^T| = |L|\,|U|$$

Recitation

Find determinants

$$A=\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix}=\begin{bmatrix}101 & 201 & 301\\ 1 & 1 & 1\\ 2 & 2 & 2\end{bmatrix}$$

$$\det A \overset{3a}{=} 2\begin{vmatrix}101 & 201 & 301\\ 1 & 1 & 1\\ 1 & 1 & 1\end{vmatrix} \overset{4}{=} 0$$

$$B=\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}= \qquad \det B=\det B^T$$

$$=\begin{bmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{bmatrix}=\begin{matrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & 0 & (c^2-a^2)-\frac{(b^2-a^2)(c-a)}{b-a}\end{matrix}$$

$$\det B=(b-a)\left((c^2-a^2)-\frac{(b^2-a^2)(c-a)}{b-a}\right)=$$

$$=(c^2-a^2)(b-a)-(b^2-a^2)(c-a)=$$

$$=(c-a)(c+a)\underbrace{(b-a)}-\underbrace{(b-a)}(b+a)(c-a)=$$

$$(c-a)(b-a)(c\cancel{+a}-b\cancel{-a})$$

$$C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \end{bmatrix}$$

Rank one matrix
Every row is multiple
of pivot row =>
rows of zero (4)
det = 0

$$D = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 4 \end{bmatrix} =$$

$$= \begin{bmatrix} -3 & -4 & 0 \\ 0 & 1 & 3 \\ 0 & 4/3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\det D = 0$