1. Consider the sequence $b_1, b_2, \dots$, where $b_1 = 1$ $b_2 = 1$ and $b_n = b_{n-1} + b_{n-2}$ for $n > 2$

Prove $\forall\, n \in \mathbb{N}$ $b_{n+1} \geqslant (3/2)^{n-1}$

Base step. If $n = 3$ $b_4 = b_3 + b_2 = b_1 + b_2 + b_2 = 3$

$\left(\frac{3}{2}\right)^{3-1} = 2,25$ $3 \geqslant 2,25$

if $n = 4$ $b_5 = b_4 + b_3 = b_4 + b_1 + b_2 = 5$ $\left(\frac{3}{2}\right)^{4-1} = 3,37$

$5 \geqslant 3,37$

Therefore proposition holds for $n = 3$ and $n = 2$

(it should be obvious that proposition holds for $n = 1, n = 2$)

Inductive step. Assume proposition holds for

$\forall n \in \{n \mid n \in \mathbb{N}\ n \geqslant 1 \wedge n \leqslant k\}$

$n = k$

$$b_{k+1} \geqslant \left(\frac{3}{2}\right)^{k-1}$$

$n = k - 1$

$$b_k \geqslant \left(\frac{3}{2}\right)^{k-2}$$

Let's show that proposition holds for $n = k+1$

$LHS = b_{(k+1)+1}$ $RHS = \left(\frac{3}{2}\right)^{(k+1)-1} = \left(\frac{3}{2}\right)^k$

$$b_{(k+1)+1} = b_{k+1} + b_k \geqslant \left(\frac{3}{2}\right)^{k-1} + \left(\frac{3}{2}\right)^{k-2} \geqslant$$

$$\geqslant \left(\frac{3}{2}\right)^k \left(\frac{2}{3} + \frac{4}{9}\right) \geqslant \left(\frac{3}{2}\right)^k \cdot \frac{10}{9} \geqslant \left(\frac{3}{2}\right)^k = RHS$$

Notice that $LHS \geqslant RHS$ which proves the proposition

$\boxed{2}$ $f: \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$

* iteration over sets Y in X and iterating Y to filter Odd numbers

$$f(X) = \{\{y \mid y \in Y,\ 2 \mid y\} \mid Y \in X\}$$

$$A = \{\emptyset, \{3k \mid k \in \mathbb{Z}\}\}$$

$$f(A) = \{\emptyset, \{3k \mid k \in \mathbb{Z},\ 2 \mid 3k\}\}$$

$B = \mathbb{N}$? $\mathbb{N} \notin \mathcal{P}(\mathbb{Z})$

$\{\mathbb{N}\} \in \mathcal{P}(\mathbb{Z})$

I think it's not possible to find f^{-1}, since there's no information of what odd numbers should be added.

$\boxed{3}$ a) $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$$f((m,n)) = 3m - 6n \quad \forall (m,n) \in \mathbb{Z} \times \mathbb{Z}$$

Let's prove injective, take two arbitrary pairs $(m_1, n_1) \neq (m_2, n_2)$

$$3m_1 - 6n_1 = 3m_2 - 6n_2$$

Let's take $(2,1)$ and $(4,2)$

$3 \cdot 2 - 6 \cdot 1 = 3 \cdot 4 - 6 \cdot 2$

$0 = 0$ Therefore it's not injective and hence not bijective

Let's show surjectiveness.

$y = 3m - 6n$

$y = 3(m-2n)$ assume $y=1$ then $m-2n$ should be equal to $\frac{1}{3} \notin \mathbb{Z}$ if $m \in \mathbb{Z}$ and $n \mathbb{Z}$ then $m-2n$ cannot be a number $\notin \mathbb{Z}$ therefore we showed that for $y=1$ $\nexists m, n \mathbb{Z}$ hence it's not surjective

b) $g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $g((m,n)) = (m-n, m+n)$ $\forall (m,n) \in \mathbb{Z} \times \mathbb{Z}$

Let's show injectiveness, let's take two arbitrary pairs $(m_1, n_1) \neq (m_2, n_2)$ such that

$(m_1 - n_1, m_1 + n_1) = (m_2 - n_2, m_2 + n_2)$

$$\begin{cases} m_1 - n_1 = m_2 - n_2 \\ m_1 + n_1 = m_2 + n_2 \end{cases} + \begin{cases} m_1 - m_2 = n_1 - n_2 \\ m_1 - m_2 = n_2 - n_1 \end{cases}$$

$2m_1 - 2m_2 = 0$

$m_1 = m_2$

$m_1 - m_1 = n_1 - n_2$

$n_1 = n_2$ we showed function is injective

let's show surjectiveness

$(k, l) = (m-n, m+n)$

$$+\begin{cases} k = m-n \\ l = m+n \end{cases} \quad \begin{cases} k+l = 2m \\ k-l = -2n \end{cases} \quad \begin{cases} m = \frac{k+l}{2} \\ n = -\frac{k-l}{2} \end{cases}$$

when $k+l$ is odd $m \notin \mathbb{Z}$ or if $k-l$ is odd then $n \notin \mathbb{Z}$ $\Rightarrow$ not surjective and therefore not bijective

$\boxed{4}$ Prove that sets $(-2,0)$ and $[0,4)$ a equivalent

a) $f: (-2,0) \to (0,2) \quad f(x) = x+2$

f is injective (obvious by linear function)

and $(0,2) \subset [0,4)$

$g: [0,4) \to \left(-\frac{5}{3}, -\frac{1}{3}\right] \quad g(x) = -\frac{x+1}{3}$

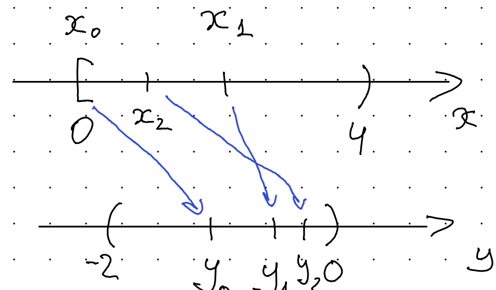
g is injective and $\left(-\frac{5}{3}, -\frac{1}{3}\right] \subset (-2,0)$

we constructed two injection functions

by Cantor - Bernstein theorem

$(-2,0) \sim [0,4)$

b) Let's define sequence in such way



$x_0 = 0$

$x_1 = \frac{0+4}{2}$

$x_2 = \frac{x_0 + x_1}{2} = \frac{x_1}{2}$

$\vdots$

$x_n = \frac{x_0 + x_{n-1}}{2} = \frac{x_{n-1}}{2} \quad X = \{x_0, x_1, \ldots\}$

let's define anothe sequence

$y_0 = \frac{-2+0}{2} \quad y_1 = \frac{y_0 + 0}{2} \quad y_2 = \frac{y_1}{2} \quad Y = \{y_0, y_1, \ldots\}$

$f: [0,4) \to (-2,0)$

$$f(x) = \begin{cases} y_i, & \exists i \in \mathbb{N} \cup \{0\},\ x = x_i \in X \quad y_i \in Y \\ -\frac{x}{2}, & x \notin X \end{cases}$$