Functions

Def: Let X and Y be sets. A function f is defined as a rule that assigns each element of X to exactly one element of Y. This is denoted as $f: X \to Y$. if an element $x \in X$ is assigned to an element $y \in Y$, we write $y = f(x)$, or $f: x \mapsto y$

The set X is called the domain of the function f, and the set Y is called the co domain.

The set $f(X) = \{f(x) : x \in X\}$ is called the image or range of the function.

Def: Two functions $f: X \to Y$ and $g: Z \to W$ are said to be equal if $X = Z$, $Y = W$ and $f(x) = g(x)$ for all $x \in X$ (or $x \in Z$, $X = Z$)

Def: Let $f: X \to Y$ be a function. The graph of f, denoted $G(f)$ is the set of ordered pairs $\{(x, f(x)) \mid x \in X\} \subset X \times Y$

Def: Let U be a universal set, and let $A \subset U$. The characteristic function of the set A is the function $\chi_A : U \to \{0, 1\}$ defines as:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in U \setminus A \end{cases}$$

Proposition

1. $\chi_A(x)\,\chi_A(x) = \chi_A(x)$
2. $\chi_{A'}(x) = \chi_{U\setminus A}(x) = 1 - \chi_A(x)$
3. $\chi_{A\cap B}(x) = \chi_A(x)\cdot\chi_B(x) = \min\{\chi_A(x), \chi_B(x)\}$
4.

Def: if $x \in X$ and $y \in Y$ such that $y = f(x)$, then y is called an image of x.

Every element of domain X has exactly one image in the codomain Y

Theorem: Let f be a function from set X to set Y. Then

1. $f(\phi) = \phi$
2. $f(X) \subset Y$
3. If $A \subset B$, then $f(A) \subset f(B)$
4. $f(A \cup B) = f(A) \cup f(B)$
5. $f(A \cap B) \subset f(A) \cap f(B)$

Def: if $x \in X$ and $y \in Y$ such that $y = f(x)$ then x is called preimage of y under f.

Def: The preimage of a set $B \subset Y$ under f is the set $f^{-1}(B)$, consisting of all elements of X that map to an element of B

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

Theorem: let $f: X \to Y$ and let $A, B \subset Y$ then

1. $f^{-1}(\phi) = \phi$
2. $f^{-1}(Y) = X$
3. $A \subset B$ then $f^{-1}(A) \subset f^{-1}(B)$

4. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
5. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
6. $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$

Theorem

Let $f: X \to Y$ be a function then

1. If $A \subset X$, then $A \subset f^{-1}(f(A))$
2. If $B \subset Y$, then $f(f^{-1}(B)) \subset B$

Def: A function $f: X \to Y$ is called even if, for all $x \in X$ the following conditions hold:

1. $-x \in X$
2. $f(-x) = f(x)$

Def: -||- odd

1) $-x \in X$
2) $f(-x) = -f(x)$

Def: fun. $f: X \to Y$ is called

a) injective or one-to-one if, for all pairs $x_1, x_2 \in X$ where $x_1 \neq x_2$ it holds that $f(x_1) \neq f(x_2)$

b) surjective or onto if $\forall y \in Y$, there exists an $x \in X$ such that $y = f(x)$

c) bijective or a one-to-one correspondence, if f is both injective and surjective

Pigeon-hole principle: Let A and B be finite sets and $f: A \to B$ a function

if $|A| > |B|$, then f is not injective

if $|A| < |B|$, then f is not surjective

Def: Let X, Y and Z be arbitrary sets. The product on composition of functions $f: X \to Y$ and $g: Y \to Z$ is the function $g \circ f: X \to Z$, for which $(g \circ f)(x) = g(f(x))$ for all $x \in X$

interior

exterior

Proposition Let X, Y, Z and W be sets. If $f: X \to Y$, $g: Y \to Z$ and $h: Z \to W$, then $h \circ (g \circ f) = (h \circ g) \circ f$

?

Proposition

Def: Let X be a set. Similarity transformation or identity transformation $I_X : X \to X$ is a function which maps every element of X to itself, this means that $I_X(x) = x$ for all $x \in X$

Def: Let X and Y be sets. The inverse function of a bijective function $f : X \to Y$ is the function $f^{-1} : Y \to X$ which assignes to every $y \in Y$ exactly one $x \in X$ for which $f(x) = y$.