Problem Set 3
Due September 29, 2010

1. The hats of n persons are thrown into a box. The persons then pick up their hats at random (i.e., so that every assignment of the hats to the persons is equally likely). What is the probability that

    (a) every person gets his or her hat back?

    (b) the first m persons who picked hats get their own hats back?

    (c) everyone among the first m persons to pick up the hats gets back a hat belonging to one of the last m persons to pick up the hats?

    Now assume, in addition, that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). What is the probability that

    (d) the first m persons will pick up clean hats?

    (e) exactly m persons will pick up clean hats?

2. Alice plays with Bob the following game. First Alice randomly chooses 4 cards out of a 52-card deck, memorizes them, and places them back into the deck. Then Bob randomly chooses 8 cards out of the same deck. Alice wins if Bob's cards include all cards selected by her. What is the probability of this happening?

3. (a) Let X be a random variable that takes nonnegative integer values. Show that

    $$\mathbf{E}[X] = \sum_{k=1}^{\infty} \mathbf{P}(X \geq k).$$

    Hint: Express the right-hand side of the above formula as a double summation then interchange the order of the summations.

    (b) Use the formula in the previous part to find the expectation of a random variable Y whose PMF is defined as follows:

    $$p_Y(y) = \frac{1}{b - a + 1}, \qquad y = a, a + 1, \ldots, b$$

    where a and b are nonnegative integers with $b > a$. Note that for $y = a, a + 1, \ldots, b$, $p_Y(y)$ does not depend explicitly on y since it is a uniform PMF.

4. Two fair three-sided dice are rolled simultaneously. Let X be the difference of the two rolls.

    (a) Calculate the PMF, the expected value, and the variance of X.

    (b) Calculate and plot the PMF of X^2.

5. Let $n \geq 2$ be an integer. Show that

    $$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}.$$

    Hint: As one way of solving the problem, following from Example 1.31 in the text, think of a committee that includes a chair and a vice-chair.

G1[†]. A candy factory has an endless supply of red, orange, yellow, green, blue, black, white, and violet jelly beans. The factory packages the jelly beans into jars in such a way that each jar has 200 beans, equal number of red and orange beans, equal number of yellow and green beans, one more black bean than the number blue beans, and three more violet beans than the number of white beans. One possible color distribution, for example, is a jar of 50 yellow, 50 green, one black, 48 white, and 51 violet jelly beans. As a marketing gimmick, the factory guarantees that no two jars have the same color distribution. What is the maximum number of jars the factory can produce?

6.041 / 6.431 Probabilistic Systems Analysis and Applied Probability

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1) n Persons throws hats into a box. Persons pick up their hats at random. Equal probability of assignment. What is the probability

a) get own hat?

b) m first persons get their own hats

2

A. chooses 4 cards out of 52.

B chooses 8 cards randomly

3 (a)

Prove

$$E[X] = \sum_{k=1}^{\infty} P(X \geq k)$$

$$\left(P(X=1) + P(X=2) + P(X=3) + \ldots + \right) +$$

$$+ \left(P(X=2) + P(X=3) + \ldots + \right) +$$

$$+ \left(P(X=3) + \ldots + \right) =$$

$$= P(X=1) + 2P(X=2) + 3P(X=3) + \ldots$$

$$= \sum_{k=1}^{\infty} k \, P(X=k)$$

(b) Use formula in the previous for P_Y

$$P_Y(y) = \frac{1}{b-a+1} \qquad y = a, a+1, \ldots, b$$

$$a > 0$$
$$b > a$$

$$E[Y] = \sum_{k=a}^{b} k \, P_Y = \sum_{k=a}^{b} P_Y(Y \geq k) = P_Y(1) \sum_{k=a}^{b} (b-k) =$$

$$= P_Y(y) \sum_{k=a}^{b} \frac{n}{2}(k+b) = P_Y \sum_{k=a}^{b} \frac{b-k+1}{2}(k+b)$$

Constant

$$\downarrow$$

$$E[Y] = \sum_{k=a}^{b} k \, P_Y = P_Y \frac{b-a+1}{2}(a+b) =$$

$$= \frac{1}{b-a+1} \cdot \frac{b-a+1}{2}(a+b) = \frac{a+b}{2}$$

4

Two   fair   three-sided   dice   are
rolled   simultaneously

X - difference   of two rolls

a) Calculate   PMF

$$P_X(x) = \begin{cases} 1/9 & \text{if } x = 2 \\ 2/9 & \text{if } x = 1 \\ 3/9 & \text{if } x = 0 \\ 2/9 & \text{if } x = -1 \\ 1/9 & \text{if } x = -2 \\ 0 & \text{otherwise} \end{cases}$$
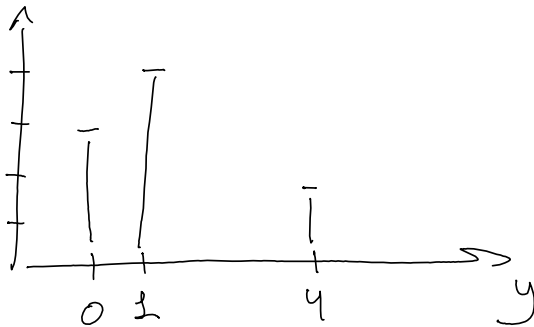
$$E[X] = 0$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = \left(4\frac{1}{9} + \frac{2}{9}\right)2 =$$

$$= \frac{12}{9} = 1\frac{1}{3}$$

b) Calculate and plot PMF X^2

$$Y = X^2$$

$$P_Y(y) = \begin{cases} 2/9 & \text{if } y=4 \\ 4/9 & \text{if } y=1 \\ 3/9 & \text{if } y=0 \\ 0 & \text{otherwise} \end{cases}$$

5 Let $n \geq 2$

Show that

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1) \cdot 2^{n-2}$$