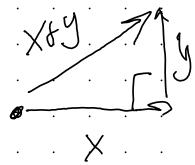
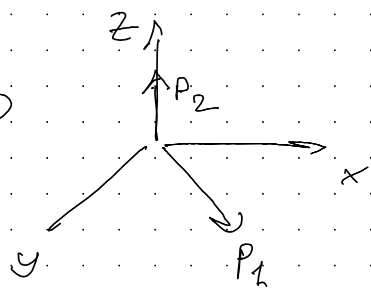
Orthogonal vectors



how to confirm that two vectors are perpendicular?

if $x^T y = 0 \Rightarrow$ orthogonal

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 0$$



Pythagoras

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$x^T x \qquad y^T y \qquad (x+y)^T (x+y)$$

What's the length of the vector

$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$\|X\|^2 = 14$ $\|y\|^2 = 5$ $\|x+y\| = 19$

$(x+y)^T(x+y) = \cancel{x^T x} + y^T x + \cancel{y^T y} + x^T y =$

$= 2x^T y$

$0 = 2x^T y$ we proved orthogonality

Subspace S is orthoganal to sub-s T

What does it mean for two sub-s to be orthogonal?

<u>means</u>: every vector in S is orth-l to every vector in T sub-s

It's true for row space and nullspace.

row space is orth-l to nullspace
Why?

$$Ax = 0 \qquad \begin{bmatrix} \text{row 1 of } A \\ \text{row 2} \\ \vdots \\ \text{row m of } A \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

x is orth-l to every row.

what else in the row space?
all row combinations

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$n = 3 \quad r = 1 \qquad \dim N(A) = 2$

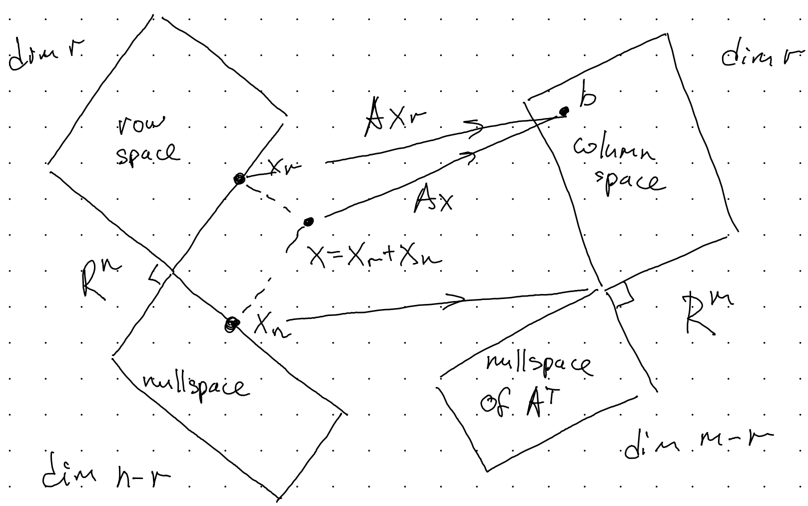
nullspace and row space are orthogonal complements in R^n
Nullspace contains all vectors $\perp$ row space

Fundamental theorem of lin. Algebra Part 2

part 1 gives dimensions of the subspace

- $N(A)$ is the orthogonal complement of the row space $C(A^T)$ (in R^n)
- $N(A^T)$ is the orthogonal complement of the column space $C(A)$ (in R^m)

Every x can be split into row space component x_r and a nullspace comp. x_n



Every vector b in the column space from one and only one vector in the row space

Proof: if $Ax_r = Ax_r'$, then $x_r - x_r'$ in nullspace. It's also in the row space where x_r and x_r' (from subspace def). This difference must be the zero vector, because nullspace and rowspace are perpendicular. $\Rightarrow x_r = x_r'$

From the row space to the column space, A is invertible. "Pseudo invertable" Section 7.3

Example for SVD.

Combining Bases from Subspaces

- Any n independent vectors in R^n must span R^n $\Rightarrow$ so they are a basis
- Any vectors that span R^n must be independent. So they are a basis.

In column space

- If the n columns of A are indep., they span $R^n \Rightarrow Ax=b$ is solvable
- If n columns span R^n, they are indep. $\Rightarrow$ $\Rightarrow Ax=b$ has only one solution.

Comming:

$Ax = b$ we want to solve when no solution (best possible way)

If m equesions $> n$

In many cases we have many equesins and noise in it. We want to separate noise from information

For chapter 4 we want to understand matrix

$$\underset{n\times m}{A^T}\ \underset{m\times n}{A}$$

symmetric

$$(A^TA)^T = A^T A^{TT} = A^T A$$

$$A^T A \hat{x} = A^T b$$ (sentral eq. of ch 4)

i hope this equetion has best solution $\hat{x}$

When is it invertable?

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$m = 3$

$n = 2$

Rank 2

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

invertable

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

$N(A^TA) = N(A)$

rank = 1

not invertabe

rank of A^TA = rank of A

Conclusion:

A^TA is invertable $\Leftrightarrow$ A has independent columns

Recitation

S is spanned by $(1\ 2\ 2\ 3)$ and $(1\ 3\ 3\ 2)$

1) Find a basis for $S^\perp$

if x in $S^\perp$, then $s_1 \cdot x = 0$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 3 \\ 0 & \boxed{1} & 1 & -1 \end{bmatrix} \cdot x = 0$$

pivots

matrix has rank 2 (two pivots)

$$x = \underbrace{x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{S^\perp} = \begin{bmatrix} -5x_4 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix} =$$

Dimention of nullspace of S is 2

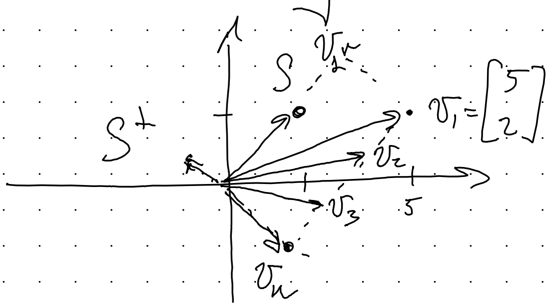
$n - r = 4 - 2$

2)

$$v = (6, 5, 4, 1)$$

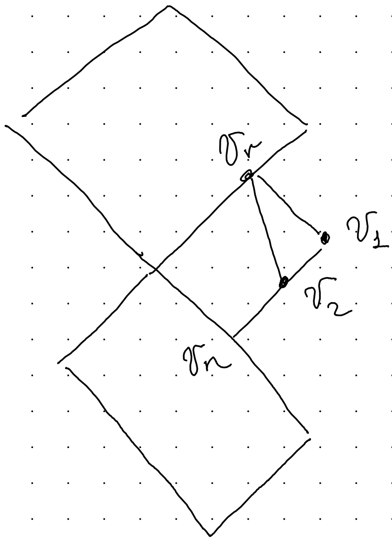
$$C(S^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 5 & -1 \end{bmatrix} \qquad N(S) = \begin{bmatrix} 0 & -5 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C(S^T) & N(S) \end{bmatrix} \cdot X = v$$



Assume $v \notin S$ and $v \in S^{\perp}$
we can split v into S and $S^{\perp}$
components v_r and v_n

Prove that exists v_1 and v_2

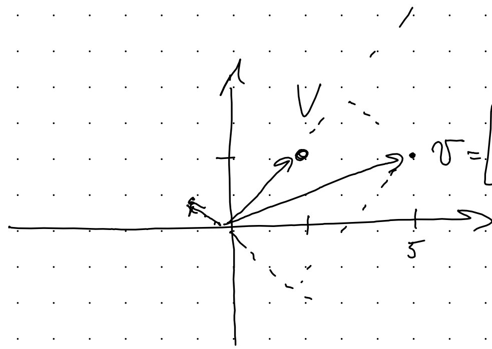


Assume exist v_1 and v_2 such that they have same components of v_r and v_n

$$v_1 = v_{r_1} + v_{n_1}$$
$$v_2 = v_{r_2} + v_{n_2}$$

Every vector can be split into row space and null space components

if $v_{r_1} = v_{r_2}$ and $v_{n_1} = v_{n2} \Rightarrow$
$v_1 = v_2$



$v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\underset{V}{\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}} \begin{bmatrix} v_r \\ v_n \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\underset{V^T V}{\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}} \begin{bmatrix} v_r \\ v_n \end{bmatrix} = \underset{V^T v}{\begin{bmatrix} 14 \\ -3 \end{bmatrix}}$$

$$I \cdot \begin{bmatrix} v_r \\ v_n \end{bmatrix} = \begin{bmatrix} 7/4 \\ -1{,}5 \end{bmatrix}$$

Проверка:

$$v_r = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot 7/4 = \begin{bmatrix} 3{,}5 \\ 3{,}5 \end{bmatrix} \qquad v_n = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot (-1{,}5) = \begin{bmatrix} 1{,}5 \\ -1{,}5 \end{bmatrix}$$

$$v = v_r + v_n = \begin{bmatrix} 3{,}5 \\ 3{,}5 \end{bmatrix} + \begin{bmatrix} 1{,}5 \\ -1{,}5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \checkmark$$

Worked Examples

4.1 B

$$\underbrace{[1, -3, -4]}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

a)

Plain P is null space of matrix A

b)

$$N(A) = \begin{bmatrix} 4 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = P$$

c) A shuld be a basis for $P^{\perp}$

d) $v = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$ split into nullspace component v_n in P

Examples

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad x = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$N(A) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$