The null space of A consists of all solutions to $Ax = 0$. These vectors x are in R^n. The null space containing all solutions of $Ax = 0$ is denoted $N(A)$

- The nullspace consists of all combinations of the special solutions

- What is singular matrix?

Example

$$\begin{bmatrix} \boxed{1} & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

pivot
so, first component of x is not free

has special solutions

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

"free" components

Example 3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

A - is inventable
no special
solutions

$$N(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Z$$

$$B = \begin{bmatrix} A \\ 2A \end{bmatrix} \qquad N(B) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Z$$

$$C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$N(C) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{pivot} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{free} \end{matrix}$$

We can make triangular matrix simpler U in two ways:

1) produce zeros above pivots
2) Produce ones in the pivots

So we reach reduced row echelon form R.

Nullspace A eq Z $\Rightarrow$ collumns of A are independent.

$$A = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to$$

$$\to \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

pivot columns

echelon

free columns

Rank of A = # of pivots echelon

2

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$2x_3 + 2x_4 = 0$$

free

special solutions

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Echelon Matrices

m by n "staircase" U is an echelon matrix

$$U = \begin{bmatrix} p & x & x & x & x & x & x \\ 0 & p & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 pivot variables
$x_1 \; x_2 \; x_6$

4 free var-s
$x_3 \; x_4 \; x_5 \; x_7$

4 special solutions in $N(U)$

Column space of U
have 4 components $\Rightarrow$ lie in $\mathbb{R}^4$

$C(U)$ consists of all vectors of form $(b_1 \; b_2 \; b_3 \; 0)$

$N(U)$ is a subspace in $\mathbb{R}^7$, combinations of 4 special solutions

Suppos $Ax=0$ has $\overset{\text{cols}}{n} > \overset{\text{rows}}{m} \Rightarrow$ there are nonzero solutions. There must be free columns, without pivots.

Reduced Row Echelon Matrix R

R = reduced row echelon form

zeros above + below pivots
and ones on pivots

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}(A)$$

$$x_1 + 2x_2 \qquad -2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

I pivot columns

F free cols

0 0 0 0

* I A is invertible, then $R = I$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$ r pivot rows

r pivot cols

n − r free cols

$Rx = 0$

nullspace matrix
(columns = special solutions)

$RN = 0$

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

free col

$r = 2$

$$x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

complete solution

Go to echelon m.

$$\begin{array}{cc} I & F \end{array}$$

$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad x = c\begin{bmatrix} -F \\ I \end{bmatrix}$$

Example 4

$U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ $\quad x_2$ is free

$$x = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

Nullspace of U is line in $\mathbb{R}^3$. It contains multiples of the special solution $s = (-5, 1, 0)$

Recitation   Solving homoginions linear system $Ax=0$

The set of points $P(xyz)$ s.t.
$x-5y+2z=9$ is a plaine in $\mathbb{R}^3$.
It's parallel to the plaine So of $P(xyz)$
s.t. $x-5y+2z=0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}}_{P_0} + C_1 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} \qquad C_1 = 0 \quad C_2 = 0$$

$$P_1 = \begin{bmatrix} 0 \\ 0 \\ 4{,}5 \end{bmatrix} \qquad C_1 = 0 \qquad C_2 = 4{,}5$$

* She considers equation as constraint. If we have 1 constraint it result in plaine (2d)

$$\begin{bmatrix} 9 & & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4{,}5 \end{bmatrix}$$

P_2

$$\begin{bmatrix} 9 & & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -9/5 \\ 0 \end{bmatrix} \qquad C_1 = 9/5$$

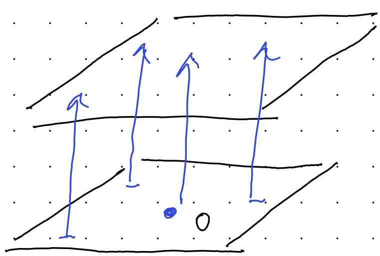
Any point in $S = P_0 + \begin{pmatrix}\text{Any point}\\ \text{in } S_0\end{pmatrix}$

$$\begin{bmatrix}\boxed{1} & -5 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=0$$

pivot

free variables

$$N(A) = y\begin{bmatrix}5\\1\\0\end{bmatrix} + z\begin{bmatrix}-2\\0\\1\end{bmatrix}$$



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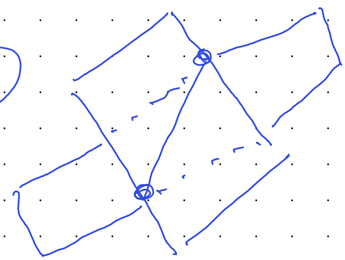

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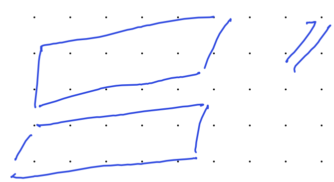


$$\begin{cases} x-5y+2z=9 \\ x-5y+2z=0 \end{cases}$$

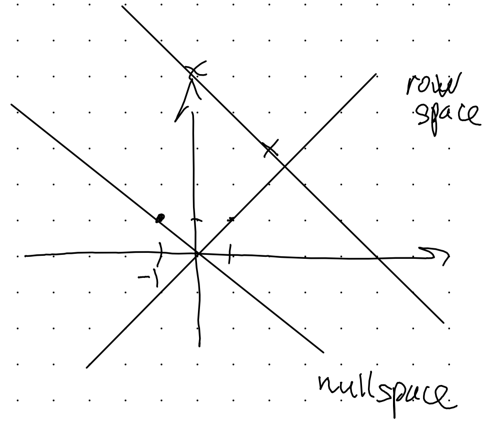
no solutions

hence 2

Let's consider simpler system

$$x + y = 5$$

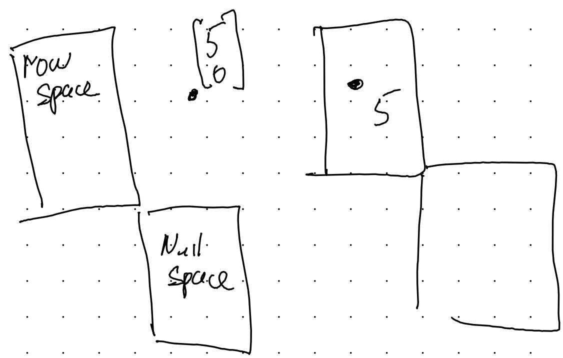
$$\underset{A}{\begin{pmatrix}1 & 1\end{pmatrix}}\begin{pmatrix}x\\y\end{pmatrix} = 5$$



Nullspase $= y\begin{pmatrix}-1\\1\end{pmatrix}$

$$\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\cdot X =$$

$C(A)$ $N(A)$



$$x = \begin{bmatrix}0\\5\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix}$$

$$X = \begin{bmatrix}0\\5\end{bmatrix} + \begin{bmatrix}-2\\2\end{bmatrix}$$