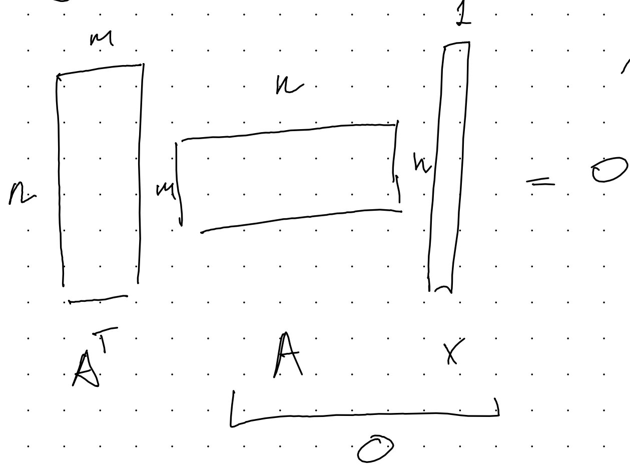
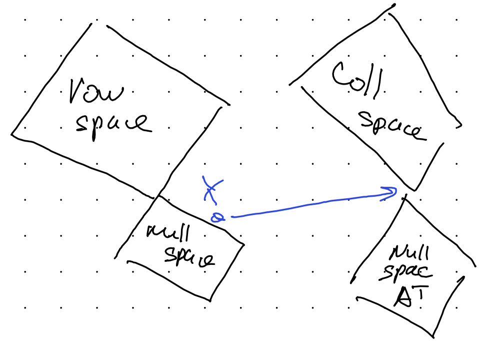
$\boxed{\text{PS } 4.1}$

$\boxed{9}$





Ax is in $C(A)$

$C(A) \perp N(A^T)$

Ax is in nullspace of A^T
and also in colum space of A
and $\perp$

Conclusion: $A^T A$ has the same nullspace as A

$\boxed{7}$

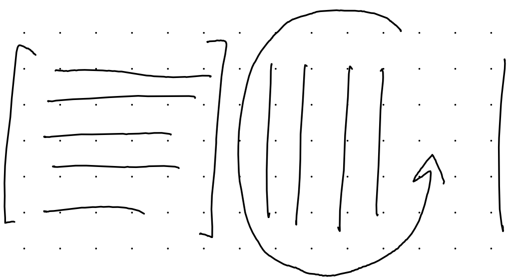
$\boxed{3.1}$

$$\underset{n\times(n-r)}{N} = \text{null}(\underset{m\times n}{A})$$

$$B = \text{null}(N^T_{(n-r)\times n})$$

$$Ax = 0$$

$$(Ax)^T \quad x$$



$A_{m\times n}$ $\qquad$ $n\times \underbrace{(n-r)}$ $\qquad$ $n\times(n-r)$

6×12 $\qquad$ $12\times(12-6)$ $\qquad$ 12×6

$\boxed{32}$ V

r   n   c   l R^2

a) $C(A^T) \perp N(A)$

assume $r = l$

$\begin{bmatrix} r & 2r \end{bmatrix}$ $\begin{bmatrix} c \end{bmatrix}$ $r \cdot c = 0$

$C(A) \perp N(A^T)$

$\begin{bmatrix} n \end{bmatrix}$ $\begin{bmatrix} l \end{bmatrix}$ $n \cdot l = 0$

b) $A = n\, r^T$ $A = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$

$r^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$

r

$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

n

$N(A) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ c $N(A^T) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ l

$\boxed{33}$ ✓

a) $A = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$R \cdot N = 0$

$C \cdot L = 0$

b) $A = \underset{4 \times 2}{\begin{bmatrix} c_1 & c_2 \end{bmatrix}} \underset{2 \times 4}{\begin{bmatrix} r_1 & r_2 \end{bmatrix}^T}$

$\boxed{\text{PS } 4.2}$

$\boxed{13}$ ✓

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$(3\times 4 \cdot 4\times 3)^{-1} \; 3\times 4 \cdot 4\times 1 = 3\times 1$$

$$p = A\hat{x} = 4\times 3 \cdot 3\times 1 = 4\times 1$$

$$P = A(A^T A)^{-1} A^T = 4\times 3\,(3\times 3)^{-1}\, 3\times 4 =$$

$$4\times 4$$

In order to find proj. matrix, simpler would be to find orthogonal compliment for $C(A)$, which is $N(A^T)$

$$A = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} y = 0$$

$$y = 0$$

$$P = A(A^T A)^{-1} A^T$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} I = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{16}$ v

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

if b is not in C(A), then closest vector to b shold be its projection onto A.

p - ? $A\hat{x} = p$

$$\hat{x} = (A^T A)^{-1} A^T b$$

! I chacked that b is in the span of two vectors

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} x_1 &= 0{,}5 \\ x_2 &= 1{,}5 \end{aligned}}$$

↑ solved before
wrong

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 6 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1/6 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1/6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/3 & -1/6 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1/6 & 1/3 & -1/6 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0{,}5 \\ 3 \end{bmatrix}$$

$$p = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 3 \end{bmatrix} = \begin{bmatrix} 3{,}5 \\ 1 \\ +2{,}5 \end{bmatrix}$$

$\boxed{17}$ ✓

$$(I-P)^2 = I^2 - 2PI + P^2 =$$

$$= I - 2P + P^2 = I - P$$

Normal vector of A (left nullspace)

When P projects onto $C(A)$, $I-P$ projects onto its orthogonal compliment which should be $N(A^T)$

$\boxed{30}$

a) $P_C = \dfrac{a a^T}{a^T a} = \dfrac{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix}}{\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \dfrac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ ✓

$A\hat{x} = p$

$\begin{bmatrix} 2 \times 3 \end{bmatrix} \begin{bmatrix} 3 \times 1 \end{bmatrix} = \begin{bmatrix} 2 \times 1 \end{bmatrix}$

$Pb = p$

$\begin{bmatrix} 2 \times 2 \end{bmatrix} \times \begin{bmatrix} 2 \times 1 \end{bmatrix} = \begin{bmatrix} 2 \times 1 \end{bmatrix}$

b) $P_R = \dfrac{\begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 6 & 6 \end{bmatrix}}{\begin{bmatrix} 3 & 6 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}} = \dfrac{1}{81} \begin{bmatrix} 9 & 18 & 18 \\ 18 & 36 & 36 \\ 18 & 36 & 36 \end{bmatrix}$

$B = \dfrac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix} \dfrac{1}{81} \begin{bmatrix} 9 & 18 & 18 \\ 18 & 36 & 36 \\ 18 & 36 & 36 \end{bmatrix}$

$B = \dfrac{a a^T}{a^T a} \begin{bmatrix} a & 2a & 2a \end{bmatrix} \dfrac{r r^T}{r^T r}$

$$B = \frac{a\,a^T}{a^T a}\left[\quad r^2 \quad\right]\frac{r\,r^T}{r^T r}$$

$$B = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}\begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}\frac{1}{81}\begin{bmatrix} 9 & 18 & 18 \\ 18 & 36 & 36 \\ 18 & 36 & 36 \end{bmatrix}$$

$$\frac{1}{25}\begin{bmatrix} 75 & 150 & 150 \\ 100 & 200 & 200 \end{bmatrix}$$

$$B = A$$

$$x = n + r \qquad n \text{ in } N(A) \qquad r \text{ in } C(A^T)$$

$$Ax = \cancel{An}^{\,0} + Ar$$

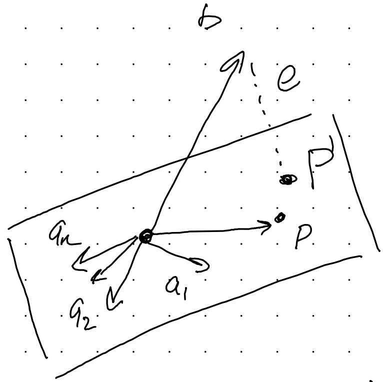
$$P_R x = \cancel{P_R n}^{\,0} + P_R r$$

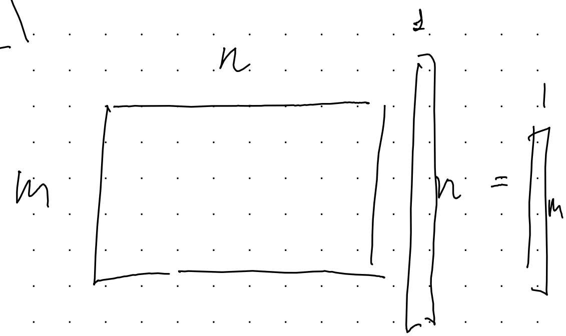
$\boxed{31}$

R^m

p is combination of $a_1 \dots a_n$

How to check $b \perp p$





$e = b - p$

$$p = A(A^TA)^{-1}A^Tb$$

$C(A) \perp e$

If p is lin. combination of A, then

$A\hat{x} = p$ How would you test to see that if p is projection of b?

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix}\hat{x} = p$$

$$[m \times n][n \times 1] = [m \times 1]$$

If p is projetion of b onto a then $b-p$, which is error shold be orthogonal to A. $(p \neq b)$ =>

$$A^T(b-p) = 0$$

$[n \times m]$ $[m \times 1]$

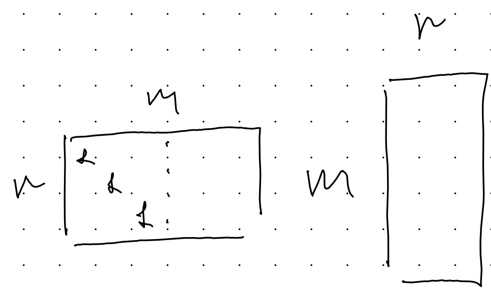
In assignments the task is a bit different than in the book. I the book we are given that p is in the span of a's. So in the assignment we cannot assume p is in the span of a's. We need to satisfy two conditions a) p is in the span of a's b) error $b-p$ is orth. to to all column in A.

It's not enough to satisfy only b) because if $p=b$ then 0 is orthogonal but p is not in span.

$\boxed{34}$ If A has r indep-t cols and B has r indep. rows, AB is invert-l

Proof:

When A is $m \times r$ with indep. cols, we know that A^TA is invertable. If B is r by n with indep rows, BB^T is invertable (replace $A = B^T$)



if A^TA is invertable

$\underline{A^TA}\, x = 0$ then the only solution when $x = 0$

$\boxed{\text{PS } 8.2}$

$\boxed{13}$