Mathematical Statistics

Lab 10, 2024

Note. This lab covers the following topics: 1) chi-square criterion to test hypotheses about proportions (see Lecture 8); 2) power function of a statistical test; 3) test about the difference of two population variances; 4) (nonparametric) sign test for related data sets.

1. The litter size of Bengal tigers is typically two or three cubs, but it can vary between one and four. Based on long-term observations, the litter size of Bengal tigers in the wild has the distribution given in the table provided. A zoologist believes that Bengal tigers in captivity tend to have different (possibly smaller) litter sizes from those in the wild. To verify this belief, the zoologist searched all data sources and found 316 litter size records of Bengal tigers in captivity. The results are given in the table provided.

   a) Formulate the H_0 and H_1. b) Calculate the expected frequencies of litter sizes under H_0. 3) Calculate the chi squared statistics value. c) Look up the appropriate test critical value at 5% level of significance. d) Test whether there is sufficient evidence in the data to conclude that the distribution of litter sizes in captivity differs from that in the wild. e) Calculate the p-value of the test, using full chi-squared table from the Lecture materials in Moodle or R command *pchisq*.

Number of cubs	1	2	3	4
Probability in wild	0.11	0.69	0.18	0.02
Observed frequency in captivity	41	243	27	5
Expected frequency	?	?	?	?

	Probability less than the critical value				
df	0.90	0.95	0.975	0.99	0.999
1	2.706	3.841	5.024	6.635	10.828
2	4.605	5.991	7.378	9.210	13.816
3	6.251	7.815	9.348	11.345	16.266
4	7.779	9.488	11.143	13.277	18.467
5	9.236	11.070	12.833	15.086	20.515
6	10.645	12.592	14.449	16.812	22.458
7	12.017	14.067	16.013	18.475	24.322
8	13.362	15.507	17.535	20.090	26.125
9	14.684	16.919	19.023	21.666	27.877
10	15.987	18.307	20.483	23.209	29.588

Table: Critical values of chi-square distribution with *df* degrees of freedom

2. Chocolate producer Lakev wants to choose between 3 different designs for their new chocolate bar. They ask a group of 25 customers to chose their favorite design among the three. It was found that the design number 3 was the most beloved among the customers (see the table below). But does this result provide a sufficient proof that design 3 is better? Formulate appropriate hypotheses and do the test at 5% significance level. Report also the test statistic and p-value.

Design	1	2	3
Frequency	8	7	10

3. Each of 15 students were asked to rate two methods of learning: the traditional lecture-based method A and the active assignment-based method B. The students

were asked to rate 10 statements related to learning achievements in a 5-point likert scale (1: strongly disagree; 2: somewhat disagree; 3: no preference; 4: somewhat agree; 5: totally agree). The score for each of the methods was acquired by summing answers to all the 10 questions.

	Rating of traditional lecture	Rating of assignment-based method
Student (N= 15)	A	B
Griffiths	23	33
Ashford	14	22
Woodlock	35	38
Jamalzadeh	26	30
Manku	28	31
Masih	19	17
Salisbury	42	42
Maman	30	25
Quinliven	26	34
Blay	31	24
Harrison	18	21
Ramakrishnan	25	46
Apostolou	23	29
Dingley	31	40
Milloy	30	41

a) Discuss whether the assumption of normal distribution could or could not be appropriate in this situation, and whether it is testable. b) Test whether there is a difference between the two learning method scores using the sign test.

Note that Salisbury rated both lectures with the same score. Such a situation is called *ties*. In case of sign test, the most often used approach is to discard the observation with ties (ie, you are left with 14 students and their answers). Please use this approach to arrive at the result.

4. Consider the example on measurement methods A and B described in the Lecture 10 slide 27.

	Object										
	1	2	3	4	5	6	7	8	9	10	11
A	20.2	22.0	19.7	21.4	16.3	17.0	24.5	15.6	16.0	13.2	19.0
B	21.6	22.9	20.0	23.6	15.7	17.5	27.8	19.8	16.4	14.7	20.1

(Note that $\bar{d} = 1.38$, $s_d = 1.39$, and $s_d/sqrt11 = 0.42$, the statistics on Slide 30 where given with some errors).

a) Assume that both measurements A and B come from normally distributed populations, $N(\mu_A, \sigma)$ and $N(\mu_B, \sigma)$, and assume $\sigma = 3.5$. Denote $\Delta := \mu_B - \mu_A$. Suppose that the researcher wants to test $H_0 : \Delta = 0$ vs. $H_0 : \Delta > 0$. Derive the power function $h_1(\Delta)$ based on the paired t-test.

b) Now suppose that the measurements relate to some health parameter, and there is a clinical interest to make the measurement with no more than 0.5 units error.

What is the power at $\Delta = 1$, i.e., $h_1(1)$, is it enough? Calculate the sample size n needed to achieve 80% power when testing $H_1 : \Delta > 0.5$.

b) Suppose now we have $H_0 : \Delta = 0$ vs. $H_0 : \Delta \neq 0$. Derive the power function $h_2(\Delta)$ based on the paired t-test.

c) Now suppose you don't have the information about the population distribution, so you decide to use the sign test to detect whether there is a systematic difference between the two samples. Derive the power function $h_3(p)$ for the sign test, where p is the proportion of positive differences $(B - A)$ in the population.

d) Plot the three functions using the R statistical program, compare and comment.

5. A medical researcher wishes to see whether the variance of the heart rates (in beats per minute) of smokers is different from the variance of heart rates of people who do not smoke. Two samples are selected, and the data are as shown. Using $\alpha = 0.05$, is there enough evidence to support the claim? To answer the question, calculate the test p-value. Assume that the variable is normally distributed.

Smokers	**Nonsmokers**
$n_1 = 26$	$n_2 = 18$
$s_1^2 = 36$	$s_2^2 = 10$

6. The CEO of an airport hypothesizes that the variance in the number of passengers for American airports is greater than the variance in the number of passengers for foreign airports. At α= 0.10, is there enough evidence to support the hypothesis? The data in millions of passengers per year are shown for selected airports. Use the P-value method. Assume the variable is normally distributed.

American airports		**Foreign airports**	
36.8	73.5	60.7	51.2
72.4	61.2	42.7	38.6
60.5	40.1		

[1]

P_i = prob. of i in captivity , $i = 1, 2, 3, 4$

$H_0: P_1 = 0{,}11 , \dots , P_4 = 0{,}02$

H_1: at least 1 i for what

$P_{captivity}(i) \neq P \quad (i)$

Number of cubes V	1	2	3	4	Σ
Prob. in wild	0,11	0,69	0,18	0,02	1
Observed frequency in captivity	41	243	27	5	316
Expected E_i np_i	35	218	57	6	316
$\frac{(O_i - E_i)^2}{E_i}$	1,029	2,867	15,789	0,167	

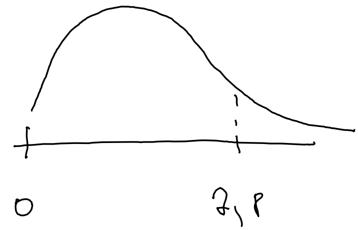
$$Q = \sum_{i=1}^{r} \frac{(O_i - E_i)^2}{E_i} = 19{,}852$$

$\chi^2_{5\%}(f) = 7{,}815$ $\quad Q > \chi^2 \Rightarrow$ rejecting H_0

$f = r - 1 = 4 - 1 = 3$

$q_{0{,}05, df=3} = 7{,}8$

$21 > 7{,}8 \Rightarrow H_1$



2

$n = 25$

$H_0: \; P_1 = P_2 = P_3 = \frac{1}{3}$

$H_1: \; P_i \neq P_j$

Design	1	2	3	Σ
Observed	8	7	10	25
E_i	$25/3$	-11-	-11-	
$\frac{(O_i - E_i)^2}{E_i}$				

? do we need to round up

$E_x = 25/3 = 8{,}333$

$$\left.\begin{aligned} X_1^2 &= \frac{(8-8{,}3)^2}{8{,}3} = 0{,}01 \\ X_2 &= 0{,}4 \\ &= 2{,}89 \end{aligned}\right\}$$

$q_{0{,}05}$ $df = 2$

$3{,}1 < 5{,}991 \Rightarrow$

cannot reject H_0

3

a) Sample size is too small to assume normal distribution => we can use a sign test

$H_0: \mu_A \geq \mu_B$

$H_1: \mu_A < \mu_B$

Sign(B-A) + + + + + - = - + - + ... +

$n = 14$ (remove tie)

$Z = 11$ "number of + signs"

$Z \overset{H_0}{\sim} Bin(14, 0,5)$

$P(Z \geq 11) = P(Z=11) + P(Z=12) + \dots + P(Z=14) =$

$= \left(\frac{14!}{11!\,3!} + \frac{14!}{12!\,2!} + \frac{14!}{13!} + 1\right)\frac{1}{2}^{14} = (7 \cdot 13 \cdot 6 + 7 \cdot 13 + 14 + 1)\frac{1}{2}^{14}$

$= 0,028$ reject H_0

$\boxed{5}$

Smokers	Non smoke
$n_1 = 26$	$n_2 = 18$
$S_1^2 = 36$	$S_2^2 = 10$

$\alpha = 0{,}05$

$$H_0: \sigma_1^2 = \sigma_2^2$$

$$H_1: \sigma_1^2 \neq \sigma_2^2$$

$$Z = \frac{S_1^2}{S_2^2} = \frac{36}{10} = 3{,}6$$

$$F_{\alpha/2}\left(\overset{25}{n_1 - 1}, \overset{17}{n_2 - 1}\right) \approx 2{,}55$$

$$Z > F_{\alpha/2} \Rightarrow \text{reject } H_0$$

$$\text{Pvalue} = 2\left(1 - \Phi(3{,}6)\right) \quad ?$$

$\boxed{6}$

$H_0: \sigma_A^2 \leq \sigma_F^2$

$H_1: \sigma_A^2 > \sigma_F^2$

$\overline{x}_A = 57{,}42$

$S_A^2 = \frac{1}{n-1} \sum^{6} (x_i - \overline{x})^2 = 246{,}4$

$\overline{x}_F = 48{,}3$

$\overline{S}_F = 95{,}9$

$F = \frac{S_A^2}{S_F^2} = 2{,}569$

$F_{\alpha/2} = 9{,}01$