Method of Moments

Let $X_1, \dots, X_n$ $X_i \sim X$, be a theoretical sample from a distribution with r unknown param-s $(\theta_1, \dots, \theta_r)$ and define the k-th theoretical moment as

$$\mu_k = E X^k \quad , \quad k = \{1, \dots, r\}$$

Given observations $x_1, \dots, x_n$ the k-th sample moment is defined as

$$m_k = \frac{1}{n} \sum_{i=1}^{k} x_i^k$$

The method of moments chooses as estimators of parameters $(\theta_1, \theta_2, \dots, \theta_r)$ the values $(\hat{\theta}_1, \dots, \hat{\theta}_r)$ that render population moments equal to sample moments

ie.

$\mu_k(\hat{\theta})$

System of r eq. with r unknowns

$\mu_k(\hat{\theta})$

MM Example

$X_1, \dots, X_n \sim U(0, \theta)$ θ is unknown

First theoretical moment is

$E X_i = \frac{\theta}{2}$ sample mean $\bar{X}$ is the first sample moment

$$\frac{\theta}{2} = \bar{X}$$

yields the MM estimator

$$\hat{\theta} = 2\bar{X}$$

Some distr. F with μ and σ^2

Theoretical moment   Sample moments

$\mu_1 = EX = \mu$ $\bar{X} \Rightarrow \hat{\mu}_1 = \bar{X}$

$\mu_2 = E(X^2) = \operatorname{Var} X + (EX)^2 = \sigma^2 + \mu^2$

$$\hat{\sigma}^2 + \bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum^{n} X_i^2 - \bar{X}^2 = \frac{1}{n}\sum (X_i - \bar{X})^2$$

$$\frac{1}{n}\sum^{n}(x_i-\bar{x})^2=\frac{1}{n}\sum\left(x_i^2-2x_i\bar{x}+\bar{x}^2\right)=\frac{1}{n}\sum x_i^2-\bar{x}^2$$

$$2\bar{x}\frac{1}{n}\sum x_i=2\bar{x}^2$$

$$(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2$$

$$x_1^2-2x_1\bar{x}+\bar{x}^2+x_2^2 \quad =$$

$$\sum_{i=1}^{n}x_i^2-2\bar{x}\sum^{n}x_i+n\bar{x}^2=$$

$$\sum_{i=1}^{n}x_i^2-2n\bar{x}^2+n\bar{x}^2=$$

$$\sum x_i^2-n\bar{x}^2$$

Application of Maximum Likelihood to the Normal Distribution

1. A Single sample

$$f(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$L(\mu,\sigma^2)=\prod_{i=1}^{n} f(x_i;\mu,\sigma^2)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x_i-\mu)^2}{2\sigma^2}}=$$

$$=(2\pi\sigma^2)^{-n/2}\,e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\ln(L(x;\mu\sigma^2))=-\frac{n}{2}\ln 2\pi\sigma^2-\frac{1}{2\sigma^2}\sum^{n}(x_i-\mu)^2$$

$$\frac{d}{d\mu}\ln L(x;\mu,\sigma^2)=-\frac{1}{2\sigma^2}2\sum^{n}(x_i-\mu)(-1)$$

$$\frac{d}{d\sigma^2}=-\frac{n}{2}\frac{1}{2\pi\sigma^2}\cdot 2\pi+\frac{1}{2\sigma^4}\sum^{n}(x_i-\mu)^2=$$

$$=\frac{1}{2\sigma^4}\sum^{n}(x_i-\mu)^2-\frac{n}{2\sigma^2}$$

$$\hat{\mu} = \frac{1}{n}\sum x_i = \bar{X}$$ unbiased

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2$$ biased

Suppose I want σ_{MLE}

Due to invariance

$$\hat{\sigma}_{MLE} = \sqrt{\hat{\theta}_{MLE}} = \sqrt{\frac{1}{n}\sum^{n}(x_i - \mu)^2}$$