Recitation after lecture 6

1) Round die $\Rightarrow N \in \{0,1,2,3\}$

2) Toss coin N times $\Rightarrow K$ heads

$N=3 \Rightarrow HHT \Rightarrow K=2$

A) Calculate PMF P_N

$$P_N(n) = \begin{cases} 1/4 & \text{if } n \in \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

B) $P_{N,K}(n,k) = P(K=k \mid N=n) \cdot P(N=n) =$

$= \frac{1}{4} P(K=k \mid N=n) \quad , n \in \{0,1,2,3\}$

$n=0 \Rightarrow K=0 \Rightarrow P(K=0 \mid N=0) = 1$

$n \in \{1,2,3\} \Rightarrow K \stackrel{d}{=} \text{Binomial}(n, \frac{1}{2})$

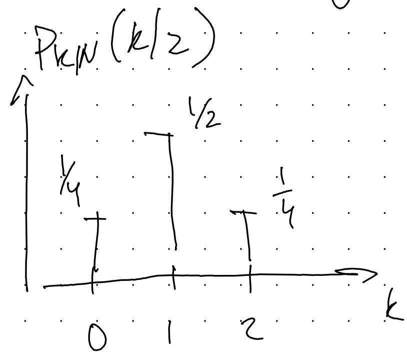
$P_{N,K}(n,k)$

	$k=0$	$k=1$	$k=2$	$k=3$
$n=0$	$1/4$	0	0	0
$n=1$	$1/8$	$1/8$	0	0
$n=2$	$1/16$	$1/8$	$1/16$	0
$n=3$	$1/32$	$3/32$	$3/32$	$1/32$

c) $P_{K|N}(k \mid N=2)$

Binomial $\left(2, \frac{1}{2}\right)$

$$P_{K|N}(k|2) = \begin{cases} 1/4 & \text{if } k=0 \\ 1/2 & k=1 \\ 1/4 & k=2 \\ 0 & \text{otherwise} \end{cases}$$



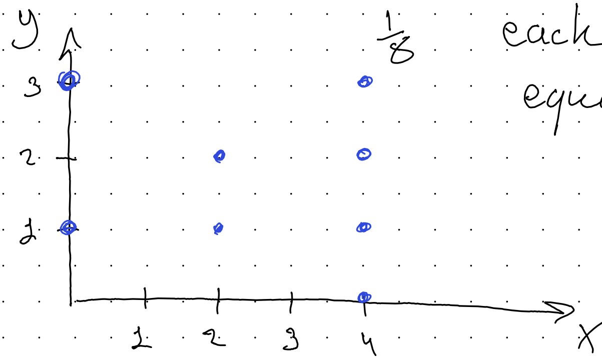
D) $P_{N|K}(N=n \mid K=2) = \dfrac{P_{N,K}(n,2)}{\boxed{\mathbb{P}(K=2)}} =$

$\dfrac{1}{16} + \dfrac{3}{32}$

$$= \begin{cases} \dfrac{1/16}{1/16 + 3/32} & n=2 \\ \dfrac{3/32}{1/16 + 3/32} & n=3 \end{cases}$$

Lecture 6 recitation 2

Joint PMF



each point is equally likely

a) Which values of x maximize $E[Y|X=x]$?

visually when x is 0 E should be 2 which is maximum among all x.

$$E[X|Y=y] = \sum_x x P_{X|Y}(x|y) = \sum_x x \frac{P(x,y)}{P(y)}$$

$$E[Y|X=x] = \sum_y y \frac{P(x,y)}{P(x)}$$

$$E[Y|X=0] = \left(1 \cdot \frac{1}{8} + 3 \frac{1}{8}\right) \cdot \frac{1}{2/8} = \frac{1}{2} + \frac{3}{2} = 2$$

$X=0$

b) Which values of y maximize $\text{var}(X|Y=y)$?

$$\text{var}(X|Y=y) = \sum_x \left(x - E[X|Y=y]\right)^2 P_{X|Y}(x|y) =$$

$$\text{var}(X|Y=1) = \left((0-2)^2 \cdot \frac{1}{8} + (2-2)^2 \cdot \frac{1}{8} + (4-2)^2 \cdot \frac{1}{8}\right) \cdot \frac{1}{3/8} =$$

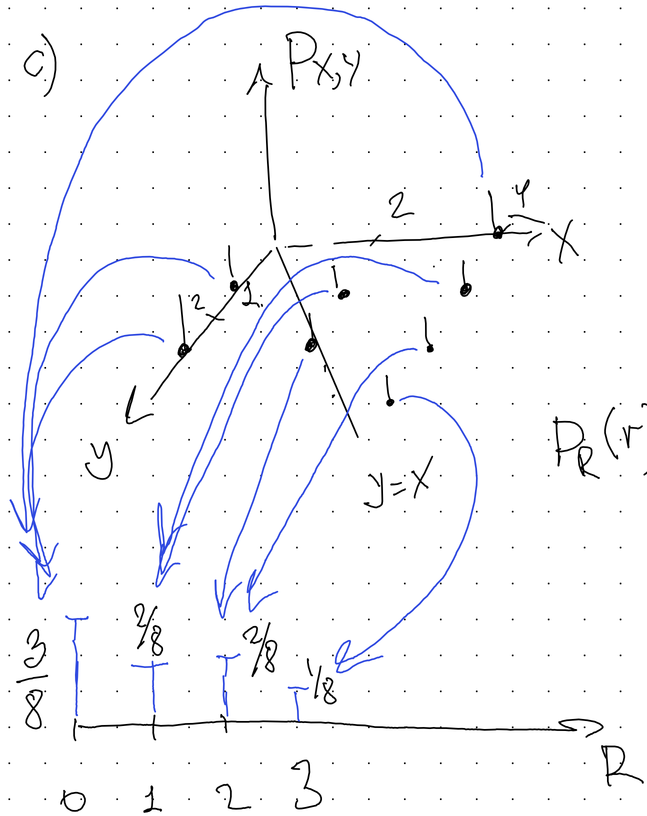
$$= (4+4) \cdot \frac{1}{3} = \frac{8}{3} = 2\frac{2}{3}$$

$$\text{var}(X|Y=2) = \left((2-3)^2 \cdot \frac{1}{8} + (4-3)^2 \frac{1}{8}\right) \cdot \frac{1}{2/8} = 1$$

$$\text{var}(X|Y=3) = \left((0-2)^2 \cdot \frac{1}{8} + (4-2)^2\right) \cdot \frac{1}{2/8} = 4$$

$$\boxed{y=3}$$

c)



$R = \min(X, Y)$

$$P_R(r) = \begin{cases} \frac{3}{8} & r = 0 \\ \frac{2}{8} & r = 1 \text{ or } r = 2 \\ \frac{1}{8} & r = 3 \\ 0 & \text{otherwise} \end{cases}$$

d)

$$E[XY] = \sum_x \sum_y x\, y\; p(x\, y) =$$

$$2(1+2)\cdot\frac{1}{8} + 4(1+2+3)\cdot\frac{1}{8} = \frac{1}{4}(3+12) =$$

$$= \frac{15}{4} \approx 3{,}75$$

Let A event $X^2 \geq Y$

$$E[XY|A] = \sum_x \sum_y xy \, P_{X,Y|A}(x,y|A) =$$

$$= 2(1+2)\cdot\frac{1}{6} + 4(1+2+3)\cdot\frac{1}{6} = \frac{1}{3}(3+12) =$$

$$= \frac{15}{3} \approx 5$$

$$W \triangleq XY$$

$$E[XY] = E[W]$$

Lecture 6 recitation 3

Coupon collector problem

We have K coupons, with equal probability to be choosen.
What's the expected value of number of draws to get all K coupons?

X - number of draws to collect all coupons

Assume $K = 3$

$$P_X \quad \begin{cases} 0 & x < K \\ \left(\frac{1}{k}\right)^k & x = k \end{cases}$$

$\boxed{3}$ ✓

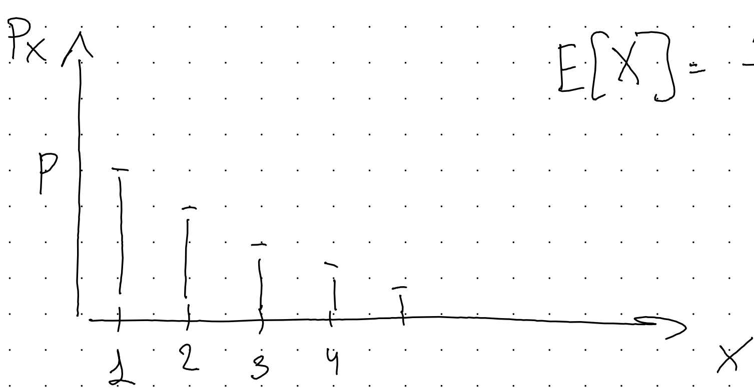
p - probability software works correctly (independent of previous attempt)

X - number of tries until program works

$var(X)$ - ?

$$P_X(x) = (1-p)^{x-1} \cdot p \quad \begin{cases} x \in \mathbb{N} \\ x > 0 \end{cases}$$

$$E[X] = \sum_{k=1} x \, (1-p)^{k-1} \cdot p$$



$$E[X] = \frac{1}{p}$$

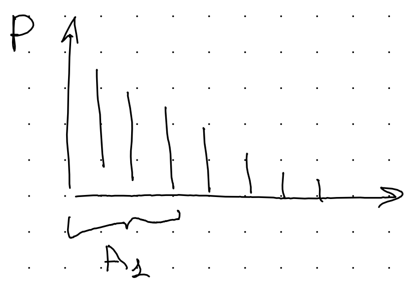
$$\mathrm{var}(X) = E[X^2] - (E[X])^2 =$$

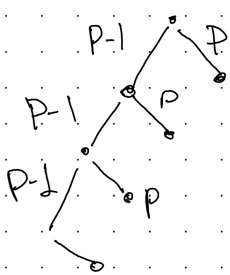
$$= E[(X - E[X])]$$

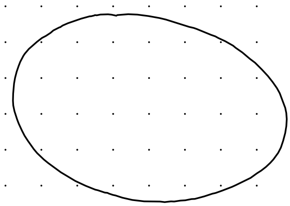
A_1 - first try failure

$$E[X|A_1] = \sum_{k=2}^{\infty} k \; p_{X|A_1}(k) =$$

$=$





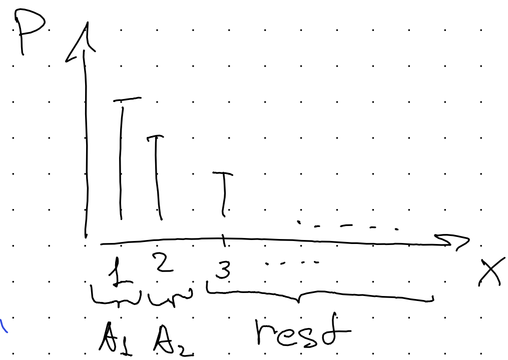


$$E[X] = P(A_1)\, E[X|A_1]$$

$$E[X] = \sum_{i=1}^{\infty} P(A_i)\, E[X \mid A_i]$$

$$E[X] = E[X-1 \mid X>0] + 1 =$$

$$= E[X-2 \mid X>1] + 2 + 1$$



from total expectation theorem and memorylessnes property

$$E[X] = \frac{1}{p}$$

$$E[X^2] = P(X^2=1)\, E[X^2 \mid X^2=1] + P(X^2>1)\, E[X^2 \mid X^2>1]$$

$$E[X^2] = p \cdot 1 + (1-p)\left(E[X^2] + 1\right) = \frac{1}{p}$$

$$\text{var}(X) = \frac{1}{p} - \frac{1}{p^2} = \frac{p-1}{p^2}$$

probably incorrect but proved it another way below ↓

$$E[X^2] = 1^2 p + 2^2 (1-p) p + \dots + n^2 (1-p)^{n-1} p$$

$$(1-p) E[X^2] = 1^2 (1-p) p + 2^2 (1-p)^2 p + \dots + n^2 (1-p)^n p$$

$$E[X^2] - (1-p) E[X^2] =$$

$$(n-1)^2 (1-p)^{n-2} p - n - {}'$$

$$n^2 (1-p)^{n-1} \cdot p - (n-1)^2 (1-p)^{n-1} p =$$

$$n^2 (1-p)^{n-1} \cdot p - \left(n^2 (1-p)^{n-1} p - 2n (1-p)^{n-1} p + (1-p)^{n-1} p\right) =$$

$$= (2n-1) (1-p)^{n-1} p$$

$$p E[X^2] = 1^2 \cdot p + (2 \cdot 2 - 1)(1-p) p + \dots + \quad =$$

$$= \cancel{1^2 p} + 2\left(2(1-p)p + 3(1-p)^2 p + \dots + n(1-p)^{n-1} p + 1 \cdot p\right) - \cancel{2p} -$$

not sure

$$- p\left((1-p) + (1-p)^2 + \dots + (1-p)^{n-1} + 1\right) + \cancel{1p} \quad - n^2 (1-p)^n \cdot p$$

$$= 2E[X] - p \frac{1}{1-(1-p)} = 2E[X] - 1$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 2E[X] - 1 - (E[X])^2$$

$$= \frac{2}{p} - 1 - \frac{1}{p^2} = \frac{2p - p - 1}{p^2} = \frac{p-1}{p^2}$$