$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} =$$

$$\cancelto{0}{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \cancelto{0}{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}$$

$$ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} +$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$

$$+ \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{31} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} +$$

$$- a_{12}a_{21}a_{33} \quad + a_{12}a_{23}a_{31} \quad + a_{31}a_{21}a_{32}$$

$$+ \begin{vmatrix} 0 & 0 & a_{31} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$- a_{31}a_{22}a_{31}$$

Big formula

$$\det A = \sum_{n! \text{ terms}} \pm\, a_{1\alpha}\, a_{2\beta}\, a_{3\gamma} \dots a_{n\omega}$$

$(\alpha\, \beta\, \gamma \dots \omega)$ = permutations of $(1, 2, \dots, n)$

Example

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$(4\,3\,2\,1) \Rightarrow +1$

$(3\,2\,1\,4) \Rightarrow -1$

det is 0 $\Rightarrow$ matrix is singular

Cofactors 3×3

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(\dots) + a_{13}(\dots)$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} +$$

$$a_{11}(\quad) + a_{12}(\underbrace{-a_{21}a_{33} + a_{23}a_{31}}_{\text{cofactor}})$$

Cofactor of $a_{ij} = C_{ij}$

$$\pm \det\begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \\ \text{col } j \end{pmatrix} \text{ erased}$$

if $i+j$ even

$i+j$ odd

Cofactor formula

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}$$

$$C_{ij} = (-1)^{i+j} \det M_{ij}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b \cdot (-c)$$

Ex

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1 \quad |A_2| = 0$

$|A_3| = -1$

$|A_4| = 1\,|A_3| - 1\,|A_2|$

Recitation find determinants

$$A = \begin{bmatrix} x & y & & & \\ & x & y & & \\ & & x & y & \\ & & & x & y \\ y & & & & x \end{bmatrix} \qquad \begin{bmatrix} x & y & & & \\ & x & y & & \\ & & x & y & \\ & & & x & y \\ y & & & & x \end{bmatrix}$$

by cofactors

$$|A| = x \begin{vmatrix} x & y & & \\ & x & y & \\ & & x & y \\ & & & x \end{vmatrix} - y \begin{vmatrix} \boxed{y} & & & \\ x & y & & \\ & x & y & \\ y & & & x \end{vmatrix} =$$

$$x^4$$

$$= x^5 - y \cdot (-y) \begin{vmatrix} \boxed{y} & & \\ x & y & \\ y & & x \end{vmatrix} = x^5 - y \cdot (-y) \cdot (-y) \begin{vmatrix} & y \\ y & x \end{vmatrix}$$

$$= x^5 - y^3 \, y \cdot (-y) = x^5 + y^5$$

by premutations

$x^5 + x\,y\,y\,y$ can't choose $+ y^5$

$$B = \begin{bmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{bmatrix}$$

i can transform to

$$B = \begin{bmatrix} x & y & y & y & y \\ y-x & x-y & 0 & 0 & 0 \\ 0 & y-x & x-y & 0 & 0 \\ 0 & 0 & y-x & x-y & 0 \\ 0 & 0 & 0 & y-x & x-y \end{bmatrix} =$$

$$= \begin{bmatrix} y & y & y & y & x \\ y-x & 0 & 0 & 0 & x-y \\ 0 & y-x & 0 & 0 & x-y \\ 0 & 0 & y-x & 0 & x-y \\ 0 & 0 & 0 & y-x & x-y \end{bmatrix}$$

*

$$|B| = x(y-x)^4 + y(y-x)^4 + y(y-x)^4 + y(y-x)^4 + y(y-x^4) = (x+4y)(y-x)^4$$