Mathematical Statistics
Week 2 exercises

1. Often some type of water quality monitoring is desired, for example to test for faecal contamination.

   For frequent monitoring, a relatively cheap test is required. A cheap test for testing faecal contamination is hydrogen sulphide test. In case of serious contamination (the presence of E.coli bacteria colony forming units >1000 CFU/100ml) the hydrogen sulphide test has specificity 0.94 and sensitivity 0.97 (`http://www-eawag.emp-eaw.ch/fileadmin/Domain1/Abteilungen/sandec/publikationen/WST/usefulness_hydrogen.pdf`).

   Let's assume just 0.1% of the samples tested to be contaminated. What is the probability of contamination given the hydrogen sulphide test yielded a positive test result?

2. A random variable X assumes the values a and b ($a < b$) with the same probability. Find the distribution function of X and draw its graph.

3. Suppose a random variable X can take the values -1, 0, 1 with corresponding probabilities such as $1/4$, $1/2$, $1/4$. Create the distribution table for rv X. Compute
   a) $P\{X \leq -1\}$,
   b) $P\{X < 0\}$,
   c) $F_X(0)$,
   d) $F_X(0.5)$,
   e) EX.

4. An insurance company sells a life insurance policy with a face value of \$1000 and a yearly premium of \$20. If 0.1% of the policyholders can be expected to die in the course of a year, what would be the company's expected earnings per policyholder in any year?

5. You buy a Euro Jackpot lottery ticket. In this lottery the goal is to match 5 correct numbers out of 50, plus another 2 supplementary numbers out of another 10. The "jackpot" is 18 million Euros and the ticket costs 2€. What are your winnings on average?

6. Five coin tosses is our random trial. Let X be the number of "tails" from the tosses. Find the distribution table. Find EX and DX.

7. Probability that after 4 independent random trials event A will occur at least once is 0.4. Find the probability of event A happening after a single trial.

8. Baker made 1000 muffins with 3000 raisins in the dough. Find the probability that in a randomly selected muffin:

   a) there are no raisins;

   b) there are exactly 3 raisins;

   c) there are more than 3 raisins.

9. Let $X \sim Po(\lambda)$. Show that $EX = DX = \lambda$.

10. Let $X \sim Po(3)$ and $Y = 9X - 4$. Find EY and DY.

11. The St. Petersburg paradox - a paradox where theoretical reasoning clashes hard with practical reasoning and decision making. A short video & Wikipedia page.

$\boxed{1}$ todo calculate the answer

hydrogen sulphide (HS)

$$\text{specificity (HS)} = \frac{TN}{TN + FP} = 0{,}94$$

$$\text{sencitivity (HS)} = \frac{TP}{TP + FN} = 0{,}97$$

	Disorder	No Disorder
Positive test	TP	FP
Negative test	FN	TN

C - contaminated

$T^+, T^- = \{\text{Test positive / negative}\}$

$P(C) = 0{,}001$

$P(T^+ | C) = 0{,}97$

$P(T^- | \overline{C}) = 0{,}94 \rightarrow P(T^+ | \overline{C}) = 0{,}06$

By Baye's theorem

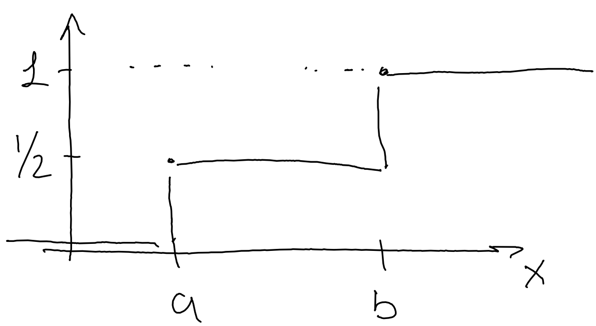
$$P(C \mid T^+) = \frac{P(C, T^+)}{P(T^+)} = \frac{P(C)P(T^+ \mid C)}{-\|- + P(\bar{C})P(T^+ \mid \bar{C})}$$

DONE

2 RV assumes values a and b $(a < b)$ with the same prob

Find distr function of X F_X - ?

$$F_X(x) = P_X(X \le x) = \begin{cases} 0 & \text{if } x < a \\ 1/2 & \text{if } a \le x < b \\ 1 & \text{if } x \ge b \end{cases}$$



DONE

$\boxed{4}$

A = „Policy holder expected to die in course of a year"

$$P(A) = 0{,}001$$

X = Random variable represent amount of money inshurance company gain per policy maker

$$P_X(x) = \begin{cases} P(A) & \text{if } x = -980 \\ 1 - P(A) & \text{if } x = 20 \end{cases}$$

$$E[X] = -980 \cdot 0{,}001 + 20 \cdot 0{,}999 = 19$$

6 Done

X - number of tails from the tosses

Find distribution table

prob-ty of tail

$$P_X(x) \sim Bin(n, p) = Bin(5, \tfrac{1}{2})$$

number of trial

$* \; p = \frac{1}{2}$

$$= \binom{5}{x} p^x (1-p)^{5-x} = \binom{5}{x} p^5$$

X_i	0	1	2	3	4	5
$P_X(x_i)$	0,03125	0,15625	0,3125	0,3125	0,15625	0,03125

$E[X] = 2,5$

$$D[X] = \sqrt{var(X)} = \sqrt{E[X^2] - (E[X])^2} =$$

$$= \sqrt{(0,15625 + 4 \cdot 0,3125 + 9 \cdot 0,3125 + 16 \cdot 0,15625 + 25 \cdot 0,03125) - (2,5)^2} =$$

$$= \sqrt{7,5 - 6,25} = \sqrt{1,25} = 1,118$$

$\boxed{7}$ Done

$X \sim$ event A happens after once after n independant trials

$P_X(x) \sim$ geometric prob. distr function

$(1-p)^k \cdot p$

$P\left(\begin{Bmatrix}\text{Event } A \text{ happens} \\ \text{at least once in} \\ \text{4 indep. trials}\end{Bmatrix}\right) = 0{,}4$

$P\left(\begin{Bmatrix}\text{Event } A \text{ happens} \\ \text{never in 4 indep} \\ \text{trials}\end{Bmatrix}\right) = 1 - 0{,}4 = (1-p)^4$

$P(A) = p$

requires calculations

$\boxed{8}$ DONE

X = num of rasins in muffin

$X \sim Po = e^{-\lambda} \frac{\lambda^k}{k!}$

$EX = \frac{3000}{1000} = 3 \Rightarrow \lambda = 3$

$P_X(x=0) = e^{-3} \frac{3^0}{0!} = e^{-3} \approx 0{,}05$

$P_X(x=3) = e^{-3} \frac{3^3}{3!} = \frac{e^{-3} \cancel{27}^{9}}{\cancel{6}_{2}} \approx 0{,}224$

$P_X(x>3) = 1 - P(0) - P(1) - P(2) - P(3)$

$= 1 - e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right) =$

$= 1 - e^{-3}\left(1 + 3 + \frac{9}{2} + \frac{27}{6}\right) =$

$= 1 - e^{-3}\left(\frac{6 + 18 + 27 + 27}{6}\right) \approx 0{,}353$

$$P(\{\text{left handed}\}) = 0{,}1$$

$$P(1) = \binom{5}{1} \cdot P(L)^1 \, P(L^c)^4 \approx 0{,}328$$

$$P(0) = P(L)^0 \, P(L^c)^5 \approx 0{,}59$$