Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem Set 4
Due October 6, 2010

1. Random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} c(x^2+y^2), & \text{if } x \in \{1,2,4\} \text{ and } y \in \{1,3\}, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?
   (b) What is $\mathbf{P}(Y < X)$?
   (c) What is $\mathbf{P}(Y > X)$?
   (d) What is $\mathbf{P}(Y = X)$?
   (e) What is $\mathbf{P}(Y = 3)$?
   (f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.
   (g) Find the expectations $\mathbf{E}[X]$, $\mathbf{E}[Y]$ and $\mathbf{E}[XY]$.
   (h) Find the variances $\text{var}(X)$, $\text{var}(Y)$ and $\text{var}(X+Y)$.
   (i) Let A denote the event $X \geq Y$. Find $\mathbf{E}[X \mid A]$ and $\text{var}(X \mid A)$.

2. The newest invention of the 6.041/6.431 staff is a three-sided die with faces numbered 1, 2, and 3. The PMF for the result of any one roll of this die is

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 1, \\ 1/4, & \text{if } x = 2, \\ 1/4, & \text{if } x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

   Consider a sequence of six independent rolls of this die, and let X_i be the random variable corresponding to the ith roll.

   (a) What is the probability that exactly three of the rolls have result equal to 3?
   (b) What is the probability that the first roll is 1, given that exactly two of the six rolls have result of 1?
   (c) We are told that exactly three of the rolls resulted in 1 and exactly three resulted in 2. Given this information, what is the probability that the sequence of rolls is 121212?
   (d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's.

3. Suppose that X and Y are independent, identically distributed, geometric random variables with parameter p. Show that

$$\mathbf{P}(X = i \mid X + Y = n) = \frac{1}{n-1}, \qquad \text{for } i = 1, 2, \ldots, n-1.$$

4. Consider 10 independent tosses of a biased coin with a probability of heads of p.

   (a) Let A be the event that there are 6 heads in the first 8 tosses. Let B be the event that the 9th toss results in heads. Show that events A and B are independent.

   (b) Find the probability that there are 3 heads in the first 4 tosses and 2 heads in the last 3 tosses.

   (c) Given that there were 4 heads in the first 7 tosses, find the probability that the 2nd head occurred during the 4th trial.

   (d) Find the probability that there are 5 heads in the first 8 tosses and 3 heads in the last 5 tosses.

5. Consider a sequence of independent tosses of a biased coin at times $t = 0, 1, 2, \ldots$. On each toss, the probability of a 'head' is p, and the probability of a 'tail' is $1 - p$. A reward of one unit is given each time that a 'tail' follows immediately after a 'head.' Let R be the total reward paid in times 1, 2, ..., n. Find $\mathbf{E}[R]$ and $\mathrm{var}(R)$.

G1†. A simple example of a random variable is the *indicator* of an event A, which is denoted by I_A:

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Prove that two events A and B are independent if and only if the associated indicator random variables, I_A and I_B are independent.

   (b) Show that if $X = I_A$, then $\mathbf{E}[X] = \mathbf{P}(A)$.

†Required for 6.431; optional for 6.041

6.041 / 6.431 Probabilistic Systems Analysis and Applied Probability
Fall 2010

$\boxed{1}$ seems easy

a) c should normalize value to probabity boundaries.

$$C = \frac{1}{\sum\limits_{x \in X} \sum\limits_{y \in Y} (x^2 + y^2)}$$

b)

$\boxed{2}$ Skipping, come back when learning counting

$$P_X(x) = \begin{cases} 1/2 & \text{if } x = 1 \\ 1/4 & \text{if } x = 2 \\ 1/4 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

a) A - exactly 3 rolls have result 3?

$$P(A) = \binom{n}{k} \cdot P$$

[3] Suppose X, Y are independent identically distributed geometric random variables with parameter p

Show that

$$P(X=i \mid X+Y=n) = \frac{1}{n-1}$$

for $i = 1, 2, \ldots, n-1$

if X and Y are indep-t and conditioned on sum of two RVs then it's equally likely

$$P(X=i \mid X+Y=n) = \frac{P(X=i, Y=n-i)}{P(X+Y=n)} =$$

$$\overset{\text{by indep}}{=} \frac{P(X=i)\, P(Y=n-i)}{P(X+Y=n)}$$

$$= \frac{(1-p)^{i-1}\, p \cdot (1-p)^{n-i-1} \cdot p}{P(X+Y=n)}$$

$$P(X+Y=n) \overset{\text{by total prob. theorem}}{=} \sum_{i=1}^{n-1} P(X=i) \cdot P(X+Y=n \mid X=i)$$

$$= \sum_{i=1}^{n-1} P(X=i)\, P(i+Y=n \mid X=i) =$$

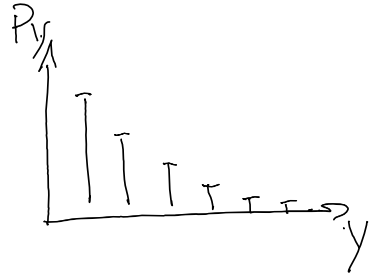
$$= \sum P(X=i)\, P(Y=n-i \cancel{\mid X=i})$$

remove since two events are indpep.

By symmetry $P(X=i)\,P(Y=n-i) = C$
$\forall\, i < n \quad i \in \mathbb{N}$

Total number of combinations to sum two RVs to n are $n-1$.

$$P(X=i \mid X+Y=n) = \frac{\cancel{c}}{\cancel{c}(n-1)} = \frac{1}{n-1}$$



$A = X + Y = n$

$$P(X=i) = (1-p)^{i-1} \cdot p$$

$$P(Y=n-i) = (1-p)^{(n-i)-1} \cdot p$$

[4] ✓

10 independent tosses of a biased coin with probability of heads of p

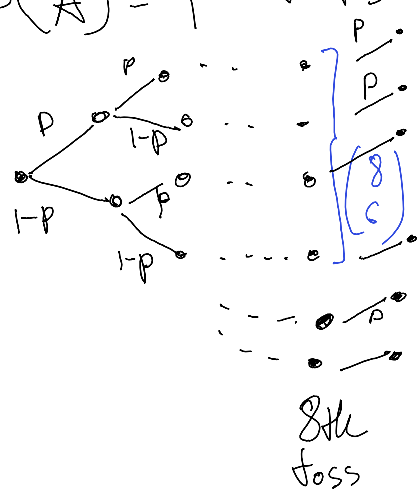
a) A - event that there are 6 heads in the first 8 tosses

B - 9th toss results in head.

Show event A and B are independent

$$P(A \cap B) = P(A)\,P(B)$$

$$P(A) = p^6 (1-p)^2 \cdot \binom{8}{6}$$



What if we have a smaller problem? 3 tosses and event B is 3rd toss is head

What is the prob-ty of B?

Event B

$$P(B) = \begin{cases} (1-p)^2 p & TTH \\ (1-p) p^2 & THH \\ (1-p) p^2 & HTH \\ p^3 & HHH \end{cases}$$

TTT
THT
HTT
HHT

incorrect since B is independent from previous tosses $\Rightarrow$ $P(B) = p$

What if i tell you that event A 2 first tosses are T and H

HH

Event A $\begin{cases} TH - (1-p)p \\ HT - (1-p)p \end{cases}$

TT

$$P(A \cap B) = P(B|A)\,P(A)$$

$$P(B|A) = \frac{(1-p)\cdot p^2 + (1-p)\,p^2}{2(1-p)\,p} = p = P(B)$$

~~I can draw a conclusion~~ $P(B|A) \neq P(B)$
~~hence two events are not independent~~

b) Find the prob. that 3 heads in the first 4 tosses and 2 heads the last 3 tosses

A – 3 heads in first 4 tosses
B – 2 heads in the 3 last tosses

$P(A) = \binom{4}{3} p^3 (1-p)$ $\quad P(A \cap B) = P(A)\,P(B)$

$P(B) = \binom{3}{2} p^2 (1-p)$

A, B – are independent events

c) Given
E - 4 heads in the first 7 tosses

Find probability that 2nd head occured during 4th trial - F

$$P(F|E) = \frac{P(F \cap E)}{P(E)}$$

H

1 2 3 4 5 6 7

at least 1 head

second Head

2 heads

$$P(E) = \binom{7}{4} p^4 (1-p)^3$$

$$P(F \cap E) = \binom{3}{1} p(1-p)^2 \cdot p \cdot \binom{3}{2} p^2(1-p)$$

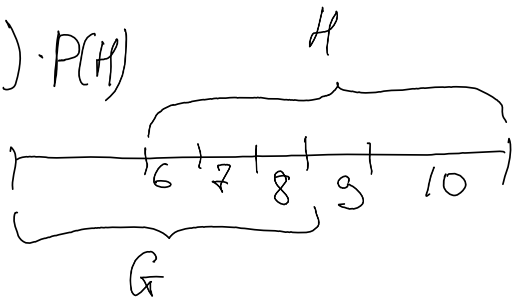
$$P(F|E) = \frac{\binom{3}{1} p(1-p)^2 \cdot p \cdot \binom{3}{2} p^2 (1-p)}{\binom{7}{4} p^4 (1-p)^3}$$

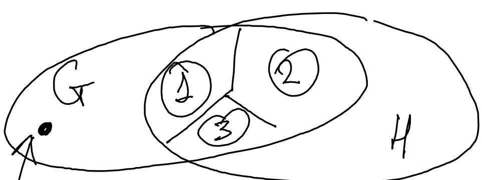
d)

G - 5 heads in the first 8 tosses

H - 3 heads in the last 5 tosses

$$P(G\cap H) = P(G|H)\cdot P(H)$$





all 5 Heads in the first 5 tosses

I was thinking if i can consider sum of independent events 1 2 3

1 $\binom{5}{4}\cdot p^4(1-p)\cdot\binom{3}{1}p(1-p)^2\cdot 1\, p^2$

2 $\binom{5}{3}p^3(1-p)^2\cdot\binom{3}{2}p^2(1-p)\cdot 2\,p(1-p)$

3 $\binom{5}{2}p^2(1-p)^3\ 1\ p^3\cdot(1-p)^2$

① $\binom{5}{4} \cdot p^4(1-p) \cdot \binom{3}{1} p(1-p)^2 \cdot 1\, p^2$

② $\binom{5}{3} p^3(1-p)^2 \cdot \binom{3}{2} p^2(1-p) \cdot 2\, p(1-p)$

③ $\binom{5}{2} p^2(1-p)^3 \; 1\, p^3 \cdot (1-p)^2$

$$5 \cdot 3 \; p^7 (1-p)^3 + 10 \cdot 3 \cdot 2 p^6 (1-p)^4 +$$

$$+ 10\, p^5 (1-p)^5$$

From solution independent events introduced could be written:

① $P(G \cap H \mid 1$ head in tosses $6-8)$

② 2

③ 3

$\boxed{5}$

$$\underbrace{\underline{H}\ \underline{T}}_{\text{reward}}\ \underline{T}\ \underbrace{\underline{H}\ \underline{T}}_{\text{reward}}$$

R - total reward paid in times

times $1, 2, \dots n$

$E[R]$ - ? $\quad var(R)$ - ?

$$\underbrace{H H H T T T T}_{\text{Reward}}\ \underbrace{H T}_{R}\ \underbrace{H T}_{R}$$

T T T H

H H H T T T H

$$P(R) = \sum_{i=1} p^{i-1}(1-p)$$

$\boxed{G1}$

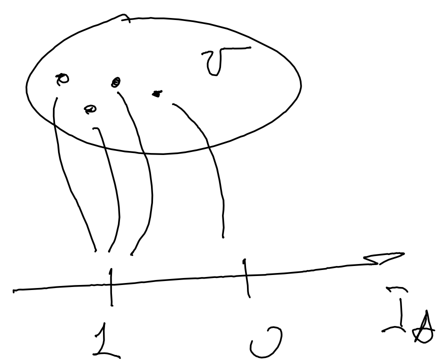
indecator of event A which is I_A

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

a) Prove that 2 events A and B are independent iff associated indicator var-s I_A and I_B are independent

$$P(A \cap B) = P(A) \cdot P(B) \quad \forall\, \omega \in \Omega$$

Assume A and B are independent events



$$P(A \cap B) = P(A)\, P(B)$$

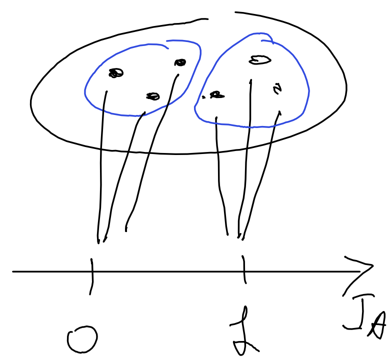
if we are given $I_A = 1$ meaning event A has occured, but because event A and B are independent it should not affect our belief of event B

Assume I_A and I_B are independent $\Rightarrow$

$$\underbrace{E[I_A I_B]}_{P(A\cap B)} = \underbrace{E[I_A]\,E[I_B]}_{P(A)\cdot P(B)}$$

$$E[I_A I_B] = \sum_{I_A=0}^{1} \sum_{I_B=0}^{1} I_A I_B \, P(A,B) = P(A\cap B)$$

b) show that if $X = I_A \Rightarrow E[X] = P(A)$



$$E[X] = 0\cdot(1-P(A)) + 1\,P(A) = $$
$$= P(A)$$