Orthonormal vectors

matrix $Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$

vectors $q_1, \dots q_n$ are orthonormal if $q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

vectors q are orthogonal unit vectors

$$Q^T Q = \begin{bmatrix} - & q_1 & - \\ & \vdots & \\ - & q_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

if Q is square $\Rightarrow$ $Q^T Q = I$ $\Rightarrow$

$Q^T = Q^{-1}$ inverse from both sides

Example let's take any permutation matrix

$$\text{perm } Q = \underset{\text{permutation matrix}}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{Q^T}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = I$$

rotation matrix

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$ Hadamar matrix

What operations orthogonal m. make easy

Q has orthonormal columns

Project onto its column space

Projections using orthogonal bases
Q replaces A

$$Q^T Q \hat{x} = Q^T b$$

$\hat{x} = Q^T b$ $\qquad$ $\hat{x}_i = q_i^T b$ $\quad$ component

$p = Q\hat{x}$ $\qquad$ $P = Q(Q^TQ)^{-1}Q^T = QQ^T\{=$
$\{= I$ if Q is square$\}$

Props of proj matrices
- P m. is symmeric
- $P^2 = P$ $\qquad$ $(Q\underbrace{Q^T)(Q}Q^T) = QQ^T$

if Q is square then

$b = q_1(q_1^T b) + \dots + q_n(q_n^T b)$ $\quad$ important for fourier series

?

Example

$$Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \qquad Q^T Q = Q Q^T = I$$

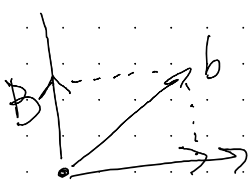
$$b = (0 \; 0 \; 1)$$

$$q_2 (q_2^T b) = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix}$$

Gram - Schmidt Process

independent vectors $a, b \rightarrow$ orthogonal $A \perp B \rightarrow q_1 = \frac{A}{\|A\|}$

error in previous lectures



$a = A$

$$B = b - \frac{A^T b}{A^T A} A$$

Check if formula is correct:

$$A^T B = A^T\left(b - \frac{\cancel{A^T} b}{\cancel{A^T}\cancel{A}} \cancel{A}\right) = 0$$

Now we're looking for third vector C

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\parallel$$

$$A$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

Q and original column space are the same

Final point:

- Elimination for matrices $A = LU$
- $A = QR$ ⇐ upper triangular

$$\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \Big[$$

Recitation

Find $q_1\ q_2\ q_3$ (orthonormal) from cols $a\ b\ c$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$
$$\quad\ \ a \quad b \quad c$$

vector

$$\vec{A} = a$$

$$\vec{B} = \left(b - \frac{A^T b}{A^T A} A\right) = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{C} = c - \frac{B^T c}{B^T B} B - \frac{A^T c}{A^T A} A = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{\cancel{18}^{\,2}}{\cancel{9}}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} - \frac{4}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$q_1 = \frac{A}{\|A\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad q_2 = \frac{B}{\|B\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Write A as QR

$$A = QR$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}}_{R} = \begin{array}{c} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} \\ \begin{matrix} a & b & c \end{matrix} \end{array}$$

you can see it as permutation matrix