Mathematical Statistics
Week 3 exercises

1. A random variable X has probability density function
$$f(x) = \begin{cases} cx^2(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$
Determine c. Find $E(X)$ and $Var(X)$.

2. Let the random variable X have a p.d.f.
$$f(x) = \begin{cases} -\frac{x}{4} & \text{if } -2 \le x < 0, \\ x & \text{if } 0 \le x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$
Derive the c.d.f. of rv X.

3. A random variable X has distribution function
$$F_X(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1 - \frac{1}{x^2} & \text{if } x \ge 1. \end{cases}$$
Determine the p.d.f and median of the distribution.

4. The lifetime of an automobile battery is described by a rv X having the Exponential distribution with parameter $\lambda = \frac{1}{3}$ ($\Rightarrow EX = 3$).
a) Determine the expected lifetime of the battery and the variation around this mean.
b) Calculate the probability that the lifetime will be between 2 and 4 time units.

5. If the rv X is distributed as $U(0,1)$, compute the expectations:
a) $E(3X^2 - 7X + 2)$
b) $E(2e^X)$.

6. Suppose that the diameter of a pinetree has a p.d.f.
$$f(x) = \begin{cases} 1.5 - 6(x - 50.2)^2 & \text{for } 49.5 \le x \le 50.5, \\ 0 & \text{elsewhere.} \end{cases}$$
Find the probability that a pinetree has a diameter between 49.8 and 50.1 cm.

7. Let $X \sim N(\mu, \sigma^2)$. Find the probability that X will not deviate from mean μ more than:
   - σ;
   - 2σ;
   - 3σ.

8. Benford's law - a weird phenomenon of randomness that has been used to get hints of fraud. For example, determining if liberal voting has been rigged, if somebody is evading taxes, or test for anomalies.

   A short video & blog post that explains it very thoroughly.

$\boxed{1}$ X is RV, which has PDF:

$$f(x) = \begin{cases} c x^2 (1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

c - ?

$u = 1 - x$

$du = -1 dx$

$dv = x^2 dx$

$v = \int x^2 dx = \frac{x^3}{3} + C$

$\int u\,dv = uv - \int v\,du = (1-x)\frac{x^3}{3} + \frac{1}{12}x^4 + C$

$= \frac{x^3}{3} - \frac{x^4}{3} + \frac{\cancel{1} x^4}{12} = \frac{x^3}{3} - \frac{x^4}{4}$

$\Big|_0^1 \left(\frac{x^3}{3} - \frac{x^4}{4}\right) dx = \left(\frac{1^4}{3} - \frac{1^3}{4}\right) - 0 = \frac{1}{12}$

$C = 12$

$EX = 12\int_0^1 x f(x)\,dx = 12\int_0^1 (x^3 - x^4)\,dx =$

$= 12\left(\frac{x^4}{4}\Big|_0^1 - \frac{x^5}{5}\Big|_0^1\right) = 12\left(\frac{1^5}{4} - \frac{1^4}{5}\right) = \frac{\cancel{12}^{3}}{\cancel{20}_{5}}$

$$\operatorname{Var} X = EX^2 - (EX)^2$$

$$EX^2 = \int_0^1 x^2 \cdot 12x^2(1-x)\,dx =$$

$$= 12\left(\int_0^1 (x^4 - x^5)\,dx\right) = 12\left(\frac{x^5}{5}\Big|_0^1 - \frac{x^6}{6}\Big|_0^1\right) =$$

$$= 12\left(\frac{1}{5}^{\backslash 6} - \frac{1}{6}^{\backslash 5}\right) = \frac{\cancel{12}^{\,2}}{\cancel{6}\cdot 5} = \frac{2}{5}$$

$$\operatorname{Var} X = \frac{2}{5}^{\backslash 5} - \frac{9}{25} = \frac{10-9}{25} = \frac{1}{25}$$