Differential Equns $\frac{du}{dt} = Au$

Example

u_0

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

- matrix is singular => $\lambda = 0$
- from trace $\lambda = -3$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad$ $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

pure solutions

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

use $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ initial condition to find c_1, c_2

at $t = 0$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow Sc = u(0)$$

$$c_1 = \frac{1}{3} \qquad c_2 = -\frac{1}{3}$$

Solution:

$$u(t) = \frac{1}{3} e^{\lambda_1 t} + \left(-\frac{1}{3}\right) e^{\lambda_2 t}$$

$$u(\infty) \to \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

when do we get stability?

1. $u(t) \to 0$ / need $e^{\lambda t} \to 0$ $\quad \mathrm{Re}\,\lambda < 0$

2. Steady state
   $\lambda_1 = 0$ and other $\mathrm{Re}\,\lambda < 0$

3. Blow up if any $\mathrm{Re}\,\lambda > 0$

2x2 stability $\text{Re}\,\lambda_1 < 0$

$\text{Re}\,\lambda_2 < 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; trace $a+d < 0$

negative

trace < 0, but still blow up

$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

det > 0 $(\lambda_1 \cdot \lambda_2)$ second condition for stability

Write solution in terms of S and λ

$$\frac{du}{dt} = \underbrace{Au}_{\text{coupled}}$$

introduce set

$$u = Sv$$

$$S\frac{dv}{dt} = ASv$$

$$\frac{dv}{dt} = S^{-1}ASv = \Lambda v$$

$$\begin{cases} \dfrac{dv_1}{dt} = \lambda_1 v_1 \\ \vdots \end{cases}$$

$$v(t) = e^{\Lambda t} v(0)$$

$$u(t) = \underbrace{S e^{\Lambda t} S^{-1}} u(0) = \underbrace{e^{At}} u(0)$$

How to show are the same?

$$e^{At} = S e^{\Lambda t} S^{-1}$$

Matrix exponential

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots + \frac{(At)^n}{n!} + \dots + =$$

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

$$= I + S\Lambda S^{-1} t + \frac{S\Lambda^2 S^{-1} t^2}{2} + \dots$$

SS^{-1}

$$= S e^{\Lambda t} S^{-1}$$

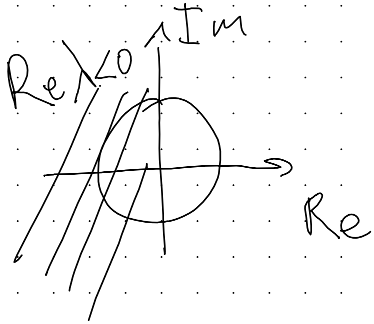
$$\frac{1}{1-x} = \sum_0^\infty x^n$$

$$(I - At)^{-1} = I + At + (At)^2 + \dots$$

* Assumption that A can be diagonilized

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \end{bmatrix}$$

? why zerose but not 1s ?



$$y'' + by' + ky = 0$$

How to change 1 second order to 2x2 1st order

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \qquad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix}$$

Recitation