Mathematical Statistics
Lab 11

Reminder of estimates and its properties

1. Let $X \sim Bin(m, p)$, where p is unknown. We have an observation x on X. Find an unbiased estimate of the variance $Var(X)$.

2. Let $x = 16$ be an observation on $X \sim Bin(25, p)$.
   (a) Estimate p.
   (b) Find the standard deviation of the estimate.

Lab 11 - Estimation methods

1. The random variable X has density function $f(x) = \theta(1+x)^{-\theta-1}$ for $x \geq 0$. It is known beforehand that θ is either 2,3 or 4. Let 0.2, 0.8 be a random sample of two values from this distribution.
   a) Find the likelihood function for all three possible values of θ.
   b) Determine the maximum likelihood estimate of θ.
   c) If values of θ are not known beforehand, find MLE.

2. Let $x_1, \ldots, x_n$ be a random sample from a distribution with density function

$$f(x) = \theta x^{\theta-1} \text{ for } 0 < x < 1.$$

   Find the least square, maximum likelihood and method of moments estimate of θ.

3. Suppose $x_1 = 3, x_2 = 4, x_3 = 3, x_4 = 7$ come from a Poisson distribution with λ unknown. The probability function is $p(x; \lambda) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$. Find the ML estimate of λ.

4. The lifetime of an automobile tire is measured in kilometres rather than time. Suppose a tire company produces three versions of a tire: a standard tire whose lifetime X_s has an exponential distribution with mean $1/\lambda > 0$, an economy version whose lifetime X_e has an exponential distribution with mean $0.77/\lambda$, and a premium tire with a lifetime X_p whose distribution is exponential with mean $1.25/\lambda$. Suppose one tire of each type is chosen randomly and independently and tested to find its lifetime and the lifetime of each is $x_s = 28, x_e = 25$, and $x_p = 31$ (in thousands of kilometres). Find the maximum likelihood estimate of λ.

5. Suppose we want to estimate the wavelength θ and for that the measurements $x_1, \ldots, x_5$ are obtained which are the realisations of independent random variables $X_1, \ldots, X_5$, where $EX_i = \theta$ and $\sqrt{VarX_i} = \sigma_i$. We have the following data:

   Find the least square estimate of θ.

x_i	79,1	80,0	81,3	81,9	81,7
σ_i	0,02	0,01	0,02	0,03	0,01

6. Let m be the number of crimes in a month and p probability detecting a crime, both parameters are unknown. For simplicity, let's assume m and p to be constant in each month. Let x_i be the number of detected crimes in a month i, $i = 1, \ldots, n$. Assuming that we have a random sample from $B(m, p)$ distribution, find the method of moments estimates for m and p.

7. Cauchy distribution - a distribution with undefined EX, i.e. $EX = \infty$.

Reminder

$X \sim Bin(m, p)$

1) p unknown

m is known

$Var(X) = m.p(1-p)$

$EX = mp$

$$\hat{p} = \frac{x}{m}$$

$$E\hat{p} = \frac{Ex}{m} = \frac{mp}{m} = p$$

$$\hat{Var} X = m\hat{p}(1-\hat{p}) = \frac{mX}{m}\left(1-\frac{X}{m}\right) =$$

$$= x\left(1-\frac{X}{m}\right)$$

$$E[\hat{Var} X] = E\left[x\left(1-\frac{X}{m}\right)\right] = EX - \frac{1}{m}EX^2 =$$

$$= mp - \frac{1}{m}\left(VarX + (EX)^2\right) = mp - \frac{1}{m}\left(mp(1-p) + (mp)^2\right)$$

$$= mp - \left(p(1-p) + mp^2\right) = mp - p + p^2 - mp^2 =$$

$$(mp - p) - p(mp - p) =$$

$= (mp - p)(1 - p) = p(m-1)(1-p)$

Unbiased estimate of Var X is

$$\frac{m}{m-1}\, x\left(1 - \frac{x}{m}\right)$$

2) $x = 16$ $\qquad X \sim Bin(25, p)$

$m = 25$

a)

$$\hat{p} = \frac{x}{m} = \frac{16}{25}$$

b)

$$Var(\hat{p}) = Var\left(\frac{x}{m}\right) = \frac{1}{m^2} Var(X) =$$

$$= \frac{1}{m^2}\, mp(1-p) = \frac{p}{m}(1-p) = \frac{16}{25^2}\left(1 - \frac{16}{25}\right)$$

$=$

$\boxed{1}$

a)

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) =$$

$$= f(0{,}2, \theta) \cdot f(0{,}8, \theta) =$$

$$= \theta (1 + 0{,}2)^{-\theta - 1} \cdot \theta (1 + 0{,}8)^{-\theta - 1}$$

$$= \theta^2 (1{,}2 \cdot 1{,}8)^{-\theta - 1} = \theta^2 \, 2{,}16^{-\theta - 1}$$

$$L(2) = 2^2 \cdot 2{,}16^{-3} = 0{,}397$$

$$L(3) = 3^2 \cdot 2{,}16^{-4} = 0{,}413$$

$$L(4) = 4^2 \cdot 2{,}16^{-5} = 0{,}340$$

c)

$$l(\theta) = \ln(L(\theta)) = 2\ln\theta + (-\theta - 1)\ln 2{,}16 =$$

$$= 2\ln\theta - \theta\ln 2{,}16 - \ln 2{,}16$$

$$\frac{dl(\theta)}{d\theta} = \frac{2}{\theta} - \ln 2{,}16 = 0 \qquad | \cdot \theta \neq 0$$

$$\theta = \frac{2}{\ln 2{,}16} \approx 2{,}597$$

2 a) ML

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

$$\ell(\theta) = \ln(L(\theta)) = n\ln\theta + \theta \ln \prod_{i=1}^{n} x_i - \ln\prod^{n} x_i$$

$$\frac{d\ell(\theta)}{d\theta} = \frac{n}{\theta} + \ln\prod_{i=1}^{n} x_i = 0$$

$$\theta = -\frac{n}{\ln\prod_{i=1}^{n} x_i}$$

b) LS

$$EX = E\left[\sum^{n} \theta x_i^{\theta-1}\right] = \theta \sum_{i=1}^{n} x_i^{\theta-1}$$

$$EX = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot \theta x^{\theta-1} dx =$$

$$= \theta \int_0^1 x^{\theta} dx = \frac{\theta x^{\theta+1}}{\theta+1}\Big|_0^1 = \frac{\theta}{\theta+1} - 0$$

$$Q(\vartheta) = \sum_{i=1}^{n} (x_i - EX)^2 = \sum_{i=1}^{n} \left(x_i - \frac{\vartheta}{\vartheta+1}\right)^2$$

$$\frac{dQ(\vartheta)}{d\vartheta} = \sum_{i=1}^{n} 2\left(x_i - \frac{\vartheta}{\vartheta+1}\right)\cdot(-1)\left(\frac{1}{\vartheta+1} + \vartheta\cdot(-1)\frac{1}{(\vartheta+1)^2}\right)$$

$$= 2\sum^{n}\left(x_i - \frac{\vartheta}{\vartheta+1}\right)\cdot(-1)\left(\frac{1}{(\vartheta+1)^2}\right) = 0$$

$$\cancel{2}\left(\sum^{n} x_i - \frac{n\vartheta}{\vartheta+1}\right)\cdot\cancel{\frac{-1}{(\vartheta+1)^2}} = 0$$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{vu' - v'u}{v^2}}$$

$$\boxed{\begin{aligned} & u'\cdot\left(\frac{1}{v}\right) + u\cdot\left(\frac{1}{v}\right)' = \\ & u'\cdot\left(\frac{1}{v}\right) + u\left(-\frac{1}{v^2}\cdot v'\right) = \\ & = \frac{u'v - uv'}{v^2} \end{aligned}}$$

$$(\vartheta+1)\sum^{n} x_i = n\vartheta$$

$$\vartheta \sum x_i + \sum x_i = n\vartheta$$

$$\hat{\vartheta} = \frac{\sum x_i}{n - \sum x_i} = \frac{\sum x_i}{n(1-\bar{x})} = \frac{\bar{x}}{1-\bar{x}}$$

c) $E(X^k) = m_k$

$$m_k = \frac{1}{n} \sum^n x_i^k$$

$$m_1 = \frac{1}{n} \sum^n x_i^1 = \overline{X}$$

$$\frac{\theta}{\theta+1} = \overline{X}$$

$$\theta = \overline{X}\theta + \overline{X}$$

$$\hat{\theta}_{MM} = \frac{\overline{X}}{1-\overline{X}}$$

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$$L(\lambda) = \frac{(e^{-\lambda})^4 \lambda^{x_1} \cdot \lambda^{x_2} \cdot \lambda^{x_3} \cdot \lambda^{x_4}}{x_1! \cdot x_2! \; x_3! \; x_4!}$$

$$\ell(\lambda) = \ln(L(\lambda)) = -4\lambda + \ln\lambda \left(x_1 + x_2 + x_3 + x_4\right)$$

$$+ \ln \frac{1}{x_1! \; x_2! \; x_3! \; x_4!}$$

$$\frac{d\ell(\lambda)}{d\lambda} = -4 + \frac{1}{\lambda}\left(x_1 + x_2 + x_3 + x_4\right) = 0$$

$$\lambda = \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{3 + 4 + 3 + 7}{4} =$$

$$= \frac{17}{4} = 4{,}25$$

$\boxed{4}$

$\frac{1}{28}$

$$f_X(x) = \lambda e^{-\lambda x}$$

$$L(\lambda) = \frac{\lambda^3}{1{,}25 \cdot 0{,}77} e^{-\lambda^3 x^3}$$

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$$Q(\theta) = \sum_{i=1}^{5} \frac{1}{\sigma_i^2}(x_i - \theta)^2 = \sum_{i=1}^{5} \left(x_i^2 - 2\theta \sum^5 x_i + 5\theta^2\right)\frac{1}{\sigma_i^2}$$

$$\frac{dQ(\theta)}{d\theta} = -2\sum^5 \frac{x_i}{\sigma_i^2} + 2\theta \sum^5 \frac{1}{\sigma_i^2} = 0$$

$$\hat{\theta}_{LS} = \frac{\sum^5 x_i/\sigma_i^2}{\sum \frac{1}{\sigma_i^2}} = \frac{2109\,000}{26111{,}1} = 80{,}77$$

6 $X \sim B(m, p)$

$$EX = mp$$

$$EX^2 = VarX - (EX)^2 = mp(1-p) + (mp)^2$$

$$\begin{cases} mp = \frac{1}{n}\sum^n x_i \\ mp(1-p) + (mp)^2 = \frac{1}{n}\sum^n x_i^2 \end{cases}$$

$$\hat{m} = \frac{\bar{x}}{p}$$

$$\bar{x} - \bar{x}p + \bar{x}^2 = \frac{1}{n}\sum x_i^2 \qquad \hat{p} = \frac{\bar{x} + \bar{x}^2 - \frac{1}{n}\sum x_i^2}{\bar{x}} = 1 - \frac{\sigma^2}{\bar{x}}$$