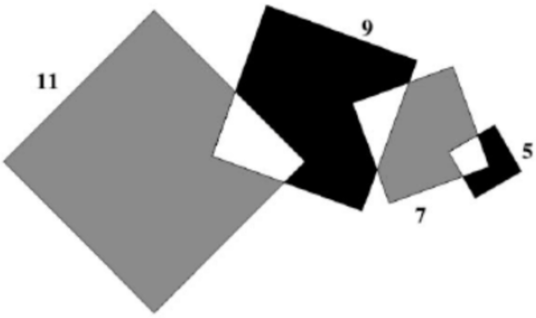
TAGES ANZEIGER MATH PROBLEMS

The Zürich daily *Tages Anzeiger* has a math column edited by Dmitrij Nikolenkov of ETH. Here are some problems (translated by me from German to English) and my solutions.

2024-05-02: Overlapping squares

Three squares with the given side lenghts overlap as follows:



The gray surface is twice as large as the black surface. What is the area of the white surface?

solution

The area of gray + white is $11^2 + 7^2 = 170$. The area of black + white is $9^2 + 5^2 = 106$.

Let w be the white area. Then $170 - w = 2(106 - w)$ and it follows that $w = 42$.

2024-05-02: Interesting boxes

The width and height of a box (a rectangular parallelepiped) are two consecutive positive integers. Its depth is equal to the product of those two integers. Prove that the diagonal of such a box is always an integer.

SOLUTION

Denote the bottom face of the box by the vertices $ABCD$ and let the corresponding vertices of the top face be $WXYZ$. The front face is given by $ABXW$.

Then, given the problem description, $|AB| = x$, $|BX| = x + 1$, $|BC| = x(x + 1)$.

Define a diagonal AC across the bottom face, and let $d_1 = |AC|$. By the Pythagorean theorem,

$$d_1^2 = x^2 + x^2(x + 1)^2 = x^4 + 2x^3 + 2x^2.$$

A diagonal of the parallelepiped is AY. Let $d_2 = |AY|$. By the Pythagorean theorem,

$$\begin{aligned} d_2^2 &= d_1^2 + (x + 1)^2 \\ &= x^4 + 2x^3 + 3x^2 + 2x + 1 \\ &= (x^2 + x + 1)^2 \end{aligned}$$

Since x is an integer, $d_2 = (x^2 + x + 1)$ is an integer also.

2024-05-16: Numbers game

Alf and Bettina play the following game. They take it in turns to write *different* digits from left to right until they have a 9-digit number. Alf starts (and finishes). If the resulting number is divisible by 4, Alf wins, otherwise Bettina wins. Do Alf or Bettina have a winning strategy? If so, what does it look like?

SOLUTION

A number of two or more digits is divisible by 4 iff its last two digits are divisible by 4. (A number whose last two digits are 00 is divisible by 4 because 100 is divisible by 4.)

The two-digit numbers with non-repeating digits that are divisible by 4 are:

04 08 12 16 20 24 28 32 36 40 48 52 56 60 64 68 72 76 80 84 92 96

Bettina can have a winning strategy. She can use her first three turns to ensure that 2, 4, and 6 are used (if Alf has not already chosen them). As a result, only the following divisible-by-4 sequences can appear in the last two digits:

08 80

On her fourth turn, Bettina can pick any remaining odd number. (Since Alf has had four turns, there is guaranteed to be at least one.)

Now, even if 0 and 8 are still available, Alf cannot choose a final digit such that the last two digits are 08 or 80. Bettina wins.

2024-05-16: Fractions

Find three different irreducible fractions with numerators and denominators not equal to 1 whose sum is an integer and the sum of the reciprocals is also an integer. There are many solutions.

solution

2024-06-27: Digit sum times 2024

Let $Q(n)$ be the sum of digits of a natural number n. For all n, $Q(2n) \leq 2Q(n)$.

Find a number n such that $Q(n) = 2024 \cdot Q(3n)$.

solution

Restating the problem, we know that for all n:

$$\frac{Q(2n)}{Q(n)} \leq 2$$

and we want some n such that:

$$\frac{Q(3n)}{Q(n)} = \frac{1}{2024} \leq 2$$

Let's use the notation $x\{k\}y$ to denote a natural number in base 10 where the digit x is repeated k times and followed by the digit y.

Playing around a bit to build intuition, for all positive k, $2(9\{k\}9) = 19\{k\}8$ and

$$\frac{Q(19\{k\}8)}{Q(9\{k\}9)} = \frac{1 + 9k + 8}{9k + 9} = 1$$

Similarly, $2(1\{k\}) = 2\{k\}$ and

$$\frac{Q(2\{k\})}{Q(1\{k\})} = 2$$

Intuitively, we want to find some increasing sequence of numbers n_i that contain a repeating pattern of digits such that $Q(n_i)$ grows much more quickly than $Q(3n_i)$ as the pattern of digits increases in length. Can we find a $3n$ that consists mostly of 0s while n does not?

The simplest pattern for which this is true is numbers of the form $n = 3\{k\}y$ with $y \in \{4, 5, 6\}$. Then $3n = 10\{k\}z$, where $z \in \{2, 5, 8\}$, respectively.

We may be onto something! Let's build a table:

n	$3\{k\}4$	$3\{k\}5$	$3\{k\}6$
$3n$	$10\{k\}2$	$10\{k\}5$	$10\{k\}8$
$Q(n)$	$3k+4$	$3k+5$	$3k+6$
$Q(3n)$	3	6	9

$\frac{3k+4}{3} = 2024$ has no integer solution (for any m, $3m - 4 \mod 3 = 2$).

$\frac{3k+5}{6} = 2024$ has no integer solution (for any m, $6m - 5 \mod 3 = 1$).

But $\frac{3k+6}{9} = 2024$ does have an integer solution ($9m - 6 \mod 3 = 0$). $k = 6070$, so $n = 3\{6070\}6$ (and $3n = 10\{6070\}8$).