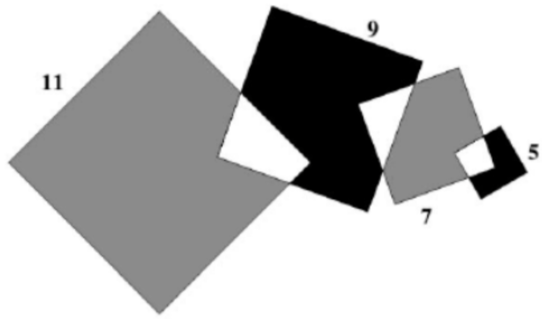
TAGES ANZEIGER MATH PROBLEMS

The Zürich daily *Tages Anzeiger* has a weekly math column. Here are some problems (translated from German to English) and my solutions.

1. 2024-05-02: Overlapping squares

Three squares with the given side lenghts overlap as follows:



The gray surface is twice as large as the black surface. What is the area of the white surface?

2. 2024-06-27: Digit sum times 2024

Let $Q(n)$ be the sum of digits of a natural number n. For all n, $Q(2n) \leq 2Q(n)$.

Find a number n such that $Q(n) = 2024 \cdot Q(3n)$.

Restating the problem, we know that for all n:

$$\frac{Q(2n)}{Q(n)} \leq 2$$

and we want some n such that:

$$\frac{Q(3n)}{Q(n)} = \frac{1}{2024} \leq 2$$

Let's use the notation $x\{k\}y$ to denote a natural number in base 10 where the digit x is repeated k times and followed by the digit y.

Playing around a bit to build intuition, for all positive k, $2(9\{k\}9) = 19\{k\}8$ and

$$\frac{Q(19\{k\}8)}{Q(9\{k\}9)} = \frac{1 + 9k + 8}{9k + 9} = 1$$

Similarly, $2(1\{k\}) = 2\{k\}$ and

$$\frac{Q(2\{k\})}{Q(1\{k\})} = 2$$

Intuitively, we want to find some increasing sequence of numbers n_i that contain a repeating pattern of digits such that $Q(n_i)$ grows much more quickly than $Q(3n_i)$ as the pattern of digits increases in length. Can we find a $3n$ that consists mostly of 0s while n does not?

The simplest pattern for which this is true is numbers of the form $n = 3\{k\}y$ with $y \in \{4, 5, 6\}$. Then $3n = 10\{k\}z$, where $z \in \{2, 5, 8\}$, respectively.

We may be onto something! Let's build a table:

n	$3\{k\}4$	$3\{k\}5$	$3\{k\}6$
$3n$	$10\{k\}2$	$10\{k\}5$	$10\{k\}8$
$Q(n)$	$3k+4$	$3k+5$	$3k+6$
$Q(3n)$	3	6	9

$\frac{3k+4}{3} = 2024$ has no integer solution (for any m, $3m - 4 \mod 3 = 2$).

$\frac{3k+5}{6} = 2024$ has no integer solution (for any m, $6m - 5 \mod 3 = 1$).

But $\frac{3k+6}{9} = 2024$ does have an integer solution ($9m - 6 \mod 3 = 0$). $k = 6070$, so $n = 3\{6070\}6$ (and $3n = 10\{6070\}8$).