Max Proft U5190335 Assignment 3 - Magnetism

Let $\vec{x}$ represent a point on a unit sphere

a) The grand canonical partition function is given by

$$Z = \frac{1}{(h^3)^N N!}\int d^3p_1 \ldots d^3p_N \, d^2x_1 \ldots d^2x_N \, e^{-\hat{H}\beta + \mu N(\vec{p}_1 \ldots \vec{x}_n)\beta}$$

$$= \frac{1}{N!\,h^{3N}}\int d^3p_1 \ldots d^3p_N \, d^2x_1 \ldots d^2x_N \, e^{\left[-\sum_{i=1}^{N}\left(\frac{p_i^2}{2m}\beta - M_0 \vec{H}\cdot\hat{n}_i\beta\right) + \mu\beta N(\vec{p}_i \ldots \vec{x}_n)\right]}$$

Since each i represents one particle we can move $N(\vec{p}_i, \ldots \vec{x}_n)$ inside the summation.

$$= \frac{1}{N!\,h^{3N}}\int d^3p_1 \ldots d^3p_N \, d^2x_1 \ldots d^2x_N \, e^{\sum_{i=1}^{N}\left(\mu\beta + M_0\beta\vec{H}\cdot\hat{n}_i - \frac{p_i^2}{2m}\beta\right)}$$

$$= \frac{1}{N!\,h^{3N}}\left(\int d^3p \, d^2x \, e^{\mu\beta + M_0\beta\vec{H}\cdot\hat{n} - \frac{p^2}{2m}\beta}\right)^N$$

Pulling out the factor of $e^{\mu\beta}$ and performing the integral with respect to d^3p we get:

$$= \frac{1}{N!}\left(\sqrt{\frac{2\pi m}{\beta h^2}}\right)^{3N} e^{\mu\beta N}\left(\int d^2x \, e^{M_0\beta\vec{H}\cdot\hat{n}}\right)$$

Orienting our axes so that $\vec{H}$ lies in the $\hat{z}$ direction, and using spherical coordinates (Unit sphere $\Rightarrow$ radius $= 1$)

$$= \frac{1}{N!}\left(\sqrt{\frac{2\pi m}{\beta h^2}}\right)^{3N} e^{\mu\beta N}\left(\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, e^{M_0\beta H\cos\theta} \sin\theta\right)^N$$

$$= \frac{(2\pi)^N}{N!} \left(\sqrt{\frac{2\pi m}{\beta h^2}} \right)^{3N} e^{M\beta N} \left(\frac{e^{H\beta\mu_0}}{H\beta\mu_0} - \frac{e^{H\beta\mu_0}}{H\beta\mu_0} \right)^N$$

$$= \frac{(4\pi)^N}{N!} \left(\sqrt{\frac{2\pi m}{\beta h^2}} \right)^{3N} e^{M\beta N} \left(\frac{\sinh(H\beta\mu_0)}{H\beta\mu_0} \right)^N$$

Using Mathematica for the rest.

b)

We have found the Grand Partition Function to be:

In[15]:=
$$Z[\beta\_] = 4\ \frac{\pi}{n!}\left(\frac{2\,\pi\,m}{\beta\,h^2}\right)^{3\,n/2}\ \mathrm{Exp}[\mu\,\beta\,n]\ \left(\frac{\mathrm{Sinh}[H\,\beta\,\mu 0]}{H\,\beta\,\mu 0}\right)^{n};$$

The Grand Potential is given by:

$\Omega = \frac{-1}{\beta}\,\mathrm{Ln}(Z)$

Expanding out all of the terms we get:

In[27]:=
$$\Omega[\beta\_] = \mathrm{Expand}\left[\mathrm{PowerExpand}\left[\frac{-1}{\beta}\ \mathrm{Log}[Z[\beta]]\right]\right]$$

Out[27]=
$$-n\,\mu - \frac{2\,\mathrm{Log}[2]}{\beta} - \frac{3\,n\,\mathrm{Log}[2]}{2\,\beta} + \frac{3\,n\,\mathrm{Log}[h]}{\beta} + \frac{n\,\mathrm{Log}[H]}{\beta} - \frac{3\,n\,\mathrm{Log}[m]}{2\,\beta} - \frac{\mathrm{Log}[\pi]}{\beta} - \frac{3\,n\,\mathrm{Log}[\pi]}{2\,\beta} + \frac{5\,n\,\mathrm{Log}[\beta]}{2\,\beta} + \frac{n\,\mathrm{Log}[\mu 0]}{\beta} + \frac{\mathrm{Log}[n!]}{\beta} - \frac{n\,\mathrm{Log}[\mathrm{Sinh}[H\,\beta\,\mu 0]]}{\beta}$$

c)

Finding the magnetism:

$M = -\frac{\partial\Omega}{\partial H} = -\frac{\partial}{\partial H}\left(\frac{n\,\mathrm{Log}[H]}{\beta} - \frac{n\,\mathrm{Log}[\mathrm{Sinh}[H\,\beta\,\mu 0]]}{\beta}\right)$

$M = -\frac{n}{H\,\beta} + n\,\mu 0\ \mathrm{Coth}[H\,\beta\,\mu 0]$