Max Proft u5190335 Assignment 5

Q6.17

Under a duality transformation, we get:

E=**E'** Cosξ +Z_0 $\boldsymbol{H'}$ Sinξ

$\boldsymbol{B} = -Z_0\, \boldsymbol{D}' \sin\xi + \boldsymbol{B}' \operatorname{Cos}\xi$

$Z_0\, \rho_e = Z_0\, \rho'_e \cos\xi + \rho'_m \sin\xi$

$\rho_m = -Z_0\, \rho'_e \sin\xi + \rho'_m \cos\xi$

I choose this transformation (from **E'→E, B'→B, etc.**) so that we get the standard Lorentzian force from these equations.

$\Rightarrow \rho_m = 0$

Taking the charge distributions to be single points (i.e. replace all ρ with charge for a single point, q).

Simplifying the equations we get:

$0 = -Z_0\, q'_e \sin\xi + q'_m \cos\xi \Rightarrow Z_0 = \frac{q'_m}{q_e' \operatorname{Tan}\xi}$

$Z_0\, q_e = Z_0\, q'_e \cos\xi + q'_m \sin\xi$

$\Rightarrow q_e = q_e' \cos\xi + \frac{q'_m \sin\xi}{Z_0} = q_e' \cos\xi + q'_e \frac{\sin^2\xi}{\cos\xi} = \frac{q_e'}{\cos\xi}$

E=**E'** Cosξ +Z_0 $\boldsymbol{H'}$ Sinξ $= \boldsymbol{E'} \operatorname{Cos}\xi + \boldsymbol{H'} \frac{q'_{m\cos\xi}}{q_e'}$

$\boldsymbol{B} = -Z_0\, \boldsymbol{D}' \sin\xi + \boldsymbol{B}' \operatorname{Cos}\xi = -\boldsymbol{D'} \frac{q'_{m\cos\xi}}{q_e'} + \boldsymbol{B}' \operatorname{Cos}\xi$

Note: since we are just performing a change of variables, the velocity of the particle does not change.

So if we substitute this into the Lorentz force we get:

F=**E** q_e +q_e(**v**×**B**)

$\boldsymbol{F} = \left(\boldsymbol{E'} \operatorname{Cos}\xi + \boldsymbol{H'} \frac{q'_{m\cos\xi}}{q_e'}\right)\left(\frac{q_e'}{\cos\xi}\right) + \left(\frac{q_e'}{\cos\xi}\right)\left(\boldsymbol{v} \times \left(-\boldsymbol{D'} \frac{q'_{m\cos\xi}}{q_e'} + \boldsymbol{B}' \operatorname{Cos}\xi\right)\right)$

$\boldsymbol{F} = \boldsymbol{E'}\, q'_e + \boldsymbol{H'}\, q'_m - q'_m\, \boldsymbol{v} \times \boldsymbol{D'} + q_e'\, \boldsymbol{v} \times \boldsymbol{B'}$

Since $\boldsymbol{H'} = \frac{\boldsymbol{B'}}{\mu}$, and $\boldsymbol{D'} = \epsilon_0\, \boldsymbol{E'}$

$\boldsymbol{F} = \boldsymbol{E'}\, q'_e + \boldsymbol{B'} \frac{q'_m}{\mu} - q'_m\, \epsilon_0\, \boldsymbol{v} \times \boldsymbol{E'} + q'_e\, \boldsymbol{v} \times \boldsymbol{B'}$

Q 7.1

In the linear polarisation basis:

Let $E_1 = a_1\, e^{i\delta_1}$ and $E_2 = a_2\, e^{i\delta_2}$

$s_0 + s_1 = 2\, a_1{}^2 \Rightarrow a_1 = \sqrt{\frac{s_0+s_1}{2}}$

$s_0 - s_1 = 2\, a_2{}^2 \Rightarrow a_2 = \sqrt{\frac{s_0-s_1}{2}}$

$\cos(\delta_2 - \delta_1) = \frac{s_2}{2\, a_1\, a_2}$

$\sin(\delta_2 - \delta_1) = \frac{s_3}{2\, a_1\, a_2}$

(Note: we need both of the last 2 equations so that we can determine which quadrent $\delta_2 - \delta_1$ lies in)

In the circular polarisation basis:

Let $E_+ = a_+\, e^{i\delta_+}$ and $E_- = a_-\, e^{i\delta_-}$

$s_0 + s_3 = 2\, a_+{}^2 \Rightarrow a_+ = \sqrt{\frac{s_0+s_3}{2}}$

$s_0 - s_3 = 2\, a_-{}^2 \Rightarrow a_- = \sqrt{\frac{s_0-s_3}{2}}$

$\cos(\delta_- - \delta_+) = \frac{s_1}{2\, a_+\, a_-}$
$\sin(\delta_- - \delta_+) = \frac{s_2}{2\, a_+\, a_-}$
(Note: we need both of the last 2 equations so that we can determine which quadrent $\delta_- - \delta_+$ lies in)

a)

$s_0 = 3,\;\; s_1 = -1,\;\; s_2 = 2,\; s_3 = -2$
$a_1 = 1$
$a_2 = \sqrt{2}$
$\cos(\delta_2 - \delta_1) = \frac{1}{\sqrt{2}}$
$\sin(\delta_2 - \delta_1) = \frac{-1}{\sqrt{2}}$
$\Rightarrow \delta_2 - \delta_1 = -\frac{\pi}{3}$

$a_+ = \frac{1}{\sqrt{2}}$
$a_- = \sqrt{\frac{5}{2}}$
$\cos(\delta_- - \delta_+) = \frac{-1}{\sqrt{5}}$
$\sin(\delta_- - \delta_+) = \frac{2}{\sqrt{5}}$
$\Rightarrow \delta_- - \delta_+ = \pi - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

b)

$s_0 = 25,\;\; s_1 = 0,\;\; s_2 = 24,\; s_3 = 7$
$a_0 = \frac{5}{\sqrt{2}}$
$a_1 = \frac{5}{\sqrt{2}}$
$\cos(\delta_2 - \delta_1) = \frac{24}{25}$
$\sin(\delta_2 - \delta_1) = \frac{7}{25}$
$\Rightarrow \delta_2 - \delta_1 = \sin^{-1}\left(\frac{7}{25}\right)$

$a_+ = 4$
$a_- = 3$
$\cos(\delta_- - \delta_+) = 0$
$\sin(\delta_- - \delta_+) = 1$
$\Rightarrow \delta_- - \delta_+ = \frac{\pi}{2}$

Pictures attached at the end.

Q7.19

Note: all integrals are over (-∞,∞)

$u(x, 0) = f(x)\, e^{i k_0 x} = \int A(k)\, e^{i(kx-\omega(k)\,t)}\, dk$

$\Rightarrow A(k) = \frac{1}{\sqrt{2\pi}} \int u(x, 0)\, e^{-ikx}\, dx$

$A(k) = \frac{1}{\sqrt{2\pi}} \int f(x)\, e^{i(k_0-k)\,x}\, dx$

a)

$f(x)=N\, e^{\frac{-\alpha|x|}{2}}$

$A(k) = \frac{N}{\sqrt{2\pi}} \int e^{\frac{-\alpha|x|}{2} - i(k_0-k)\,x}\, dx$

In[22]:=
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Integrate[N/Sqrt[2 π] Exp[(-α Abs[x])/2 - I (k0 - k) x],
 {x, -∞, ∞}, Assumptions → {α > 0, k ∈ Reals, k0 ∈ Reals}]
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Out[22]= $\frac{2\, N \sqrt{\frac{2}{\pi}}\, \alpha}{(2\,k - 2\,k0 - i\,\alpha)\,(2\,k - 2\,k0 + i\,\alpha)}$

If we simplify this then we get:

$A(k) = \frac{2\sqrt{\frac{2}{\pi}}\, N\alpha}{4\,(k-k0)^2+\alpha^2}$

And so we get:

$|\, A(k)\, |^2 = \frac{8\, N^2\, \alpha^2}{\pi\, \left(4\, (k-k0)^2+\alpha^2\right)^2}$

$|\, u(x, 0)\, |^2 = |\, f(x)\, |^2 = N^2\, e^{-\alpha|x|}$

$\langle k \rangle = \frac{\int k\, |A(k)|^2 dk}{\int |A(k)|^2 dk} =$

In[75]:=
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Integrate[(8 N^2 α^2)/(π (4 (k-k0)^2+α^2)^2) k, {k, -∞, ∞}, Assumptions → {α > 0, L > 0, k0 > 0}]
/
Integrate[(8 N^2 α^2)/(π (4 (k-k0)^2+α^2)^2), {k, -∞, ∞}, Assumptions → {α > 0, L > 0, k0 > 0}]
```

Out[75]= k0

$\langle k^2 \rangle = \frac{\int k^2\, |A(k)|^2 dk}{\int |A(k)|^2 dk}$

In[77]:=
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Integrate[(8 N^2 α^2)/(π (4 (k-k0)^2+α^2)^2) k^2, {k, -∞, ∞}, Assumptions → {α > 0, L > 0, k0 > 0}]
/
Integrate[(8 N^2 α^2)/(π (4 (k-k0)^2+α^2)^2), {k, -∞, ∞}, Assumptions → {α > 0, L > 0, k0 > 0}]
```

Out[77]= $\frac{1}{4}\left(4\, k0^2 + \alpha^2\right)$

And we get:

$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2 = \frac{\alpha^2}{4}$

$<x>= \frac{\int x\,|u(x)|^2 dx}{\int |u(x)|^2 dx} =$

In[1]:=
$$\frac{\text{Integrate}\left[x\, N^2\, \text{Exp}\left[-\alpha\, \text{Abs}[x]\right],\ \{x, -\infty, \infty\},\ \text{Assumptions} \to \{\alpha > 0\}\right]}{\text{Integrate}\left[N^2\, \text{Exp}\left[-\alpha\, \text{Abs}[x]\right],\ \{x, -\infty, \infty\},\ \text{Assumptions} \to \{\alpha > 0\}\right]}$$

Out[1]= 0

$< x^2 >= \frac{\int x^2\,|u(x)|^2 dx}{\int |u(x)|^2 dx} =$

In[4]:=
$$\frac{\text{Integrate}\left[x^2\, N^2\, \text{Exp}\left[-\alpha\, \text{Abs}[x]\right],\ \{x, -\infty, \infty\},\ \text{Assumptions} \to \{\alpha > 0\}\right]}{\text{Integrate}\left[N^2\, \text{Exp}\left[-\alpha\, \text{Abs}[x]\right],\ \{x, -\infty, \infty\},\ \text{Assumptions} \to \{\alpha > 0\}\right]}$$

Out[4]= $\frac{2}{\alpha^2}$

And we get

$\sigma_x{}^2 = < x^2 > - < x >^2 = \frac{2}{\alpha^2}$

And hence $\sigma_x{}^2\, \sigma_\Delta{}^2 = \frac{1}{2} \Rightarrow \sigma_x\, \sigma_\Delta = \frac{1}{\sqrt{2}} > 0.5$

b)

$f(x)=N\, e^{\frac{-\alpha^2 x^2}{4}}$

$A(k) = \frac{N}{\sqrt{2\pi}} \int e^{\frac{-\alpha^2 x^2}{4} - i(k_0 - k)\, x}\, dx$

In[23]:=
$$\text{Integrate}\left[\frac{N}{\sqrt{2\,\pi}}\, \text{Exp}\left[\frac{-\alpha^2\, x^2}{4} - I\, (k_0 - k)\, x\right],\ \{x, -\infty, \infty\},\ \text{Assumptions} \to \{\alpha > 0\}\right]$$

Out[23]=
$$\frac{\sqrt{2}\, e^{-\frac{(k-k_0)^2}{\alpha^2}}\, N}{\alpha}$$

And so we get:

$|\, A(k)\, |^2 = \frac{2\, N^2}{\alpha^2}\, e^{-\frac{2\,(k-k_0)^2}{\alpha^2}}$

$|\, u(x, 0)\, |^2 = |\, f(x)\, |^2 = N^2\, e^{\frac{-\alpha^2 x^2}{2}}$

$<k>= \frac{\int k\,|A(k)|^2 dk}{\int |A(k)|^2 dk} =$

In[4]:=
$$\frac{\text{Integrate}\left[\frac{2\, N^2}{\alpha^2}\, \text{Exp}\left[-\frac{2\,(k - k0)^2}{\alpha^2}\right] k,\ \{k, -\infty, \infty\},\ \text{Assumptions} \to \{\alpha > 0,\ k0 > 0\}\right]}{\text{Integrate}\left[\frac{2\, N^2}{\alpha^2}\, \text{Exp}\left[-\frac{2\,(k - k0)^2}{\alpha^2}\right],\ \{k, -\infty, \infty\},\ \text{Assumptions} \to \{\alpha > 0,\ k0 > 0\}\right]}$$

Out[4]= k0

$< k^2 >= \frac{\int k^2\,|A(k)|^2 dk}{\int |A(k)|^2 dk} =$

In[5]:= $$\frac{\text{Integrate}\left[\frac{2 N^2}{\alpha^2}\,\text{Exp}\left[-\frac{2 (k-\text{k0})^2}{\alpha^2}\right] k^2\ \ , \{\text{k}, -\infty, \infty\}, \text{Assumptions} \to \{\alpha > 0, \text{k0} > 0\}\right]}{\text{Integrate}\left[\frac{2 N^2}{\alpha^2}\,\text{Exp}\left[-\frac{2 (k-\text{k0})^2}{\alpha^2}\right]\ \ , \{\text{k}, -\infty, \infty\}, \text{Assumptions} \to \{\alpha > 0, \text{k0} > 0\}\right]}$$

Out[5]= $$\frac{1}{4}\left(4\,\text{k0}^2 + \alpha^2\right)$$

$\sigma_k{}^2 = <k^2> - <k>^2 = \frac{\alpha^2}{4}$

$<\text{X}>= \frac{\int x\,|u(x)|^2 \text{dx}}{\int |u(x)|^2 \text{dx}} =$

In[11]:= $$\frac{\text{Integrate}\left[\text{N}^2\,\text{Exp}\left[\frac{-\alpha^2 \text{x}^2}{2}\right] \text{x}, \{\text{x}, -\infty, \infty\}, \text{Assumptions} \to \{\alpha > 0\}\right]}{\text{Integrate}\left[\text{N}^2\,\text{Exp}\left[\frac{-\alpha^2 \text{x}^2}{2}\right], \{\text{x}, -\infty, \infty\}, \text{Assumptions} \to \{\alpha > 0\}\right]}$$

Out[11]= 0

$<x^2>= \frac{\int x^2\,|u(x)|^2 \text{dx}}{\int |u(x)|^2 \text{dx}} =$

In[12]:= $$\frac{\text{Integrate}\left[\text{N}^2\,\text{Exp}\left[\frac{-\alpha^2 \text{x}^2}{2}\right] \text{x}^2, \{\text{x}, -\infty, \infty\}, \text{Assumptions} \to \{\alpha > 0\}\right]}{\text{Integrate}\left[\text{N}^2\,\text{Exp}\left[\frac{-\alpha^2 \text{x}^2}{2}\right], \{\text{x}, -\infty, \infty\}, \text{Assumptions} \to \{\alpha > 0\}\right]}$$

Out[12]= $$\frac{1}{\alpha^2}$$

And we get
$\sigma_x{}^2 = <x^2> - <x>^2 = \frac{1}{\alpha^2}$
And hence $\sigma_x{}^2\,\sigma_\Delta{}^2 = \frac{1}{4} \Rightarrow\ \sigma_x\,\sigma_\Delta = \frac{1}{2}$

Pictures attached at the end.