Assignment 4

Tank Question:

Consider a cylindrical tank with a hollow core - a toroid. The inner diameter is 2a and the outer diameter is 2b. Many current carrying coils are wrapped around the tank (through the core) so as to generate a toroidal field B_\phi.

(a) What equation governs B in the tank?

(b) What is the functional dependence of B_\phi in steady state?

Jackson Q5.3, 6.1

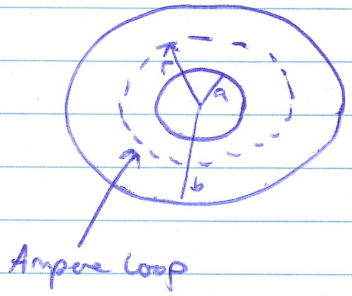
2. a) Diffusion equation governs $\underset{\sim}{B}$ inside tank, ie.

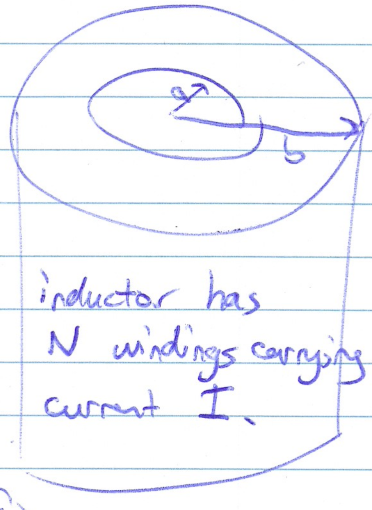
$$\nabla^2 \underset{\sim}{B} = \mu\sigma \frac{\partial \underset{\sim}{B}}{\partial t}$$

This follows from Jackson pg 218-219

2

b) In steady state apply Ampere's law: 1





$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 I_{enc}$$

$d\underset{\sim}{l} = r\,d\phi\,\hat{\phi}$ $\qquad I_{enc} = NI$ 1

by symmetry $\underset{\sim}{B}$ is independent of ϕ 1

~~and~~

$\Rightarrow \underset{\sim}{B} \cdot \hat{\phi} \times 2\pi r = \mu_0 NI$ $\qquad$ provided $a < r < b$

$$B_\phi = \frac{\mu_0 NI}{2\pi r} \qquad a < r < b$$ 1

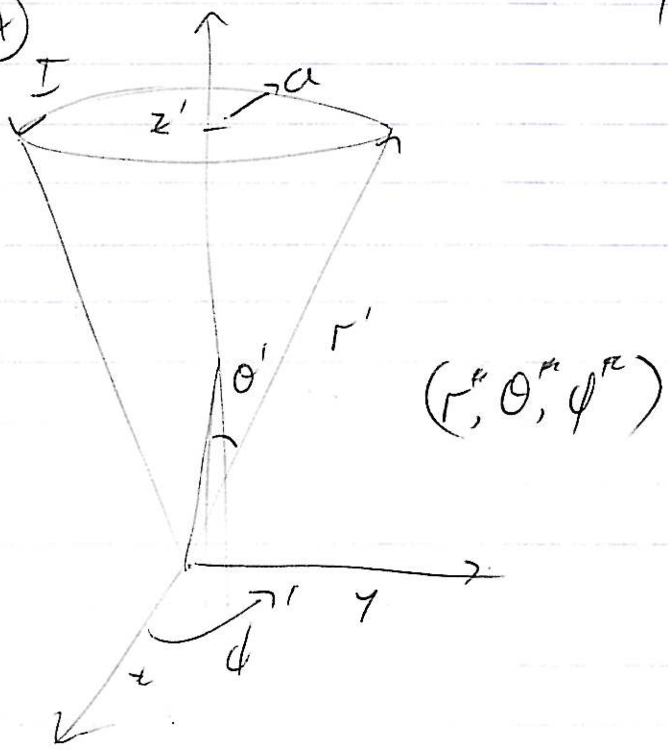
For r outside this range, $I_{enc} = 0 \Rightarrow B_\phi = 0$ 4

6

total

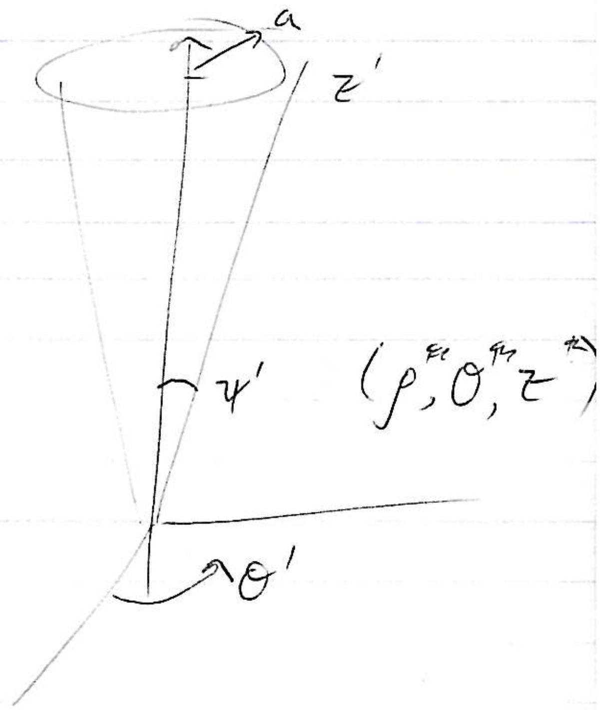
5.3 $\quad d\underset{\sim}{B} = \frac{\mu_0 I}{4\pi}\, d\underset{\sim}{l} \times \frac{\underset{\sim}{x} - \underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}$

A



or

B



$\underset{\sim}{x}' = r'\,\hat{r}(\theta')$ $\qquad\qquad$ $\underset{\sim}{x}' = a\hat{\rho} + z'\hat{z}$

$d\underset{\sim}{l} = a\, d\phi'\, \hat{\phi}$ $\qquad\qquad$ $d\underset{\sim}{l} = a\, d\theta'\, \hat{\theta}$. $\quad$ 1

Using B

$$\underset{\sim}{x} - \underset{\sim}{x}' = -a\hat{\rho} - z'\hat{z}.$$

$$d\underset{\sim}{l} \times (\underset{\sim}{x} - \underset{\sim}{x}') = \begin{vmatrix} \rho & \theta & z \\ 0 & a\,d\theta' & 0 \\ a & 0 & z' \end{vmatrix}$$

$$= \hat{\rho}\, a\, d\theta'\, z' + \hat{z}(-a^2 d\theta') \quad 1$$

$$\& \quad |\underset{\sim}{x} - \underset{\sim}{x}'| = (a^2 + z'^2)^{\frac{1}{2}} \quad 1$$

$$d\underset{\sim}{B} = \frac{\mu_0 I}{4\pi} \cdot \frac{z'\hat{\rho}(\theta')a\, d\theta' - a^2 d\theta'\hat{z}}{(a^2 + z'^2)^{\frac{3}{2}}}. \quad 1$$

$$\cancel{\underset{\sim}{B} = \frac{\mu_0 I}{4\pi} \int \left(\frac{\hat{\rho}\, a z' - a^2 \hat{z}}{(a^2 + z'^2)^{3/2}} \right) d\theta' = \frac{\mu_0 I}{2} \left(\frac{\hat{\rho}\, a z'}{(a^2 + z'^2)^{3/2}} - \frac{a^2 \hat{z}}{(a^2 + z'^2)^{3}} \right)}$$

$$\therefore \underset{\sim}{B} = \frac{\mu_0 I}{4\pi} \int \frac{a z'}{(a^2+z'^2)^{\frac{3}{2}}} \hat{\rho}(\theta) d\theta' - \frac{\mu_0 I}{2} \oint \frac{a^2}{(a^2+z'^2)^{\frac{3}{2}}} \hat{z} \cancel{d\theta'}$$ (1)

The radial component will sum to zero. Hence

$$\underset{\sim}{B} = -\frac{\mu_0 I}{2} \oint \frac{a^2}{(a^2+z'^2)^{\frac{3}{2}}} \hat{z}$$ (1).

The solenoid has finite length, and N turns per unit length. Hence

$$dB_z = -\frac{\mu_0 I N dz' a^2}{2(a^2+z'^2)^{\frac{3}{2}}}$$ (1)

$$B_z = -\frac{\mu_0 I N a^2}{2} \int_{z_1}^{z_2} \frac{dz'}{(a^2+z'^2)^{\frac{3}{2}}}$$

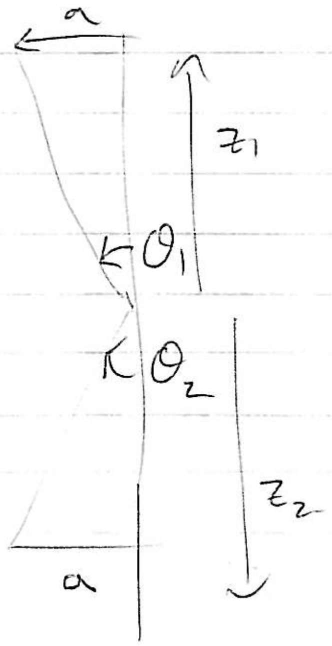
$$= -\frac{\mu_0 I N \cancel{a^2}}{2} \left[\frac{z'}{(a^2+z'^2)^{\frac{1}{2}}} \right]_{z_1}^{z_2}$$

$$\left[\frac{d}{dz'} \left(\frac{z'}{(a^2+z'^2)^{\frac{1}{2}}} \right) = \frac{(a^2+z'^2)^{\frac{1}{2}} - \frac{2z'^2}{2}(a^2+z'^2)^{-\frac{1}{2}}}{(a^2+z'^2)^{\frac{2}{2}}} \right.$$

$$= \frac{(a^2+z'^2)^{-\frac{1}{2}} (a^2 + \cancel{z'^2} - \cancel{z'^2})}{(a^2+z'^2)}$$

$$\left. = \frac{a^2}{(a^2+z'^2)^{\frac{3}{2}}} \right]$$

$$\therefore B_z = -\frac{\mu_0 I N}{2} \left(\frac{z_2}{(a^2+z_2^2)^{\frac{1}{2}}} - \frac{z_1}{(a^2+z_1^2)^{\frac{1}{2}}} \right)$$ (1)



So $\frac{z_1}{(a^2+z_1^2)^{1/2}} = \cos\theta_1$.

& $z_2 < 0$ $\therefore$

$$\left| \frac{z_2}{(a^2+z_2^2)^{3/2}} \right| = \cos\theta_2$$

Thus

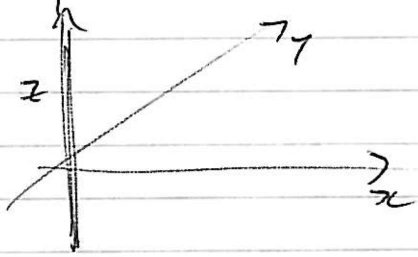
$$B_z = \frac{\mu_0 NI}{2}\left(\cos\theta_1 + \cos\theta_2 \right) \quad (1)$$

1. (a) ~~Soln~~. Show that $f(\underline{x}', t') = \delta(x')\delta(y')\delta(t')$ produces solution

$$\underline{u}(x, y, t) = \frac{z_c \, H(ct - \rho)}{\sqrt{c^2 t^2 - \rho^2}} \qquad H(\xi) = \begin{cases} 0; & \xi < 0 \\ 1; & \xi > 0 \end{cases}$$

with $\rho^2 = x^2 + y^2$

The problem is a line source flash on the z axis:



Plug in source into

$$\underline{u}(\underline{x}, t) = \int \frac{\left[f(\underline{x}', t')\right]_{ret}}{|\underline{x} - \underline{x}'|} d^3x'$$

$$= \iiint \frac{f(x')\delta(y')\,\delta(t - \frac{R}{c})}{R}\, dx'\, dy'\, dz' \qquad (1)$$

$$= \iiint \frac{f(x')\delta(y')\,\delta\!\left(t - \frac{((x-x')^2 + (y-y')^2 + (z-z')^2)^{\frac{1}{2}}}{c}\right)}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{\frac{1}{2}}}\, dx'\, dy'\, dz'$$

Use

$$= \int_{-\infty}^{\infty} \frac{\delta\!\left(t - \frac{(x^2 + y^2 + (z-z')^2)^{\frac{1}{2}}}{c}\right)}{(x^2 + y^2 + (z-z')^2)^{\frac{1}{2}}}\, dz' \qquad (1)$$

Use $\bar{z} = z' - z$; $d\bar{z} = dz'$, so

$$\underline{u}(\underline{x}, t) = \int_{-\infty}^{\infty} \frac{\delta\!\left(t - \frac{(\rho^2 + \bar{z}^2)^{\frac{1}{2}}}{c}\right)}{(\rho^2 + \bar{z}^2)^{\frac{1}{2}}}\, d\bar{z} \qquad (1)$$

$$= \int_{-\infty}^{\infty} \sum_{\bar{z}_i} \frac{c\sqrt{\rho^2 + \bar{z}_i^2}}{|\bar{z}_i|}\, \delta(\bar{z} - \bar{z}_i)\, \frac{d\bar{z}}{(\rho^2 + \bar{z}^2)^{\frac{1}{2}}}$$

where we have used

$$\delta(g(x)) = \sum_{\substack{a \\ g(a) = 0 \\ g'(a) \neq 0}} \frac{\delta(x - a)}{|g'(a)|}$$

with a zeroes of g. Here $\bar{z}_i$ are zeroes of $t - \frac{(\rho^2 + \bar{z}^2)^{\frac{1}{2}}}{c}$

That is

$$\bar{z}_i = \pm\sqrt{c^2 t^2 - \rho^2} \qquad (0.5)$$

The integral is over $\bar{z}$ real. Hence

$$u(\underline{x}_1, t) = \begin{cases} \dfrac{2c}{\sqrt{c^2t^2-\rho^2}} & \text{if } c^2t^2-\rho^2 > 0 \\ 0 & \text{if } c^2t^2-\rho^2 < 0. \end{cases}$$

0.5

This can be written:

$$u(\underline{x}_1, t) = 2c\, H(ct-\rho)$$

[NB: $c^2t^2-\rho^2 > 0 \Rightarrow c^2t^2 > \rho^2 \therefore \quad t > \rho/c$ or $t < -\rho/c$

But $\rho > 0$ and so the second cond is $t<0$, but we need a retarded greens' fn. soln., ~~so~~ at ~~$t<0$~~ ~~4~~ $t < -\rho/c$, $u(\underline{x}_1, t) = 0$]

Q6.1(b)	6.3(a)	Total
5	5	10

Assignment 8

6.1(b) This question asks for the solution of the wave equation

$$\nabla^2 \underline{u} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -4\pi f(\underset{\sim}{x}, t)$$

using a Green's function technique. It suggests use of the retarded Greens function, with $t' = t - |\underset{\sim}{x} - \underset{\sim}{x}'|/c$ &

$$\underline{u}(\underset{\sim}{x}, t) = \int \frac{[f(\underset{\sim}{x}', t')]_{ret}}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d^3x'$$

where $f(\underset{\sim}{x}', t')$ is the source function.

1

For a sheet source $f(\underset{\sim}{x}', t') = \delta(x')\delta(t')$, s.t. the sheet is in the y-z plane at $x' = 0$, & turns on & off at $t' = 0$. Thus

$$\underline{u}(\underset{\sim}{x}, t) = \int \frac{\delta(x')\,\delta(t - |\underset{\sim}{x} - \underset{\sim}{x}'|/c)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} dx'dy'dz'$$ 1

$$= \int \frac{\delta(x')\,\delta(t - \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}/c)}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \times dx'dy'dz'$$

$$= \int \frac{\delta(t - \sqrt{x^2 + (y-y')^2 + (z-z')^2}/c)}{\sqrt{x^2 + (y-y')^2 + (z-z')^2}} dy'dz'$$ 1

Shift ~~$y' \to \bar{y}$~~ $\bar{y} = y' - y$ (ie $y' \to y' + y$) & $\bar{z} = z' - z$ (ie $z' \to z' + z$). Then $d\bar{y} = dy'$; $d\bar{z} = dz'$ &

$$\underline{u}(\underset{\sim}{x}, t) = \int \frac{\delta(t - \sqrt{x^2 + \bar{y}^2 + \bar{z}^2}/c)}{\sqrt{x^2 + \bar{y}^2 + \bar{z}^2}} d\bar{y}\,d\bar{z}$$

Transform to polar coordinates in the $\bar{y}, \bar{z}$ plane, then $\bar{y}^2 + \bar{z}^2 = \bar{r}^2$ & $d\bar{y}\,d\bar{z} = \bar{r}\,d\bar{r}\,d\bar{\theta}$. Then -

$$\underline{u}(\underset{\sim}{x}, t) = \int \frac{\delta(t - \sqrt{x^2 + \bar{r}^2}/c)}{\sqrt{x^2 + \bar{r}^2}} \bar{r}\,d\bar{r}\,d\bar{\theta}$$

$$= 2\pi \int \frac{\delta\left(t - \sqrt{x^2 + r^2}/c\right)}{\sqrt{x^2 + r^2}}\, r\,dr \qquad (1)$$

Using $\delta(f(\xi)) = \sum_i \frac{\delta(\xi - \xi_i)}{|f'(\xi)|}$ where ξ_i are the zeroes of $f(\xi)$

then

$$u(\underset{\sim}{x}, t) = 2\pi \int \sum_{r_i} \left[\frac{\delta(r - r_i)}{\cancel{\sqrt{x^2 + r^2}}} \cdot \frac{\cancel{\sqrt{x^2 + r^2}}}{\cancel{r_i}/c}\, \cancel{r}\,dr \right]$$

$$= 2\pi \int_0^\infty \sum_{r_i} c\,\delta(r - r_i)\, dr$$

where the zeros r_i occur at $t - \sqrt{x^2 + r^2}/c = 0$

ie

$$r = \pm\sqrt{c^2 t^2 - x^2} \qquad (0{\cdot}5)$$

$r > 0$, so only one zero exists, $r_i = \sqrt{c^2 t^2 - x^2}$.
Also, for r_i to be real $c^2 t^2 - x^2 > 0$, and so (0·5)

$$u(\underset{\sim}{x}, t) = 2\pi c\, \Theta(ct - |x|)$$

where $\Theta(ct - |x|) = \begin{cases} 1 & ct - |x| > 0 \\ 0 & ct - |x| < 0 \end{cases}$.