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Assignment 2

3.1

$\nabla^2 \Phi = 0 \Rightarrow \frac{\nabla^2 \Phi}{\Phi} r^2 = 0$

Taking $\Phi(r, \theta) = \frac{R(r)}{r} T(\theta)$ (The system is symmetrical in the phi direction)

In spherical coordinates this gives us:

$$\frac{1}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{T(\theta)\sin(\theta)} \frac{d}{d\theta}\left(\sin(\theta) \frac{dT(\theta)}{d\theta}\right) = 0$$

Since both terms are independent of each other, we must get that:

$$\frac{r^2}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} = -\frac{1}{T(\theta)\sin(\theta)} \frac{d}{d\theta}\left(\sin(\theta) \frac{dT(\theta)}{d\theta}\right) = \text{Constant} = l(l+1)$$

Solving for R(r) we get:

$$\frac{\partial^2 R(r)}{\partial r^2} = l(l + 1) \frac{R(r)}{r^2}$$

Which has solutions:

$R(r) = \mathrm{A}r^{l+1} + \mathrm{B}r^{-l}$

Solving for T(θ) we get:

$$\frac{d}{d\theta}\left(\sin(\theta) \frac{dT(\theta)}{d\theta}\right) + l(l+1)\, T(\theta)\sin(\theta) = 0$$

Let x=cos(θ) $\Rightarrow$ dx = -sin(θ) dθ (Note: $0<\theta<\pi \Rightarrow -1<x<1$)

Substituting in for x we get:

$$-\sin(\theta) \frac{d}{dx}\left(-\sin^2(\theta)\, T'(x)\right) + l(l+1)\, T(x)\sin(\theta) = 0$$

Since $1 - x^2 = 1 - \cos^2(\theta) = \sin^2(\theta)$ we get:

$$\frac{d}{dx}\left((1 - x^2)\, T'(x)\right) + l(l+1)\, T(x) = 0$$

This has solutions $U_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{dx^l}\left(x^2 - 1\right)^l$, where l is a non-negative integer.

So we can write the general solution as:

$$\Phi(r, \theta) = \sum^{\infty}{}_{l=0} \left(A_l\, r^l + B_l\, r^{-l-1}\right) U_l = \sum^{\infty}{}_{l=0} \left(A_l\, r^l + B_l\, r^{-l-1}\right) \frac{1}{2^l\, l!} \frac{d^l}{dx^l}\left(x^2 - 1\right)^l$$

If we multiply both sides by $U_{l'}(x)$ and integrate over x (using the orthogonality of the solution):

$$A_l\, r^l + B_l\, r^{-l-1} = \frac{2\,l+1}{2} \int_{-1}^{1} V(r, x) \frac{1}{2^l\, l!} \frac{d^l}{dx^l}\left(x^2 - 1\right)^l dx$$

$\mathrm{V(a,x)}=\begin{cases} V & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

And hence:

$$A_l\, a^l + B_l\, a^{-l-1} = V \frac{2\,l+1}{2} \int_0^1 \frac{1}{2^l\, l!} \frac{d^l}{dx^l}\left(x^2 - 1\right)^l dx$$

$\mathrm{V(b,x)}=\begin{cases} 0 & 0 < x < 1 \\ V & \text{otherwise} \end{cases}$

And hence:

$$A_l\, b^l + B_l\, b^{-l-1} = \frac{2\,l+1}{2} V \int_{-1}^{0} \frac{1}{2^l\, l!} \frac{d^l}{dx^l}\left(x^2 - 1\right)^l dx$$

Solving the for a and b with mathematica we get:

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AB[l_] := FullSimplify[
  Solve[{A a^l + B a^(-l-1) == V (2 l + 1)/2 Integrate[1/(2^l l!) D[(x^2 - 1)^l, {x, l}], {x, 0, 1}],
    A b^l + B b^(-l-1) == (2 l + 1)/2 V Integrate[1/(2^l l!) D[(x^2 - 1)^l, {x, l}], {x, 0, 1}]}, {A, B}]]
AB[0]
AB[1]
AB[2]
AB[3]
AB[4]
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$$\left\{\left\{A \to \frac{V}{2},\ B \to 0\right\}\right\}$$

$$\left\{\left\{A \to \frac{3\ (a + b)\ V}{4\ \left(a^2 + a\ b + b^2\right)},\ B \to \frac{3\ a^2\ b^2\ V}{4\ \left(a^2 + a\ b + b^2\right)}\right\}\right\}$$

$$\{\{A \to 0,\ B \to 0\}\}$$

$$\left\{\left\{A \to \frac{7\ \left(a^4 - b^4\right)\ V}{16\ \left(-a^7 + b^7\right)},\ B \to -\frac{7\ a^4\ b^4\ \left(a^2 + a\ b + b^2\right)\ V}{16\ \left(a^6 + a^5\ b + a^4\ b^2 + a^3\ b^3 + a^2\ b^4 + a\ b^5 + b^6\right)}\right\}\right\}$$

$$\{\{A \to 0,\ B \to 0\}\}$$

As a→0 and b→∞ we get:

l=0:

$A = \frac{V}{2},\ B = 0$

l=1:

$A = \frac{3\,b\,V}{4\,b^2} = 0,\ B = \frac{3\,a^2\,b^2\,V}{4\,b^2} = 0$

l=2:

A=0, B=0

l=3:

$A = \frac{-7\,b^4\,V}{16\,b^7} = 0,\ B = -\frac{7\,a^4\,b^6\,V}{16\,b^6} = 0$

Remember that we have $\Phi \propto \frac{R(r)}{r} = A_l\, r^l + B_l\, r^{-l-1}$

In order to stop Φ becoming infinite, we need $B_l \to 0$ (when r is small) and $A_l \to 0$ for $l \geq 1$. This only leaves one term remaining, A_0.

3.9

$\nabla^2 \Phi = 0 \Rightarrow \frac{\nabla^2 \Phi}{\Phi} = 0$

$\Phi(s,\phi,z) = R(s)\mathrm{T}(\phi)\mathrm{Z}(z)$

$\frac{1}{s\,R(s)} \frac{\partial}{\partial s}(s\,R'(s)) + \frac{1}{s^2\,T(\phi)} \frac{\partial^2 T(\phi)}{\partial \phi^2} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = 0$

$\Rightarrow \frac{\partial^2 Z(z)}{\partial z^2} = -a^2\, Z(z)$, for some constant a.

This gives solutions

$Z(z) = A\, e^{\mathrm{iaz}} + B\, e^{-\mathrm{iaz}}$

The potential is zero when z=0,L

$\Rightarrow$ A=-B

$Ae^{iaL} - Ae^{-iaL} = 0 \Rightarrow e^{2\,iaL} = 1 \Rightarrow a = \frac{k\pi}{L}$, for a positive integer k

$\Rightarrow Z(z) = A\, e^{i\,k\,\pi\,z/L} - A\, e^{-i\,k\,\pi\,z/L} = 2\,A\,\mathrm{Sin}(k\pi\,z/L)$

Rearranging the laplacian again:

$\frac{s}{R(s)}\frac{\partial}{\partial s}(s\,R'(s)) + \frac{1}{T(\phi)}\frac{\partial^2 T(\phi)}{\partial\phi^2} + s^2\,a^2 = 0$

And so we get:

$\frac{\partial^2 T(\phi)}{\partial\phi^2} = -d^2\,T(\phi)$

$\Rightarrow T(\phi) = Ce^{id\phi} + De^{-id\phi}$

From continuting of the laplacian:

T(0)=T(2π)

$\Rightarrow C + D = Ce^{id\,2\,\pi} + De^{-id\,2\,\pi}$

$\Rightarrow C\left(e^{id\,2\,\pi}\right)^2 - (C + D)\left(e^{id\,2\,\pi}\right) + D = 0$

$e^{id\,2\,\pi} = \frac{C+D\pm\sqrt{C^2+D^2+2\,CD-4\,CD}}{2\,C}$

$e^{id\,2\,\pi} = \frac{C+D\pm(C-D)}{2\,C} = 1$ OR $\frac{D}{C}$

Using $e^{id\,2\,\pi} = \frac{D}{C}$ (As we can take D=C to get the other solution)

$\Rightarrow T(\phi) = Ce^{id\phi} + Ce^{-id(\phi-2\,\pi)}$

From the continutity of the laplacian:

T(π)=T(-π)

$\Rightarrow Ce^{id\pi} + Ce^{id\pi} = Ce^{-id\pi} + Ce^{id3\pi}$

$\Rightarrow 1 = \frac{e^{-id2\pi}+e^{id2\pi}}{2} = \cos(2\,\pi\,d)$

$\Rightarrow$ b is a positive integer.

$\Rightarrow T(\phi) = Ce^{id\phi} + Ce^{-id\phi} = 2\,C\,\mathrm{Cos}(d\phi)$

Rearranging the Laplacian Once More:

$\frac{s}{R(s)}\frac{\partial}{\partial s}(s\,R'(s)) + d^2 + s^2\,a^2 = 0$

$s^2\,R''(s) + s\,R'(s) + \left(s^2\,a^2 + d^2\right)R(s) = 0$

Solving with mathematica:

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DSolve[s² R''[s] + s R'[s] + (s² a² + d²) R[s] == 0, R[s], s]
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{{R[s] → BesselJ[ⅈ d, a s] C[1] + BesselY[ⅈ d, a s] C[2]}}
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$R(s) = E\,J_{id}(a\,s) + F\,Y_{id}(a\,s)$

Since the potential is finite , we must have F=0

Hence we get:

$\Phi(s,\,\phi,\,z) = \sum_{d,k=1}^{\infty} G_{dk}\,J_{id}(k\pi\,s/L)\,\mathrm{Cos}(d\phi)\,\mathrm{Sin}(k\pi z/L)$

Since Φ(b,ϕ,z)=V(ϕ,z), if we multiply both sides by Cos(d' ϕ) Sin(k'πz/L) and integrate we get (note that we get delta functions):

$G_{dk}\,J_{id}(k\pi\,b/L)\,\frac{\pi\,L}{2} = \int_0^L dz\int_0^{2\,\pi} d\phi\,V(\phi,\,z)\,\mathrm{Cos}(d\phi)\,\mathrm{Sin}(k\pi z/L)$

Rearranging:

$G_{dk} = \frac{2}{\pi\,L\,J_{id}(k\pi\,b/L)}\int_0^L dz\int_0^{2\,\pi} d\phi\,V(\phi,\,z)\,\mathrm{Cos}(d\phi)\,\mathrm{Sin}(k\pi z/L)$

3.10 a)

$G_{\mathrm{dk}} = \frac{2}{\pi L\, J_{\mathrm{id}}(\mathrm{k}\pi\, b/L)} \int_0^L \mathrm{dz} \int_0^{2\pi} \mathrm{d}\phi\, V(\phi, z)\, \mathrm{Cos}(\mathrm{d}\phi)\, \mathrm{Sin}(\mathrm{k}\pi \mathrm{z}/L)$

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FullSimplify[Integrate[Integrate[V Cos[d ϕ] Sin[k π z / L], {z, 0, L}], {ϕ, -π / 2, π / 2}] -
  Integrate[Integrate[V Cos[d ϕ] Sin[k π z / L], {z, 0, L}], {ϕ, π / 2, 3 π / 2}]]
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$$\frac{8\,L\,V\,\mathrm{Sin}\left[\frac{d\pi}{2}\right]^3 \mathrm{Sin}\left[\frac{k\pi}{2}\right]^2}{d\,k\,\pi}$$

And so we get the coefficients to be:

$G_{\mathrm{dk}} = \frac{2}{\pi L\, J_{\mathrm{id}}(\mathrm{k}\pi\, b/L)} \frac{8\,L\,V \mathrm{Sin}\left[\frac{d\pi}{2}\right]^3 \mathrm{Sin}\left[\frac{k\pi}{2}\right]^2}{d\,k\,\pi} = \frac{16\,V \mathrm{Sin}\left[\frac{d\pi}{2}\right]^3 \mathrm{Sin}\left[\frac{k\pi}{2}\right]^2}{d\,k\,\pi^2\, J_{\mathrm{id}}(\mathrm{k}\pi\, b/L)}$