Max Proft U5190335 Assignment 4.

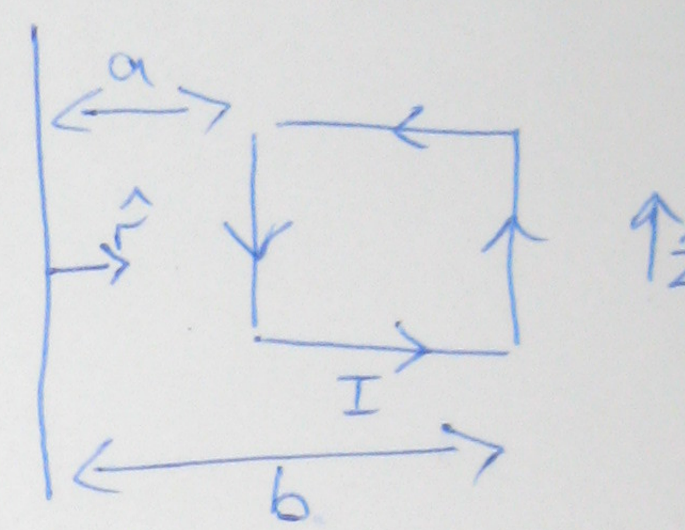
Tank Question:

a) Ampere's Law $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

b) Steady state $\Rightarrow \frac{\partial \vec{E}}{\partial t} = 0$

$\vec{B}$ is in the $\hat{\phi}$ direction as the opposite sides of the loop

x-section



The z component of $\nabla \times \vec{B}$ is: $\frac{1}{r}\frac{\partial}{\partial r}(r B_\phi)$

The z component of the current density (for an infintesimal wedge) $dI\left(\delta(r-b) - \delta(r-a)\right)\frac{\delta(\phi)}{r}$

Notice that $\frac{1}{r}$ cancels from both sides.

since $\int_0^R \left[\delta(r-b) - \delta(r-a)\right] dr = \begin{cases} 1 & a < R < b \\ 0 & \text{otherwise} \end{cases}$

we get $B_\phi \propto \int \frac{dI}{r}$ $\quad a < r < b$

$$\left(\frac{dI}{d\theta} = N I \right)$$

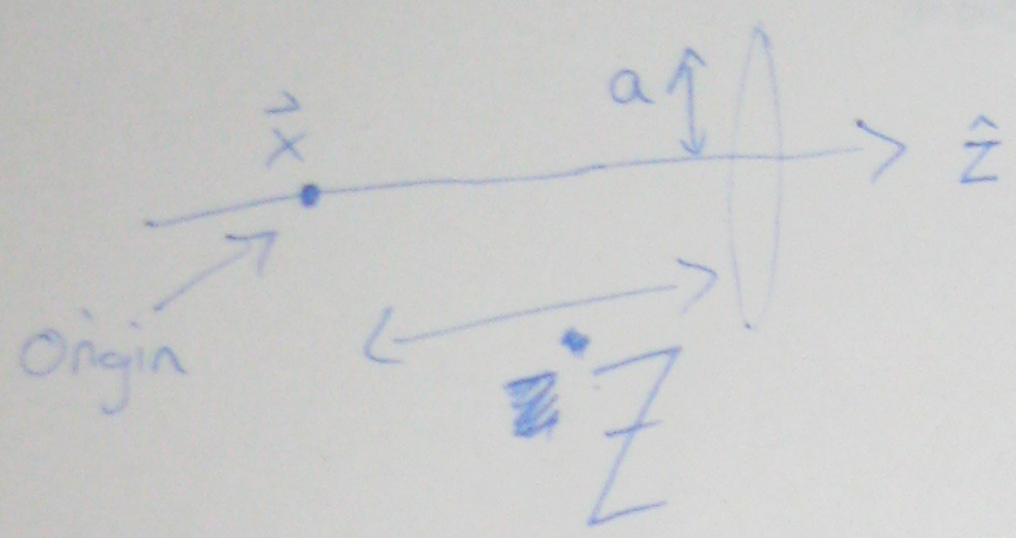
loops per angle

current per wedge

And so the functional dependence of B_ϕ is:

$$B_\phi \propto \begin{cases} \frac{1}{r} & a < r < b \\ 0 & \text{otherwise} \end{cases}$$

Q 5.3 First I will consider one loop carrying a current dI. (cylindrical coordinates)



Current density:

$$\vec{J}(\vec{x}') = \hat{\phi}\, dI\, \delta(r'-a)\, \delta(z'-Z)$$

$$\vec{x} = 0$$

$$\vec{x}' = r'\hat{r} + z'\hat{z}$$

From the Biot-Savart Law

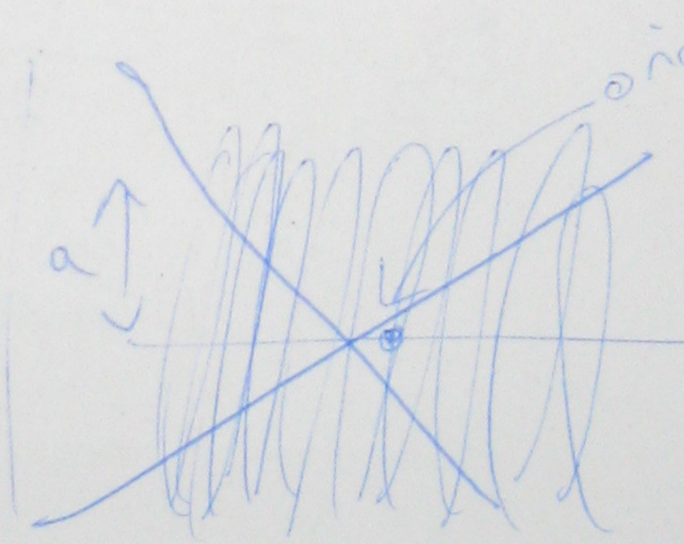
$$d\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\iiint \vec{J}(\vec{x}') \times \frac{-r'\hat{r} - z'\hat{z}}{(r'^2 + z'^2)^{3/2}}\, r'\,dr'\,d\phi'\,dz'$$

$d\vec{B}$ since we have an infintesimal current.

Since we are only interested in the magnetic field in the z-direction, and since $\hat{\phi} \times \hat{r} = -\hat{z}$, $\hat{\phi} \times \hat{z} = \hat{r}$

$$dB_z(\vec{x}) = \frac{\mu_0}{4\pi}\iiint \frac{dI\, \delta(r'-a)\, \delta(z'-Z)\, r'}{(r'^2 + z'^2)^{3/2}}\, r'\,dr'\,d\phi'\,dz'$$

$$= \frac{\mu_0}{4\pi}\, dI\, \frac{a^2\, 2\pi}{(a^2 + Z^2)^{3/2}}$$



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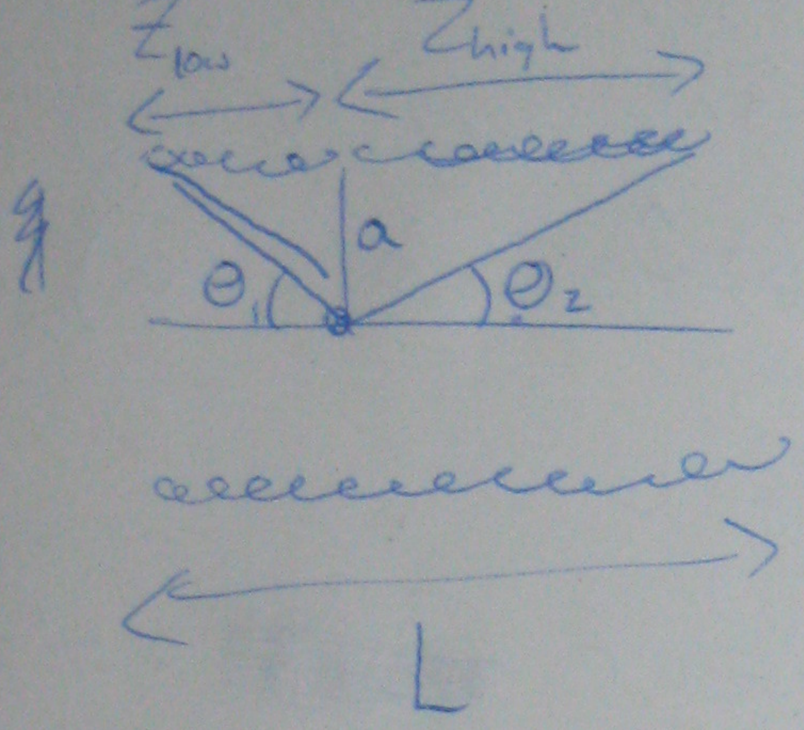
Now consider a large number of these rings next to each other (So large that the net magnetic field can be considered an integral).

The current per length is given by:

$$\frac{dI}{dZ} = NI$$

$$\Rightarrow B_z(z) = \frac{N I \mu_0 a^2}{2} \int_{-Z_{low}}^{Z_{high}} (a^2 + Z^2)^{-3/2} \, dZ$$



Using mathematica we get this integral to be:

(#1 integral)

$$B_z = \frac{I N \mu_0}{2} \left(\frac{Z_{high}}{\sqrt{a^2 + Z_{high}^2}} + \frac{Z_{low}}{\sqrt{a^2 + Z_{low}^2}} \right)$$

From our diagram we can see that

$$\cos\theta_1 = \frac{Z_{low}}{\sqrt{a^2 + Z_{low}^2}} \qquad \cos\theta_2 = \frac{Z_{high}}{\sqrt{a^2 + Z_{high}^2}}$$

$$\Rightarrow B_z = \frac{I N \mu_0}{2} (\cos\theta_1 + \cos\theta_2),$$

as required

Q 6.1 a)

$$\Psi(\vec{x},t) = \int \frac{[f(\vec{x}',t')]_{ret}}{|\vec{x}-\vec{x}'|} d^3\vec{x}' \quad , \quad f(\vec{x},t) = \delta(x')\delta(y')\delta(t)$$

$$\Rightarrow \Psi(\vec{x},t) = \int \frac{\delta(x')\delta(y')\delta\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|} dx'\,dy'\,dz'$$

Doing the x' and y' integrals, and using $x^2+y^2=\rho^2$

$$= \int \frac{\delta\left(t - \frac{\sqrt{\rho^2+(z-z')^2}}{c}\right)}{\sqrt{\rho^2+(z-z')^2}} dz'$$

Since $\delta(g(z')) = \sum_{\substack{\tilde{z} \text{ is a zero of } g(z') \\ g'(\tilde{z}) \neq 0}} \frac{\delta(z'-\tilde{z})}{|g'(\tilde{z})|}$

If we solve for the roots of $g(z') = t - \frac{\sqrt{\rho^2+(z-z')^2}}{c}$:

$$\Rightarrow c^2t^2 = \rho^2 + (z-z')^2$$

$$\Rightarrow z' = z \pm \sqrt{c^2t^2-\rho^2}$$

finding $|g'(z')| = \left| \frac{1}{c} \frac{(\rho^2+(z-z')^2)^{-1/2}}{2} \times 2(z'-z) \right|$

$$= \left| \frac{\sqrt{c^2t^2-\rho^2}}{c\sqrt{\rho^2+c^2t^2-\rho^2}} \right| = \frac{\sqrt{c^2t^2-\rho^2}}{c^2t}$$

Note: This solution is only valid if $c^2t^2-\rho^2 > 0$ $\Rightarrow ct-\rho > 0$ $(t \geq 0)$ and so it ~~may~~ must be zero at all other times. Hence we can multiply our solution by $\Theta(ct-\rho) := \begin{cases} 0 & ct-\rho < 0 \\ 1 & ct-\rho > 0 \end{cases}$

$$\Rightarrow \quad \Psi(\vec{x},t) = \int \frac{\delta\left(z' - \left(z + \sqrt{c^2t^2-\rho^2}\right)\right) + \delta\left(z' - \left(z - \sqrt{c^2t^2-\rho^2}\right)\right)}{\sqrt{\rho^2+(z-z')^2}\,\sqrt{c^2t^2-\rho^2}\,/\,c^2 t}\, dz' \times \Theta(ct-\rho)$$

$$= \frac{2c^2 t\, \Theta(ct-\rho)}{\sqrt{\rho^2+c^2t^2-\rho^2}\,\sqrt{c^2t^2-\rho^2}}$$

$$= \frac{2c\, \Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}$$, as required.

#1 Integral

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Integrate[n i μ a^2 / (2 (z^2 + a^2)^(3/2)), {z, -Zlow, Zhigh}, Assumptions → {Zlow > 0, Zhigh > 0, a > 0}]
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$$\frac{1}{2} i n \left(\frac{\text{Zhigh}}{\sqrt{a^2 + \text{Zhigh}^2}} + \frac{\text{Zlow}}{\sqrt{a^2 + \text{Zlow}^2}}\right) \mu$$

Q6.1 b)

Note: all integrals are over (-∞,∞) unless otherwise specified.

$$\psi(\vec{x},\ t) = \int \frac{[f(\vec{x}',t)]_{\text{ret}}}{|\vec{x}-\vec{x}'|} d^3 x'$$

where $f(\vec{x}',\ t') = \delta(x')\,\delta(t')$

$$\Rightarrow \psi(\vec{x},\ t) = \int \frac{\delta(x')\,\delta\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|} \text{dx}'\,\text{dy}'\,\text{dz}'$$

doing the integral with respect to x' we get:

$$\Rightarrow \psi(\vec{x},\ t) = \int \frac{\delta\left(t - \frac{\sqrt{x^2+(y-y')^2+(z-z')^2}}{c}\right)}{\sqrt{x^2+(y-y')^2+(z-z')^2}} \text{dy}'\,\text{dz}'$$

Using $\delta(g(y')) = \sum_{\text{zeros}} \frac{\delta(y'-y_0)}{|g'(y_0)|}$

Solving for the zeros we get:

$$y_0 = y \pm \sqrt{c^2 t^2 - x^2 - (z - z')^2}$$

and $g'(y_0)$ is given by:

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D[t - Sqrt[x^2 + (y - yp)^2 + (z - zp)^2]/c, yp] /. yp -> y ± Sqrt[c^2 t^2 - x^2 - (z - z')^2]
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$$\frac{y - \left(y \pm \sqrt{c^2\ t^2 - x^2 - (z - z')^2}\right)}{c \sqrt{x^2 + (z - zp)^2 + \left(y - \left(y \pm \sqrt{c^2\ t^2 - x^2 - (z - z')^2}\right)\right)^2}}$$

And so we can simplify this to get:

$$|\ g'(y_0)\ | = \frac{\sqrt{c^2 t^2 - x^2 - (z-z')^2}}{c^2 t}$$

Which is only valid for $c^2 t^2 - x^2 - (z - z')^2 > 0$

Defining Θ as the step function as we did in a) we get:

$$\psi(\vec{x},\ t) = \int\!\!\int \left(\delta\left(y' - \left(y + \sqrt{c^2 t^2 - x^2 - (z-z')^2}\right)\right) + \delta\left(y' - \left(y - \sqrt{c^2 t^2 - x^2 - (z-z')^2}\right)\right)\right) \Big/ \left(\sqrt{x^2 + (y - y')^2 + (z - z')^2}\ \frac{\sqrt{c^2 t^2 - x^2 - (z-z')^2}}{c^2 t}\right)$$
$$\Theta(c^2 t^2 - x^2 - (z - z')^2)\,\text{dy}'\,\text{dz}'$$

Doing the integral with respect to y' we get:

$$\psi(\vec{x},\ t) = \int \frac{2\,c\,\Theta(c^2 t^2 - x^2 - (z-z')^2)}{\sqrt{c^2 t^2 - x^2 - (z-z')^2}} \text{dz}'$$

I will do a variable transformation ξ=z'-z$\Rightarrow$ dξ=dz'
From the definition of Θ we know that the function will only be non-zero for $\xi^2 < c^2\,t^2 - x^2$
$\Rightarrow -\sqrt{c^2\,t^2 - x^2}\;\;\xi < \sqrt{c^2\,t^2 - x^2}$
This only makes sense if $c^2\,t^2 - x^2 > 0 \Leftrightarrow$ ct-|x|>0, and so it must be zero at all other times.
Hence we can multiply the solution by the step function: Θ(ct-|x|).
Doing the integral we get (A:=$c^2\,t^2 - x^2$):

$$\text{Integrate}\left[\frac{2\,c\,\text{StepFunction}}{\sqrt{A - \xi^2}},\ \left\{\xi,\ -\sqrt{A},\ \sqrt{A}\right\},\ \text{Assumptions} \to \{A > 0\}\right]$$

$$2\,c\,\pi\,\text{StepFunction}$$

And so our solution is
$\psi(\vec{x},\ t) = 2\,\pi\,c\,\Theta(\text{ct} - |\,x\,|)$