Max Proft u5190335 Assignment 5

Q6.17

Under a duality transformation, we get:

E=**E'** Cosξ +Z_0 $\boldsymbol{H'}$ Sinξ

$\boldsymbol{B} = -Z_0\,\boldsymbol{D}'\sin\xi + \boldsymbol{B}'\,\mathrm{Cos}\xi$

$Z_0\,\rho_e = Z_0\,\rho'_e\cos\xi + \rho'_m\sin\xi$

$\rho_m = -Z_0\,\rho'_e\sin\xi + \rho'_m\cos\xi$

I choose this transformation (from **E'→E, B'→B, etc.**) so that we get the standard Lorentzian force from these equations.

$\Rightarrow \rho_m = 0$

Taking the charge distributions to be single points (i.e. replace all ρ with charge for a single point, q). Simplifying the equations we get:

$0 = -Z_0\,q'_e\sin\xi + q'_m\cos\xi \Rightarrow Z_0 = \frac{q'_m}{q_e'\,\mathrm{Tan}\,\xi}$

$Z_0\,q_e = Z_0\,q'_e\cos\xi + q'_m\sin\xi$

$\Rightarrow q_e = q_e'\cos\xi + \frac{q'_m\sin\xi}{Z_0} = q_e'\cos\xi + q'_e\frac{\sin^2\xi}{\cos\xi} = \frac{q_e'}{\cos\xi}$

E=**E'** Cosξ +Z_0 $\boldsymbol{H'}$ Sinξ $= \boldsymbol{E'}\,\mathrm{Cos}\xi + \boldsymbol{H'}\,\frac{q'_{m\cos\xi}}{q_e'}$

$\boldsymbol{B} = -Z_0\,\boldsymbol{D}'\sin\xi + \boldsymbol{B}'\,\mathrm{Cos}\xi = -\boldsymbol{D'}\,\frac{q'_{m\cos\xi}}{q_e'} + \boldsymbol{B}'\,\mathrm{Cos}\xi$

Note: since we are just performing a change of variables, the velocity of the particle does not change. So if we substitute this into the Lorentz force we get:

F=**E** q_e +q_e(**v**×**B**)

$\boldsymbol{F} = \left(\boldsymbol{E'}\,\mathrm{Cos}\xi + \boldsymbol{H'}\,\frac{q'_{m\cos\xi}}{q_e'}\right)\left(\frac{q_e'}{\cos\xi}\right) + \left(\frac{q_e'}{\cos\xi}\right)\left(\boldsymbol{v}\times\left(-\boldsymbol{D'}\,\frac{q'_{m\cos\xi}}{q_e'} + \boldsymbol{B'}\,\mathrm{Cos}\xi\right)\right)$

$\boldsymbol{F} = \boldsymbol{E'}\,q'_e + \boldsymbol{H'}\,q'_m - q'_m\,\boldsymbol{v}\times\boldsymbol{D'} + q_e'\,\boldsymbol{v}\times\boldsymbol{B'}$

Since $\boldsymbol{H'} = \frac{\boldsymbol{B'}}{\mu}$, and $\boldsymbol{D'} = \epsilon_0\,\boldsymbol{E'}$

$\boldsymbol{F} = \boldsymbol{E'}\,q'_e + \boldsymbol{B'}\,\frac{q'_m}{\mu} - q'_m\,\epsilon_0\,\boldsymbol{v}\times\boldsymbol{E'} + q'_e\,\boldsymbol{v}\times\boldsymbol{B'}$

Q 7.1

In the linear polarisation basis:

Let $E_1 = a_1\,e^{i\delta_1}$ and $E_2 = a_2\,e^{i\delta_2}$

$s_0 + s_1 = 2\,a_1^2 \Rightarrow a_1 = \sqrt{\frac{s_0+s_1}{2}}$

$s_0 - s_1 = 2\,a_2^2 \Rightarrow a_2 = \sqrt{\frac{s_0-s_1}{2}}$

$\cos(\delta_2 - \delta_1) = \frac{s_2}{2\,a_1\,a_2}$

$\sin(\delta_2 - \delta_1) = \frac{s_3}{2\,a_1\,a_2}$

(Note: we need both of the last 2 equations so that we can determine which quadrent $\delta_2 - \delta_1$ lies in)

In the circular polarisation basis:

Let $E_+ = a_+\,e^{i\delta_+}$ and $E_- = a_-\,e^{i\delta_-}$

$s_0 + s_3 = 2\,a_+^2 \Rightarrow a_+ = \sqrt{\frac{s_0+s_3}{2}}$

$s_0 - s_3 = 2\,a_-^2 \Rightarrow a_- = \sqrt{\frac{s_0-s_3}{2}}$

$\cos(\delta_- - \delta_+) = \frac{s_1}{2\, a_+\, a_-}$
$\sin(\delta_- - \delta_+) = \frac{s_2}{2\, a_+\, a_-}$

(Note: we need both of the last 2 equations so that we can determine which quadrent $\delta_- - \delta_+$ lies in)

a)

$s_0 = 3,\ \ s_1 = -1,\ \ s_2 = 2,\ s_3 = -2$
$a_1 = 1$
$a_2 = \sqrt{2}$
$\cos(\delta_2 - \delta_1) = \frac{1}{\sqrt{2}}$
$\sin(\delta_2 - \delta_1) = \frac{-1}{\sqrt{2}}$
$\Rightarrow \delta_2 - \delta_1 = -\frac{\pi}{3}$

$a_+ = \frac{1}{\sqrt{2}}$
$a_- = \sqrt{\frac{5}{2}}$
$\cos(\delta_- - \delta_+) = \frac{-1}{\sqrt{5}}$
$\sin(\delta_- - \delta_+) = \frac{2}{\sqrt{5}}$
$\Rightarrow \delta_- - \delta_+ = \pi - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

b)

$s_0 = 25,\ \ s_1 = 0,\ \ s_2 = 24,\ s_3 = 7$
$a_0 = \frac{5}{\sqrt{2}}$
$a_1 = \frac{5}{\sqrt{2}}$
$\cos(\delta_2 - \delta_1) = \frac{24}{25}$
$\sin(\delta_2 - \delta_1) = \frac{7}{25}$
$\Rightarrow \delta_2 - \delta_1 = \sin^{-1}\left(\frac{7}{25}\right)$

$a_+ = 4$
$a_- = 3$
$\cos(\delta_- - \delta_+) = 0$
$\sin(\delta_- - \delta_+) = 1$
$\Rightarrow \delta_- - \delta_+ = \frac{\pi}{2}$

Pictures attached at the end.

Q7.19

Note: all integrals are over (-∞,∞)

$u(x,\ 0)\ =\ f(x)\,e^{i\,k_0\,x} = \int A(k)\,e^{i(\mathrm{kx}-\omega(k)\,t)}\,\mathrm{dk}$

$\Rightarrow A\,(k)\ =\ \frac{1}{\sqrt{2\,\pi}}\int u\,(x,\,0)\,e^{-\mathrm{ikx}}\,\mathrm{dx}$

$A(k) = \frac{1}{\sqrt{2\,\pi}}\int f(x)\,e^{i\,(k_0-k)\,x}\,\mathrm{dx}$

a)

$\mathrm{f(x)=N}\ e^{\frac{-\alpha\,|x|}{2}}$

$A(k) = \frac{N}{\sqrt{2\,\pi}}\int e^{\frac{-\alpha\,|x|}{2}-i(k_0-k)\,x}\,\mathrm{dx}$

In[22]:=
```
Integrate[N/Sqrt[2 π] Exp[(-α Abs[x])/2 - I (k0 - k) x],
  {x, -∞, ∞}, Assumptions → {α > 0, k ∈ Reals, k0 ∈ Reals}]
```

Out[22]= $\frac{2\,\mathrm{N}\,\sqrt{\frac{2}{\pi}}\,\alpha}{(2\,\mathrm{k}-2\,\mathrm{k0}-\mathrm{i}\,\alpha)\,(2\,\mathrm{k}-2\,\mathrm{k0}+\mathrm{i}\,\alpha)}$

If we simplify this then we get:

$\mathrm{A\,(k)}\ =\ \frac{2\,\sqrt{\frac{2}{\pi}}\,\mathrm{N}\,\alpha}{4\,(\mathrm{k}-\mathrm{k0})^2+\alpha^2}$

And so we get:

$|\ A(k)\ |^2 = \frac{8\,\mathrm{N}^2\,\alpha^2}{\pi\,\left(4\,(\mathrm{k}-\mathrm{k0})^2+\alpha^2\right)^2}$

$|\ u\,(x,\,0)\ |^2 =\ |\ f(x)\ |^2 = N^2\,e^{-\alpha\,|\,x\,|}$

$\mathrm{<k>=}\ \frac{\int k\,|\,A(k)\,|^2\mathrm{dk}}{\int |\,A(k)\,|^2\mathrm{dk}} =$

In[75]:=
```
Integrate[(8 N^2 α^2)/(π (4 (k-k0)^2+α^2)^2) k, {k, -∞, ∞}, Assumptions → {α > 0, L > 0, k0 > 0}]
/
Integrate[(8 N^2 α^2)/(π (4 (k-k0)^2+α^2)^2), {k, -∞, ∞}, Assumptions → {α > 0, L > 0, k0 > 0}]
```

Out[75]= k0

$<k^2>=\ \frac{\int k^2\,|\,A(k)\,|^2\mathrm{dk}}{\int |\,A(k)\,|^2\mathrm{dk}}$

In[77]:=
```
Integrate[(8 N^2 α^2)/(π (4 (k-k0)^2+α^2)^2) k^2, {k, -∞, ∞}, Assumptions → {α > 0, L > 0, k0 > 0}]
/
Integrate[(8 N^2 α^2)/(π (4 (k-k0)^2+α^2)^2), {k, -∞, ∞}, Assumptions → {α > 0, L > 0, k0 > 0}]
```

Out[77]= $\frac{1}{4}\left(4\,\mathrm{k0}^2+\alpha^2\right)$

And we get:

$\sigma_k{}^2 = <k^2> - <k>^2 =\ \frac{\alpha^2}{4}$

$<x> = \frac{\int x\,|u(x)|^2 dx}{\int |u(x)|^2 dx} =$

In[1]:=

$$\frac{\texttt{Integrate}\left[x\,N^2\,\texttt{Exp}\left[-\alpha\,\texttt{Abs}[x]\right],\ \{x, -\infty, \infty\},\ \texttt{Assumptions} \to \{\alpha > 0\}\right]}{\texttt{Integrate}\left[N^2\,\texttt{Exp}\left[-\alpha\,\texttt{Abs}[x]\right],\ \{x, -\infty, \infty\},\ \texttt{Assumptions} \to \{\alpha > 0\}\right]}$$

Out[1]= 0

$< x^2 > = \frac{\int x^2\,|u(x)|^2 dx}{\int |u(x)|^2 dx} =$

In[4]:=

$$\frac{\texttt{Integrate}\left[x^2\,N^2\,\texttt{Exp}\left[-\alpha\,\texttt{Abs}[x]\right],\ \{x, -\infty, \infty\},\ \texttt{Assumptions} \to \{\alpha > 0\}\right]}{\texttt{Integrate}\left[N^2\,\texttt{Exp}\left[-\alpha\,\texttt{Abs}[x]\right],\ \{x, -\infty, \infty\},\ \texttt{Assumptions} \to \{\alpha > 0\}\right]}$$

Out[4]= $\frac{2}{\alpha^2}$

And we get

$\sigma_x{}^2 = < x^2 > - < x >^2 = \frac{2}{\alpha^2}$

And hence $\sigma_x{}^2\,\sigma_\Delta{}^2 = \frac{1}{2} \Rightarrow \sigma_x\,\sigma_\Delta = \frac{1}{\sqrt{2}} > 0.5$

b)

$f(x) = N\,e^{\frac{-\alpha^2 x^2}{4}}$

$A(k) = \frac{N}{\sqrt{2\pi}} \int e^{\frac{-\alpha^2 x^2}{4} - i(k_0 - k)\,x}\,dx$

In[23]:=

$$\texttt{Integrate}\left[\frac{N}{\sqrt{2\pi}}\,\texttt{Exp}\left[\frac{-\alpha^2\,x^2}{4} - I\,(k_0 - k)\,x\right],\ \{x, -\infty, \infty\},\ \texttt{Assumptions} \to \{\alpha > 0\}\right]$$

Out[23]=

$$\frac{\sqrt{2}\,e^{-\frac{(k-k_0)^2}{\alpha^2}}\,N}{\alpha}$$

And so we get:

$|\,A(k)\,|^2 = \frac{2N^2}{\alpha^2}\,e^{-\frac{2(k-k_0)^2}{\alpha^2}}$

$|\,u(x, 0)\,|^2 = |\,f(x)\,|^2 = N^2\,e^{\frac{-\alpha^2 x^2}{2}}$

$<k> = \frac{\int k\,|A(k)|^2 dk}{\int |A(k)|^2 dk} =$

In[4]:=

$$\frac{\texttt{Integrate}\left[\frac{2\,N^2}{\alpha^2}\,\texttt{Exp}\left[-\frac{2\,(k-k0)^2}{\alpha^2}\right]\,k,\ \{k, -\infty, \infty\},\ \texttt{Assumptions} \to \{\alpha > 0,\ k0 > 0\}\right]}{\texttt{Integrate}\left[\frac{2\,N^2}{\alpha^2}\,\texttt{Exp}\left[-\frac{2\,(k-k0)^2}{\alpha^2}\right],\ \{k, -\infty, \infty\},\ \texttt{Assumptions} \to \{\alpha > 0,\ k0 > 0\}\right]}$$

Out[4]= k0

$< k^2 > = \frac{\int k^2\,|A(k)|^2 dk}{\int |A(k)|^2 dk} =$

In[5]:=
$$\frac{\text{Integrate}\left[\frac{2\,N^2}{\alpha^2}\,\text{Exp}\left[-\frac{2\,(k-k0)^2}{\alpha^2}\right]\,k^2\ ,\ \{k,-\infty,\infty\},\ \text{Assumptions}\to\{\alpha>0,\ k0>0\}\right]}{\text{Integrate}\left[\frac{2\,N^2}{\alpha^2}\,\text{Exp}\left[-\frac{2\,(k-k0)^2}{\alpha^2}\right]\ ,\ \{k,-\infty,\infty\},\ \text{Assumptions}\to\{\alpha>0,\ k0>0\}\right]}$$

Out[5]=
$$\frac{1}{4}\left(4\,k0^2+\alpha^2\right)$$

$\sigma_k{}^2 = <k^2> - <k>^2 = \frac{\alpha^2}{4}$

$<X> = \frac{\int x\,|u(x)|^2 dx}{\int |u(x)|^2 dx} =$

In[11]:=
$$\frac{\text{Integrate}\left[N^2\,\text{Exp}\left[\frac{-\alpha^2\,x^2}{2}\right]\,x,\ \{x,-\infty,\infty\},\ \text{Assumptions}\to\{\alpha>0\}\right]}{\text{Integrate}\left[N^2\,\text{Exp}\left[\frac{-\alpha^2\,x^2}{2}\right],\ \{x,-\infty,\infty\},\ \text{Assumptions}\to\{\alpha>0\}\right]}$$

Out[11]= 0

$<x^2> = \frac{\int x^2\,|u(x)|^2 dx}{\int |u(x)|^2 dx} =$

In[12]:=
$$\frac{\text{Integrate}\left[N^2\,\text{Exp}\left[\frac{-\alpha^2\,x^2}{2}\right]\,x^2,\ \{x,-\infty,\infty\},\ \text{Assumptions}\to\{\alpha>0\}\right]}{\text{Integrate}\left[N^2\,\text{Exp}\left[\frac{-\alpha^2\,x^2}{2}\right],\ \{x,-\infty,\infty\},\ \text{Assumptions}\to\{\alpha>0\}\right]}$$

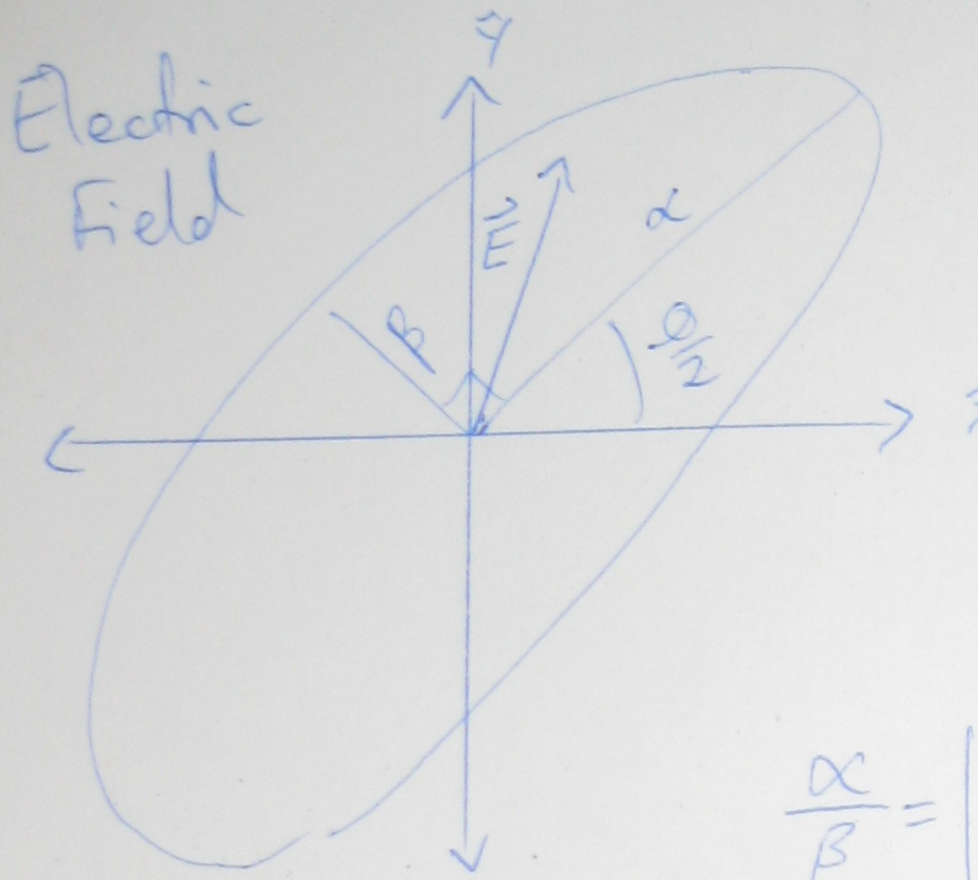
Out[12]=
$$\frac{1}{\alpha^2}$$

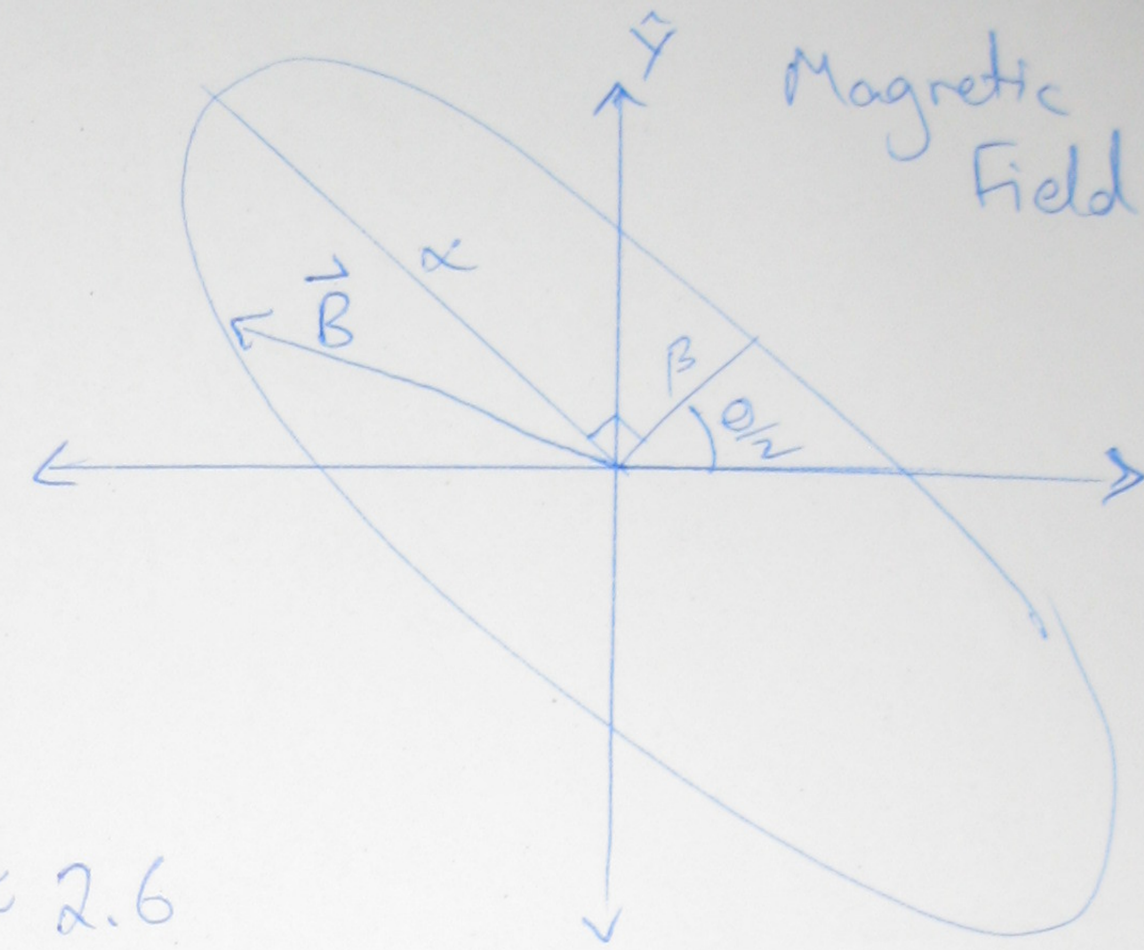
And we get
$\sigma_x{}^2 = <x^2> - <x>^2 = \frac{1}{\alpha^2}$
And hence $\sigma_x{}^2\,\sigma_\Delta{}^2 = \frac{1}{4} \Rightarrow \sigma_x\,\sigma_\Delta = \frac{1}{2}$

Pictures attached at the end.

Q 7.1

a) $\frac{E_-}{E_+} = \sqrt{5}\, e^{i\theta}$, $\theta = \pi - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \approx 117°$

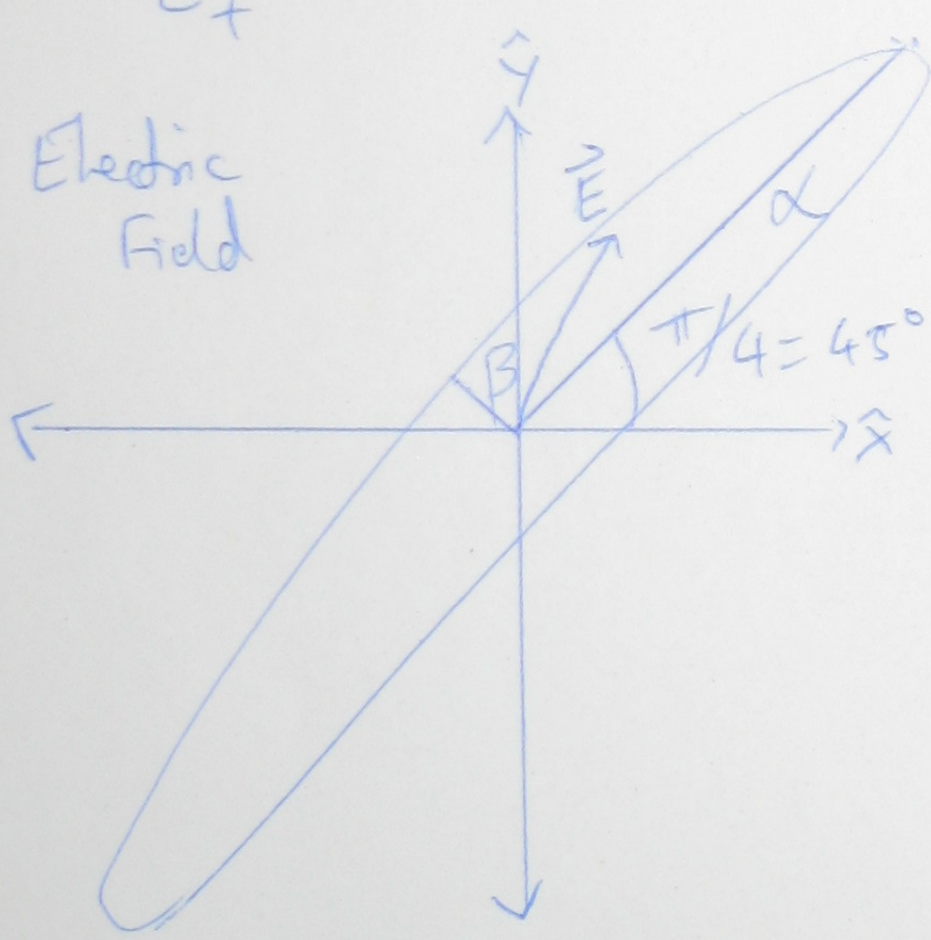


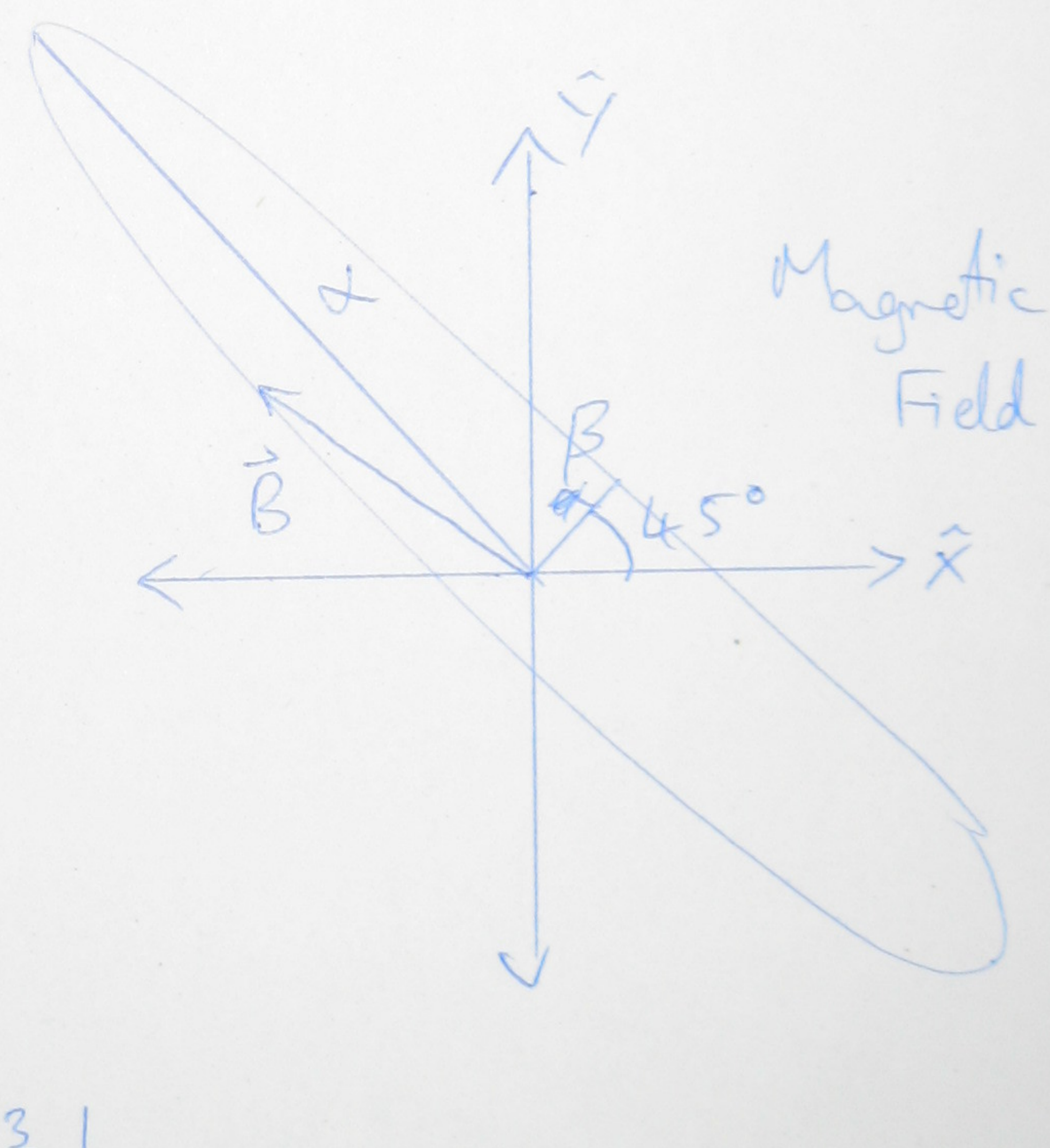


$$\frac{\alpha}{\beta} = \left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right| \approx 2.6$$

$\hat{x}$ in direction of $\vec{E}_1$
$\hat{y}$ in direction of $\vec{E}_2$

b) $\frac{E_1}{E_+} = \frac{3}{4}\, e^{i\theta}$ $\theta = \frac{\pi}{2}$

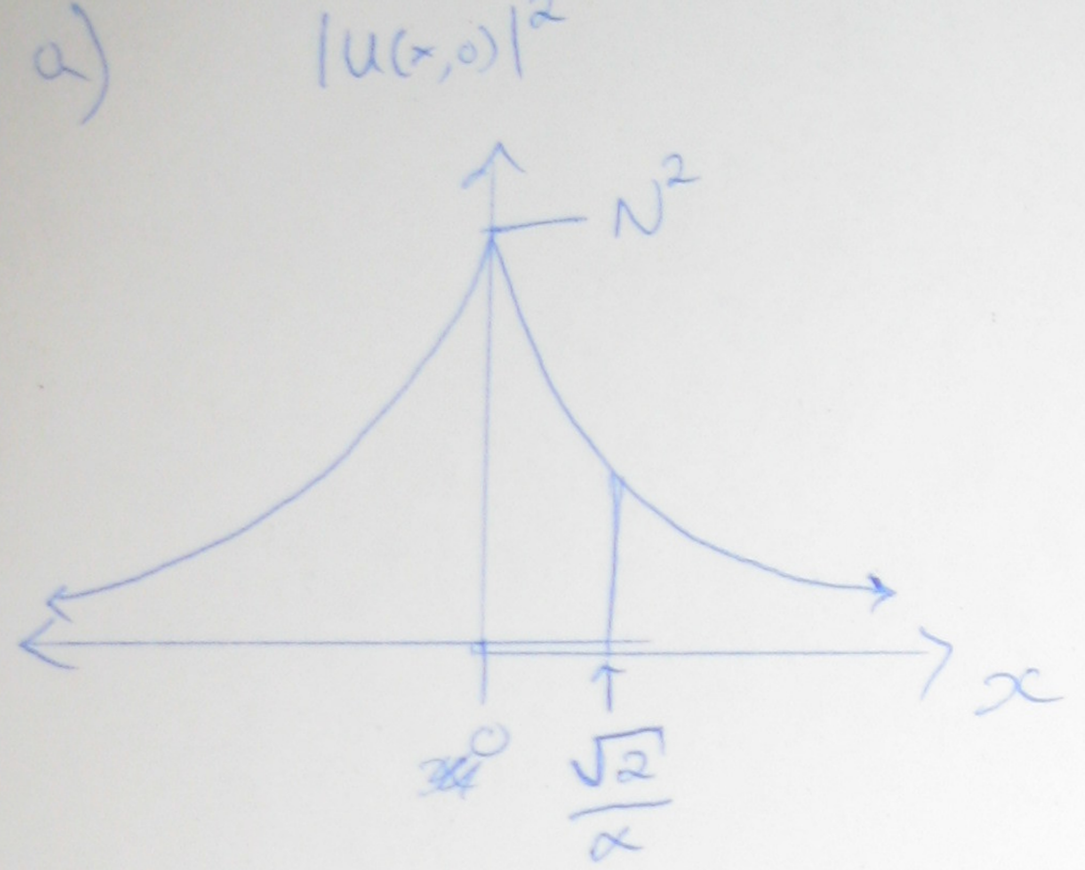




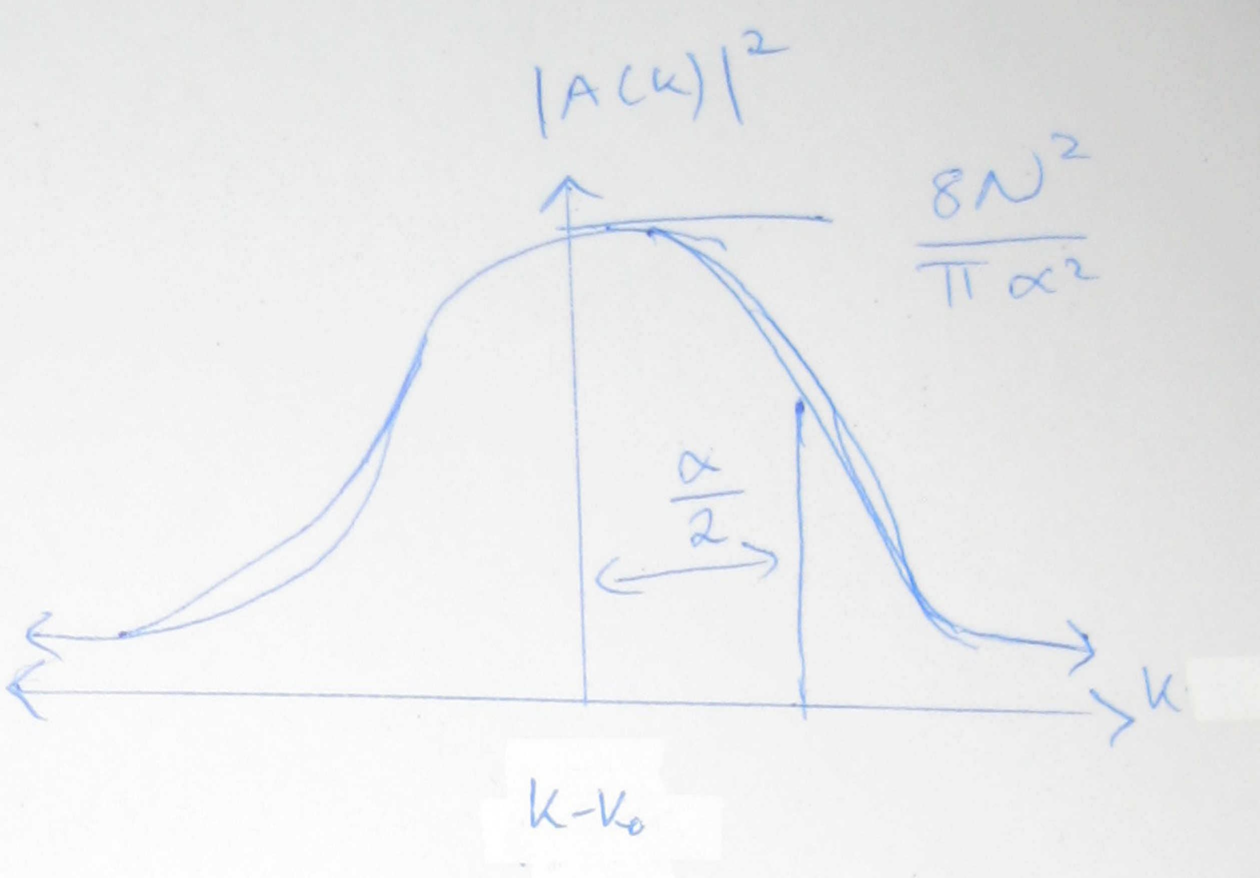
$$\frac{\alpha}{\beta} = \left|\frac{1+\frac{3}{4}}{1-\frac{3}{4}}\right| = 7$$

Q 7.19

a)

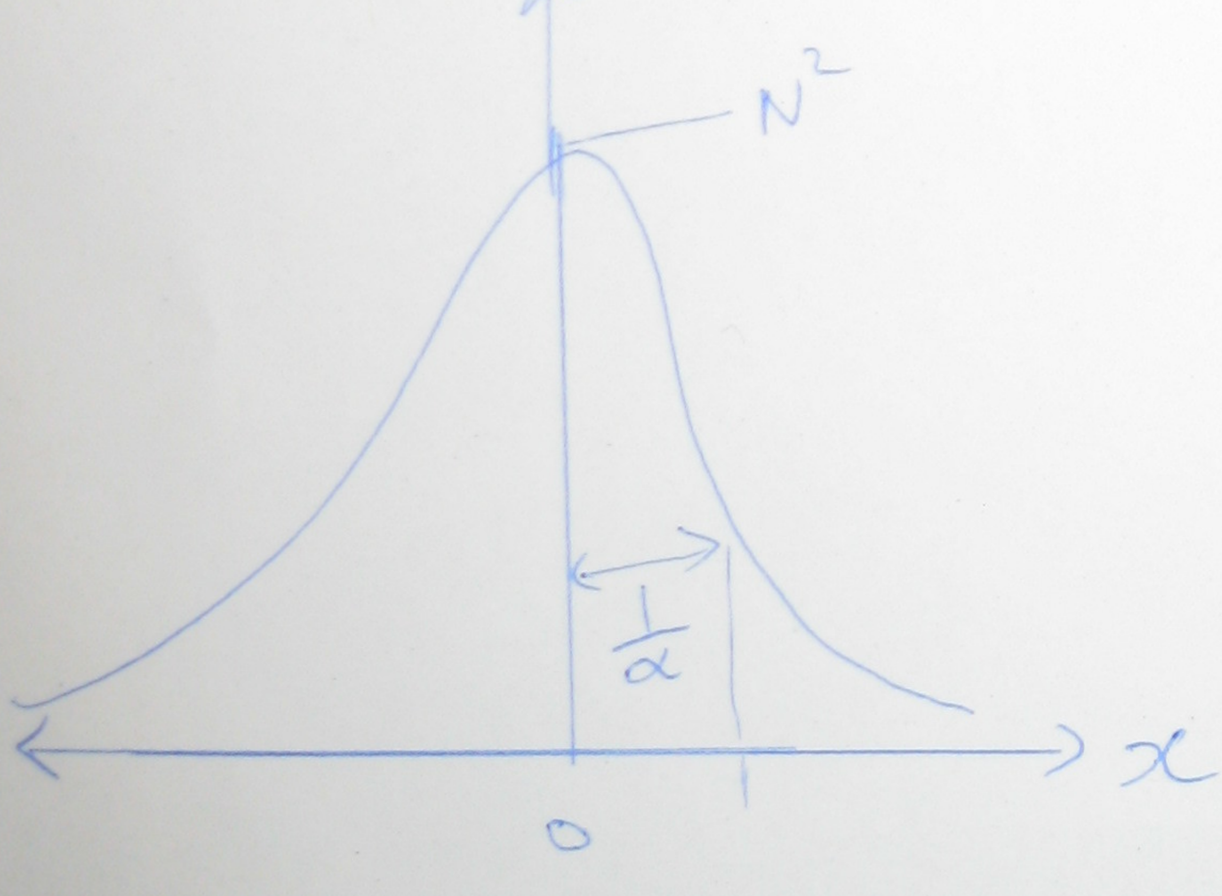


$|u(x,0)|^2$
N^2
x
0
$\frac{\sqrt{2}}{\alpha}$

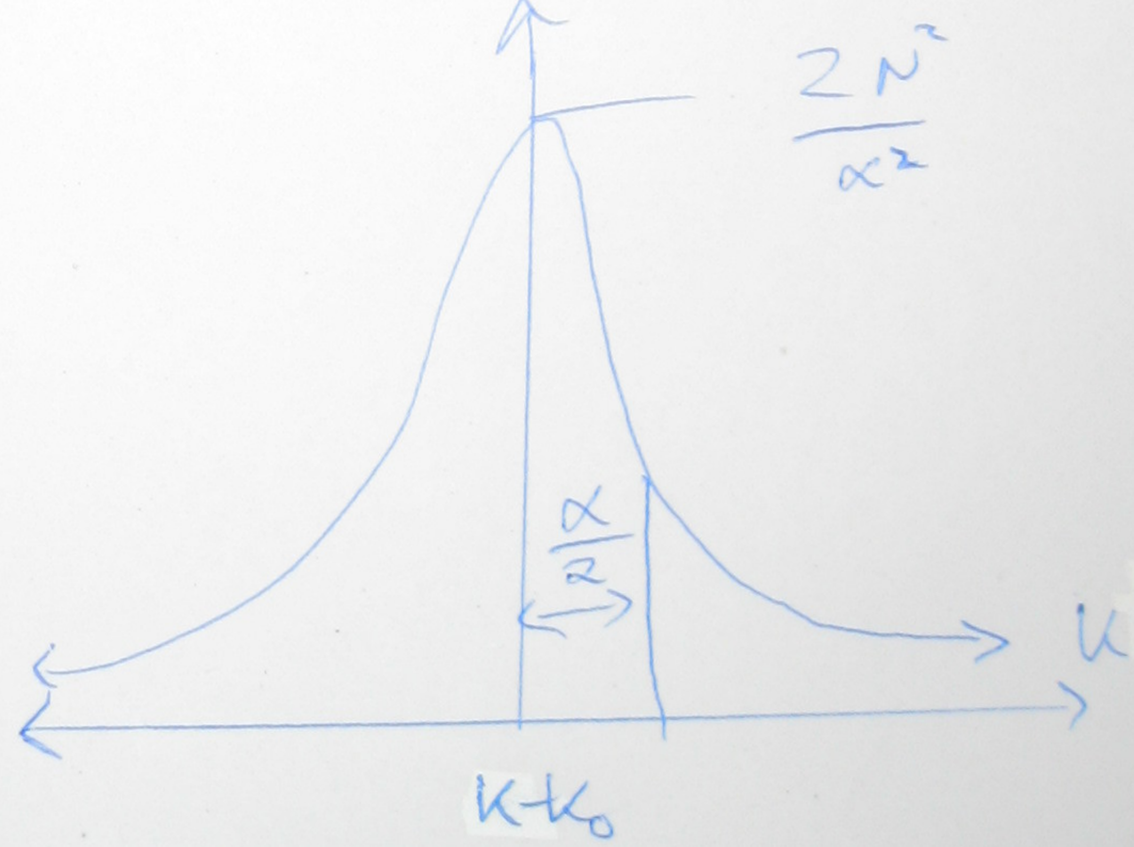


$|A(k)|^2$
$\frac{8N^2}{\pi\alpha^2}$
$\frac{\alpha}{2}$
k
$k-k_0$

b)



$|u(x,0)|^2$
N^2
$\frac{1}{\alpha}$
x
0



$|A(k)|^2$
$\frac{2N^2}{\alpha^2}$
$\frac{\alpha}{2}$
k
$k-k_0$