Max Proft U5790335 Assignment 3

1. The probability a particle is at a position x is given by:

$Pr(x) = \langle x | \hat{\rho} | x \rangle$. To see if this changes in time:

↖ position basis.

$$\frac{d}{dt} \langle x | \hat{\rho} | \hat{x} \rangle = \langle x | \frac{d\hat{\rho}}{dt} | x \rangle$$

(since $\hat{x}$ is hermetian)

$$= \gamma \langle x | \hat{x} \hat{\rho} \hat{x} - \frac{1}{2} \hat{x}^2 \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{x}^2 | x \rangle$$

Since $\hat{x} | x \rangle = x | x \rangle$

$$= \gamma \left(x^2 \langle x | \hat{\rho} | x \rangle - \frac{1}{2} x^2 \langle x | \hat{\rho} | x \rangle - \frac{1}{2} x^2 \langle x | \hat{\rho} | x \rangle \right)$$

$= 0$, as required.

2. The coefficients of the density matrix are given by

$$c_{nm} = \langle n | \hat{\rho} | m \rangle$$

$$\frac{d c_{nm}}{dt} = \gamma \langle n | \frac{d\hat{\rho}}{dt} | m \rangle$$

$$= \gamma \left(\langle n | n \hat{\rho} m | m \rangle - \frac{1}{2} \langle n | n^2 \hat{\rho} | m \rangle - \frac{1}{2} \langle n | \hat{\rho} m^2 | m \rangle \right)$$

$$= \frac{-\gamma}{2} (n-m)^2 \langle n | \hat{\rho} | m \rangle$$

$$= \frac{-\gamma}{2} (n-m)^2 c_{nm}$$

$$\Rightarrow \quad c_{nm} = e^{\frac{-\gamma}{2}(n-m)^2 t} c_{nm}(0)$$

↖ initial state.

3. The Heisenberg equation of motion gives us:

$$i\hbar \frac{d}{dt}\langle \hat{\Psi}_m(\vec{x})\rangle = \left\langle \left[\hat{\Psi}_m(\vec{x}), \hat{H}\right]\right\rangle$$

From the mean field approximation, $\langle \hat{\Psi}_m(x)\rangle = \Psi_m(x)$

so we now need to find a $\hat{H}$ which ~~g~~ gives the equation. ~~give~~

The first term is kinetic energy:

$$\hat{H}_1 = \sum_{n=-2}^{2} \int d^3x\, \hat{\Psi}_n^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\hat{\Psi}_n(\vec{x}).$$

$$\left[\hat{\Psi}_m(\vec{y}), \hat{H}_1\right] = \sum_{n=-2}^{2}\int d^3x \left[\hat{\Psi}_m(\vec{y})\hat{\Psi}_n^\dagger(\vec{x})\left(\frac{-\hbar^2}{2m}\nabla^2\right)\hat{\Psi}_n(\vec{x}) - \Psi_n^\dagger(\vec{x})\left(\frac{-\hbar^2}{2m}\nabla^2\right)\underbrace{\hat{\Psi}_n(\vec{x})\hat{\Psi}_m(\vec{y})}\right]$$

Since $\left[\hat{\Psi}_n(x), \hat{\Psi}_m(y)\right] = 0$, we can re-arrange this.

the derivative acts on x, not y, so they commute.

$$= \sum_{n=-2}^{2}\int d^3x \left[\hat{\Psi}_m(\vec{y})\hat{\Psi}_n^\dagger(\vec{x})\left(\frac{-\hbar^2}{2m}\nabla^2\right)\hat{\Psi}_n(\vec{x}) - \underbrace{\hat{\Psi}_n^\dagger(\vec{x})\hat{\Psi}_m(\vec{y})}\left(\frac{-\hbar^2}{2m}\nabla^2\right)\hat{\Psi}_n(\vec{x})\right]$$

~~These terms~~ ~~also~~ equal

~~that~~

~~the~~a.

$$= \sum_{n=-2}^{2}\int d^3x \left[\hat{\Psi}_m(y), \hat{\Psi}_n(\vec{x})\right]\left(\frac{-\hbar^2}{2m}\nabla^2\right)\hat{\Psi}_n(\vec{x})$$

$$= \sum_{n=-2}^{2}\int d^3x\, \delta_{nm}\,\delta(x-y)\,\frac{-\hbar^2}{2m}\nabla^2\hat{\Psi}_n(\vec{x})$$

$$= \frac{-\hbar^2}{2m}\nabla^2\hat{\Psi}_m(\vec{y})$$

From the mean field approximation

$$\left\langle\left[\hat{\Psi}_m(\vec{x}), \hat{H}\right]\right\rangle = \left\langle\frac{-\hbar^2}{2m}\nabla^2\hat{\Psi}_m(\vec{y})\right\rangle = \frac{-\hbar^2}{2m}\nabla^2\Psi_m(\vec{y})$$

The second, non-linear term:

$$\hat{H}_2 = \sum_{p,q=-2}^{2} \int d^3x\ \hat{\psi}^+_p(\vec{x}) \hat{\psi}^+_q(\vec{x}) \hat{\psi}_p(\vec{x}) \hat{\psi}_q(\vec{x})$$

$$\left[\hat{\psi}_m(\vec{y}), \hat{H}_2\right] = \frac{U}{2} \sum_{p,q=-2}^{2} \int d^3x \left(\hat{\psi}_m(\vec{y})\ \hat{\psi}^+_p(\vec{x})\ \hat{\psi}^+_q(\vec{x}) \hat{\psi}_p(\vec{x}) \hat{\psi}_q(\vec{x}) - \hat{\psi}^+_p(\vec{x}) \hat{\psi}^+_q(\vec{x}) \hat{\psi}_p(\vec{x}) \hat{\psi}_q(\vec{x}) \hat{\psi}_m(\vec{y}) \right)$$

since $\psi_m(y)$ commutes with $\psi_n(x)$

$$= \frac{U}{2} \sum_{p,q=-2}^{2} \int d^3x \left(\left[\hat{\psi}_m(\vec{y}), \hat{\psi}^+_p(\vec{x})\ \hat{\psi}^+_q(\vec{x})\right] \right) \left(\hat{\psi}_p(\vec{x})\ \hat{\psi}_q(\vec{x}) \right)$$

$$= \frac{U}{2} \sum_{p,q=-2}^{2} \int d^3x \left(\hat{\psi}^+_p(\vec{x})\ \delta_{qm} \delta(\vec{y}-\vec{x}) + \delta_{pm} \delta(\vec{y}-\vec{x})\ \hat{\psi}^+_q(\vec{x}) \right) \left(\hat{\psi}_p(\vec{x})\ \hat{\psi}_q(\vec{x}) \right)$$

$$= \frac{U}{2} \left(\sum_{p=-2}^{2} \hat{\psi}^+_p(\vec{y}) \hat{\psi}_p(\vec{y}) \hat{\psi}_m(\vec{y}) + \sum_{q=-2}^{2} \hat{\psi}^+_q(\vec{y}) \hat{\psi}_m(\vec{y}) \hat{\psi}_q(\vec{y}) \right)$$

since $\psi_m(y)$ commutes with $\psi_q(y)$

$$= U \sum_{p=-2}^{2} \hat{\psi}^+_p(\vec{y}) \hat{\psi}_p(\vec{y})\ \hat{\psi}_m(\vec{y})$$

relabeling the summation.

From the mean field approximation:

$$\left\langle \left[\hat{\psi}_m(\vec{y}), \hat{H}_2\right] \right\rangle = U \left\langle \sum_{p=-2}^{2} \hat{\psi}^+_p(\vec{y})\ \hat{\psi}_p(\vec{y})\ \hat{\psi}_m(\vec{y}) \right\rangle$$

$$= U \sum_{p=-2}^{2} \psi^*_p(\vec{y})\ \psi_p(\vec{y})\ \psi_m(\vec{y})$$

$$= U\, \psi_m(\vec{y}) \sum_{p=-2}^{2} |\psi_p(\vec{y})|^2$$

$$= U \rho(\vec{y})\ \psi_m(\vec{y})$$

For the matrix:

$$\hat{H}_3 = \sum_{j,k=-2}^{2} \int d^3x \, \hat{\psi}_j^\dagger(\vec{x}) \hat{\psi}_k(\vec{x}) V_{jk}(\vec{x})$$

$$[\hat{\psi}_m(\vec{y}), \hat{H}_3] = \sum_{j,k=-2}^{2} \int d^3x \left(\hat{\psi}_m(\vec{y}) \hat{\psi}_j^\dagger(\vec{x}) V_{jk}(\vec{x}) \hat{\psi}_k(\vec{x}) - \hat{\psi}_j^\dagger(\vec{x}) V_{jk}(\vec{x}) \hat{\psi}_k(\vec{x}) \hat{\psi}_m(\vec{y}) \right)$$

since $\hat{\psi}_k(x)$ and $\hat{\psi}_m(y)$ commute,
and since V_{jk} is a constant

$$= \sum_{j,k=-2}^{2} \int d^3x \left([\psi_m(\vec{y}), \psi_j^\dagger(\vec{x})] \right) \left(V_{jk}(\vec{x}) \hat{\psi}_k(\vec{x}) \right)$$

Since $[\psi_m(\vec{y}), \psi_j^\dagger(\vec{x})] = \delta_{mj} \, \delta(\vec{x} - \vec{y})$

$$= \sum_{k=-2}^{2} V_{mk}(\vec{y}) \hat{\psi}_k(\vec{y})$$

Using the mean field approximation:

$$\langle [\hat{\psi}_m(\vec{y}), \hat{H}_3] \rangle = \left\langle \sum_{k=-2}^{2} V_{mk}(\vec{y}) \, \hat{\psi}_k(\vec{y}) \right\rangle$$

$$= \sum_{k=-2}^{2} V_{mk}(\vec{y}) \psi_k(\vec{y})$$

so if we take $V_{mk}(\vec{y})$ to be the element of the matrix in the m^{th} row and k^{th} column we get the final term.

If we take $\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3$, by linearity we get the desired equations

Hence $\hat{H} = \sum_{n=-2}^{2} \int d^3x \, \hat{\psi}_n^\dagger(\vec{x}) \left(\frac{-\hbar^2}{2m} \nabla^2 \right) \hat{\psi}_n(x)$

~~Hence~~ $\cancel{\hat{H} \sum \frac{-\hbar^2}{2m} \nabla^2 \psi_n(\vec{x})}$

$$+ \sum_{p,q=-2}^{2} \int d^3x \, \hat{\psi}_p^\dagger(\vec{x}) \hat{\psi}_q^\dagger(\vec{x}) \hat{\psi}_p(\vec{x}) \hat{\psi}_q(\vec{x}) + \sum_{j,k=-2}^{2} \int d^3x \, \hat{\psi}_j^\dagger(\vec{x}) \hat{\psi}_k(\vec{x}) V_{jk}(\vec{x})$$

Q4. I will define:

$$\hat{\sigma}_{n++} = |e, n-1\rangle\langle e, n-1|$$

$$\hat{\sigma}_{n--} = |g, n\rangle\langle g, n|$$

$$\hat{\sigma}_{n+-} = |e, n-1\rangle\langle g, n|$$

$$\hat{\sigma}_{n-+} = |g, n\rangle\langle e, n-1|$$

On resonance, so $\omega_L = \omega_a = \omega$

The Hamiltonian is given by:

$$\hat{H}_n = \hbar\omega n(\underset{①}{\hat{\sigma}_{n++}} + \underset{②}{\hat{\sigma}_{n--}}) + G\sqrt{n}(\underset{③}{\hat{\sigma}_{n+-}} + \underset{④}{\hat{\sigma}_{n-+}})$$

The time evolution operator is given by:

$$U_n(t) = \exp\left(\frac{-it}{\hbar}\hat{H}_n\right)$$ (initially it is at $t_0 = 0$).

From linearity, we can break this into 4 pieces.

① $$\exp(-it\omega n\,\hat{\sigma}_{n++}) = \sum_{k=0}^{\infty}\frac{(-it\omega n)^k}{k!}(\hat{\sigma}_{n++})^k$$, note: $(\hat{\sigma}_{n++})^k = \sigma_{n++}$, $k \geq 1$

$$= \hat{1} + \hat{\sigma}_{n++}\sum_{k=1}^{\infty}\frac{(-it\omega n)^k}{k!}$$

$$= \hat{1} + \hat{\sigma}_{n++}(-1 + e^{-it\omega n})$$

② $$\exp(-it\omega n\,\hat{\sigma}_{n--}) = \sum_{k=0}^{\infty}\frac{(-it\omega n)^k}{k!}(\hat{\sigma}_{n--})^k$$ note: $(\hat{\sigma}_{n--})^k = \sigma_{n--}$, $k \geq 1$

$$= \hat{1} + \hat{\sigma}_{n--}\sum_{k=1}^{\infty}\frac{(-it\omega n)^k}{k!}$$

$$= \hat{1} + \hat{\sigma}_{n--}(-1 + e^{-it\omega n})$$

③ $$\exp(-itG\sqrt{n}/\hbar\,\hat{\sigma}_{n+-}) = \sum_{k=0}^{\infty}\frac{(-itG\sqrt{n}/\hbar)^k}{k!}(\hat{\sigma}_{n+-})^k$$ note: $(\sigma_{n+-})^k = 0$, $k \geq 2$

$$= \hat{1} - \frac{itG\sqrt{n}}{\hbar}\hat{\sigma}_{n+-}$$

④ $$\exp(-itG\sqrt{n}/\hbar\,\hat{\sigma}_{n-+}) = \sum_{k=0}^{\infty}\frac{(-itG\sqrt{n}/\hbar)^k}{k!}(\hat{\sigma}_{n-+})^k$$ note $(\sigma_{n-+})^k = 0$, $k \geq 2$

$$= \hat{1} - \frac{itG\sqrt{n}}{\hbar}\hat{\sigma}_{n-+}$$

The time evolution operator is the product of these 4 terms.

$$U_n(t) = \left(\hat{1}+\hat{\sigma}_{n++}\left(-1+e^{-it\omega n}\right)\right)\left(\hat{1}+\hat{\sigma}_{n--}\left(-1+e^{-it\omega n}\right)\right)\left(\hat{1}-\frac{itG\sqrt{n}}{\hbar}\hat{\sigma}_{n+-}\right)\times$$

$$\left(\hat{1}-\frac{itG\sqrt{n}}{\hbar}\hat{\sigma}_{n-+}\right)$$

$$= \left(\hat{1}+\hat{\sigma}_{n--}\left(-1+e^{-i\omega tn}\right)+\hat{\sigma}_{n++}\left(-1+e^{-it\omega n}\right)\right)\left(\hat{1}-\frac{itG\sqrt{n}}{\hbar}\left(\hat{\sigma}_{n+-}+\hat{\sigma}_{n-+}\right)-\frac{t^2G^2n}{\hbar^2}\hat{\sigma}_{n++}\right)$$

$$= \hat{1}-\frac{itG\sqrt{n}}{\hbar}\left(\hat{\sigma}_{n+-}+\hat{\sigma}_{n-+}\right)-\frac{t^2G^2n}{\hbar^2}\hat{\sigma}_{n++}+\hat{\sigma}_{n--}\left(-1+e^{-i\omega tn}\right)$$

$$-\frac{itG\sqrt{n}}{\hbar}\left(-1+e^{-i\omega tn}\right)\hat{\sigma}_{n-+}+\hat{\sigma}_{n++}\left(-1+e^{-it\omega n}\right)$$

$$-\frac{itG\sqrt{n}}{\hbar}\left(-1+e^{-it\omega n}\right)\hat{\sigma}_{n+-}-\frac{t^2G^2n}{\hbar^2}\left(-1+e^{-it\omega n}\right)\hat{\sigma}_{n++}$$

$$= \hat{1}-\frac{itG\sqrt{n}}{\hbar}\left(\hat{\sigma}_{n+-}+\hat{\sigma}_{n-+}\right)+\left(e^{-i\omega tn}-1\right)\hat{\sigma}_{n--}$$

$$+\hat{\sigma}_{n++}\left(-\frac{t^2G^2n}{\hbar^2}+\frac{t^2G^2n}{\hbar^2}+e^{-it\omega n}-1-\frac{t^2G^2n}{\hbar^2}e^{-it\omega n}\right)$$

$$= \hat{1}-\frac{itG\sqrt{n}}{\hbar}\left(\hat{\sigma}_{n+-}+\hat{\sigma}_{n-+}\right)+\left(e^{-i\omega tn}-1\right)\hat{\sigma}_{n--}-\left(1+\frac{t^2G^2n}{\hbar^2}e^{-it\omega n}\right)\hat{\sigma}_{n++}$$

a) excited state population $= \langle\hat{\sigma}_{n++}\rangle$ for an optical field in a number state

$$\langle\hat{\sigma}_{n++}\rangle = \langle g,n|\hat{U}_n^{\dagger}(t)\,\hat{\sigma}_{n++}\,\hat{U}_n(t)|g,n\rangle$$

$$= \left(e^{i\omega tn}\langle g,n|+\frac{itG^*\sqrt{n}}{\hbar}\langle e,n-1|\right)\left(-\frac{itG\sqrt{n}}{\hbar}|e,n-1\rangle\right)$$

$$= \frac{t^2|G|^2n}{\hbar^2}$$

Note:

$$U(t)|g,n\rangle = |g,n\rangle-\frac{itG\sqrt{n}}{\hbar}|e,n-1\rangle+\left(e^{-i\omega tn}-1\right)|g,n\rangle$$

$$= e^{-i\omega tn}|g,n\rangle-\frac{itG\sqrt{n}}{\hbar}|e,n-1\rangle$$

~~$\Rightarrow$ excited state population $\propto |\langle\sigma_{n++}\rangle|^2 = \frac{t^4G^4n^2}{\hbar^4}$~~

b) The total hamiltonian is $\hat{H} = \sum_{n=1}^{\infty} \hat{H}_n$,

← no $n=0$ term, as you cannot excite an atom with no photons.

$$\Rightarrow \hat{U}(t) = \prod_{n=1}^{\infty} \hat{U}_n(t)$$

$$\Rightarrow \hat{U}(t)|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \hat{U}(t)|g,n\rangle$$

Summation now goes from $n=1$.

Since $\hat{\sigma}_{n\pm\pm}|g,j\rangle = 0$ for $n \neq j$:

$$= e^{-|\alpha|^2/2} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \hat{U}_n(t)|g,n\rangle$$

For the same reason, if we act $\sum_{n=1}^{\infty} \hat{\sigma}_{n++}$ on the above we get:

$$= e^{-|\alpha|^2/2} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \hat{\sigma}_{n++} \hat{U}_n(t)|g,n\rangle$$

And from orthogonality we get:

$$\left\langle \sum_{n=1}^{\infty} \hat{\sigma}_{n++} \right\rangle = \sum_{m=1}^{\infty} \langle g,n|\hat{U}_n^{\dagger}(t) \sum_{n=1}^{\infty} \frac{(\alpha^*)^n \alpha^n}{(n!)^2} \hat{\sigma}_{n++} \hat{U}_n(t)|g,n\rangle e^{-|\alpha|^2}$$

$$= \sum_{n=1}^{\infty} \frac{\alpha^{*n}\alpha^n}{(n!)^2} \langle g,n|\hat{U}_n^{\dagger}(t) \hat{\sigma}_{n++} \hat{U}_n(t)|g,n\rangle e^{-|\alpha|^2}$$

Using a)

$$= \sum_{n=1}^{\infty} \frac{|\alpha|^{2n}}{(n!)^2} \frac{t^2|G|^2 n}{\hbar^2} e^{-|\alpha|^2}$$

$$\cancel{= \frac{t^2|G|^2}{\hbar^2} \alpha e^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(n-1)!}}$$

$$\cancel{= \frac{t^2|G|^2}{\hbar^2} \alpha e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}}$$

$$\cancel{= \frac{t^2|G|^2}{\hbar^2} \alpha e^{-|\alpha|^2} e^{\alpha}}$$

$$\cancel{= \frac{t^2|G|^2}{\hbar^2} \alpha e^{\alpha-|\alpha|^2}}$$

Using Mathematica: (see attached)

$$= \frac{t^2|G|^2|\alpha|}{\hbar^2} I_1(2|\alpha|) e^{-|\alpha|^2}$$

Modified Bessel function of the first kind

Q4.

Solving the summation ($a=|\alpha|$):

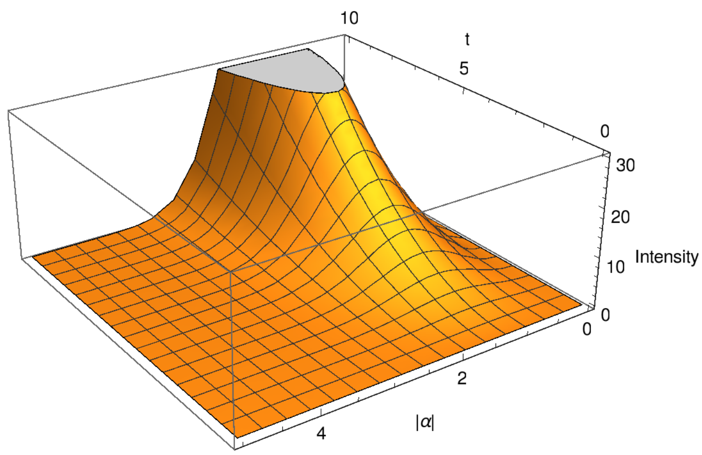
In[1]:=
```
Sum[a^(2 n)/(n!)^2 n, {n, 1, ∞}]
```
$$\text{Sum}\left[\frac{a^{2n}}{(n!)^2}n,\ \{n, 1, \infty\}\right]$$

Out[1]= a BesselI[1, 2 a]

c) Taking $\frac{|G|^2}{\hbar^2} = 1$

In[34]:=
```
Plot3D[t^2 a BesselI[1, 2 a] Exp[-a^2], {t, 0, 10},
 {a, 0, 5}, AxesLabel → {"t", "|α|", "Intensity"}]
```

Out[34]=



For $|\alpha|>4$ or $|\alpha|<0.01$ we have the number of particles in the excited state to neglegably small. Additionally, if you increase $|\alpha|$, we get the intensity approaching zero. Hence Rabi oscillations will be too small to see,