Assignment 4: Statistical Mechanics

Q1. The Dieterici equation of state for a gas is

$$p = \frac{k_B T}{v - b} \exp\left(-\frac{a}{k_B T v}\right)$$

where v = V/N. Find the critical point and compute the ratio $p_c v_c / k_B T_c$. Calculate the critical exponents β, δ and γ.

Q2. Consider the free energy

$$F = a(T)m^2 + b(T)m^4 + c(T)m^6$$

where b(T) < 0 and, for stability, c(T) > 0 for all T. Sketch the possible behaviours of the free energy as a(T) varies and, in each case, identify the ground state and metastable states. Show that the system undergoes a phase transition at some temperature T_c. Determine the value $a(T_c)$ and the discontinuity in m at the transition.

Q3. To model a warm plasma, assume that the electron and ion distribution functions are given by

$$f_{e0}(v) = \frac{a_e}{\pi} \frac{1}{v^2 + {a_e}^2}, \quad f_{i0}(v) = \frac{a_i}{\pi} \frac{1}{v^2 + {a_i}^2}$$

Consider a plasma with B_0=0, and E_0=0.

(a) Derive the dispersion relation (in integral form) in the Vlasov formalism assuming an electrostatic perturbation of angular frequency ω.

BONUS question

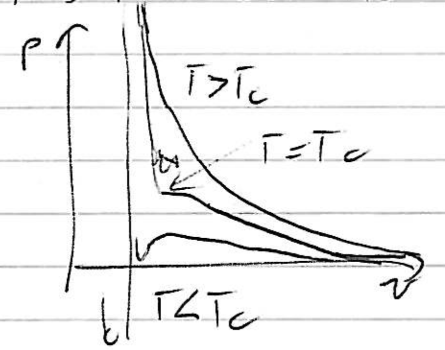
Q3(b) Obtain an approximate expression for the dispersion relation if $\omega \leq \Omega_p$, with Ω_p the plasma frequency. Under what conditions are the waves weakly damped?

Q1 Dieteric eqn. of state for a gas:

$$p = \frac{k_B T}{v-b} \exp\left(\frac{-a}{k_B T v}\right)$$

with $v = V/N$. Find the critical pt. & compute $p_c v_c / k_B T_c$

This fn. looks like



At critical pt. $\frac{dp}{dv} = \frac{d^2p}{dv^2} = 0$.

$$\frac{dp}{dv} = \frac{k_B T}{v-b} \exp\left(\frac{-a}{k_B T v}\right)\left(\frac{-1}{v-b} + \frac{a}{k_B T v^2}\right) \quad (1)$$

$$\frac{d^2p}{dv^2} = \exp\left(\frac{-a}{k_B T v}\right)\frac{k_B T}{(v-b)^3}\left(2 + \frac{a^2(v-b)^2}{k_B^2 T^2 v^4} - \frac{(v-b)(2v-b)2a}{v^3 k_B T}\right) \quad (2)$$

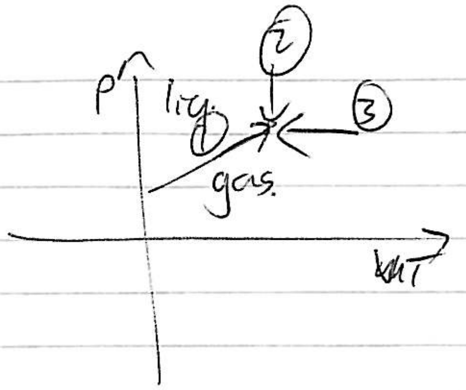
Solving (1) & (2) $= 0$ for T_c, v_c gives

$$k_B T_c = \frac{a}{4b} \quad 0.5, \qquad v_c = 2b \quad 0.5 \qquad P_c = \frac{a^2}{4b^2 e^2} \quad 0.5$$

$$\therefore \frac{P_c v_c}{k_B T_c} = \frac{a^2}{4b^2 e^2} \cdot \frac{2b}{(a/4b)} = \frac{2}{e^2} \quad 0.5$$

Critical Exponents:

① Along the co-existence line.



Write $\bar T = T/T_c$; $\bar v = v/v_c$; $\bar p = p/p_c$

Hence

$$\bar p p_c = \frac{k_B \bar T T_c}{(\bar v v_c - b)} \cdot \exp\left(\frac{-a}{k_B \bar T T_c \bar v v_c}\right)$$

$$\Rightarrow \quad \frac{\bar p}{e^2} = \frac{\bar T}{1-2\bar v} \cdot e^{-2/\bar T \bar v}$$

or $$\bar p = \frac{\bar T}{1-2\bar v} \exp\left(+2\left(1 - \frac{1}{\bar T \bar v}\right)\right). \quad 1$$

This is the law of corresponding states. At $\bar p = \bar v = \bar T = 1$.

& $$\frac{p_c v_c}{k_B T_c} = \frac{\frac{a}{4b^2 e^2} \cdot 2b}{k_B\left(\frac{a}{4b k_B}\right)} = 2/e^2$$

For $\bar T < 1$ $$\bar p = \frac{\bar T}{1-2\bar v_{liq}} \exp\left(+2\left(1 - \frac{1}{\bar T \bar v_{liq}}\right)\right)$$

$$= \frac{\bar T}{1-2\bar v_{gas}} \exp\left(+2\left(1 - \frac{1}{\bar T \bar v_{gas}}\right)\right). \quad 1$$

& solve for $\bar T$. Eq. is symmetric in $\bar v_{gas}$ & $\bar v_{liq}$ so write

$$\bar v_{gas} = 1 + \varepsilon/2 \ ; \quad \bar v_{liq} = 1 - \varepsilon/2 . \quad 1$$

& solve to leading order.

$$\frac{1-2\bar v_{gas}}{1-2\bar v_{liq}} = \exp\left(2\left(\cancel{1} - \frac{1}{\bar T \bar v_{liq}}\right) - 2\left(\cancel{1} - \frac{1}{\bar T \bar v_{gas}}\right)\right)$$

$$= \exp\left(\frac{2}{\bar T}\left(-\frac{1}{\bar v_{liq}} + \frac{1}{\bar v_{gas}}\right)\right).$$

So.

$$\bar T = \left(\ln\left(\frac{1-2\bar v_{gas}}{1-2\bar v_{liq}}\right)\right)^{-1} 2\left(\frac{-1}{\bar v_{liq}} + \frac{1}{\bar v_{gas}}\right)$$

Performing series & substituting:

$$\bar{T}_{\cancel{d}} = 1 - \frac{\varepsilon^2}{12} + O(\varepsilon^3)$$

$$= 1 - (v_{gas} - v_{liq.})^2/12 .$$ (1)

& so along co-existence line ~~$v_{gas} - v_{liq} \sim (T_c - T)^{\frac{1}{2}}$~~

$\bar{T} < 1 .$

$$= \frac{T - T_c}{T_c} \qquad \left(\frac{T - T_c}{T_c}\right) = \cancel{\xi} - \frac{(v_{gas} - v_{liq})^2}{12} .$$ (1)

$$v_{gas} - v_{liq} \sim (T_c - T)^{\frac{1}{2}}$$

(2) At $\bar{T} = \cancel{4t} 1$

$$\bar{p} = \frac{\bar{T}}{1 - 2\bar{v}} \cdot \exp\left(2\left(1 - \frac{1}{\bar{T}\bar{v}}\right)\right) \qquad (\#)$$

Sub. $\bar{T} = 1$ & write $\bar{v} = v/v_c = 1 + \varepsilon$. then expand (#) to $O(\varepsilon^4)$ =>. (1)

$$\bar{p} = p/p_c = 1 - \frac{2}{3}\varepsilon^3 + O(\varepsilon^4)$$

$$= 1 - \frac{2}{3}\left(\frac{v - v_c}{v_c}\right)^3$$

so

$$\frac{p - p_c}{p_c} = -\frac{2}{3}\left(\frac{v - v_c}{v_c}\right)^3$$ (1)

or/

$$p - p_c \sim (v - v_c)^3$$ (1)

(3) Compressibility. $k = -\frac{1}{v}\frac{\partial v}{\partial p}\Big|_T$ (1) for $T > T_c$.

At crit. pt. $\frac{\partial p}{\partial v} = 0$.

$$\frac{\partial p}{\partial v}\Big|_{T} = \exp\left(\frac{-a}{k_B T v}\right)\frac{k_B T}{v - b}\left(\frac{a}{k_B T v} - \frac{1}{v - b}\right)$$

Expand $\left.\frac{\partial p}{\partial v}\right|_{T, v=v_c}$ with $T = T_c + \varepsilon T_c = T_c(1+\varepsilon)$.

$$\left.\frac{\partial p}{\partial v}\right|_{T, v=v_c} = \frac{-a}{4 t^3} e^{-2/(1+\varepsilon)} \varepsilon$$ (1)

$$\simeq \frac{-a}{4 t_c^3 e^2} \varepsilon + O(\varepsilon^2)$$

& so $k = \frac{-1}{v} \left.\frac{\partial v}{\partial p}\right|_T \simeq (T - T_c)^{-1}$ (1)

Critical exponents

$$v_{gas} - v_{liq} \sim (T_c - T)^{\beta} \qquad \beta = \frac{1}{2}$$

$$(p - p_c) \sim (v - v_c)^{1/\delta} \qquad \delta = \frac{1}{3}$$

$$k \sim (T - T_c)^{\gamma} \qquad \gamma = -1.$$

These are the same as the Van der Waals eqn.

Q2: Consider
$$F = a(T)m^2 + b(T)m^4 + c(T)m^6$$

with $b(T) < 0$ & $c(T) > 0$.

Find solutions of $\frac{\partial F}{\partial m} = 0$ (equilibrium cond.)

$$\frac{\partial F}{\partial m} = 2a(T)m + 4b(T)m^3 + 6c(T)m^5$$
$$= m(2a + 4bm^2 + 6cm^4)$$
$$= 0$$

Solns are $m = 0$, $m^2 = \frac{-4b \pm \sqrt{16b^2 - 48ac}}{12c}$ (1)

If $16b^2 - 48ac > 0$ $\Rightarrow$ real solns of m^2 exist.
$b^2 > 3ac$

$$\& \quad m^2 = \frac{-b \pm \sqrt{b^2 - 3ac}}{3c}$$

We are given $b < 0$. If $b^2 - 3ac > b^2$, solutions are

(A) $a < 0$ $\quad m_0 = 0$, $m_1 = \pm\left(\frac{|b| + \sqrt{b^2 - 3ac}}{3c}\right)^{\frac{1}{2}}$ (1)

(B) If $0 < b^2 - 3ac < b^2$, solns are

$$m_0 = 0, \quad m_1 = \pm\left(\frac{|b| + \sqrt{b^2 - 3ac}}{3c}\right)^{\frac{1}{2}}$$
$$m_2 = \pm\left(\frac{|b| - \sqrt{b^2 - 3ac}}{3c}\right)^{\frac{1}{2}}$$ (1)

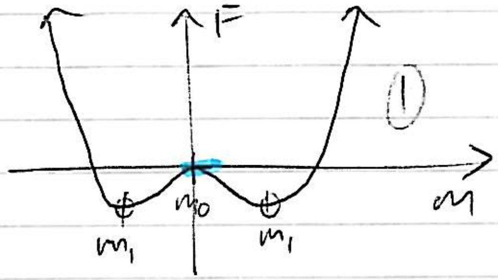
~~This condition e~~
The condition $0 < b^2 - 3ac < b^2 \Rightarrow 0 < a < b^2/3c$

(C) Finally, if $b^2 - 3ac < 0 \Rightarrow a > b^2/3c$ there is only one solution

$$m_0 = 0.$$ (1)

There are hence 3 regimes. The system behaviour is as follows:

(A) $a<0$



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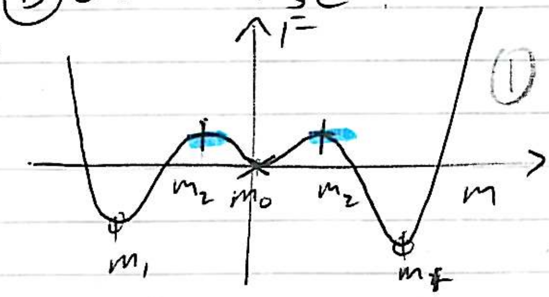
(B) $0<a<b^2/3c$



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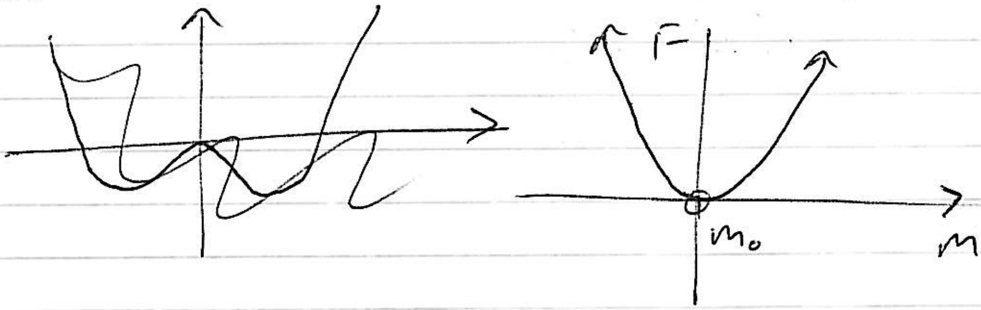
(C) $a>b^2/3c$



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o = ground state
x = metastable state
+ = unstable state

The system undergoes a transition at $a^* = b^2/3c$ 1
For $a^*(T) > b^2/3c$ $m_0 = 0$. is the grnd state.

For $a(T) < b^2/3c$ m_1 $(\neq 0)$ is the grnd state. 1

Q3 To model a warm plasma assume that

$$\hat{f}_{e0}(v) = \frac{a_e}{\pi}\frac{1}{v^2+a_e^2} \quad ; \quad \hat{f}_{i0}(v) = \frac{a_i}{\pi}\cdot\frac{1}{v^2+a_i^2}$$

Consider a plasma with $E_0 = B_0 = 0$.

(a) Vlasov eqn. for species α

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f + \frac{q}{m}\left(\underline{E} + \underline{v}\times\underline{B}\right)\cdot\frac{\partial f}{\partial \underline{v}} = 0 \quad ①$$

There are two species: ions and electrons. For both species can write

$$f = f_0(v) + f_1(x, v, t) \quad ①$$

as problem is 1D. To first order, perturbed Vlasov eqn. is

$$\frac{\partial f_1}{\partial t} + \underline{v_0}\cdot\nabla f_1 + \frac{q}{m}\left(\underline{E_1}\cdot\frac{\partial f_0}{\partial \underline{v}}\right) = 0 \quad ①$$

Assume $f_{e1} \propto e^{i(kx-\omega t)}$, $f_{i1} \propto e^{i(kx-\omega t)}$, so.

$$-i\omega f_1 + ikv f_1 + \frac{q}{m}E_1\frac{\partial f_0}{\partial v} = 0$$

For ions:

$$f_{i1} = \frac{-\frac{ie}{m_i}E_1\frac{\partial f_{i0}}{\partial v}}{\omega - kv} \quad ① \quad (1)$$

For electrons:

$$f_{e1} = \frac{\frac{ie}{m_e}E_1\frac{\partial f_{e0}}{\partial v}}{\omega - kv} \quad ① \quad (2)$$

Poisson's eqn:

$$\varepsilon_0\underline{\nabla}\cdot\underline{E} = -en_e + en_i \quad ①$$

As $E_0 = 0$ $\therefore$ $-en_{e0} + en_{i0} = 0 \Rightarrow n_{e0} = n_{i0}$.

$$\int_{-\infty}^{\infty}\hat{f}_{e0}(v)\,dv = \int_{-\infty}^{\infty}\frac{a_e}{\pi}\frac{1}{v^2+a_e^2}\,dv = 1$$

$$\int_{-\infty}^{\infty}\hat{f}_{i0}(v)\,dv = \int_{-\infty}^{\infty}\frac{a_i}{\pi}\frac{1}{v^2+a_i^2}\,dv = 1$$

[Note that the question gave $f_{eo}(v)$ & $f_{io}(v)$ – but by integrating it is obvious that this was the probability, no the distn. fn.]

To 1st order:
$$\varepsilon_0 ikE_1 = -en_{e1} + en_{i1}$$
$$= n_0 e\left(\int_{-\infty}^{\infty}\hat{f}_{i1}\,dv - \int_{-\infty}^{\infty}\hat{f}_{e1}\,dv\right) \quad (3)$$

with $n_0 = n_{eo} = n_{io}$

Substitute (1), (2) into (3) gives:

$$\varepsilon_0 ikE_1 = n_0 e\left[-\frac{ie}{m_i}E_1\int\frac{\frac{\partial\hat{f}_{io}}{\partial v}}{\omega-kv}dv - \frac{ie}{m_e}E_1\int\frac{\frac{\partial\hat{f}_{eo}}{\partial v}dv}{\omega-kv}\right]$$

$$1 = -\frac{n_0 e^2}{\varepsilon_0 m_e}\cdot\frac{1}{k}\left[\frac{m_e}{m_i}\int\frac{\frac{\partial\hat{f}_{io}}{\partial v}}{\omega-kv}dv + \int\frac{\frac{\partial\hat{f}_{eo}}{\partial v}dv}{\omega-kv}\right]$$

$$= -\frac{\Omega_p^2}{k}\left[\frac{m_e}{m_i}\int\frac{-2va_i}{\pi(v^2+a_i^2)^2}\frac{1}{\omega-kv}dv + \int\frac{-2va_e}{\pi(v^2+a_e^2)^2}\frac{1}{\omega-kv}dv\right].$$

where $\Omega_p = \left(\frac{n_0 e^2}{\varepsilon_0 m_e}\right)^{\frac{1}{2}}$ is the plasma frequency.

If $a_i \sim a_e$ (comparable) then

$$1 \simeq \frac{\Omega_p^2}{k}\cdot\frac{2a_e}{\pi}\int_{-\infty}^{\infty}\frac{v}{(v^2+a_e^2)^2}\cdot\frac{1}{\omega-kv}dv. \quad (4)$$

(b) Compute dispersion relation if $\omega \lesssim \Omega_p$.

Eq. (4) has a pole @ ~~$\omega = kv$~~ $v = \omega/k$. If $\frac{\omega^i}{k} > 0$ then the contour integral can be performed

$$+\int\frac{v}{(v^2+a_e^2)^2}\,\frac{1}{v-\omega/k}\,dv = \frac{k^2\pi}{2a_e(a_e k - i\omega)^2}$$

Hence, $$1 = \frac{\Omega_p^2}{\cancel{k}}\cdot\frac{\cancel{2a_e}}{\pi}\left(-\frac{1}{k}\right)\frac{\cancel{k}^2\pi}{\cancel{2a_e}(a_e k - i\omega)^2}$$
$$= \frac{-\Omega_p^2}{(a_e k - i\omega)^2}$$

& so $$-\omega^2 - 2a_e k\omega i + a_e^2 k^2 = -\Omega_p^2.$$

with solns $$\omega = \frac{-2a_e k i \pm \sqrt{-4a_e^2 k^2 + 4(a_e^2 k^2 + \Omega_p^2)}}{2}$$
$$= -a_e k i \pm \Omega_p$$

& $$\omega/k = -a_e i \pm \frac{\Omega_p}{k}.$$

But $a_e > 0$ & so $\text{Im}(\omega/k) < 0$.

This violates our assumption.

Search for solns. with $\omega^i/k < 0$.

$$\int \frac{g(u)}{u-\omega/k}\,du = \int_{-\infty}^{\infty} \frac{g(u)\,du}{u-\omega/k} + 2\pi i g(\omega/k)$$
$$= \frac{k^2\pi}{2a_e(a_e k - i\omega)^2} + 2\pi i \frac{\omega/k}{((\omega/k)^2 + a_e^2)^2}$$

Hence $$1 = \frac{\Omega_p^2}{k^2}\frac{2a_e}{\pi}\left[\frac{k^2\pi}{2a_e(a_e k - i\omega)^2} + \frac{2\pi i\,\omega/k}{((\omega/k)^2 + a_e^2)^2}\right]$$

Solving for $\omega \Rightarrow \quad \frac{\omega}{k} = i a_e(3 \pm 2\sqrt{2})$

These solns have $\text{Im}(\omega/k) > 0$ $\forall a_e > 0$.

$\Rightarrow$ ~~There are~~ no Solutions ~~that~~ have $\text{Im}(\omega/k) = 0$.

ie, they are all neutral.