Max Proft U5190335
Assignment 4
Q1

The critical point occurs when $\frac{\partial p}{\partial v} = \frac{\partial^2 p}{\partial v^2} = 0$

Finding $\frac{\partial p}{\partial v}$

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p[v_] = (k T)/(v - b) Exp[(-a)/(k T v)];
FullSimplify[p'[v]]
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$$-\frac{e^{-\frac{a}{k T v}} \left(a (b - v) + k T v^2\right)}{(b - v)^2 v^2}$$

Setting $\frac{\partial p}{\partial v} = 0$ we get:

$a (b - v) + k T v^2 = 0$

$\Rightarrow T = \frac{-a(b-v)}{v^2 k}$

Finding and setting $\frac{\partial^2 p}{\partial v^2} = 0$

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FullSimplify[p''[v]]
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$$\frac{e^{-\frac{a}{k T v}} \left(a^2 (b - v)^2 - 2 a k T (b - 2 v) (b - v) v + 2 k^2 T^2 v^4\right)}{k T v^4 (-b + v)^3}$$

$\Rightarrow a^2 (b - v)^2 - 2 a k T (b - 2 v) (b - v) v + 2 k^2 T^2 v^4 = 0$

Substituting in the expression for T in order to find v_c :

$\Rightarrow 3 a^2 (b - v_c)^2 + \frac{2 a^2 (b - 2 v_c) (b - v_c)^2}{v_c} = 0$

$\Rightarrow 3 v_c + 2 (b - 2 v_c) = 0$

$\Rightarrow v_c = 2 b$

Substituting this back into T to find T_c:

$T_c = \frac{a}{4 b k}$

Substituting this into p to find p_c:

$p_c = \frac{k}{2 b - b} \frac{a}{4 b k} \text{Exp}\left[\frac{-a}{k 2 b} \frac{4 b k}{a}\right]$

$p_c = \frac{a}{4 b^2 e^2}$

$\frac{p_c v_c}{k T_c} = \frac{\frac{a}{4 b^2 e^2} * 2 b}{k \frac{a}{4 b k}} = \frac{2}{e^2}$

$\overline{p} = \frac{p}{p_c} = \frac{2 \overline{T} v_c}{2 v - v_c} \frac{e^2}{2} \text{Exp}\left[\frac{-a}{k \overline{T} T_c \overline{v} v_c}\right] = \frac{2 \overline{T}}{2 \overline{V} - 1} \frac{e^2}{2} \text{Exp}\left[\frac{-2}{\overline{T} \overline{v}}\right]$

Finding the expansion coefficients:

Along the coexistance curve: $\overline{p}_g = \overline{p}_L$, and $\overline{T}_g = \overline{T}_L$

$$\frac{2\,\overline{T}}{2\,\overline{V}_g-1}\,\frac{e^2}{2}\,\mathrm{Exp}\left[\frac{-2}{\overline{T}\,\overline{v}_g}\right]=\frac{2\,\overline{T}}{2\,\overline{V}_L-1}\,\frac{e^2}{2}\,\mathrm{Exp}\left[\frac{-2}{\overline{T}\,\overline{v}_L}\right]$$

$$\mathrm{Exp}\left[\frac{-2}{\overline{T}}\left(\frac{1}{\overline{v}_g}-\frac{1}{\overline{v}_L}\right)\right]=\frac{2\,\overline{v}_g-1}{2\,\overline{V}_L-1}$$

$$\overline{T}=\frac{-2\left(\frac{1}{\overline{v}_g}-\frac{1}{\overline{v}_L}\right)}{\mathrm{Ln}\left(\frac{2\,\overline{v}_g-1}{2\,\overline{V}_L-1}\right)}$$

Setting $\overline{v}_g \approx 1+\epsilon/2$ and $\overline{v}_L \approx 1-\epsilon/2$

$$\overline{T}=\frac{-2\left(\frac{\overline{v}_L-\overline{v}_g}{\overline{v}_g\,\overline{v}_L}\right)}{\mathrm{Ln}(1+\epsilon)-\mathrm{Ln}(1-\epsilon)}=\frac{-2\left(\frac{1-\epsilon/2-(1+\epsilon/2)}{(1-\epsilon/2)\,(1+\epsilon/2)}\right)}{\epsilon-(-\epsilon)}=\frac{-2\left(\frac{-\epsilon}{1-\epsilon^2/4}\right)}{2\,\epsilon}=\frac{1}{1-\epsilon^2/4}$$

From Binomial Expansion:

$\overline{T}\approx 1+\epsilon^2/4$

Since $\epsilon = v_g - v_L$, if we rearrange we get:

$(v_g - v_L) \sim (T - T_c)^{-0.5}$

And so we get β=-0.5

Along an isotherm:

$$p = \frac{k\,T}{v-b}\,\mathrm{Exp}\left[\frac{-a}{k\,T\,v}\right]$$

If we taylor expand this about $v = v_c$ we get (By definition of the critical point, the first and second derivatives are zero):

$$p = p_c + \frac{d^3\left(\frac{k\,T}{v-b}\,\mathrm{Exp}\left[\frac{-a}{k\,T\,v}\right]\right)}{dv^3}\Bigg|_{v=v_c}(v-v_c)^3$$

$$p = p_c - \frac{\mathrm{Exp}\left[-\frac{a}{k\,T\,v_c}\right]\left(a\,(b-v_c)+k\,{v_c}^2\right)}{(b-v_c)^2\,{v_c}^2}(v-v_c)^3$$

$$p - p_c = \frac{4\,k\,(-1+T_c)}{e^2\,{v_c}^2}(v_c - v)^3$$

And so we get δ=3

$$\kappa = \frac{-1}{v}\,\frac{\partial v}{\partial p}\Bigg|_T$$

Finding $\frac{\partial p}{\partial v}$, and evaluating this at $v = v_c$

$$\mathrm{FullSimplify}\left[\mathrm{D}\left[\frac{\mathrm{k\,T}}{\mathrm{v-vc}/2}\,\mathrm{Exp}\left[\frac{-\mathrm{Tc}*2\,\mathrm{vc\,k}}{\mathrm{k\,T\,v}}\right],\ \mathrm{v}\right]\right]$$

$$-\frac{4\,e^{-\frac{2\,\mathrm{Tc\,vc}}{\mathrm{T\,v}}}\,\mathrm{k}\left(\mathrm{T\,v}^2+\mathrm{Tc\,vc}\,(-2\,\mathrm{v}+\mathrm{vc})\right)}{\mathrm{v}^2\,(-2\,\mathrm{v}+\mathrm{vc})^2}$$

$$\mathrm{FullSimplify}\left[-\frac{4\,e^{-\frac{2\,\mathrm{Tc\,vc}}{\mathrm{T\,v}}}\,\mathrm{k}\left(\mathrm{T\,v}^2+\mathrm{Tc\,vc}\,(-2\,\mathrm{v}+\mathrm{vc})\right)}{\mathrm{v}^2\,(-2\,\mathrm{v}+\mathrm{vc})^2}\ /.\ \mathrm{v}\to\mathrm{vc}\right]$$

$$\frac{4\,e^{-\frac{2\,\mathrm{Tc}}{\mathrm{T}}}\,\mathrm{k}\,(-\mathrm{T}+\mathrm{Tc})}{\mathrm{vc}^2}$$

Hence we get:

$\kappa = \frac{-1}{v_c} \frac{v_c}{4 e^{-\frac{2\,\mathrm{Tc}}{T}} k (-T+\mathrm{Tc})} = \frac{e^{2/\overline{T}}}{4 k} \frac{1}{T-T_c}$

And hence γ=1

Q2

Throughout this question, assume that a, b, c are functions of T.

b < 0

c > 0

Stable solutions occur when $\frac{\partial F}{\partial m} = 0$ and $\frac{\partial^2 F}{\partial m^2} > 0$

Solving $\frac{\partial F}{\partial m} = 0$ we get either m=0 or:

$a + 2\,b\,m^2 + 3\,c\,m^4 = 0$

$m^2 = \frac{-b \pm \sqrt{b^2 - 3\,a\,c}}{3\,c}$

If $b^2 - 3\,\mathrm{ac} < 0$, we will only have 1 real solution, m=0.

$a > \frac{b^2}{3\,c} > 0$

Suppose that $b^2 - 3\,\mathrm{ac} > 0$ and that $-b - \sqrt{b^2 - 3\,\mathrm{ac}} < 0$.

$\Rightarrow b^2 - 3\,\mathrm{ac} > b^2$

$\Rightarrow \mathrm{ac} < 0$

$\Rightarrow a < 0$

This means we have 3 solutions:

m=0

$m = \pm \sqrt{\frac{-b + \sqrt{b^2 - 3\,a\,c}}{3\,c}}$

Suppose that $b^2 - 3\,\mathrm{ac} > 0$ and that $-b - \sqrt{b^2 - 4\,\mathrm{ac}} > 0$.

$\Rightarrow b^2 - 3\,\mathrm{ac} < b^2$

$\Rightarrow \mathrm{ac} > 0$

$\Rightarrow a > 0$

This means we have 5 solutions:

m=0 or:

$m = \pm \sqrt{\frac{-b \pm \sqrt{b^2 - 3\,a\,c}}{3\,c}}$

I assume that a ≠ 0, as this case can be considered as the limit of the above cases.

Finding out which of these solutions are stable:

i.e. $\frac{\partial^2 F}{\partial m^2} > 0$

$\frac{\partial^2 F}{\partial m^2} = 2\,a + 12\,b\,m^2 + 30\,c\,m^4$

1 solution:

$m=0 \Rightarrow \frac{\partial^2 F}{\partial m^2} = 2\,a$

Since $a > \frac{b^2}{3\,c} > 0$, this solution is stable.

3 Solutions:

$m=0 \Rightarrow \frac{\partial^2 F}{\partial m^2} = 2\,a$

Since a<0, this is unstable.

$$m = \pm\sqrt{\frac{-b+\sqrt{b^2-3\,a\,c}}{3\,c}}$$

$$\frac{\partial^2 F}{\partial m^2} = 2\,a + 12\,b\,\frac{-b+\sqrt{b^2-3\,a\,c}}{3\,c} + 30\,c\left(\frac{-b+\sqrt{b^2-3\,a\,c}}{3\,c}\right)^2$$

$$= 2\,a + 4\,b\,\frac{-b+\sqrt{b^2-3\,a\,c}}{c} + 10\left(\frac{b^2-2\,b\,\sqrt{b^2-3\,a\,c}+b^2-3\,a\,c}{3\,c}\right)$$

$$= 2\,a - \frac{4\,b^2}{c} + \frac{4\,b}{c}\sqrt{b^2-3\,a\,c} + \frac{20\,b^2}{3\,c} - \frac{20\,b}{3\,c}\sqrt{b^2-3\,a\,c} - 10\,a$$

$$= -8\,a + \frac{8\,b^2}{3\,c} - \frac{8\,b\,\sqrt{b^2-3\,a\,c}}{3\,c}$$

Let $\Delta=\sqrt{b^2 - 3\,a\,c}$

$$\frac{\partial^2 F}{\partial m^2} = \frac{8\,\Delta^2}{3\,c} - 8\,\frac{b\,\Delta}{3\,c}$$

Since $\Delta > 0$, $c > 0$, $b < 0$, both of these solutions have $\frac{\partial^2 F}{\partial m^2} > 0$, and so they are stable.

5 Solutions:

$m=0 \Rightarrow \frac{\partial^2 F}{\partial m^2} = 2\,a$

Since a>0, this is stable.

$$m^2 = \frac{-b\pm\sqrt{b^2-3\,a\,c}}{3\,c}$$

The + solution is the same as in the 3 solution case, and so we get these solutions to be stable.

Now consider only the - solution

$$\frac{\partial^2 F}{\partial m^2} = 2\,a + 12\,b\,\frac{-b-\sqrt{b^2-3\,a\,c}}{3\,c} + 30\,c\left(\frac{-b-\sqrt{b^2-3\,a\,c}}{3\,c}\right)^2$$

$$= 2\,a + 4\,b\,\frac{-b-\sqrt{b^2-3\,a\,c}}{c} + 10\left(\frac{b^2+2\,b\,\sqrt{b^2-3\,a\,c}+b^2-3\,a\,c}{3\,c}\right)$$

$$= 2\,a - \frac{4\,b^2}{c} - \frac{4\,b}{c}\sqrt{b^2-3\,a\,c} + \frac{20\,b^2}{3\,c} + \frac{20\,b}{3\,c}\sqrt{b^2-3\,a\,c} - 10\,a$$

$$= -8\,a + \frac{8\,b^2}{3\,c} + \frac{8\,b\,\sqrt{b^2-3\,a\,c}}{3\,c}$$

Let $\Delta=\sqrt{b^2 - 3\,a\,c}$

$$\frac{\partial^2 F}{\partial m^2} = \frac{8\,\Delta^2}{3\,c} + 8\,\frac{b\,\Delta}{3\,c} = \frac{8\,\Delta}{3\,c}\,(\Delta + b)$$

This is stable if $\Delta > -b$

$\Rightarrow -3\,a\,c > 0$

$\Rightarrow a<0$

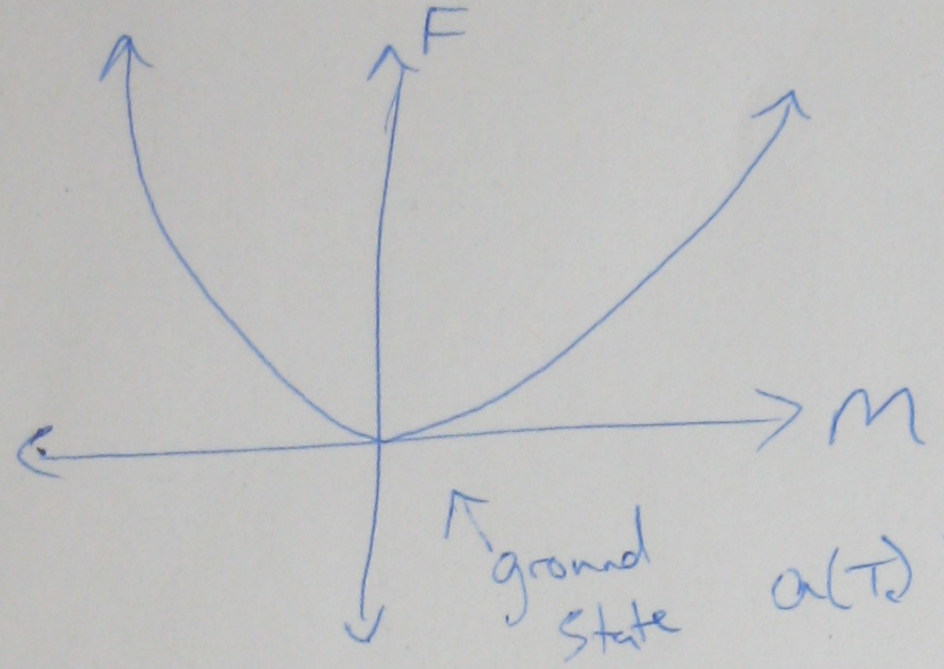
But for this solution, we have $a > 0$.

Hence it cannot be stable.

Sketches attached at the end.

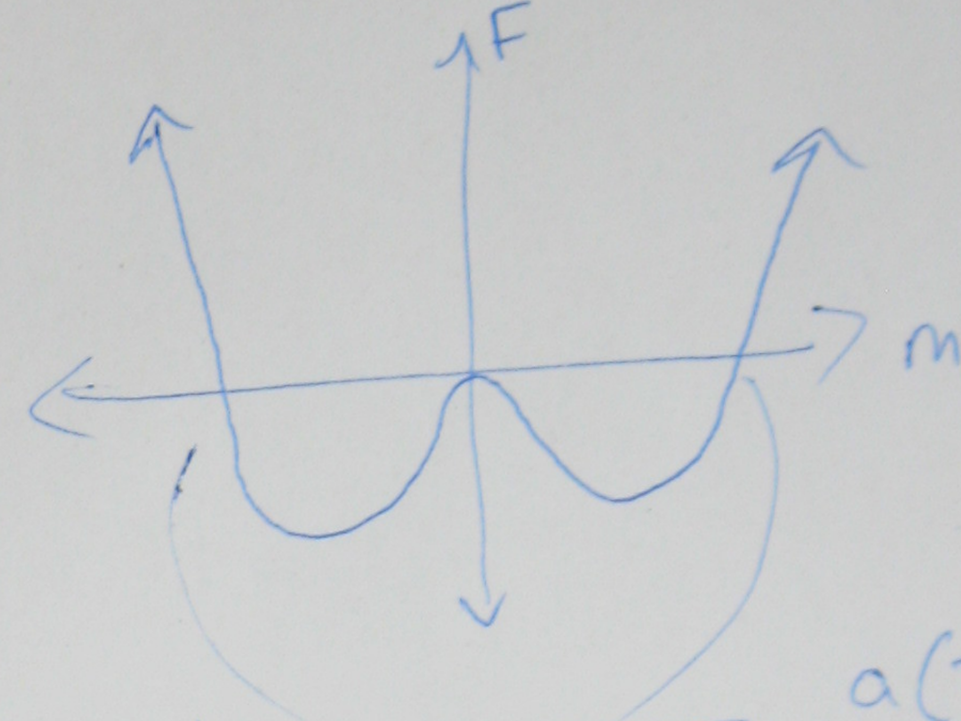
Q2

1 Solution



$a(T_c) > \frac{b^2}{3c}$

3 solutions

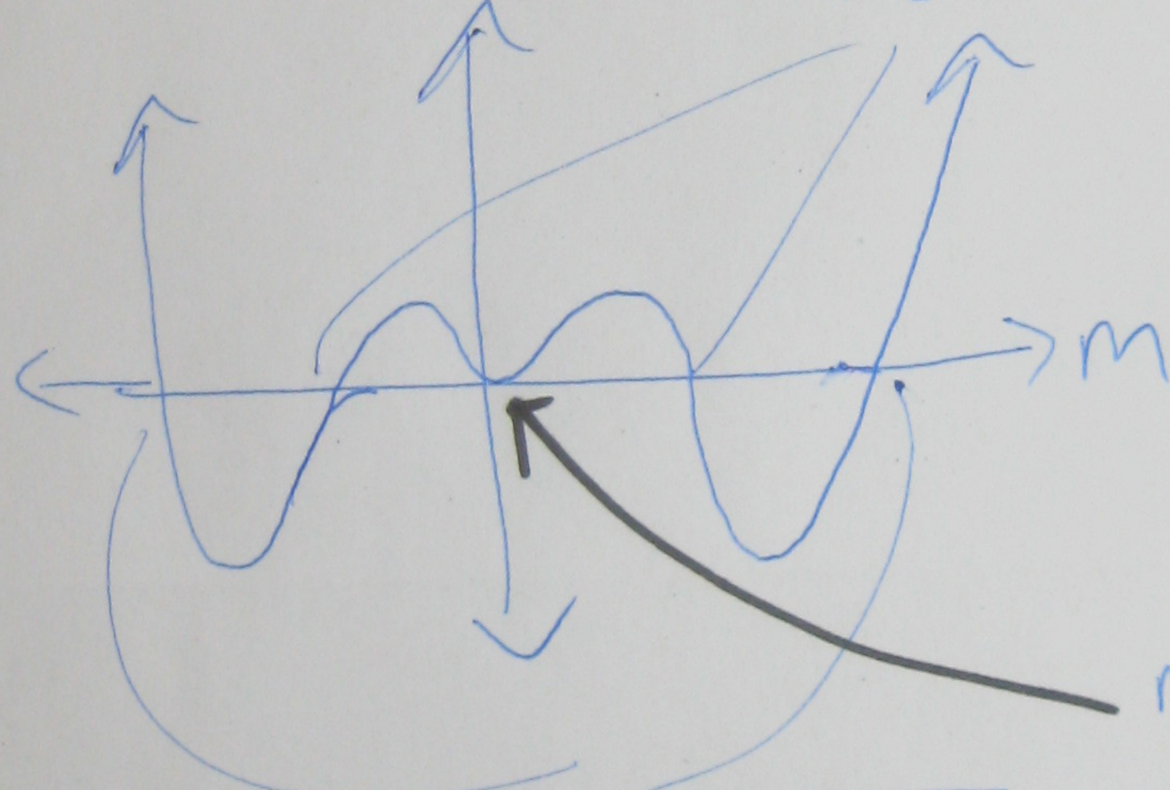


$a(T_c) < 0$

2 ground states: $\pm\sqrt{\frac{-b+\sqrt{b^2-3ac}}{3c}}$

5 solutions

$\pm\sqrt{\frac{-b-\sqrt{b^2-3ac}}{3c}}$



Metastable state

2 ground states: $\pm\sqrt{\frac{-b+\sqrt{b^2-3ac}}{3c}}$

$0 < a(T_c) < \frac{b^2}{3c}$

Since Isotropic, we can treat this as a 1D problem.

Q3. let $f = f_0(v) + f_1(x, v, t)$

$E = E_0 + E_1$, $E_0 = 0$, $B = 0$.

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \frac{q}{m}(E + v \times B) \cdot \frac{\partial f}{\partial v} = 0$$

Keeping 1st order terms only:

$$\Rightarrow \partial_t f_1 + v \partial_x f_1 + \frac{q}{m} E_1 \partial_v f_0 = 0$$

$$\Rightarrow \left(\partial_t + v \partial_x\right) f_1 = \frac{-q}{m} E_1 \partial_v f_0$$

$$= \frac{-q}{m} E_1 \frac{-2va}{\pi(a^2+v^2)^2}$$

$$= \frac{2v\pi q E_1}{am} f_0(v)^2$$

$$\Rightarrow \frac{am}{2\pi v q f_0(v)^2}\left(\partial_t + v\partial_x\right) f_1 = E_1$$

If there is a perturbation of angular frequency ω,

$$E_1 \propto e^{-(i\omega t - ikx)}$$

$$\Rightarrow f_1 \propto e^{-(i\omega t - ikx)}$$

$$\Rightarrow E_1 = \frac{am}{2\pi v q f_0(v)^2}\left(-i\omega + ikv\right) f_1(x, v, t)$$

$$\Rightarrow f_1(x, v, t) = \frac{2\pi v q E_1 f_0(v)^2}{iam(kv - \omega)}$$

From $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ we get:

To first order we get $\nabla \cdot \vec{E}_0 = e n_{i0} - e n_{e0} \Rightarrow n_{i0} = n_{e0}$

$ik E_1 \varepsilon_0 = e n_i - e n_e$ (assuming ion has charge $+e$)

~~Since f_{i0} and f_{e0} are normalised~~

~~(see normalisation)~~

To second order we get:

$$ik E_1 \varepsilon_0 = e n_i - e n_e$$

$$= e(n_{i0} + n_{i1}) - e(n_{e0} + n_{e1}) = e(n_{i1} - n_{e1})$$

$$= e \int f_{i1}\, dv - e \int f_{e1}\, dv$$

substituting in for f_{i1} and f_{e1}, and dividing both sides by $\frac{E_1}{e}$ we get our dispersion relation:

$$\frac{ik\varepsilon_0}{e} = \frac{2\pi e}{i a m_i} \int_{\infty}^{\infty} \frac{v f_{0i}(v)^2}{kv - \omega}\, dv + \frac{2\pi e}{i a m_e} \int_{-\infty}^{\infty} \frac{v f_{0i}(v)}{kv - \omega}\, dv$$