The Ginzburg-Landau Equations for Superconductivity with Random Fluctuations

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Dedicated to the memory of Sergey L'vovich Sobolev, one of the greatest mathematicians of the twentieth century

Abstract Thermal fluctuations and material inhomogeneities have a large effect on superconducting phenomena, possibly inducing transitions to the non-superconducting state. To gain a better understanding of these effects, the Ginzburg–Landau model is studied in situations for which the described physical processes are subject to uncertainty. An adequate description of such processes is possible with the help of stochastic partial differential equations. The boundary value problem of Neumann type for the stochastic Ginzburg–Landau equations with additive and multiplicative white noise is investigated. We use white noise with minimal restriction on its independence property. The existence and uniqueness of weak and strong statistical solutions are proved. Our approach is based on using difference schemes for the Ginzburg–Landau equations.

1 Introduction

This paper is dedicated to the memory of Sergey L'vovich Sobolev. His outstanding contributions to the theory for the equations of mathematical physics are extremely deep and influential. Indeed, since the 1960s, practically all investigations in the aforementioned field of mathematics use Sobolev

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spaces and, at the least, are thereby guided by Sobolev's ideas. The present paper, of course, is no exception to this common rule. Moreover, the use of Sobolev spaces in complicated functional constructions for stochastic partial differential equations is especially successful and effective. Note also that being the closest aide to I.V. Kurchatov in the realization of the nuclear project in the Soviet Union after 1943, S.L. Sobolev took part in the numerical solution of huge problems of mathematical physics. From that time on to the end of his life, he had an invariable interest in the discrete approximation of continuum objects, especially in cubature formulas. In the present paper, discrete approximations are not only used, they play a crucial role in obtaining the main results.

This paper is devoted to the mathematical study of a boundary value problem for the stochastic Ginzburg–Landau model of superconductivity; we hope it will promote a better understanding of the transitions that occur between the superconducting and nonsuperconducting states.

In 1908, Kamerlingh–Onnes discovered that when metals such as mercury, lead, and tin are cooled to an absolute temperature below some small but positive critical value, their electrical resistivity completely disappears. This was a great surprise since what was expected is that the resistivity of metals would smoothly tend to zero as the temperature also tended to zero. In addition to this *zero resistance* property, superconductors are characterized by the property of *perfect diamagnetism*. This phenomenon was discovered in 1933 by Meissner and Ochsenfeld and is also known as the Meissner effect. What they observed is that not only is a magnetic field excluded from a superconductor, i.e., if a magnetic field is applied to a superconducting material at a temperature below the critical temperature, it does not penetrate into the material, but also that a magnetic field is expelled from a superconductor, i.e., if a superconductor subject to a magnetic field is cooled through the critical temperature, the magnetic field is expelled from the material. One of the consequences of the Meissner effect is that superconductors cannot be "perfect conductors" which are idealized (and unattainable) materials that have zero resistivity and that can be described by the linear Maxwell equations of electromagnetism. For such materials the magnetic field would not be expelled from the material when it is cooled through the critical temperature.

Superconductivity was not adequately explained until, in 1957, Bardeen, Cooper, and Schrieffer (BCS) [1] published their landmark paper describing a microscopic theory of superconductivity. However, even earlier, several phenomenological *continuum* theories were proposed, most notably by Ginzburg and Landau [20] in 1950. The Ginzburg–Landau theory was itself based on a general theory, introduced by Landau in 1937, for second-order phase transitions in fluids. Ginzburg and Landau thought of conducting electrons as being a "fluid" that could appear in two phases, namely superconducting and normal (non-superconducting). Through a stroke of intuitive genius, Ginzburg and Landau added to the theory of phase transitions certain effects, motivated by quantum-mechanical considerations, to account for how the electron "fluid" motion is affected by the presence of magnetic fields. In 1959, Gor'kov [21] showed that, in an appropriate limit, the macroscopic Ginzburg–Landau theory can be derived from the microscopic BCS theory. Details about the Ginzburg–Landau model can be found in [7, 13, 12, 41], the last of which may also be consulted for details about the BCS model.

The dependent variables of the Ginzburg–Landau model are the complex-valued order parameter ψ and the vector-valued magnetic potential A. Physically interesting variables such as the density of superconducting electrons, the current, and the induced magnetic field can be easily deduced from ψ and A. The Ginzburg–Landau model itself can be expressed as a system of two coupled partial differential equations from which ψ and A can be determined. One of these equations is a vector-valued, nonlinear Maxwell equation that relates the supercurrent, i.e., the current that flows without resistance, to a nonlinear function of ψ, $\nabla\psi$, and A. The second equation is a complex-valued equation that relates spatial and temporal variations of ψ to a nonlinear potential energy term. After appropriate non-dimensionalizations, there are two non-dimensional parameters appearing in the differential equations. One is the ratio of the relaxation times of ψ and A, the other, known as the Ginzburg–Landau parameter, is the ratio of the characteristic lengths over which ψ and A vary. These two length scales are referred to as the coherence and penetration lengths respectively.

In this paper, we consider a simplified Ginzburg–Landau system for ψ in which A is assumed to be a given vector-valued field. There are two situations of paramount practical interest for which the use of this simplified Ginzburg–Landau system can be justified. First, for high values of the Ginzburg–Landau parameter, it can be shown [6, 12] that, to leading order, the magnetic field in a superconductor is simply that given by the linear Maxwell equations so that A may be determined from these equations. Thus, insofar as the other component equation of the Ginzburg–Landau model is concerned, A can be viewed as a given vector field. A similar uncoupling can be shown to occur for thin film samples [5] for all values of the Ginzburg–Landau parameter. Most superconductors of practical interest are characterized by "high" values of the Ginzburg–Landau parameter and superconducting films are of very substantial technological interest; the simplified Ginzburg–Landau system we study can be used to model both of these situations. Furthermore, in the more general case where one has to consider the fully coupled Ginzburg–Landau equations for ψ and A, random fluctuations enter into the system in very much the same way as they do for the simplified system, so much of what is learned about stochastic versions of the simplified system applies to stochastic versions of the full system.

The Ginzburg–Landau theory is applicable only to highly idealized physical contexts that do not take into account factors such as material inhomogeneities and thermal fluctuations due to molecular vibrations. Both these factors play a crucial role in practical superconductivity since the former enables large currents to flow through a superconductor without resistance while the latter can have the opposite effect, especially at temperatures close to critical transition temperature (see, for example, [30, 39]). In [22], it is shown that, within the Ginzburg–Landau framework, thermal fluctuations are properly modeled by an additive white noise term in the Ginzburg–Landau equation for ψ; the amplitude of the noise term grows as the temperature approaches the critical temperature. In [4, 30], it is shown that, again in the Ginzburg–Landau framework, material inhomogeneities can be correctly modeled through the coefficient of the linear (in ψ) term in the Ginzburg–Landau equation for ψ; random variations in the material properties can thus be modeled as random perturbations in this coefficient which results in a multiplicative white noise term in the Ginzburg–Landau equations. In this paper, we treat both the additive and multiplicative noise cases. Studies of the physics of superconductors in the presence of white noise perturbations can be found in [11, 15, 23, 35, 39, 42, 43]; computational studies of the Ginzburg–Landau equations with additive and multiplicative noise are given in [9, 10].

In this paper, we study the stochastic Ginzburg–Landau equation written in the following dimensionless form:

$$d\psi(t,x) + \big((i\nabla + A(x))^2\psi - \psi + |\psi|^2\psi\big)dt = \widehat{r}[\psi]dW, \quad t > 0,\ x \in G \subset \mathbb{R}^d\,, \tag{1.1}$$

where G is a bounded domain, $d = 2, 3$, and an explanation of the notation employed on the right-hand side of (1.1) is given below in (1.3) and (1.4). On the boundary ∂G of G, we set

$$\big(i\nabla + A(x)\big)\psi(t,x)\cdot n = 0, \quad t > 0,\ x \in \partial G\,, \tag{1.2}$$

where n denotes the unit outer normal vector to ∂G.

From the view of the general theory of dynamical systems, the superconducting state is a stable steady-state solution of (1.1) (with zero right-hand side). The disappearance of the superconducting state (when some parameter of the system changes) means that some other steady-state solution of (1.1) arises and becomes stable or either time-periodic or chaotic behavior is realized.

We emphasize that when the dynamical system became unstable, the classical derivation of the equation for the superconducting state, rigorously speaking, looses its correctness. Indeed, in that derivation, as well as in other derivations of such a kind, only the main "forces" controlling the situation are taken into account and all relatively small and unessential "forces" are omitted, implicitly assuming stability in the sense that small fluctuations of "forces" lead to small fluctuations of the state. In unstable situations, this argument is evidently incorrect. The alternative is to replace, in the unstable situation, all small and unessential "forces" by white noise forcing (additive white noise) or perhaps by white noise multiplied by a function proportional to the state (multiplicative white noise). The physical basis of this approach is that, since "values" of white noise at different times are statistically independent, white noise renders a "smoothing" influence on the dynamical system. In more rigorous terms, this means the addition of white noise to the right-hand side of (1.1) leads to the substitution of many steady-state solutions of (1.1) by the unique (ergodic) statistical steady-state solution of (1.1) that is stable, i.e., that satisfies the *mixing property.* We also note that, in stable situations, replacing unessential "forces" by additive (multiplicative) white noise means taking into account thermal (material inhomogenety) fluctuations, as was noted above.

Very important arguments that can be used to justify the physical adequateness of the aforementioned modeling of superconductivity effects with the help of the stochastic problem (1.1) and (1.2) are given by recent results about ergodicity for abstract dynamical systems, including the two-dimensional Navier–Stokes and Ginzburg–Landau equations with random kick forces or additive white noise. The first results in this direction were obtained in [14, 16, 29]. In these papers, ergodicity was proved in stable situations, i.e., when the corresponding dynamical system with random forces omitted is stable. In the case of an unstable dynamical system, ergodicity was established in [36, 37, 38]. A detailed exposition of this topic can be found in [28].

Taking into account all of the above discussion, the following plan for the mathematical investigation of the superconducting state and its possible disappearance in industrial conditions is possible.

- Proof of the existence and uniqueness of weak and strong solutions of the stochastic boundary value problem (1.1) and (1.2).
- Proof of the ergodicity property for the random dynamical system generated by (1.1) and (1.2).
- Investigation of the disappearance of the supercoducting state in terms of the ergodic measure P that corresponds to the stochastic problem (1.1) and (1.2).

This paper is devoted to the proof of the first of these assertions.

The list of investigations of stochastic parabolic partial differential equations is huge because equations of such type arise in many problems of mathematics, physics, biology, and other applications. Here, we cite only the earliest papers in this field and papers closely connected with our paper. Investigations of linear parabolic stochastic partial differential equations were begun in the middle of 1960s [8]. Nonlinear stochastic parabolic equations were studied in [2, 33] and the stochastic Navier–Stokes system was studied in [3, 44, 45]. The paper [27] and the book [34] contain many deep results on these topics as well as a detailed historical review. Lastly, we note the works [25, 32].

In this paper, we study the stochastic boundary value problem (1.1) and (1.2) for the Ginzburg–Landau equation. Note that the right-hand side in

(1.1) should be written in a more detailed way as follows:

$$\widehat{r}[\psi]dW = r\big(\mathrm{Re}\psi(t,x)\big)\, d\mathrm{Re}W(t,x) + ir\big(\mathrm{Im}\psi(t,x)\big)\, d\mathrm{Im}W(t,x), \qquad (1.3)$$

where $dW = dW(t,x)$ is a complex-valued white noise and, as usual, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real and imaginary parts of a complex number z respectively. In addition, $r(\lambda)$, $\lambda \in \mathbb{R}$, is, roughly speaking,[1] the following function:

$$r(\lambda) = \max\left(\rho_1,\ \rho_2|\lambda|\right), \qquad \rho_1 > 0,\ \rho_2 \geqslant 0. \qquad (1.4)$$

In particular, when $\rho_2 = 0$, (1.3) reduces to complex-valued additive white noise. Note immediately that the main difficulties we are forced to overcome in this paper are connected with the case $\rho_2 > 0$ which results in some kind of multiplicative white noise. The form (1.3) of the random fluctuations for the Ginzburg–Landau equation is reasonable from our point of view when, describing Ginzburg–Landau flow in instable situation, one replaces all small and unessential "forces" by stochastically independent fluctuations, i.e., by white noise. Indeed, since by the definition of complex-valued white noise $dW(t,x)$, its real ($d\mathrm{Re}W(t,x)$) and imaginary ($d\mathrm{Im}W(t,x)$) parts are mutually independent white noises [19, Chapt. III, Sect. 1]), (1.3) gives the maximal independent form of multiplicative white noise.

In this paper, we provide a detailed exposition of the proof of the existence and uniqueness of weak and strong statistical solutions of the stochastic boundary value problem (1.1) and (1.2). The main feature of our exposition is that, to prove the existence of a weak solution, we use, instead of Galerkin approximations, *approximations by method of lines*, i.e., we introduce a finite difference approximation of the Ginzburg–Landau equation with respect to the spatial variables. Although the method of lines is more complicated in realization than Galerkin's method, it has one important advantage: method of lines approximations inherit the structure of the Ginzburg–Landau equation much better than do Galerkin ones and therefore we can obtain many estimates for method of line approximations that cannot be obtained for Galerkin approximations. All these estimates we essentially use in our proof in order to overcome difficulties arising mostly because of the multiplicative structure of white noise. Nevertheless, one important a priori estimate which can be derived (formally) for the Ginzburg–Landau equation we cannot yet derive for its method of lines approximation. That is why for the three-dimensional Ginzburg–Landau equation with multiplicative white noise, we have proved here only the existence of a weak solution. For the two-dimensional Ginzburg-landau equation with multiplicative white noise as well as for the two- and three-dimensional Ginzburg–Landau equation with additive white noise, we can prove the existence and uniqueness of both weak and strong solutions.

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[1] In fact, $r(\lambda)$, is the function (1.4) smoothed at points of discontinuity of its derivative. See the exact definition given below in (3.19).

The structure of the paper can be deduced from its content as described above.

2 The Ginzburg–Landau Equation and Its Finite Difference Approximation

In this section, we formulate the boundary value problem for the (simplified) Ginzburg–Landau equations without fluctuations and define an approximation by the method of lines that will play an important role in our analysis.

2.1 Boundary value problem for the Ginzburg–Landau equation

Let $G \subset \mathbb{R}^d$, $d = 2, 3$, denote a bounded domain with C^∞-boundary ∂G, and let $Q_T = (0, T) \times G$ denote a space-time cylinder. In Q_T, we consider the Ginzburg–Landau equation for the complex-valued function $\psi(t, x)$, referred to as the order parameter,

$$\frac{\partial \psi}{\partial t} + \bigl(i\nabla + A\bigr)^2 \psi - \psi + |\psi|^2 \psi = 0 \quad \text{for } (t, x) \in Q_T \tag{2.1}$$

along with the boundary condition

$$(i\nabla + A)\psi \cdot n = 0 \quad \text{on } (0, T) \times \partial G \tag{2.2}$$

and the initial condition

$$\psi(0, x) = \psi_0(x) \quad \text{in } G\,, \tag{2.3}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$ denotes the gradient operator and $A(x) = (A^1, \ldots, A^d)$, the magnetic potential, is a given real-valued vector field such that $\operatorname{div} A = \sum\limits_{j=1}^{d} \frac{\partial A^j}{\partial x_j} = 0$. Also, $n = (n_1, \ldots, n_d)$ denotes the unit outer normal vector to the boundary ∂G and $\psi_0(x)$ is a given initial condition. We have

$$\begin{aligned}(i\nabla + A)^2 \psi &= \bigl(i\nabla + A, i\nabla + A\bigr)\psi \\ &= \sum_{j=1}^{d} \Bigl(i\frac{\partial}{\partial x_j} + A^j(x)\Bigr)\Bigl(i\frac{\partial \psi(x)}{\partial x_j} + A^j(x)\psi(x)\Bigr)\,.\end{aligned} \tag{2.4}$$

We assume that $A(x) \in (C^2(\overline{G}))^d$ and, for any fixed time, $\psi(t,x) \in L^2(G)$.

We want to introduce function spaces within which it is natural to look for the solution of the problem (2.1)–(2.3). The Sobolev space of complex-valued functions defined in G and square integrable there together with their derivatives up to order k is denoted by $H^k(G)$, $k \in \mathbb{N}$. Here, $\mathbb{N}$ denotes the set of positive integers. In addition, we define the space

$$H_A^2(G) = \{\phi(x) \in H^2(G) \ : \ (i\nabla + A)\phi \cdot n = 0 \ \text{ on } \partial G\}\,. \tag{2.5}$$

The space of solutions of (2.1)–(2.3) is defined as follows:

$$\mathcal{Y} = \Big\{\psi(t,x) \in L^2\big(0,T; H_A^2(G)\big) \cap L^6(Q_T) \ : \ \frac{\partial \psi}{\partial t} \in L^2(Q_T)\Big\}\,. \tag{2.6}$$

We also study generalized solutions of the problem (2.1)–(2.3). To obtain a weak formulation, we multiply (2.1) by the complex conjugate of ϕ, denoted by $\overline{\phi}$, and integrate over Q_T. Using the boundary condition (2.2) and integration by parts, we obtain

$$\int\limits_{Q_T} \left[\frac{\partial \psi}{\partial t}\overline{\phi} + \big(i\nabla + A\big)\psi \cdot \overline{\big(i\nabla + A\big)\phi} - \psi\overline{\phi} + |\psi|^2\psi\overline{\phi}\right] dxdt = 0\,. \tag{2.7}$$

Here, we will not make more precise the function space used for generalized solutions, defined by (2.3) and (2.7) with arbitrary $\overline{\phi} \in L_2(0,T;H^1(G))$, of the problem (2.1)–(2.3) because just at this moment it is not necessary.

2.2 Approximation by the method of lines

The approximation of the solution of a partial differential equation by the method of lines means that we approximate the continuous space variables $x = (x_1, \ldots, x_d)$ by a discrete grid or mesh so that we approximate the partial differential equation problem by a system of ordinary differential equations. In our case, we use finite difference quotients to approximate spatial derivatives. We assume that the grid is uniform and the scale of the grid, $h > 0$, is a fixed, sufficiently small number. Let an arbitrary point on the grid be denoted by kh, where $k \in \mathbb{Z}^d$, $kh = (k_1h, \ldots, k_dh)$, and $\mathbb{Z}$ denotes the set of integers. Since $\psi(x)$ is a function of the continuous variable x, we let ψ_k, defined on the given grid, denote the approximation to ψ at the point kh.

We now define the corresponding discrete "derivatives" or difference quotients; we distinguish the discrete derivatives from the continuous derivatives $\frac{\partial}{\partial x_j}$ by using the notation $\partial_{j,h}$. Let δ_{jk} denote the Kronecker delta, and let $e_j = (\delta_{j1}, \ldots, \delta_{jd})$, $j = 1, \ldots, d$. We can approximate the derivative $\frac{\partial \psi}{\partial x_j}$ by the forward difference quotient $\partial_{j,h}^{+}\psi_k = \frac{1}{h}(\psi_{k+e_j} - \psi_k)$ or by the backward difference quotient $\partial^-_{j,h}\psi_k = \frac{1}{h}(\psi_k - \psi_{k-e_j})$. The discrete divergence operator $\mathrm{div}_h{}^{\pm}$, the discrete gradient operator $\nabla^{\pm}_h$, and the discrete Laplace operator $\Delta_h = \mathrm{div}^-_h \nabla^+_h$ are then defined in an obvious manner.

Analogous to (2.4), we define

$$
\begin{aligned}
(i\nabla_h + A_k)^2\psi_k &= (i\nabla^-_h + A_k, i\nabla^+_h + A_k)\psi_k \\
&= \sum_{j=1}^{d} \left(i\partial^-_{j,h} + A^j_k\right)\left(i\partial^+_{j,h}\psi_k + A^j_k\psi_k\right),
\end{aligned}
\tag{2.8}
$$

where $A_k = A(kh)$ and A^j_k denotes the jth component of the vector $A_k = (A^1(kh), \dots, A^d(kh))$.

We now approximate the domain G and its boundary ∂G.

Definition 2.1. The approximate boundary ∂G_h is the subset of the grid kh, $k \in \mathbb{Z}^d$, that consists of two parts $\partial G_h = \partial G^+_h \cup \partial G^-_h$, where

(i) ∂G^-_h is the set of points $kh \in G$ such that $(k + e_j)h \in \mathbb{R}^d \setminus G$ or $(k - e_j)h \in \mathbb{R}^d \setminus G$ for some $j = 1, \dots, d$

and

(ii) ∂G^+_h the set of points $kh \in \mathbb{R}^d \setminus G$ such that $(k+e_j)h \in G$ or $(k-e_j)h \in G$ for some $j = 1, \dots, d$.

Definition 2.2. The approximate domain G_h is the subset of points $kh \in G$, $k \in \mathbb{Z}^d$; we also set $G^0_h = G_h \backslash \partial G^-_h$.

We introduce the following subsets of the approximate boundary ∂G_h:

$$
\left.
\begin{aligned}
\partial G^+_h(-j) &= \{kh \in \partial G^+_h \,:\, (k + e_j)h \in \partial G^-_h\} \\
\partial G^+_h(+j) &= \{kh \in \partial G^+_h \,:\, (k - e_j)h \in \partial G^-_h\}
\end{aligned}
\right\} \quad \text{for } j = 1, \dots, d
\tag{2.9}
$$

and

$$
\left.
\begin{aligned}
\partial G^-_h(-j) &= \{kh \in \partial G^-_h \,:\, (k + e_j)h \in \partial G^+_h\} \\
\partial G^-_h(+j) &= \{kh \in \partial G^-_h \,:\, (k - e_j)h \in \partial G^+_h\}
\end{aligned}
\right\} \quad \text{for } j = 1, \dots, d.
\tag{2.10}
$$

The sets $\partial G^+_h(\pm j)$ and $\partial G^-_h(\pm j)$ are illustrated in Fig. 2.1 for a domain in $\mathbb{R}^2$. In addition, we note that the sets $\partial G^-_h(\pm j)$, $j = 1, \dots, d$, can possess nontrivial pairwise intersections.

We now turn to the approximation of the boundary value problem (2.1)–(2.3) by the method of lines. We have

$$
\frac{\partial \psi_k}{\partial t} + (i\nabla_h + A_k)^2\psi_k - \psi_k + |\psi_k|^2\psi_k = 0 \quad \text{for } kh \in G_h
\tag{2.11}
$$

and

$$
\psi_k\big|_{t=0} = \psi_{0,k} \quad \text{for } kh \in G_h,
\tag{2.12}
$$

where the notation $(i\nabla_h + A_k)^2$ is defined by (2.8).

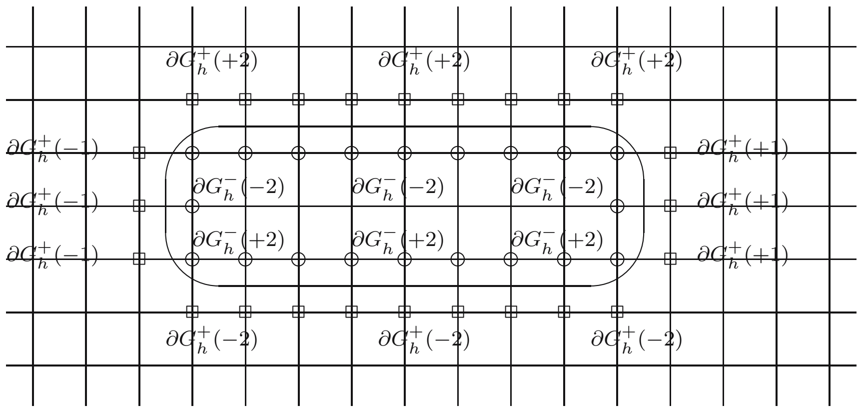

Fig. 2.1 The approximate boundary ∂G_h^+ is denoted by squares, and ∂G_h^- is denoted by circles.

In order to define the analogue of the boundary condition (2.2), we first note that the key property of this condition is that it implies the following formula for integration by parts:

$$\int_G (i\nabla + A)^2\psi(x)\overline{\phi(x)}\,dx = \int_G (i\nabla + A)\psi(x)\overline{(i\nabla + A)\phi(x)}\,dx \qquad \forall\,\psi \in \mathcal{H}_A^2(G), \phi \in \mathcal{H}^1(G)\,. \tag{2.13}$$

Using (2.13), one can define a weak solution of our problem (2.1)–(2.3) with the aid of (2.7). To define the weak solution for the system (2.11) and (2.12), we need the following discrete analogue of (2.13):

$$h^d \sum_{kh\in G_h} \left(i\nabla_h^- + A_k, i\nabla_h^+ + A_k\right)\psi_k\overline{\phi_k} = h^d \sum_{j=1}^d \sum_{kh\in G_h\cup\partial G_h^+(-j)} (i\partial_{j,h}^+\psi_k + A_k^j\psi_k)\overline{(i\partial_{j,h}^+\phi_k + A_k^j\phi_k)}\,. \tag{2.14}$$

We take this formula, which will be proved in the next subsection, as the foundation for the definition of the discrete analogue of the boundary condition (2.2).

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