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Exact solutions for the cubic–quintic nonlinear Schrödinger equation

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Abstract

In this paper, the cubic–quintic nonlinear Schrödinger equation is solved through the extended elliptic sub-equation method. As a consequence, many types of exact travelling wave solutions are obtained which including bell and kink profile solitary wave solutions, triangular periodic wave solutions and singular solutions.

1. Introduction

The cubic–quintic nonlinear Schrödinger equation (CQNLSE) with nonlinearity management presents practical interest since it appears in many branches of physics such as nonlinear optics and Bose–Einstein condensate (BEC). In nonlinear optics it describes the propagation of pulses in double-doped optics fibers [1], in BEC it models the condensate with two and three body interactions [2,3]. In optical fibers periodic variation of the nonlinearity can be achieved by varying the type of dopants along the fiber. In BEC the variation of the atomic scattering length by the Feshbach resonance technique leads to the oscillations of the mean field cubic nonlinearity [4]. The CQNLSE when the cubic term is equal to zero is the critical quintic NLSE. The quintic Townes soliton is an unstable solution of the quintic NLSE [5].

In nonlinear science, the construction of exact solutions for nonlinear partial differential equations (NLPDEs) is one of the most important and essential tasks. With the help of exact solutions, the phenomena modelled by these NLPDEs such as the stability of optical soliton propagation can be well understood. In recent years, many powerful methods to construct exact analytical solutions have been proposed, such as the inverse scattering method, the Bäcklund transformation and Darboux transformation, the Painlevé truncation expansion, the homogeneous balance method, the sine–cosine function method, the tanh-function method, and the Jacobian elliptic function method [6–11]. Very recently, an extended elliptic sub-equation method has been developed to obtained new bell and kink profile solitary wave solutions, triangular periodic wave solutions and singular solutions [12].

The cubic–quintic nonlinear Schrödinger equation (CQNLSE) which describes the propagation of pulses in the double-doped optic fibers [1,3] can be written as

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$$iq_z + q_{tt} + \gamma|q|^2 q + \alpha|q|^4 q = 0, \tag{1}$$

where $q(z,t)$ is the complex envelope of the electrical field in a comoving frame, γ is the Kerr nonlinear parameter and α is the saturation of the nonlinear refractive index (i.e. higher-order nonlinearity). They are real functions of the normalized propagation distance z, and t is the retarded time. In this paper, we investigate the CQNLSE through the extended elliptic sub-equation method, and some exact travelling wave solutions are derived.

2. The extended elliptic sub-equation method

In what follows, we outline the elliptic sub-equation method: for a given nonlinear wave equation with one physical field $u(x,t)$ in two variables x, t

$$F(u, u_t, u_x, u_{xt}, u_{xx}, u_{tt}, \ldots) = 0. \tag{2}$$

Step 1. We seek its travelling wave solution, in the form of

$$u(x,t) = u(\xi), \quad \xi = kx - ct, \tag{3}$$

where k, c are constants. Substituting (3) into Eq. (2) gives rise to a nonlinear ordinary differential equation

$$G(u, u', u'', u''', \ldots) = 0, \tag{4}$$

where $'$ denotes $\frac{d}{d\xi}$.

Step 2. To seek the travelling wave solutions of Eq. (4), we assume that Eq. (4) has the solutions in the form of

$$u(\xi) = \sum_{i=0}^{n} a_i \phi^i(\xi) \tag{5}$$

with the new variable $\phi(\xi)$ satisfied

$$\phi'(\xi)^2 = h_0 + h_2\phi^2 + h_4\phi^4 + h_6\phi^6, \quad \phi''(\xi) = h_2\phi + 2h_4\phi^3 + 3h_6\phi^5, \tag{6}$$

where h_j $(j = 0,2,4,6)$ are constants and n is an integer to be determined by balancing the highest derivative term with the nonlinear terms.

Step 3. Substituting Eq. (5) along with Eq. (6) into Eq. (4) and setting the coefficients of all powers of $\phi^k \phi'^l$ $(k = 0,1,2,\ldots; l = 0,1)$ to zero, we will obtain a system of nonlinear algebraic equations with respect to the parameters a_i, c, h_j $(i = 0,1,\ldots,n; j = 0,2,4,6)$. By solving the nonlinear algebraic equations if available, we can determine those parameters explicitly.

Step 4. By considering the different values of h_0, h_2, h_4, and h_6, we will find that Eq. (6) admits many kinds of fundamental solutions which are listed as follows:

Case A. Suppose that $h_0 = 0$, $h_6 < 0$ and $h_4^2 - 4h_2h_6 > 0$.

1. If $h_2 > 0$ and $h_4 < 0$, then Eq. (6) has a bell profile solution

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{sech}^2\sqrt{h_2}(\xi+\xi_0)}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right)\operatorname{sech}^2\sqrt{h_2}(\xi+\xi_0)} \right\}^{1/2} \tag{7}$$

and a singular solution

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{csch}^2[\pm\sqrt{h_2}(\xi+\xi_0)]}{2\sqrt{h_4^2 - 4h_2h_6} + \left(\sqrt{h_4^2 - 4h_2h_6} - h_4\right)\operatorname{csch}^2\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)} \right\}^{1/2}. \tag{8}$$

2. If $h_2 < 0$ and $h_4 > 0$, then Eq. (6) has a triangular periodic solution

$$\phi(\xi) = \left\{ \frac{-2h_2 \sec^2 \sqrt{-h_2}(\xi + \xi_0)}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} - h_4\right) \sec^2 \sqrt{-h_2}(\xi + \xi_0)} \right\}^{1/2} \tag{9}$$

and a singular triangular periodic solution

$$\phi(\xi) = \left\{ \frac{2h_2 \csc^2 \left[\pm\sqrt{-h_2}(\xi + \xi_0)\right]}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right) \csc^2 \left(\pm\sqrt{-h_2}(\xi + \xi_0)\right)} \right\}^{1/2}. \tag{10}$$

Case B. Suppose that $h_0 = \frac{8h_2^2}{27h_4}$ and $h_6 = \frac{h_4^2}{4h_2}$.

1. If $h_2 < 0$ and $h_4 > 0$, then Eq. (6) has a kink profile solution

$$\phi(\xi) = \left\{ -\frac{8h_2 \tanh^2 \left[\pm\sqrt{-\frac{h_2}{3}}(\xi + \xi_0)\right]}{3h_4\left(3 + \tanh^2 \left[\pm\sqrt{-\frac{h_2}{3}}(\xi + \xi_0)\right]\right)} \right\}^{1/2} \tag{11}$$

and a singular solution

$$\phi(\xi) = \left\{ -\frac{8h_2 \coth^2 \left[\pm\sqrt{-\frac{h_2}{3}}(\xi + \xi_0)\right]}{3h_4\left(3 + \coth^2 \left[\pm\sqrt{-\frac{h_2}{3}}(\xi + \xi_0)\right]\right)} \right\}^{1/2}. \tag{12}$$

2. If $h_2 > 0$ and $h_4 < 0$, then Eq. (6) has a triangular periodic solution

$$\phi(\xi) = \left\{ \frac{8h_2 \tan^2 \left[\pm\sqrt{\frac{h_2}{3}}(\xi + \xi_0)\right]}{3h_4\left(3 - \tan^2 \left[\pm\sqrt{\frac{h_2}{3}}(\xi + \xi_0)\right]\right)} \right\}^{1/2} \tag{13}$$

and a singular triangular periodic solution

$$\phi(\xi) = \left\{ \frac{8h_2 \cot^2 \left[\pm\sqrt{\frac{h_2}{3}}(\xi + \xi_0)\right]}{3h_4\left(3 - \cot^2 \left[\pm\sqrt{\frac{h_2}{3}}(\xi + \xi_0)\right]\right)} \right\}^{1/2}. \tag{14}$$

Case C. Suppose that $h_0 = 0$ and $h_6 = \frac{h_4^2}{4h_2}$. If $h_2 > 0$ and $h_4 < 0$, then Eq. (6) has a kink profile solution

$$\phi(\xi) = \left\{ -\frac{h_2}{h_4}\left(1 + \tanh\left[\pm\sqrt{h_2}(\xi + \xi_0)\right]\right) \right\}^{1/2} \tag{15}$$

and a singular solution

$$\phi(\xi) = \left\{ -\frac{h_2}{h_4}\left(1 + \coth\left[\pm\sqrt{h_2}(\xi + \xi_0)\right]\right) \right\}^{1/2}. \tag{16}$$

Case D. Suppose that $h_0 = \frac{h_4}{32h_6}$ and $h_6 = \frac{5h_4^2}{16h_2}$. If $h_2 < 0$, $h_4 > 0$, $h_6 < 0$ and $h_4^2 - 3h_2h_6 > 0$, then Eq. (6) has a bell profile solution

$$\phi(\xi) = \left\{ -\frac{\left(h_4 - \sqrt{h_4^2 - 3h_2h_6}\right)}{3h_6}\left(1 \pm \operatorname{sech}\left[\left(\frac{h_4}{2}\sqrt{-\frac{1}{h_6}}\right)(\xi + \xi_0)\right]\right) \right\}^{1/2}. \tag{17}$$

Step 5. Making use of the above solutions and substituting the parameters a_i, c, h_j $(i = 0, 1, \ldots, n; j = 0, 2, 4, 6)$ into Eq. (5), we can then obtain all the possible solutions of Eq. (2).

3. Exact solutions of the cubic–quintic nonlinear Schrödinger equation

In order to obtain some exact solutions of Eq. (1), we first make the transformation

$$q(z,t) = e^{i[\psi(\xi)-\omega t]}E(\xi), \quad \xi = kz - ct. \tag{18}$$

Substituting (18) into Eq. (1), and letting the real part and imaginary part be zero, respectively, We can yield

$$-cE' + 2k^2E'\psi' + k^2E\psi'' = 0, \tag{19}$$

$$cE\psi' + E\omega + k^2E'' - k^2E\psi'^2 + \gamma E^3 + \alpha E^5 = 0. \tag{20}$$

Supposing that

$$\psi' = A + BE. \tag{21}$$

To make the left hand side of Eq. (19) be zero, identically, we choose

$$A = \frac{c}{2k^2}, \quad B = 0. \tag{22}$$

Making use of (21) and (22), Eq. (20) can be reduced to

$$k^2E'' = -\frac{(c^2 + 4\omega k^2)}{4k^2}E - \gamma E^3 - \alpha E^5 = 0. \tag{23}$$

Integrating Eq. (23), we get

$$(E')^2 = h_0 - \frac{(c^2 + 4\omega k^2)}{4k^4}E - \frac{\gamma}{2k^2}E^3 - \frac{\alpha}{3k^2}E^5, \tag{24}$$

where h_0 is a arbitrary constant. Letting $h_2 = -\frac{(c^2+4\omega k^2)}{4k^4}$, $h_4 = -\frac{\gamma}{2k^2}$, $h_6 = -\frac{\alpha}{3k^2}$, $\Delta = h_4^2 - 4h_2h_6$, when h_0, h_2, h_4, h_6 satisfy different conditions, we can obtain the following travelling wave solutions of Eq. (1).

Case A. Suppose that $h_0 = 0$, $\alpha > 0$ and $3k^2\gamma^2 > 4\alpha(c^2 + 4\omega k^2)$.

1. If $-c^2 < 4\omega k^2$ and $\gamma > 0$, then Eq. (24) has a bell profile solution

$$E_1(\xi) = \left\{\frac{2h_2\operatorname{sech}^2(\sqrt{h_2}\xi)}{2\sqrt{\Delta} - (\sqrt{\Delta} + h_4)\operatorname{sech}^2(\sqrt{h_2}\xi)}\right\}^{1/2} \tag{25}$$

and the solution corresponds to Eq. (1) is

$$q_1(z,t) = \left\{\frac{2h_2\operatorname{sech}^2[\sqrt{h_2}(kz-ct)]}{2\sqrt{\Delta} - (\sqrt{\Delta} + h_4)\operatorname{sech}^2[\sqrt{h_2}(kz-ct)]}\right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]} \tag{26}$$

and a singular solution

$$E_2(\xi) = \left\{\frac{2h_2\operatorname{csch}^2(\pm\sqrt{h_2}\xi)}{2\sqrt{\Delta} + (\sqrt{\Delta} - h_4)\operatorname{csch}^2(\pm\sqrt{h_2}\xi)}\right\}^{1/2} \tag{27}$$

and the solution corresponds to Eq. (1) is

$$q_2(z,t) = \left\{\frac{2h_2\operatorname{csch}^2[\pm\sqrt{h_2}(kz-ct)]}{2\sqrt{\Delta} + (\sqrt{\Delta} - h_4)\operatorname{csch}^2[\pm\sqrt{h_2}(kz-ct)]}\right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]}. \tag{28}$$

2. If $4\omega k^2 + c^2 > 0$ and $\gamma < 0$, then Eq. (24) has a triangular periodic solution

$$E_3(\xi) = \left\{\frac{-2h_2\sec^2(\sqrt{-h_2}\xi)}{2\sqrt{\Delta} - (\sqrt{\Delta} - h_4)\sec^2(\sqrt{-h_2}\xi)}\right\}^{1/2} \tag{29}$$

and the solution corresponds to Eq. (1) is

$$q_3(z,t) = \left\{ \frac{-2h_2 \sec^2\left[\sqrt{-h_2}(kz-ct)\right]}{2\sqrt{\Delta} - \left(\sqrt{\Delta} - h_4\right)\sec^2\left[\sqrt{-h_2}(kz-ct)\right]} \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]} \tag{30}$$

and a singular triangular periodic solution

$$E_4(\xi) = \left\{ \frac{2h_2 \csc^2(\pm\sqrt{-h_2}\xi)}{2\sqrt{\Delta} - \left(\sqrt{\Delta} - h_4\right)\csc^2\left(\sqrt{-h_2}\xi\right)} \right\}^{1/2} \tag{31}$$

and the solution corresponds to Eq. (1) is

$$q_4(z,t) = \left\{ \frac{2h_2 \csc^2\left[\pm\sqrt{-h_2}(kz-ct)\right]}{2\sqrt{\Delta} - \left(\sqrt{\Delta} - h_4\right)\csc^2\left[\sqrt{-h_2}(kz-ct)\right]} \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]}. \tag{32}$$

Case B. Suppose that $h_0 = -\frac{(c^2+4\omega k^2)^2}{27k^6\gamma}$ and $3k^2\gamma^2 = 4\alpha(c^2+4k^2\omega)$.

1. If $4\omega k^2 + c^2 > 0$ and $\gamma < 0$, then Eq. (24) has a kink profile solution

$$E_5(\xi) = \left\{ -\frac{4(c^2+4\omega k^2)\tanh^2\left(\pm\sqrt{-\frac{h_2}{3}}\xi\right)}{3\gamma k^2\left[3+\tanh^2\left(\pm\sqrt{-\frac{h_2}{3}}\xi\right)\right]} \right\}^{1/2} \tag{33}$$

and the solution corresponds to Eq. (1) is

$$q_5(z,t) = \left\{ -\frac{4(c^2+4\omega k^2)\tanh^2\left[\pm\sqrt{-\frac{h_2}{3}}(kz-ct)\right]}{3\gamma k^2\left(3+\tanh^2\left[\pm\sqrt{-\frac{h_2}{3}}(kz-ct)\right]\right)} \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]} \tag{34}$$

and a singular solution

$$E_6(\xi) = \left\{ -\frac{4(c^2+4\omega k^2)\coth^2\left(\pm\sqrt{-\frac{h_2}{3}}\xi\right)}{3\gamma k^2\left(\left[3+\coth^2\left(\pm\sqrt{-\frac{h_2}{3}}\xi\right)\right]\right)} \right\}^{1/2} \tag{35}$$

and the solution corresponds to Eq. (1) is

$$q_6(z,t) = \left\{ -\frac{4(c^2+4\omega k^2)\coth^2\left[\pm\sqrt{-\frac{h_2}{3}}(kz-ct)\right]}{3\gamma k^2\left(3+\coth^2\left[\pm\sqrt{-\frac{h_2}{3}}(kz-ct)\right]\right)} \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]}. \tag{36}$$

2. If $4\omega k^2 + c^2 < 0$ and $\gamma > 0$, then Eq. (24) has a triangular periodic solution

$$E_7(\xi) = \left\{ \frac{4(c^2+4\omega k^2)\tan^2\left(\pm\sqrt{-\frac{h_2}{3}}\xi\right)}{3\gamma k^2\left[3-\tan^2\left(\pm\sqrt{-\frac{h_2}{3}}\xi\right)\right]} \right\}^{1/2} \tag{37}$$

and the solution corresponds to Eq. (1) is

$$q_7(z,t) = \left\{ \frac{4(c^2+4\omega k^2)\tan^2\left[\pm\sqrt{-\frac{h_2}{3}}(kz-ct)\right]}{3\gamma k^2\left(3-\tan^2\left[\pm\sqrt{-\frac{h_2}{3}}(kz-ct)\right]\right)} \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]} \tag{38}$$

and a singular triangular periodic solution

$$E_8(\xi) = \left\{ \frac{4(c^2+4\omega k^2)\cot^2\left(\pm\sqrt{-\frac{h_2}{3}}\xi\right)}{3\gamma k^2\left[3-\cot^2\left(\pm\sqrt{-\frac{h_2}{3}}\xi\right)\right]} \right\}^{1/2} \tag{39}$$

and the solution corresponds to Eq. (1) is

$$q_8(z,t) = \left\{ \frac{4(c^2+4\omega k^2)\cot^2\left[\pm\sqrt{-\frac{h_2}{3}}(kz-ct)\right]}{3\gamma k^2\left(3-\cot^2\left[\pm\sqrt{-\frac{h_2}{3}}(kz-ct)\right]\right)} \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]}. \tag{40}$$

Case C. Suppose that $h_0 = 0$ and $3k^2\gamma^2 = 4\alpha(c^2+4k^2\omega)$. If $4\omega k^2 + c^2 < 0$ and $\gamma > 0$, then Eq. (24) has a kink profile solution

$$E_9(\xi) = \left\{ -\frac{(c^2+4\omega k^2)}{2\gamma k^2}\left[1+\tanh\left(\pm\sqrt{h_2}\xi\right)\right] \right\}^{1/2} \tag{41}$$

and the solution corresponds to Eq. (1) is

$$q_9(z,t) = \left\{ -\frac{(c^2+4\omega k^2)}{2\gamma k^2}\left[1+\tanh\left(\pm\sqrt{h_2}(kz-ct)\right)\right] \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]} \tag{42}$$

and a singular solution

$$E_{10}(\xi) = \left\{ -\frac{(c^2+4\omega k^2)}{2\gamma k^2}\left[1+\coth\left(\sqrt{h_2}\xi\right)\right] \right\}^{1/2} \tag{43}$$

and the solution corresponds to Eq. (1) is

$$q_{10}(z,t) = \left\{ -\frac{(c^2+4\omega k^2)}{2\gamma k^2}\left[1+\coth\left(\sqrt{h_2}(kz-ct)\right)\right] \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]}. \tag{44}$$

Case D. Suppose that $4\omega k^2 + c^2 > 0$ and $\gamma < 0$, $\alpha < 0$, $h_0 = \frac{h_4}{32h_6}$ and $h_6 = \frac{5h_4^2}{16h_2}$, then Eq. (6) has a bell profile solution

$$E_{11}(\xi) = \left\{ -\frac{h_4-\sqrt{\Delta}}{2h_6}\left[1\pm\operatorname{sech}\left(-\frac{\gamma}{8k}\sqrt{-\frac{1}{\alpha}}\xi\right)\right] \right\}^{1/2} \tag{45}$$

and the solution corresponds to Eq. (1) is

$$q_{11}(z,t) = \left\{ -\frac{h_4-\sqrt{\Delta}}{2h_6}\left[1\pm\operatorname{sech}\left[-\frac{\gamma}{8k}\sqrt{-\frac{1}{\alpha}}(kz-ct)\right]\right] \right\}^{1/2} e^{i\left[\frac{c(kz-ct)}{2k^2}+\xi_0-\omega t\right]}. \tag{46}$$

4. Conclusion

In this paper, we have utilized the extended elliptic sub-equation method to study the cubic–quintic nonlinear Schrödinger equation. As a result, some new exact travelling wave solutions of the cubic–quintic nonlinear Schrödinger equation have been obtained which including bell and kink profile solitary wave solutions, singular solutions and triangular periodic solutions. These solutions may be useful for describing the propagation of pulse in the double-doped optic fibers.

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