Homework 4 - Group 23

Empirical Methods in Finance

Modeling Volatlity: GARCH Models

Master of Science (MSc) in Finance

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1.1 Optimal portfolio weights

In order to compute the optimal portfolio weights, we have to maximize the following expression with respect to α:

$$\max_{\alpha} \alpha' \mu_t + (1 - e'\alpha) R_f - \frac{\lambda}{2} \alpha' \Sigma \alpha \tag{1}$$

We compute the first-order derivative of the mean-variance criterion with respect to α:

$$\frac{\partial}{\partial \alpha} = \mu - e' R_f - \frac{\lambda}{2} 2 \Sigma \alpha = 0 \tag{2}$$

The optimal weights are given by the following equation:

$$\alpha^* = \frac{1}{\lambda} \Sigma^{-1} (\mu - e' R_f) \tag{3}$$

We can know define ER as the excess return:

$$ER = \mathrm{E}[\mu_t - R_{ft}] \tag{4}$$

Therefore, the optimal weights are computed as follow:

$$\alpha^* = \frac{1}{\lambda} \Sigma^{-1} ER \tag{5}$$

With the weight for the risk free asset equals to:

$$\alpha^*_{R_f} = 1 - e' \alpha^* \tag{6}$$

1.2 Computation of optimal weights vector for $\lambda = 2$, $\lambda = 10$

As we can see in Table 1, we compute the optimal weights of the portfolio composed of Stocks, Bonds and the Risk-free asset for $\lambda = 2$ and $\lambda = 10$. We can first notice that, in this type of optimization criterion, we allocate our portfolio utilizing a large amount of leverage. Indeed, when $\lambda = 2$, we have a leverage approximately equal to 9 times our initial wealth (we borrow 9 times the initial wealth at the risk-free rate in order to invest in stocks or bonds). However, when the λ parameter increases to $\lambda = 10$, we penalize more the increase in the volatility (i.e. we are more risk averse) and therefore we use less leverage.

Optimal Weights				
	Constant Excess Return		Constant Risk-free rate	
	Lambda=2	Lambda=10	Lambda=2	Lambda=10
Stock	1,7968	0,3594	1,7968	0,3594
Bond	8,4855	1,6971	8,4855	1,6971
Risk-free	-9,2823	-1,0565	-9,2823	-1,0565

Table 1: Optimal Weights

2 Estimation of a GARCH model

2.1 Non-Normality tests

In the context of this question, we are going to show, that for both assets (stocks and bonds), some evidences of the non-normality of the excess returns. Indeed, we perform a Kolmogorov-Smirnoff test and the auto-correlation (Ljung-Box test, with 4 lags).

As the mean μ and the variance σ^2 are unknown, we have to use the Lilliefors test (distribution of KS is not appropriate anymore in this case).

Non-Normality Test

	KS-stat	p-Value	Ljung-Box Qstat	p-Value
Stock	0,0793	0,0010	11,9435	0,0178
Bond	0,0342	0,0106	8,4638	0,0760

Table 2: Non-Normality tests

As we can see in Table 2, we can reject the normality of the excess returns of stocks and bonds at a level of 5% as KS_L is bigger than the critical value of $(0.886/\sqrt{T}, T = 938)$. We can also prove the non-autocorrelation of the excess returns by performing a Ljung-Box test. This test shows that we reject the null hypothesis of no autocorrelation for a level of 5% for stocks. However, the null is not rejected for bonds as the p-value of 7.6% is bigger than 5%. The presence of autocorrelation in bonds might be due to the fact that the assets is generally less liquid than stock.

2.2 AR(1) Model

In this sub-question, we will focus on estimating an AR(1) model for stocks and bonds. The purpose of this model is to see whether returns can be predicted.

AR(1)					
	Intercept	**t-stat**	**phi**	**t-stat**	**R squared**
Stock	0,0014	1,8381	-0,0629	-1,9275	0,0040
Bond	0,0009	4,0490	-0,0616	-1,8935	0,0038

Table 3: AR(1)

In Table 3, we can first notice that the intercept of the model is significant at 5% for the bond. However, this is not the case for the stocks, as the t-stat of 1.8381 is smaller than the critical value. Concerning the coefficients estimated ϕ, we can notice that for both assets, they are not significant at a level of 5%. Furthermore, we can see that the predictive power of this model is very weak (in fact, the R squared are close to zero).

2.3 LM test of Engle

In this subsection, we would like to test for ARCH effects. In order to perform a Langrange-Multiplier (LM) test, we have to estimate the following regression:

$$\hat{\epsilon}_t^2 = a_0 + a_1\hat{\epsilon}_{t-1}^2 + a_2\hat{\epsilon}_{t-2}^2 + a_3\hat{\epsilon}_{t-3}^2 + a_4\hat{\epsilon}_{t-4}^2 + \nu_t \tag{7}$$

Let's assume the following hypothesis:

$$\mathrm{H}_0 : a_1 = a_2 = a_3 = a_4 = 0 \tag{8}$$

$$\mathrm{H}_a : \exists a_i \geq 0 \tag{9}$$

The alternative hypothesis is valid with at least one strict inequality. The LM test statistic is equivalent to the TR^2 test statistic (T being the sample size), shown by Engle in 1982. This methodology avoids to estimate an ARCH model when we do not need to use it. We only have to run a regression on past epsilon squared. We also know that under the null of no ARCH effect, the TR^2 is asymptotically distributed as a $\chi^2(p)$. Similarly, we can test for ARCH effects using the Ljung-Box statistics for $\hat{\epsilon}_t^2$ for p lags which is also asymptotically distributed as a $\chi^2(p)$.

LM test				
	Lm stat	**p-Value**	**Ljung-Box Qstat**	**p-Value**
Stock	116,2713	0,0000	179,3059	0,0000
Bond	40,4686	0,0000	47,4103	0,0000

Table 4: LM test

As we can see in Table 4, the null hypothesis of the LM test is clearly rejected (p-value of 0 for both assets). Equivalently, the Ljung-Box test leads to the same conclusion. Therefore, we can say that there is some heteroskedasticity in the residuals. As a result, we can conclude that we need a model for the volatility such as an ARCH or a GARCH model.

2.4 GARCH(1,1)

As we noticed some heteroskedasticity in the residual terms in the last sub-section, it is interesting to estimate a GARCH(1,1) model in order to capture this effect. Moreover, it is equivalent to test for an ARCH effect or for a GARCH effect. The GARCH model is more powerfull than an ARCH as it is able to capture the persistence of the presence with small numbers of parameter while the ARCH require a lot of lag to capture this property of the volatility.

GARCH(1,1)				
	Stock		**Bond**	
	Estimate	T-stat	Estimate	T-stat
Omega	0,0000	3,4197	0,0000	1,8442
Alpha	0,2064	5,9836	0,0524	4,1618
Beta	0,7570	20,8685	0,9335	60,0511
Alpha + Beta	0,9633	-	0,9859	-

Table 5: GARCH(1,1)

In Table 5, we can notice that every coefficient in the GARCH model are significant except for the ω of the bond. It is very important to notice that the sum $\alpha + \beta$ is never equal to one because of the restrictions in the optimization process on these parameters. Hence, we can stipulate that the GARCH(1,1) processes are covariance stationary (if $\alpha + \beta = 1$, the value of the unconditional variance explodes and, hence, we don't have a covariance stationary process but we still have strictly stationnary process for ϵ_t, even when the sum is equal to 1, because of the Jensen inequality and the Nelson theorem (1990)). Furthermore, the $\alpha + \beta$ captures the persistence of the volatility process, we can see that this one is quite persistent. The effect is even more important in the case of the bonds.

2.5 Forecasts for $\sigma^2_{i,T+k}$

In this sub-question, we want to make forecasts for $\sigma^2_{i,T+k}$ for $k = 1, 2, ..., 52 weeks$. We perform a weekly volatility forecast for both security types (stocks and bonds).

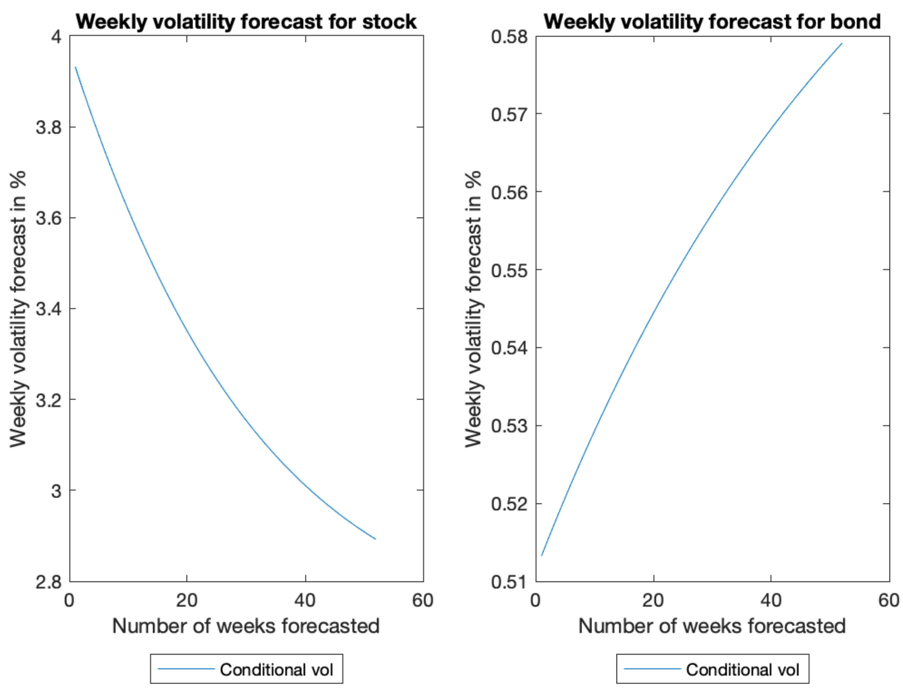

Figure 1: Weekly Volatility forecast (52 weeks)

Imposing the sum of alpha and beta is lower than 1, when we estimate them, implies that the long term forecast will converge to the unconditional mean. As we can see in Figure 1, we clearly see the trend of the convergence. After performing some algebra, we have a geometric sequence that we can express under the form of unconditional mean as the forecast goes to infinity. However, we have to increase the number of weeks in order to notice that convergence graphically. In fact, we can see in Figure 2, that when we increase the number of weeks to 250, we can clearly notice a convergence in the volatility forecasts for stocks and bonds towards the unconditional volatility, which are approximately equal to 2.7% and 0.63% respectively. Indeed, as our process is highly persistent (and covariance-stationary), the volatility forecasts should converge slowly towards the unconditional variance.

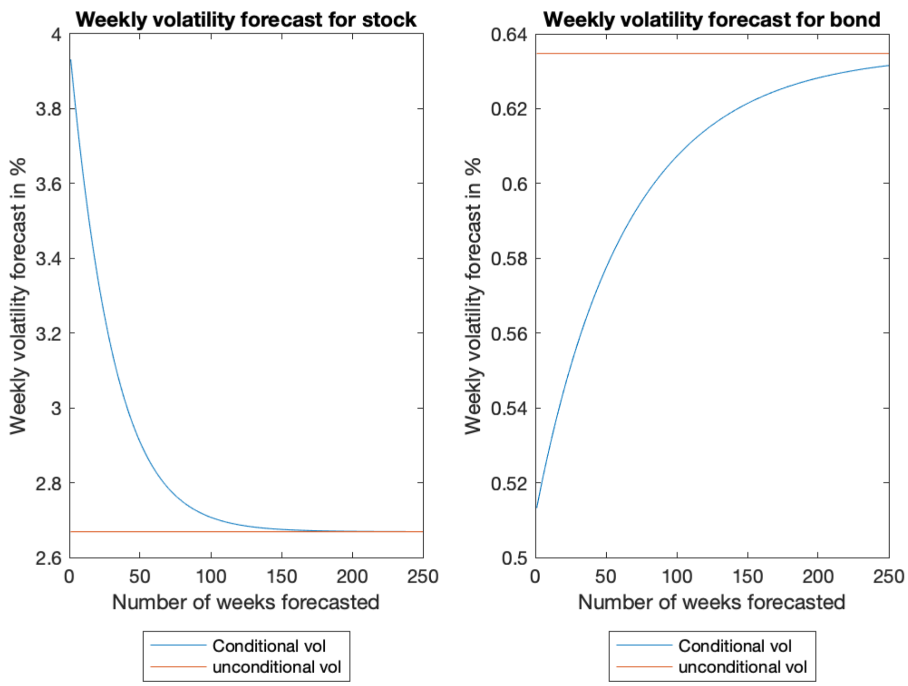

Figure 2: Weekly Volatility forecast (250 weeks)

3 Dynamic asset allocation with time-varying expected return and volatility

3.1 Sample Correlation

The sample correlation between the two residuals (for equity and bond markets forecasts) is -36.61%.

3.2 Plot the vector of optimal weight vector for stocks and bonds

The Figure 3 and Figure 4 picture the volatility of the weights associated with a risk aversion of 2 and 10, respectively. This picture shows how the portfolio is allocated in a dynamic way using conditional volatility and expected return instead of constant similar parameters which yield to a flat line over time. We can notice, looking at the Y axis of both graphs, that the lower the risk aversion is, the higher the magnitude of the weights' changes. We observe that for λ=2, the weights move within a range of w:-40 to about +30 while it is a smaller range for λ=10 (about -8 to +7). This is what we expect in that case. That is, with a lower risk aversion, an investor is able to adjust their optimal portfolio weights throughout a broader scale of weights while still meeting their risk tolerance's requirements.

In both cases, the graphs show a lot of noise in the weights adjustments, because, in reality, the rebalancing frequency of the portfolio is possible to a certain extent due to the transaction fees incurred every time the weights are adjusted. Finally, we see that the curves are similar no matter what lambda we use.

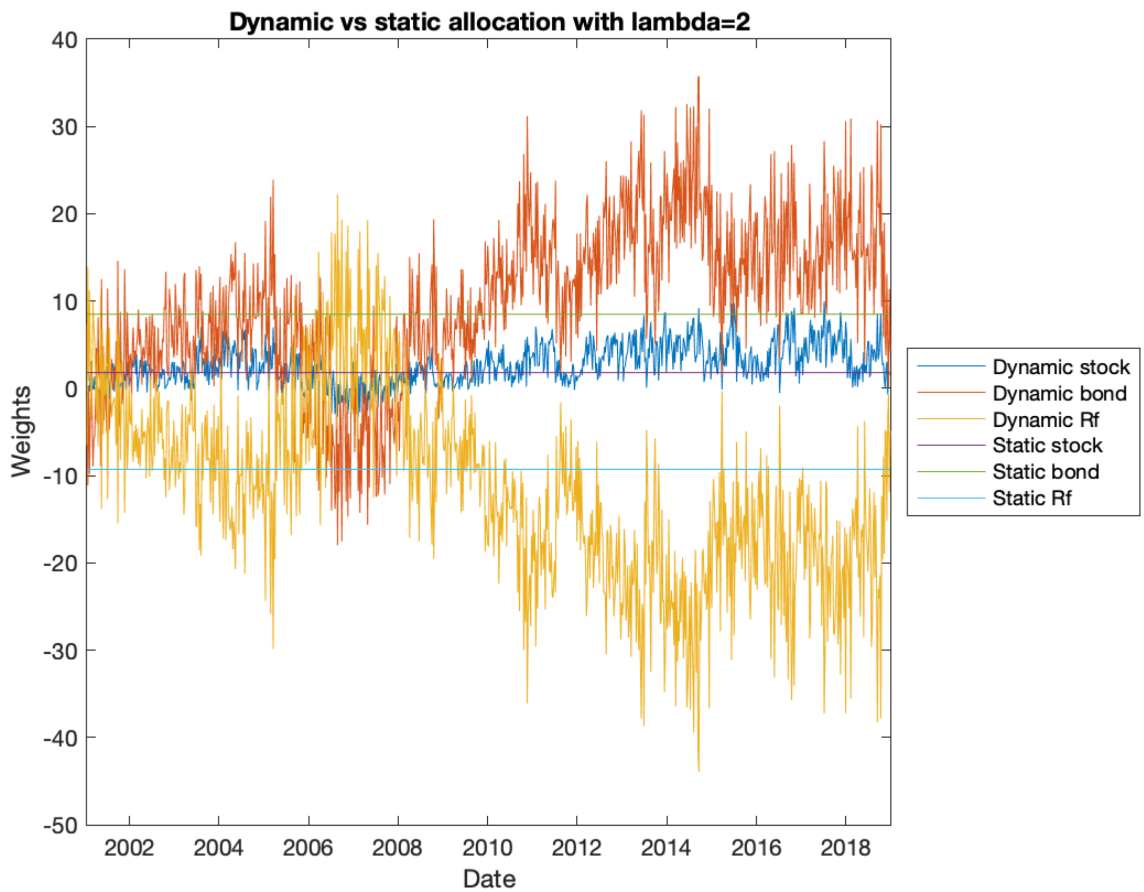

Figure 3: Dynamic Allocation - lambda=2

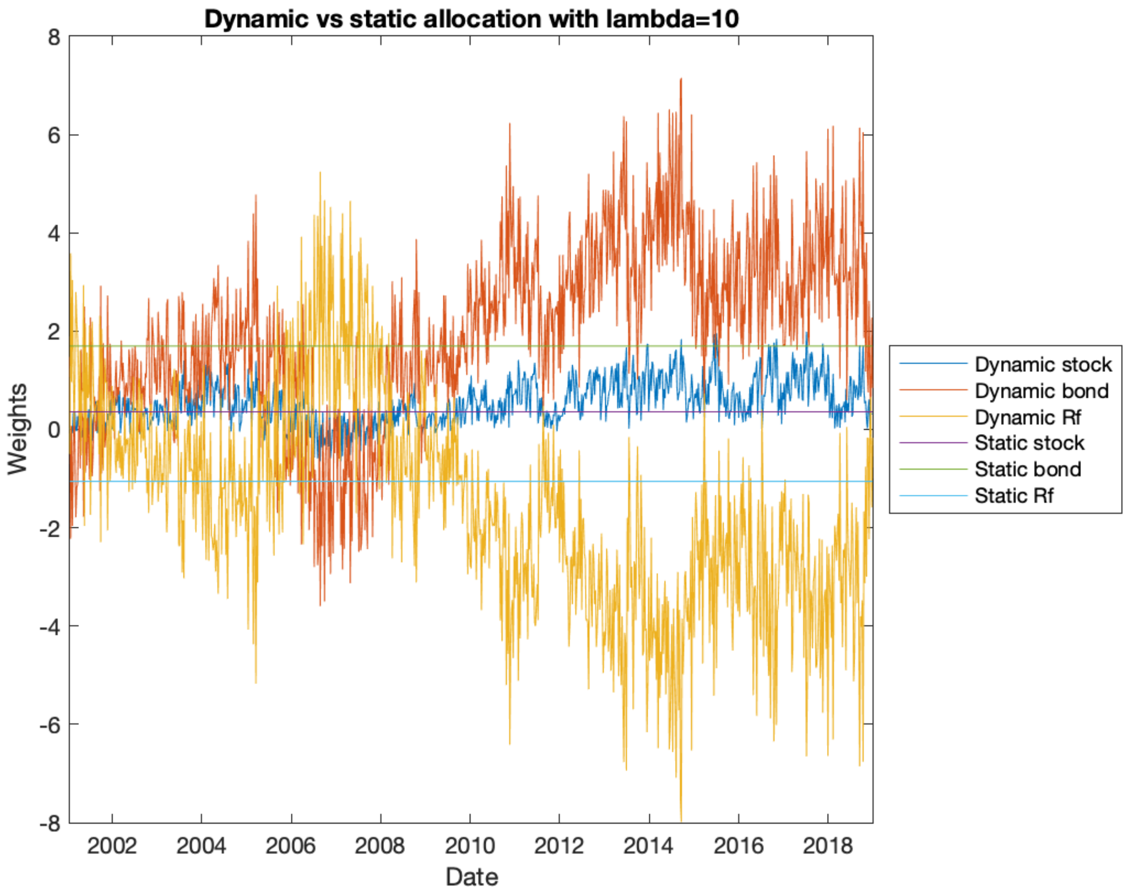

Figure 4: Dynamic Allocation - lambda=10

3.3 Cumulative returns of the optimal portfolio for the two approaches

According to Table 6, the dynamic portfolios outperform the static ones for most of the statistical measures : higher Sharpe Ratio, higher average returns, lower excess kurtosis and a skewness closer to zero. However, the volatility of the static allocation is lower than the dynamic one, which is better. Nonetheless, it is more relevant to look at the risk-reward trade-off, as a measure of risk, and this is measured by the Sharpe Ratio, better in the case of a dynamic portfolio. Now, we can pursue the analysis in including the risk aversion parameter. Obviously, the volatility rises with a lower risk aversion, but so do the the returns. Ultimately, the Sharpe Ratios are identical with a lambda equal to 2 and 10. As regards the graph of of cumulative returns, we can see that the dynamic portfolio overtook the static one's performance from 2010 on.

Summary Statistics				
	Static Allocation		**Dynamic Allocation**	
	Lambda=2	Lambda=10	Lambda=2	Lambda=10
Annualized mean	0,3414	0,0803	0,6326	0,1385
Annualized volatility	0,4001	0,0799	0,5822	0,1163
Sharpe Ratio	0,8158	0,8170	1,0608	1,0621
Skewness	-0,7161	-0,7189	-0,4570	-0,4768
Excess Kurtosis	5,1884	5,2112	3,3801	3,4203

Table 6: Summary Statistics

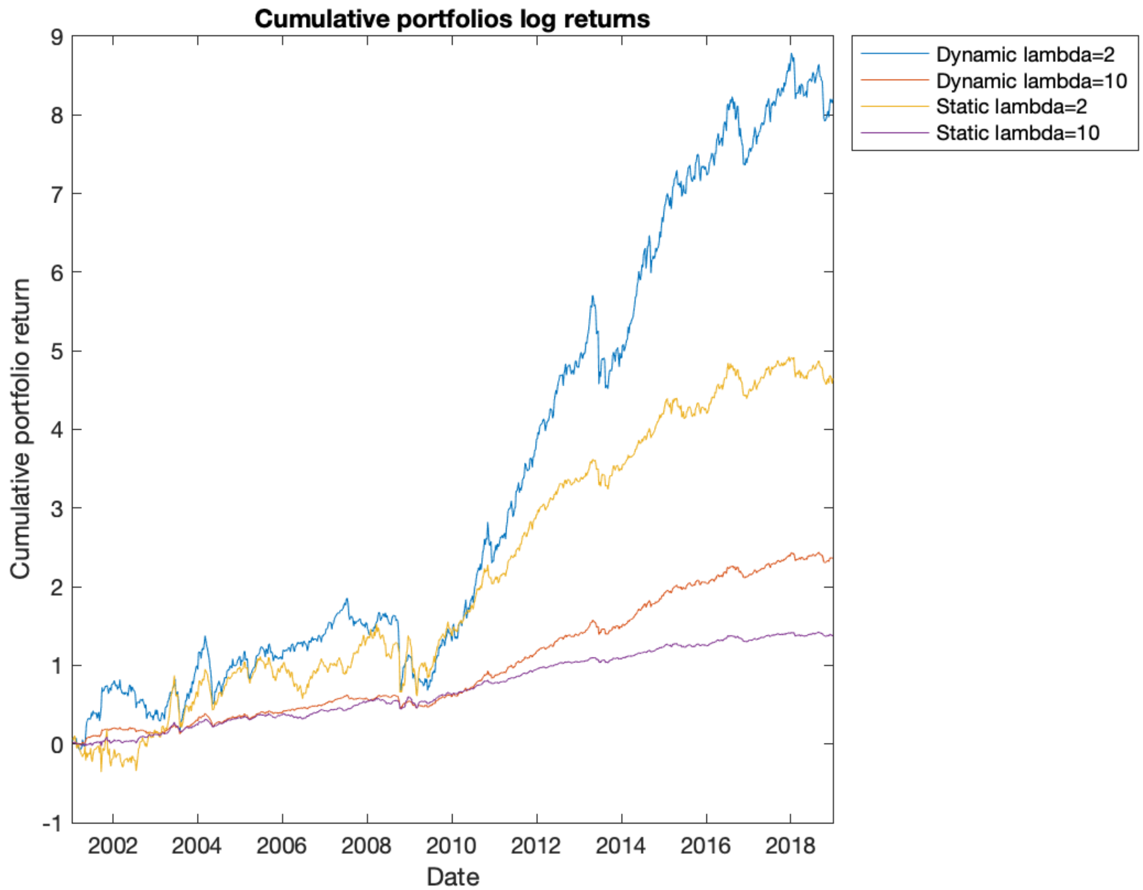

Figure 5: Cumulative portfolios log returns

3.4 Introduction of transaction costs

In this question, we compared graphically the dynamic allocation versus the static one, adding the transaction costs. As we would expect, those should be significantly higher for a dynamic portfolio because of the more frequent rebalancing that come at a cost every time it occurs. As a result, the graph with transaction costs (Figure 6) shows that the static portfolios have a higher cumulative return up until 2015. In 2014-15, we observe a convergence in cumulative returns to finally converge to the exact same value. This figure shows how harmful the transactions costs to the performance of the portfolio. Indeed, between 2008 and 2010, these fees completely wipe off all the positive returns of the dynamic portfolio with low risk aversion cumulated until then to end up into negative territories. It might be due a massive increase in volatility during the 2008-2010 period which induced a higher volatility in the weights. This is why the performance decreases (because transaction cost increases with the increase in weights). Moreover by looking at Figure 3 and Figure 4 we see that the magnitude of the changes in weights massively increases during this period.

Indeed, we need to compute τ^* such that:

$$CR_T^D - \tau \sum_{t=1}^{T} TO_t = CR_T^S \tag{10}$$

$$\tau^* = \frac{CR_T^D - CR_T^S}{\sum_{t=1}^{T} TO_t} \tag{11}$$

We found the following results of τ^* for $\lambda = 2$ and $\lambda = 10$:

$$\tau^*_{\lambda=2} = 0.0493\% \tag{12}$$

$$\tau^*_{\lambda=10} = 0.068\% \tag{13}$$

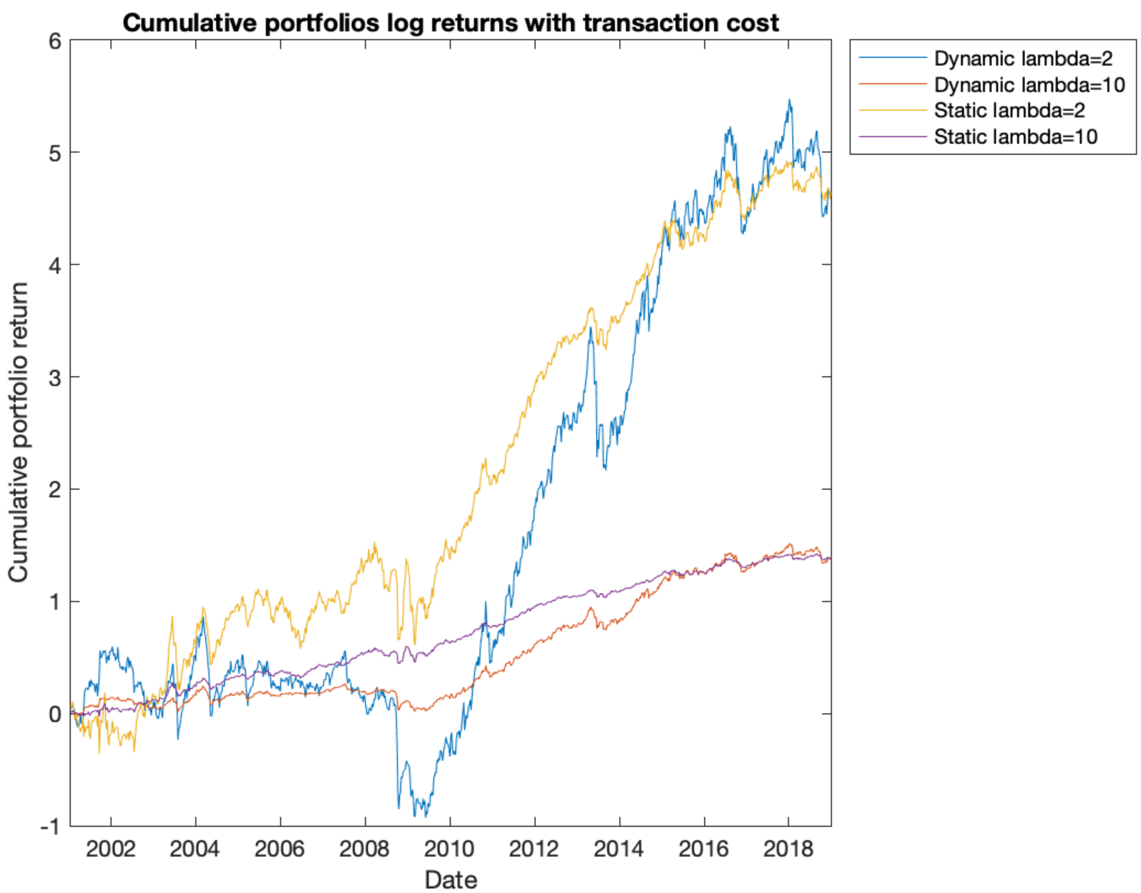

Figure 6: Cumulative portfolios log returns with transaction costs