Solutions Manual

A Book of Abstract Algebra - 2nd Edition

Charles C. Pinter

This solution manual was created by the MathLearners study group.

Contents

Chapter 1

Operations

1.1 A: Examples of Operations

3 $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, on the set $\mathbb{R}$.

Solution. From $x^2 - a^2b^2 = 0$ we get $x^2 = a^2b^2$, so $\pm ab$ is a root, which means it's not unique. Thus $a * b$ is not an operation on $\mathbb{R}$.

5 Subtraction, on the set $\{n \in \mathbb{Z} | n \geq 0\}$.

Solution. Subtraction is not an operation on that set, because if we have $k, l \in \mathbb{Z}_{\geq 0}$, where $l > k$, we get a negative result, which would not be in $\mathbb{Z}_{\geq 0}$.

1.2 B: Properties of Operations

1 $x * y = x + 2y + 4$

Solution. We follow the steps from the example.

I. Commutative: $x * y = x + 2y + 4$; $y * x = y + 2x + 4$. Thus $x * y$ is not commutative.

II. Associative: $x * (y * z) = x * (y + 2z + 4) = x + 2(y + 2z + 4) + 4$
$(x * y) * z = (x + 2y + 4) * z = x + 2y + 4 + 2z$. Thus $x * y$ is not associative.

III. Solve $x * e = x$ for e.

$$\begin{aligned} x * e &= x \\ x + 2e + 4 &= x \\ 4 &= -2e \\ -2 &= e. \end{aligned}$$

IV. Solve $x * x' = e$ for x'.

$$\begin{aligned} x * x' &= e \\ x + 2x' + 4 &= e \\ x + 2x' + 4 &= -2 \\ x + 2x' &= -6 \\ 2x' &= -6 - x \\ x' &= -\frac{6+x}{2}. \end{aligned}$$

2 $x * y = x + 2y - xy$

Solution. We follow the steps from the example.

I. Commutative:

$$\begin{aligned} x * y &= x + 2y - xy \\ y * x &= y + 2x - yx. \end{aligned}$$

Thus $x * y$ is not commutative.

II. Associative:

$$\begin{aligned} x * (y * z) &= x * (y + 2z - yz) = x + 2(y + 2z - yz) - x(y + 2z - yz) \\ (x * y) * z &= (x + 2y - xy) * z = x + 2y - xy + 2z - (x + 2y - xy)z. \end{aligned}$$

Thus $x * y$ is not associative.

III. Solve $x * e = x$ for e.

$$\begin{aligned} x * e &= x \\ x + 2e - xe &= x \\ 2e - xe &= 0 \\ e(2 - x) &= 0 \\ e &= 0. \end{aligned}$$

Check that it works: $x * 0 = x + 2 \cdot 0 - x \cdot 0 = x + 0 - 0 = x$.

IV. Solve $x * x' = e$ for x'.

$$\begin{aligned} x * x' &= e \\ x + 2x' - xx' &= 0 \\ x + x'(2 - x) &= 0 \\ x'(2 - x) &= -x \\ x' &= -\frac{x}{2 - x}. \end{aligned}$$

If $x = 2$, then the right side is undefined. Thus there is no inverse.

3 $x * y = |x + y|$

Solution. We follow the steps from the example.

I. Commutative:

$$x * y = |x + y|$$
$$y * x = |y + x| = |x + y|.$$

Thus $x * y$ is commutative.

II. Associative:

$$x * (y * z) = x * |y + z| = \big|x + |y + z|\big|$$
$$(x * y) * z = |x + y| * z = \big||x + y| + z\big|$$

Let $x = 2, y = -2$ and $z = 0$. Then $x * (y * z) = 2 * (-2 * 0) = \big|2 + |-2 + 0|\big| = |2 + 2| = 4$. But $(x * y) * z = (2 * -2) * 0 = \big||2 + (-2)| + 0\big| = |0 + 0| = 0$. Thus $x * y$ is not associative.

III. Solve $x * e = x$ for e.

$$x * e = x$$
$$|x + e| = x.$$

But if $x < 0$, then the right side is negative, but the left side is nonnegative, which is a contradiction. Thus there is no identity element.