Physics 131 Problem Set 5

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1.)

$$\langle \phi_n | \phi_m \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx$$

Note $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$. Then $\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$

$$\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((n+m)x) + \cos((n-m)x)dx$$

$$\Rightarrow \frac{\sin((n+m)x)}{2\pi(n+m)} + \frac{\sin((n-m)x)}{2\pi(n-m)}\Big|_{-\pi}^{\pi}$$

$$\Rightarrow \frac{\sin((n+m)\pi)}{\pi(n+m)} + \frac{\sin((n-m)\pi)}{\pi(n-m)}$$

For $n \neq m$, then $n+m$ and $n-m$ are integers, and $\sin(k\pi)$ for $k \in \mathbb{Z} = 0$. Thus, $\langle \phi_n | \phi_m \rangle = 0$ for $n \neq m$.

If $n = m$, then again, $n+m$ is an integer and so the lefthand term equals zero. However, this produces an indeterminant form on the right term, so we must take the limit as $n \to m$.

$$\lim_{n \to m} \frac{\sin((n-m)\pi}{\pi(n-m)} = \lim_{x \to 0} \frac{\sin(x\pi)}{x\pi} = \lim_{x \to 0} \frac{\pi\cos(x\pi)}{\pi} = 1$$

Thus, for $n = m$, then $\langle \phi_n | \phi_m \rangle = 1$. Therefore, $\langle \phi_n | \phi_m \rangle = \delta_{nm}$

$$\langle\psi_a|\psi_b\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(ax)\sin(bx)dx$$

Note $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$. Then $\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos((a-b)x) - \cos((a+b)x)dx$$

$$\Rightarrow \frac{\sin((a-b)x)}{2\pi(a-b)} - \frac{\sin((a+b)x)}{2\pi(a+b)}\Big|_{-\pi}^{\pi}$$

$$\Rightarrow \frac{\sin((a-b)\pi)}{\pi(a-b)} - \frac{\sin((a+b)\pi)}{\pi(a+b)}$$

For $a \neq b$, then $a+b$ and $a-b$ are integers, and $\sin(k\pi)$ for $k \in \mathbb{Z} = 0$. Thus, $\langle\psi_a|\psi_b\rangle = 0$ for $a \neq b$.

If $a = b$, then again, $a+b$ is an integer and so the righthand term equals zero. However, this produces an indeterminant form on the left term, so we must take the limit as $a \to b$.

$$\lim_{a\to b}\frac{\sin((a-b)\pi}{\pi(a-b)} = \lim_{x\to 0}\frac{\sin(x\pi)}{x\pi} = \lim_{x\to 0}\frac{\pi\cos(x\pi)}{\pi} = 1$$

Thus, for $a = b$, then $\langle\psi_a|\psi_b\rangle = 1$. Therefore, $\langle\psi_a|\psi_b\rangle = \delta_{ab}$

$$\langle\phi_n|\psi_a\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)\sin(ax)dx$$

Note $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ and $\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$. Then $\sin(a)\cos(b) = \frac{1}{2}(\sin(b+a) - \sin(b-a))$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin((a+n)x) + \sin((a-n)x)dx$$

$$\Rightarrow -\frac{\cos((a+n)x)}{2\pi(a+n)} - \frac{\cos((a-n)x)}{2\pi(a-n)}\Big|_{-\pi}^{\pi}$$

Since $\cos(\theta)$ is an even function, this will be zero, because its value at π will be equal to that at $-\pi$. Thus $\langle\phi_n|\psi_a\rangle = 0$.

$$a_n = \langle \phi_n | f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_n f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x) dx$$

$$a_0 = \langle \phi_0 | f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_0 f(x) dx = \frac{1}{\pi\sqrt{2}} \int_{-\pi}^{\pi} f(x) dx$$

$$b_m = \langle \psi_m | f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \psi_m f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(mx) f(x) dx$$

2.)

$$\langle \xi_n | \xi_m \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} e^{imx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix(m-n)} dx = \frac{e^{ix(m-n)}}{2i\pi(m-n)} \Big|_{-\pi}^{\pi}$$

$$\Rightarrow \frac{e^{i\pi(m-n)}}{2i\pi(m-n)} - \frac{e^{-i\pi(m-n)}}{2i\pi(m-n)} = \frac{1}{2i\pi(m-n)} (e^{i\pi(m-n)} - e^{-i\pi(m-n)})$$

For $n = m$, we have an indeterminant form, so we must evaluate it by taking the limit as $m \to n$.

$$\Rightarrow \lim_{m \to n} \frac{1}{2i\pi(m-n)} (e^{i\pi(m-n)} - e^{-i\pi(m-n)}) = \lim_{x \to 0} \frac{1}{2ix\pi} (e^{ix\pi} - e^{-ix\pi})$$

$$\Rightarrow \lim_{x \to 0} \frac{1}{2} (e^{ix\pi} + e^{-ix\pi}) = 1$$

Thus, $\langle \xi_n | \xi_m \rangle = 1$ for $n = m$. Now consider the case in which $n \neq m$. Then we can expand our complex exponentials.

$$\Rightarrow \frac{1}{2i\pi(m-n)} (\cos(\pi(m-n)) + i \sin(\pi(m-n)) - cos(\pi(m-n)) + i \sin(\pi(m-n)))$$

$$\Rightarrow \frac{\sin(\pi(m-n))}{\pi(m-n)}$$

For $n \neq m$, then $n + m$ and $n - m$ are integers, and $\sin(k\pi)$ for $k \in \mathbb{Z} = 0$. Thus, $\langle \xi_n | \xi_m \rangle = 0$ for $n \neq m$. All together, $\langle \xi_n | \xi_m \rangle = \delta_{nm}$.

$$c_n = \langle \phi_n | f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) dx$$

3.)

$$c_n = \langle e^{inx} | f \rangle = \frac{1}{P} \int_P e^{-inx} f(x) dx$$

$$c_{-n} = \langle e^{-inx} | f \rangle = \frac{1}{P} \int_P e^{inx} f(x) dx$$

$$c_{\bar{n}} = \overline{\langle e^{inx} | f \rangle} = \overline{\frac{1}{P} \int_P e^{-inx} f(x) dx}$$

Since the only complex contribution from the integral comes from the complex exponential, we can redistribute the "bar".

$$\Rightarrow c_{\bar{n}} = \frac{1}{P} \int_P \overline{e^{-inx}} f(x) dx = \frac{1}{P} \int_P e^{inx} f(x) dx = c_{-n}$$

4.)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx) f(x) dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx) f(x) dx$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx) f(x) dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} i\sin(nx) f(x) dx$$

$$\Rightarrow c_n = \frac{a_n}{2} - \frac{ib_n}{2}$$

$$c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} f(x) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx) f(x) dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} i\sin(nx) f(x) dx$$

$$\Rightarrow c_{-n} = \frac{a_n}{2} + \frac{ib_n}{2}$$

Adding c_n with c_{-n}, we get $\Rightarrow a_n = c_n + c_{-n}$ Similarly, subtracting, we arrive at $c_{-n} - c_n = ib_n$

$$\Rightarrow b_n = \frac{c_{-n} - c_n}{i}$$

5.)

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\sum_n a_n\phi_n + b_n\psi_n)^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\sum_n(a_n\phi_n + b_n\psi_n)\sum_m(a_m\phi_m + b_m\psi_m))dx$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n,m}(a_n\phi_n + b_n\psi_n)(a_m\phi_m + b_m\psi_m)dx$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n,m}(a_na_m\langle\phi_n|\phi_m\rangle + a_nb_m\langle\phi_n|\psi_m\rangle + b_na_m\langle\psi_n\phi_m\rangle + b_nb_m\langle\psi_n|\psi_m\rangle)dx$$

Using the orthonormality of the basis vectors, we arrive at

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n,m}a_na_m\delta_{nm} + b_nb_m\delta_{nm}dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_n a_n^2 + b_n^2 dx = \frac{1}{\pi}\sum_n(a_n^2 + b_n^2)x\Big|_{-\pi}^{\pi}$$

$$\Rightarrow \sum_n(a_n^2 + b_n^2)$$