Physics 131 Problem Set 1

Max Smiley

Spring 2021

1). To find the rank of the following matrices, we will put them into reduced row echelon form.

$$\begin{pmatrix}1 & 1 & 2\\ 2 & 4 & 6\\ 3 & 2 & 5\end{pmatrix}\begin{matrix} \\ -2\times I\\ -3\times I\end{matrix} \Rightarrow \begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 2\\ 0 & -1 & -1\end{pmatrix}\begin{matrix} \\ +III\\ +\frac{1}{2}\times II\end{matrix} \Rightarrow \begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{matrix} -II\\ \\ \\ \end{matrix} \Rightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$$

This matrix has 2 pivots, and thus has a rank of 2.

$$\begin{pmatrix}2 & -3 & 5 & 3\\ 4 & -1 & 1 & 1\\ 3 & -2 & 3 & 4\end{pmatrix}\begin{matrix} \\ -2\times I\\ -\frac{3}{2}\times I\end{matrix} \Rightarrow \begin{pmatrix}2 & -3 & 5 & 3\\ 0 & 5 & -9 & -5\\ 0 & \frac{5}{2} & -\frac{9}{2} & -\frac{1}{2}\end{pmatrix}\begin{matrix} \\ \\ \times 2 - II\end{matrix} \Rightarrow \begin{pmatrix}2 & -3 & 5 & 3\\ 0 & 5 & -9 & -5\\ 0 & 0 & 0 & 4\end{pmatrix}\begin{matrix}\times\frac{1}{2}\\ \times\frac{1}{5}\\ \times\frac{1}{4}\end{matrix} \Rightarrow$$

$$\begin{pmatrix}1 & -\frac{3}{2} & \frac{5}{2} & \frac{3}{2}\\ 0 & 1 & -\frac{9}{5} & -1\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{matrix}+\frac{3}{2}II\\ +III\\ \\ \end{matrix} \Rightarrow \begin{pmatrix}1 & 0 & -\frac{1}{5} & 0\\ 0 & 1 & -\frac{9}{5} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

This matrix has 3 pivots, and thus has a rank of 3.

2). Let the general 2×2 matrix have the representation $\begin{pmatrix}x & y\\ z & w\end{pmatrix}$. If its square is the zero matrix, then

$$\begin{pmatrix}x & y\\ z & w\end{pmatrix}\begin{pmatrix}x & y\\ z & w\end{pmatrix} = \begin{pmatrix}x^2+yz & xy+yw\\ xz+zw & yz+w^2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \Rightarrow \begin{aligned} x^2+yz&=0\\ xy+yw&=0\\ xz+zw&=0\\ yz+w^2&=0\end{aligned}$$

Assuming not both $y, z = 0$, we have $\begin{matrix} x^2 = -yz \\ w^2 = -yz \\ x = -w \end{matrix} \Rightarrow \begin{matrix} x = \sqrt{-yz} \\ w = -\sqrt{-yz} \end{matrix} \Rightarrow \begin{pmatrix} x & y \\ x & w \end{pmatrix} = \begin{pmatrix} \sqrt{-yz} & y \\ z & -\sqrt{-yz} \end{pmatrix}$. Now if we choose $\begin{matrix} a = \sqrt{-z} \\ b = \sqrt{y} \end{matrix}$, we see our arbitrary matrix can be represented as

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$$

Note that if both $y, z = 0$, this would still hold, as the arbitrary matrix representation would be $\begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix}$, giving us

$$\begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ 0 & w^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x = 0 \\ w = 0 \end{matrix}$$

and we could trivially choose $a = b = 0$.

3.) Suppose we have $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $det(A) = det(B) = 1 \Rightarrow det(A) + det(B) = 2$. However, $det(A+B) = det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 \neq 2$, hence, this property does not hold in general.

4.) If two nonzero vectors lie in a plane, then their cross product will produce a vector normal to the plane. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are all in the same plane, then $\vec{e} = \vec{a} \times \vec{b}$ and $\vec{f} = \vec{c} \times \vec{d}$ will both be normal vectors to the plane, i.e. pointing in the same direction. Since $\vec{e} \times \vec{f} := |\vec{e}||\vec{f}| \sin\theta$, where $\theta = 0$ as both vectors are pointing in the same direction, $\vec{e} \times \vec{f} = 0 \Rightarrow (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 0$. Note if any $\vec{a}, \vec{b}, \vec{c}, \vec{d} = 0$, then their cross product with any other vector would be the zero vector, and so the whole product would also still be the zero vector.

5.)
a.)

$$\sigma_1\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \qquad \delta_{11}\mathbf{1} + i\epsilon_{11k}\sigma_k = \mathbf{1}$$

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 \qquad \delta_{12}\mathbf{1} + i\epsilon_{123}\sigma_3 = i\sigma_3$$

$$\sigma_1\sigma_3 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\sigma_2 \qquad \delta_{13}\mathbf{1} + i\epsilon_{132}\sigma_2 = -i\sigma_2$$

$$\sigma_2\sigma_1 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} = -i\sigma_3 \qquad \delta_{21}\mathbf{1} + i\epsilon_{213}\sigma_3 = -i\sigma_3$$

$$\sigma_2\sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbf{1} \qquad \delta_{22}\mathbf{1} + i\epsilon_{22k}\sigma_k = \mathbf{1}$$

$$\sigma_2\sigma_3 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = i\sigma_1 \qquad \delta_{23}\mathbf{1} + i\epsilon_{231}\sigma_1 = i\sigma_1$$

$$\sigma_3\sigma_1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = i\sigma_2 \qquad \delta_{31}\mathbf{1} + i\epsilon_{312}\sigma_2 = i\sigma_2$$

$$\sigma_3\sigma_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = -i\sigma_1 \qquad \delta_{32}\mathbf{1} + i\epsilon_{321}\sigma_1 = -i\sigma_1$$

$$\sigma_3\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbf{1} \qquad \delta_{33}\mathbf{1} + i\epsilon_{33k}\sigma_k = \mathbf{1}$$

b.) Let $\vec{A} = (a_1, a_2, a_3)$, $\vec{B} = (b_1, b_2, b_3)$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. Then

$$(\sigma_1 \cdot \vec{A})(\sigma_1 \cdot \vec{B}) = (a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)(b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3) = \sum_{ij}^{3} a_i b_j \sigma_i \sigma_j$$

$$\Rightarrow \sum_{ij}^{3} a_i b_j (\delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma_k) = (a_1b_1 + a_2b_2 + a_3b_3)\mathbf{1} + i\sum_{ij}^{3} a_i b_j \epsilon_{ijk}\sigma_k$$

$$\Rightarrow (\vec{A}\cdot\vec{B})\mathbf{1} + i(a_1b_2\sigma_3 - a_1b_3\sigma_2 - a_2b_1\sigma_3 + a_2b_3\sigma_1 + a_3b_1\sigma_2 - a_3b_2\sigma_1)$$

$$\Rightarrow (\vec{A}\cdot\vec{B})\mathbf{1} + i(\sigma_1(a_2b_3 - a_3b_2) + \sigma_2(-a_1b_3 + a_3b_1) + \sigma_3(a_1b_2 - a_2b_1))$$

$$\Rightarrow (\vec{A}\cdot\vec{B})\mathbf{1} + i(a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1)\cdot\vec{\sigma})$$

$$\Rightarrow (\vec{A}\cdot\vec{B})\mathbf{1} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}$$

c.)

$$\sin(k\sigma_1) = k\sigma_1 + \frac{k^3\sigma_1^3}{3!} + \frac{k^5\sigma_1^5}{5!} + \dots$$

Since $\sigma_1^2 = \mathbf{1}$, then any odd power of σ_1 is σ_1, i.e. $\sigma_1^{2n+1} = \sigma_1^{2n}\sigma_1 = \mathbf{1}\sigma_1 = \sigma_1$.

$$\Rightarrow k\sigma_1 + \frac{k^3\sigma_1}{3!} + \frac{k^5\sigma_1}{5!} + \dots$$

$$\Rightarrow \sigma_1(k + \frac{k^3}{3!} + \frac{k^5}{5!} + \dots)$$

$$\Rightarrow \sigma_1\sin(k)$$

$$e^{k\sigma_3} = \mathbf{1} + k\sigma_3 + \frac{k^2\sigma_3^2}{2!} + \frac{k^3\sigma_3^3}{3!} + \dots$$

Since $\sigma_3^2 = \mathbf{1}$, all even powers of σ_3 equal one, i.e. $\sigma_3^{2n} = \mathbf{1}$, and all odd powers of σ_1 equal σ_1, i.e. $\sigma_3^{2n+1} = \sigma_3^{2n}\sigma_3 = \mathbf{1}\sigma_3 = \sigma_3$

$$\Rightarrow \mathbf{1} + k\sigma_3 + \frac{k^2\mathbf{1}}{2!} + \frac{k^3\sigma_3}{3!} + \dots$$

$$\Rightarrow (\mathbf{1} + \frac{k^2\mathbf{1}}{2!} + \dots) + (k\sigma_3 + \frac{k^3\sigma_3}{3!} + \dots)$$

$$\Rightarrow \mathbf{1}(1 + \frac{k^2}{2!} + \dots) + \sigma_3(k + \frac{k^3}{3!} + \dots)$$

$$\Rightarrow \cosh(k)\mathbf{1} + \sinh(k)\sigma_3$$

$$e^{\theta\sigma'} = \mathbf{1} + \theta\sigma' + \frac{\theta^2\sigma'^2}{2!} + \frac{\theta^3\sigma'^3}{3!} + \dots$$

Where $\sigma' = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note $\sigma'^2 = -\mathbf{1}$, and hence $\sigma'^3 = -\sigma'$, $\sigma'^4 = \mathbf{1}$, and the sigmas subsequently cycle through these values every four powers.

$$\Rightarrow \mathbf{1} + \theta\sigma' - \frac{\theta^2\mathbf{1}}{2!} - \frac{\theta^3\sigma'}{3!} + \dots$$

$$\Rightarrow (\mathbf{1} - \frac{\theta^2 \mathbf{1}}{2!} + \dots) + (\theta\sigma' - \frac{\theta^3\sigma'}{3!} + \dots)$$

$$\Rightarrow \mathbf{1}(1 - \frac{\theta^2}{2!} + \dots) + \sigma'(\theta - \frac{\theta^3}{3!} + \dots)$$

$$\Rightarrow \cos(\theta)\mathbf{1} + \sin(\theta)\sigma'$$

$$\Rightarrow \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \end{pmatrix} + \begin{pmatrix} 0 & \sin(\theta) \\ -\sin(\theta) & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

This is a rotation matrix that will rotate a vector in $\mathbb{R}^2$ by $-\theta$.

6a.) $det(A) = det\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} + det\begin{pmatrix} 4 & -1 \\ 4 & 0 \end{pmatrix} + det\begin{pmatrix} 4 & 0 \\ 4 & -2 \end{pmatrix} = -6.$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & -1 \\ 4 & -2 & 0 \end{pmatrix} \Rightarrow M = \begin{pmatrix} -2 & 4 & -8 \\ 2 & -4 & 2 \\ 1 & -5 & 4 \end{pmatrix} \Rightarrow C = \begin{pmatrix} -2 & -4 & -8 \\ -2 & -4 & -2 \\ 1 & 5 & 4 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{-6}\begin{pmatrix} -2 & -2 & 1 \\ -4 & -4 & 5 \\ -8 & -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{-1}{6} \\ \frac{2}{3} & \frac{2}{3} & \frac{-5}{6} \\ \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} \end{pmatrix}$$

$det(B) = det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + det\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = 1.$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix} \Rightarrow C = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow B^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$B^{-1}AB = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & -1 \\ 4 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 2 \\ 0 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 2 \\ -2 & -2 & -2 \\ -2 & -1 & 0 \end{pmatrix}$$

$$B^{-1}A^{-1}B = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{-1}{6} \\ \frac{2}{3} & \frac{2}{3} & \frac{-5}{6} \\ \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 2 \\ 0 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}$$

b.) $(B^{-1}AB)(B^{-1}A^{-1}B) = \begin{pmatrix} 3 & 1 & 2 \\ -2 & -2 & -2 \\ -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1}$

and $(B^{-1}A^{-1}B)(B^{-1}AB) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ -2 & -2 & -2 \\ -2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1}$

and thus $(B^{-1}AB)$ and $(B^{-1}A^{-1}B)$ are inverses of each other.

Now we claim the inverse of a product of n matricies $A_1, \dots A_n$ is equal to the product inverse of the matrices in reverse order, namely $(A_1 \dots A_n)^{-1} = A_n^{-1} \dots A_1^{-1}$. (where we assume all the inverse of A_i exists $\forall i$)

Proof By Induction: Let $P(n)$ be the preceding statement.
Base Case: $n = 2$.
Suppose we have the equation $A_1 A_2 x = y$. Then $B = (A_1 A_2)^{-1}$ is the unique matrix such that $x = By$. If we multiply both sides (on the left) first by A_1^{-1}, and then A_2^{-1}, we get

$$A_2^{-1}A_1^{-1}A_1A_2x = A_2^{-1}A_1^{-1}y \Rightarrow A_2^{-1}A_2x = A_2^{-1}A_1^{-1}y \Rightarrow x = A_2^{-1}A_1^{-1}y$$

Thus, $B = (A_1A_2)^{-1} = A_2^{-1}A_1^{-1}$

Induction Step: Assume $P(n-1)$
Suppose now we have some equation $(\prod_{i=1}^{n} A_i)x = y$ that involves the product of n

matrices. Then $B = (\prod_{i=1}^{n} A_i)^{-1}$ is the unique matrix such that $x = By$. By matrix associativity, we have

$$(\prod_{i=1}^{n} A_i)x = y = A_1(\prod_{i=2}^{n} A_i)x = y \Rightarrow (\prod_{i=2}^{n} A_i)x = A_1^{-1}y$$

However, since $(\prod_{i=2}^{n} A_i)$ is simply a product of $n-1$ matrices, by the induction hypothesis, its inverse is $A_n^{-1} \dots A_2^{-1}$.

$$\Rightarrow x = A_n^{-1} \dots A_1^{-1}y$$

$$\Rightarrow B = (A_1 \dots A_n)^{-1} = A_n^{-1} \dots A_1^{-1}$$