

CS 180

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This is an introductory course to algorithms that I took at UCLA. After the first few introductory lessons, we cover graphs, greedy algorithms, divide and conquer, dynamic programming, NP completeness, and randomized algorithms. In each section, I'll go over definitions, theorems, and proofs. This is a relatively informal set of notes, mainly intended to be referenced by myself in the future.

# The Stable Matching Problem

The *Stable Matching Problem* is an idealism of many real world problems, in which we want to find a reasonable pairing between members of two different groups. For example, algorithms developed to study this problem are implemented in medical school, where students get residencies and are matched to various institutions.

The Stable Matching Problem is presented as follows. Suppose we have two groups of equal size, for example, we take these groups to be men and women, where  $M = \{m_1, \dots, m_n\}$  and  $W = \{w_1, \dots, w_n\}$ . Each  $m \in M$  and  $w \in W$  also have a preference list of the opposite sex. The goal is to create a one to one pairing between members of the groups, such that the matching is stable.

**Def:** a *perfect matching* between  $n$  men  $M$  and  $n$  women  $W$  is a bijection  $S = \{(m_1, w_1), \dots, (m_n, w_n)\}$  between elements of  $M$  and  $W$ . That is, for each  $m_i \in M$ , there exists a unique  $w_j \in W$  such that  $(m_i, w_j) \in S$ , and vice versa.

**Def:** a *rogue pair* is a stable matching  $S$  such that  $\exists (m_i, w_j), (m'_i, w'_j) \in S$  such that  $m_i$  prefers  $w'_j$  to  $w_j$ , and  $w'_j$  prefers  $m_i$  to  $m'_i$ .

**Def:** a *stable matching* is a perfect matching such that there are no rogue pairs.

## Gale-Shapley Algorithm:

*while*  $\exists$  unmatched  $m \in M$  :

*select*  $m \in M$ , *find* highest ranking woman  $w \in W$  *not yet proposed to*

*if*  $w$  *is unmatched*

*add*  $(m, w)$  *to*  $S$

*else if*  $w$  *prefers her current man*  $m'$  *to*  $m$

*reject*  $m$

*else if*  $w$  *prefers*  $m$  *to her current man*  $m'$

*remove*  $(m', w)$  *from*  $S$ , *add*  $(m, w)$  *to*  $S$

**Lemma A:** Once  $w$  is matched, she will never be unmatched.

*Proof:* if  $w$  is unmatched, then she will either reject  $m$ , or break her relationship with  $m'$  in favor of  $m$ . In either case,  $w$  will still be matched.

**Lemma B:** Once matched,  $m$ 's partners will only decrease in favorability.

*Proof:* if  $m$  has previously had partner  $w$ , then  $m$  had proposed to  $w$ , and thereafter may only propose to women lower on his priority list. Thus, he can only match with those women.

**Lemma C:** Once matched,  $w$ 's partners will only increase in favorability.

*Proof:* suppose  $w$  is matched with  $m$ . Then, she will only accept a new man  $m'$  if she prefers  $m'$  to  $m$ .

**Claim:** The GS algorithm terminates.

*Proof:* the algorithm iterates over  $n$  men, each of which having  $n$  women to potentially propose. Thus, the algorithm terminates after at most  $n^2$  steps.

**Claim:** The GS algorithm outputs a perfect matching.

*Proof:* since the algorithm assigns 1 to 1 pairs, if there is not a perfect matching by the end of the algorithm, then  $\exists m \in M$  such that no  $(m, w) \in S$  for  $w \in W$ . However, the algorithm wouldn't have ended if there were still an  $m$  leftover at the end, it would have assigned him to  $w$ .

**Claim:** The GS algorithm outputs a stable matching.

*Proof:* assume, by way of contradiction, that the GS algorithm did not output a stable matching. Then  $\exists (m, w), (m', w') \in S$  such that  $m$  prefers  $w'$  to  $w$  and  $w'$  prefers  $m$  to  $m'$ . If this were the case, then  $m$  would have proposed to  $w'$  before  $w$ .  $w'$  would have either rejected  $m$  (if she had a better mate) or accepted him, but in either case, by Lemma C,  $w'$  would never have matched with  $m'$  thereafter, as she prefers  $m$  to  $m'$ .

**Def:** a pair  $(m, w)$  is *valid* if there exists some stable matching  $S$  such that  $(m, w) \in S$ .

**Claim:** The GS algorithm outputs the best valid pairs for men.

*Proof:* assume, by way of contradiction, that a man  $m$  is rejected by his best valid partner  $w$  in favor of another man  $m'$ . Since this many happen many times, assume this is the first instance of a man being rejected by his best valid partner. However, since  $(m, w)$  is a valid matching, there exists a stable matching  $S$  such that  $(m, w), (m', w') \in S$ . Originally, when  $m$  proposed to  $w$ , she was already matched with  $m'$ ,

meaning  $m'$  had  $w$  higher up on his list than any other valid match by Lemma B (he couldn't have been rejected by his best valid match by the original condition. Since  $m$  being rejected by  $w$  was the first instance of someone being rejected by their best valid partner, then in the original scenario, ()) We know  $w$  prefers  $m'$ , so showing  $m'$  prefers  $w$  is sufficient to show there is a rogue pair, and thus a contradiction.

**Claim:** The GS algorithm outputs the worst valid pairs for women.

*Proof:* assume, by way of contradiction, that the GS algorithm does not output the worst valid pairs for women, i.e.  $\exists(m, w) \in S$  such that for some  $m'$  ranked lower than  $m$  by  $w$ ,  $(m', w)$  is part of a stable matching. By Lemma C, if  $(m', w)$  were ever a pair, it would have been before  $(m, w)$  was a pair.