Judgment Theory: A Syntax for Virtual Equipments and Internal Category Theory

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In what follows we present an internal language for virtual equipments (TODO: cite Crutwell-Shulman, Leinster, Hermida) Despite sounding slightly obscure, virtual equipments are quite useful as "a good place to do category theory", that is a virtual equipment has enough structure to define internal notions of category, multicategory and polycategory. From a type-theoretic angle and a sprinkle of Curry-Howard-Lambek ideology, we can also see a virtual equipment as a "good place to study type theories" and so we call the type theory "judgment theory".

The following figures present the basic formation, primitive and admissible and equality rules of a type theory we call judgment theory, an internal language for virtual equipments. Note that contra Crutwell-Shulman, we define a virtual equipment to be a virtual double category with all restrictions, whereas they additionally require the category to have all *units*. Judgment theory has 4 central concepts, which we will refer to sometimes using type theoretic and at other times category-theoretic terminology. In type theoretic terminology, judgment theory has *sorts* $\mathbb{C}, \mathbb{D}, \mathbb{E}$, and *types* A, B, C which have a given sort and are also *parameterized* by an object of some other sort, and so can also be thought of as *functions* with input and output sorts. Next there are *judgments* P, Q, R, S which are parameterized by two variables. In $\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid R$, we say R depends on α *contravariantly* and in β *covariantly*. Finally, we have *terms* s, t, u that are elements of some judgment R but are also parameterized by elements of some other judgments which are given by a *context* Φ, which is a "compatible string" of judgments where the covariant variable of one judgment is the same as the contravariant variable of the next. Since these terms are parameterized by other terms, we can see these as *inference rules* for the judgments.

At other times we will want to think of these using set-theoretic terminology, in which case we have sets $\mathbb{C}, \mathbb{D}, \mathbb{E}$, functions/elements A, B, C, *spans* $R, \ldots$ and morphisms of spans s, t, u. Finally when we develop multicategory theory we will use category-theoretic terminology, in which the structure is most familiar and we have categories, objects/functors, *profunctors*/bimodules and 2-cells/homomorphisms of bimodules.

1 Internal Categories and Category Theory

Next, we introduce the notion of category internal to our syntactic virtual equipment and the *monoids and modules* translation that shows that, with the inclusion of hom sets, our syntax is already suitable for doing syntactic category theory.

The benefits of our syntax become clear when we define internal categories, functors, profunctors and transformations: the definitions read exactly as the classical definitions.

Definition 1 (Internal Category)**.** *An* (internal) category *consists of*

1. *A sort* $\mathbb{C}_0$
2. *A span* $\alpha^o : C_0; \alpha' : C_0 \mid \alpha^o \to_{\mathbb{C}} \alpha'$
3. *Identity arrows* $\alpha^o : C_0; \alpha : C_0 \mid \cdot \vdash id_\alpha : \alpha \to_{\mathbb{C}} \alpha$.
4. *Composition* $\alpha_0^o : C_0; \alpha_2 : C_0 \mid \phi_0 : \alpha_0 \to \alpha_1, \phi_1 : \alpha_1 \to \alpha_2 \vdash \phi_1 \circ \phi_0 : \alpha_0 \to \alpha_2$
5. *Satisfying Unitality:*

$$id_\alpha \circ \phi = \phi = \phi \circ id_{\alpha'}$$

$$\mathbb{C}\,\mathrm{sort}$$

$$\frac{\mathbb{C}\,\mathrm{sort} \qquad \mathbb{D}\,\mathrm{sort}}{\alpha : \mathbb{C} \vdash A : \mathbb{D}}\ \textsc{Object/Type-formation}$$

$$\frac{\mathbb{C}\,\mathrm{sort} \qquad \mathbb{D}\,\mathrm{sort}}{\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid R}\ \textsc{Relation/Judgment/Profunctor-formation}$$

$$\frac{\mathbb{C}\,\mathrm{sort} \qquad \mathbb{D}\,\mathrm{sort}}{\alpha^o : \mathbb{C}; \beta : \mathbb{D}\,\mathrm{sort} \mid \Phi\,\mathrm{context}}\ \textsc{Context-formation}$$

$$\frac{\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \Phi \qquad \alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid R}{\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \Phi \vdash t : R}\ \textsc{Element/Term/Arrow-formation}$$

$$\frac{\alpha : \mathbb{C} \vdash A : \mathbb{C}' \qquad \beta : \mathbb{D} \vdash B : \mathbb{D}' \qquad \alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \Phi \qquad \alpha'^o : \mathbb{C}'; \beta' : \mathbb{D}' \mid \Psi}{A; B \mid \Phi \vdash \vec{t} : \Psi}\ \textsc{Substitution-formation}$$

Figure 1: Formation Rules of Judgment Theory

$$\frac{}{\alpha : \mathbb{C} \vdash \alpha : \mathbb{C}} \qquad \frac{\alpha : \mathbb{C} \vdash B : \mathbb{D} \qquad \beta : \mathbb{D} \vdash A : \mathbb{E}}{\alpha : \mathbb{C} \vdash A[B/\beta] : \mathbb{E}}\ *$$

$$\frac{\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid R \qquad \alpha' : \mathbb{C}' \vdash A : \mathbb{C} \qquad \beta' : \mathbb{D}' \vdash B : \mathbb{D}'}{\alpha'^o : \mathbb{C}'; \beta' : \mathbb{D}' \mid R[A/\alpha, B/\beta]}\ \textsc{Restriction}*$$

$$\frac{}{\alpha^o : \mathbb{C}; \alpha : \mathbb{C} \mid \cdot} \qquad \frac{\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \Phi \qquad \beta^o : \mathbb{D}; \gamma : \mathbb{E} \mid R}{\alpha^o : \mathbb{C} : \gamma : \mathbb{E} \mid \Phi, \phi : R} \qquad \frac{\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \Phi \qquad \beta^o : \mathbb{D}; \gamma : \mathbb{E} \mid \Psi}{\alpha^o : \mathbb{C}; \gamma : \mathbb{E} \mid \Phi, \Psi}\ *$$

$$\frac{\Phi \vdash t : R[A/\alpha, B/\beta]}{A; B \mid \Phi \vdash t/\phi : (\phi : R)} \qquad \frac{A; B \mid \Psi \vdash \vec{t} : \Phi \qquad B; C \mid \Psi' \vdash s : R[B/\beta, C/\gamma]}{A; C \mid \Psi, \Psi' \vdash \vec{t}, s/\phi : (\Phi, \phi : R)}$$

$$\frac{A; B \mid \Psi \vdash \vec{t} : \Phi \qquad B; C \mid \Psi' \vdash \vec{s} : \Phi'}{A; C \mid \Psi, \Psi' \vdash \vec{t}, \vec{s} : \Phi, \Phi'}\ *$$

$$\frac{}{\alpha^o; \beta \mid \phi : R \vdash \phi : R} \qquad \frac{A; B \mid \Psi \vdash \vec{s} : \Phi \qquad \alpha'^o : \mathbb{C}'; \beta' : \mathbb{D}' \mid \Phi \vdash t : R}{\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \Psi \vdash t[\vec{s}] : R[A/\alpha', B/\beta']}\ *$$

Figure 2: Basic Rules of Judgment Theory, starred rules are admissible, non-starred are primitive

$$
\begin{array}{rcl}
A[\alpha/\alpha] & = & A \\
A[B/\beta][C/\gamma] & = & A[B[C/\gamma]/\beta] \\
\\
R[\alpha/\alpha, \beta/\beta] & = & R \\
R[\alpha'.A/\alpha, \beta'.B/\beta][A'/\alpha', B'/\beta'] & = & R[A[A'/\alpha']/\alpha, B[B'/\beta']/\beta] \\
\\
\Phi, \cdot & = & \Phi \\
\cdot, \Phi & = & \Phi \\
(\Phi, \Psi), \Theta & = & \Phi, (\Psi, \Theta) \\
\\
(\vec{t}, \vec{s}), \vec{u} & = & \vec{t}, (\vec{s}, \vec{u}) \\
\\
t[\phi_1/\phi_1, \ldots \phi_n/\phi_n] & = & t \\
t[\Psi_1.s_1/\phi_1, \ldots, \Psi_n.s_n/\phi_n][\vec{u}_1/\Psi_1, \ldots, \vec{u}_n/\Psi_n] & = & t[s_1[\vec{u}_1/\Psi_1]/\phi_1, \ldots, s_n[\vec{u}_n/\Psi_n]/\phi_n]
\end{array}
$$

Figure 3: Algebraic Properties of Basic Rules

$$
\frac{\beta : \mathbb{D} \vdash A : \mathbb{C} \qquad \gamma : \mathbb{E} \vdash A' : \mathbb{C}}{\beta^o : \mathbb{D}; \gamma : \mathbb{E} \mid A \to_{\mathbb{C}} A'}\ \textsc{Hom-Formation}
\qquad
\frac{}{\alpha^o : \mathbb{C}, \alpha : \mathbb{C} | \cdot \vdash \mathrm{id}_\alpha : \alpha \to_{\mathbb{C}} \alpha}\ \textsc{Hom-Introduction}
$$

$$
\frac{\beta_1^o; \beta_2 \mid \Psi \vdash f : \beta_1 \to_C \beta_2 \qquad \alpha^o; \beta \mid \Phi \qquad \beta^o; \gamma \mid \Phi' \qquad \Phi, \Phi' \vdash g : R}{\Phi[\cdot; \beta_1/\beta], \Psi, \Phi'[\beta_2/\beta; \cdot] \vdash \text{when}\ f = \mathrm{id}_\beta.\ g : R[\beta_1/\beta; \beta_2/\beta]}\ \textsc{Hom-Elim}
$$

$$
\text{when}\ \mathrm{id}_\alpha = \mathrm{id}_\beta.\ g \equiv g[\alpha/\beta]
\qquad
\frac{\ldots}{\beta_1^o : \mathbb{C}; \beta_2 : \mathbb{C} \mid \Phi \vdash f \equiv \text{when}\ f = \mathrm{id}_\alpha.\ \mathrm{id}_\alpha : \beta_1 \to_{\mathbb{C}} \beta_2}\ \textsc{Hom-}\eta
$$

Figure 4: Hom Sets

6. *And Associativity:*
$$(\phi \circ \psi) \circ \theta = \phi \circ (\psi \circ \theta)$$

Definition 2 (Internal Functor)**.** *A functor F between internal categories $\mathbb{C}$ and $\mathbb{D}$ consists of*

1. *a function on objects $\alpha : \mathbb{C}_0 \vdash F_0 : \mathbb{D}_0$*
2. *a function on arrows $\alpha_0^o : \mathbb{C}_0; \alpha_1 : \mathbb{C}_0 \mid \phi : \alpha_0 \to_{\mathbb{C}} \alpha_1 \vdash F_1(\phi) : F_0(\alpha_0) \to_{\mathbb{D}} F_0(\alpha_1)$*
3. *that preserves identity:*
$$F_1(id_\alpha) = id_{F_0(\alpha)}$$
4. *and composition:*
$$F_1(\psi \circ \phi) = F_1(\psi) \circ F_1(\phi)$$

Definition 3 (Internal Profunctor)**.** *A profunctor R between internal categories $\mathbb{C}$ and $\mathbb{D}$ consists of*

1. *A span $\alpha^o : \mathbb{C}_0 \mid \beta : \mathbb{D}_0 \mid \alpha \to_R \beta$*
2. *C-composition: $\alpha_0^o : \mathbb{C}_0; \beta : \mathbb{D}_0 \mid \phi : \alpha_0 \to_C \alpha_1, \psi : \alpha_1 \to_R \beta \vdash \psi \circ_{R\mathbb{C}} \phi : \alpha_0 \to_R \beta$*
3. *D-composition: $\alpha^o : \mathbb{C}_0; \beta_1 : \mathbb{D}_0 \mid \psi : \alpha \to_R \beta, \theta : \beta_0 \to_{\mathbb{D}} \beta_1 \vdash \theta \circ_{\mathbb{D}R} \psi : \alpha \to_R \beta_1$*
4. *Satisfying $\mathbb{C}$ associativity: $\psi \circ_{R\mathbb{C}} (\phi_1 \circ_{\mathbb{C}} \phi_0) = (\psi \circ_{R\mathbb{C}} \phi_1) \circ_{R\mathbb{C}} \phi_0$*
5. *Satisfying $\mathbb{C}$ unitality: $\psi \circ_{R\mathbb{C}} id_\alpha = \psi$*
6. *Satisfying $\mathbb{D}$ associativity: $(\theta_1 \circ_{\mathbb{D}} \theta_0) \circ_{\mathbb{D}R} \psi = \theta_1 \circ_{\mathbb{D}R} (\theta_0 \circ_{\mathbb{D}R} \psi)$*
7. *Satisfying $\mathbb{D}$ unitality: $id_\beta \circ_{\mathbb{D}R} \psi = \psi$*
8. *Satisfying mixed associativity: $(\theta \circ_{\mathbb{D}R} \psi) \circ_{R\mathbb{C}} \phi = \theta \circ_{\mathbb{D}R} (\psi \circ_{R\mathbb{C}} \phi)$*

Next, we show that Profunctors can be restricted by functors

Definition 4 (Restriction of Profunctors)**.** *If $\mathbb{C}, \mathbb{D}, \mathbb{C}', \mathbb{D}'$ are categories, and $\alpha' : \mathbb{C}' \vdash F[\alpha'] : \mathbb{C}$ and $\beta' : \mathbb{D}' \vdash G[\beta'] : \mathbb{D}$ are functors and $\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \alpha^o \to_R \beta$ is a profunctor, then $F[\alpha'] \to_R G[\beta']$ is given the structure of a profunctor by:*

1. *$\mathbb{C}'$ composition:*
$$\phi : \alpha_0 \to_{\mathbb{C}'} \alpha_1, \psi : F[\alpha_1] \to_R G[\beta] \vdash \psi \circ_{R\mathbb{C}} F_1[\phi] : F[\alpha_1] \to_R G[\beta]$$
2. *$\mathbb{D}'$ composition:*
$$\alpha^o : C'; \beta_1 : \mathbb{D}' \mid \psi : F[\alpha] \to_R G[\beta], \theta : \beta_0 \to_{\mathbb{D}'} \beta_1 \vdash G[\theta] \circ_{\mathbb{D}R} \psi : F[\alpha] \to_R G[\beta_1]$$

Definition 5 (Natural Transformation)**.** *A transformation $F : \Phi \vdash: R$ where each of the sorts in $\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \Phi$ has a given category structure and each of the spans in Φ has a profunctor structure on those category structures is given by*

1. *A function $\Phi \vdash F_1 : \alpha \to_R \beta$*
2. *Internal Naturality: For any decomposition $\Phi = \Phi_0, \phi : \gamma \to_S \delta, \psi : \delta \to_T \epsilon, \Phi_1$ where $\gamma : \mathbb{E}$ and $\delta : \mathbb{F}$, $\epsilon : \mathbb{G}$, an equality*
$$\Phi_0, \phi : \gamma \to_S \delta_0, \theta : \delta_0 \to_{\mathbb{F}} \delta_1, \psi : \delta_1 \to_T \epsilon, \Phi_1 \vdash F_1[\theta \circ_{\mathbb{F}S} \phi/\phi] = F_1[\psi \circ_{T\mathbb{F}} \theta/\psi]$$
3. *Left naturality: If $\Phi = \phi : \alpha \to_s \gamma, \Phi'$ then*
$$F_1 \circ_{R\mathbb{C}} \psi = F_1[\phi \circ_{S\mathbb{C}} \psi]$$

4. Right naturality: If $\Phi = \Phi', \phi : \gamma \to_S \beta$*, then*

$$\phi \circ_{\mathbb{D}R} F_1 = F_1[\psi \circ \phi/\psi]$$

This last definition is probably unfamiliar to the reader that is not experienced with profunctors. However, we can show that the traditional definition is equivalent to a natural transformation $\alpha : \mathbb{C}_0 \mid \cdot \vdash t_\alpha : F(\alpha) \to_{\mathbb{D}} G(\alpha')$.

Definition 6 (Traditional Natural Transformation)**.** *A traditional natural transformation from* $\alpha : \mathbb{C}_0 \vdash F(\alpha) : \mathbb{D}_0$ *to* $\alpha : \vdash G(\alpha') : \mathbb{D}_0$ *consists of*

1. A term $\cdot \vdash t_\alpha : F(\alpha) \to_{\mathbb{D}} G(\alpha)$

2. Such that

$$\phi : \alpha \to_{\mathbb{C}} \alpha' \vdash t_{\alpha'} \circ F_1(\phi) \equiv G_1(\phi) \circ t_\alpha : F(\alpha) \to_{\mathbb{D}} G(\alpha')$$

Theorem 1 (Equivalent Presentations of Natural Transformations)**.** *There is a bijection between traditional natural transformations* $t : F \to G$ *and profunctor homomorphisms* $h : \alpha \to_{\mathbb{C}} \alpha' \vdash F(\alpha) \to_{\mathbb{D}} G(\alpha')$ *given by*

1. From a natural transformation t *we define a homomorphism* $h(t)$ *by composing on the* F *side. The choice of this side is arbitrary and we could just as easily have used* G *since they are equal by naturality.*

$$\phi : \alpha \to_{\mathbb{C}} \alpha' \vdash t_{\alpha'} \circ F_1(\phi) : F(\alpha) \to_{\mathbb{D}} G_1(\alpha')$$

2. From a homomorphism $\phi.h$*, we define a natural transformation* $t(h)$ *as*

$$\alpha \mid \cdot \vdash h[id_\alpha/\phi]$$

Proof. First we show that the maps $h(\cdot), t(\cdot)$ actually produce homomorphisms and natural transformations.

1. On the left, we need to show $h(t)_\phi \circ F_1(\psi) = h(t)_{|phi \circ \psi}$. Expanding the definition, we just need associativity and functoriality of F_1:

$$\begin{aligned} h(t)_\phi \circ F_1(\psi) &\equiv (t_{\alpha'} \circ F_1(\phi)) \circ F_1(\psi) \\ &\equiv t_{\alpha'} \circ F_1(\phi \circ \psi) \\ &\equiv h(t)_{\phi \circ \psi} \end{aligned}$$

on the right, we use naturality:

$$\begin{aligned} G_1(\theta) \circ h(t)_\phi &\equiv G_1(\theta) \circ (t_{\alpha'} \circ F_1(\phi)) \\ &\equiv (t_{\alpha''} \circ F_1(\theta)) \circ F_1(\phi) \\ &\equiv t_{\alpha''} \circ F_1(\theta \circ \phi) \\ &\equiv t_{\theta \circ \phi} \end{aligned}$$

2. In the other direction, to prove naturality:

$$\begin{aligned} h[\mathrm{id}_\alpha/\phi] \circ F_1(\psi) &\equiv h[\mathrm{id}_\alpha \circ \psi/\phi] \\ &\equiv h[\psi/\phi] \\ &\equiv h[\psi \circ \mathrm{id}_{\alpha'}/\phi] \\ &\equiv G_1(\psi) \circ h[\mathrm{id}_\alpha/\phi] \end{aligned}$$

Next we seek to prove that the functions are mutually inverse.

1. Round trip for homomorphisms:

$$\begin{aligned} h(t(h))_\phi &= t(h)_\alpha \circ F_1[\phi] \\ &= h[\mathrm{id}_\alpha] \circ F_1[\phi] \\ &= h[\mathrm{id} \circ \phi] \\ &= h[\phi] \end{aligned}$$

2. Round trip for natural transformations uses the fact that functors preserve identity.

$$\begin{aligned} t(h(t))_\alpha &= h(t)_{\mathrm{id}_\alpha} \\ &= t_\alpha \circ F_1[\mathrm{id}_\alpha] \\ &= t_\alpha \circ \mathrm{id}_{F_0\alpha} \\ &= t_\alpha \end{aligned}$$

□

1.1 Universal Properties, Internally

In a sense, the very purpose of category theory is to study universal properties and so it is essential that our syntax provide a nice presentation of universal properties.

Fortunately, the central placement of spans/profunctors in our syntax makes the definition of a functor satisfying a universal property very natural (should I introduce the unit category earlier?).

Definition 7 (Right Representability)**.** *Let $\mathbb{C}, \mathbb{D}$ be categories and $\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \alpha \to_Q \beta$ be a profunctor between them. Then a* right representation *of Q consists of*

1. *A functor $\beta : \mathbb{D} \vdash G[\beta] : \mathbb{C}$*
2. *An "introduction rule" homomorphism $\phi : \alpha \to_Q \beta \vdash I_G[\phi] : \alpha \to_{\mathbb{C}} G[\beta]$.*
3. *An inverse homomorphism $\psi : \alpha \to_{\mathbb{C}} G[\beta] \vdash I_G^{-1}[\psi] : \alpha \to_Q \beta$. That is, it satisfies*

$$\begin{aligned} I_G[I_G^{-1}[\psi]/\phi] &= \psi \\ I_G^{-1}[I_G[\phi]/\psi] &= \phi \end{aligned}$$

While this is a fairly standard definition, it is likely unfamiliar to type theorists, who may be surprised that Right Representability is essentially the same as the definition of a negative type, which has an introduction rule, but instead of an inverse, there is an elimination rule, and the functoriality of the type is not primitive. Fortunately, we can prove the equivalence between these notions entirely syntactically.

Definition 8 (Negative Type)**.** *A negative type (connective) N between categories $\mathbb{C}$,$\mathbb{D}$ consists of*

1. *A judgment it represents $\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \alpha \to_Q \beta$*
2. *A type $\beta : \mathbb{D} \vdash N[\beta] : \mathbb{C}$*
3. *An "introduction rule" $\phi : \alpha \to_Q \beta \vdash I_N[\phi] : \alpha \to_{\mathbb{C}} N[\beta]$ that is a homomorphism* on the left*:*

$$\psi : \alpha' \to_{\mathbb{C}} \alpha, \phi : \alpha \to_Q \beta \vdash I_N[\phi] \circ_{\mathbb{C}} \psi \equiv I_N[\phi \circ_Q \psi] : \alpha' \to_{\mathbb{C}} N[\beta]$$

4. *An "elimination rule" $\beta^o : \mathbb{D}; \beta : \mathbb{D} \mid \cdot \vdash \epsilon_N[\beta] : N[\beta] \to_Q \beta$*
5. *Such that $\psi : \alpha \to_{\mathbb{C}} N[\beta] \vdash \epsilon_N[\beta] \circ_Q \psi : \alpha \to_Q \beta$ is an inverse to $I_N[\phi]$ in that*

$$\begin{aligned} I_n[\epsilon_N[\beta] \circ \psi/\phi] &= \psi && (\eta) \\ \epsilon_N[\beta] \circ I_n[\phi] &= \phi && (\beta) \end{aligned}$$

Part of this equivalence is the *Yoneda Lemma*, which we state now in its internal, parameterized form. The usual non-parameterized Yoneda lemma can be formed if we have a unit sort.

Lemma 1 (Parameterized Yoneda Lemma). *Given categories* $\mathbb{C}, \mathbb{D}$ *and a profunctor* $\alpha^o : \mathbb{C}; \beta : \mathbb{D} \vdash \alpha \to_Q \beta$ *and a function* $\beta : \mathbb{D} \vdash G[\beta] : \mathbb{C}$*, then there is an isomorphism between*

1. *Terms* $\cdot \vdash t : G[\beta] \to_Q \beta$
2. *left-homomorphisms* $\phi : \alpha \to_{\mathbb{C}} G[\beta] \vdash u : \alpha \to_Q \beta$ *i.e. they satisfy homomorphism on the left (the other side doesn't make sense if* G *is not a functor):* $u[\phi \circ \psi] = u[\phi] \circ \psi$

Proof. First, the constructions

1. Given t, we define $u(t)$ to be
$$\phi : \alpha \to_{\mathbb{C}} G[\beta] \vdash t \circ_Q \phi : \alpha \to_Q \beta$$
Which is a left-homomorphism because:
$$(t \circ_Q \phi) \circ_Q \psi = t \circ_Q (\phi \circ_{\mathbb{C}} \psi)$$
2. Given u, we define $t(u)$ to be
$$\cdot \vdash u[\mathrm{id}_{G[\beta]}/\phi] : G[\beta] \to_Q \beta$$

Now we show they are inverse

1. First, the very easy case, $\cdot \vdash t(u(t)) : G[\beta] \to_Q \beta$:
$$t \circ_Q \mathrm{id}_{G[\beta]} = t$$
by the fact that Q is a profunctor.
2. Next, $\phi : \alpha \to_{\mathbb{C}} G[\beta] \vdash u(t(u)) : \alpha \to_Q \beta$:
$$\begin{aligned} u[\mathrm{id}_{G[\beta]}/\phi] \circ_Q \phi &= u[\mathrm{id} \circ_{\mathbb{C}} \phi/\phi] \\ &= u[\phi/\phi] \\ &= u \end{aligned}$$
which uses the fact that u is a left-homomorphism.

□

Theorem 2 (Negative Types and Right Representables are Equivalent). *For any profunctor* $\alpha^o : \mathbb{C}; \beta : \mathbb{D} \mid \alpha \to_Q \beta$*, there is a bijection between*

1. *Right Represntables* $\beta : \mathbb{D} \vdash G[\beta] : \mathbb{C}$ *for* Q*.*
2. *Negative Types* $\beta : \mathbb{D} \vdash N[\beta] : \mathbb{C}$ *representing* Q*.*

Proof.

1. Given a right representable $G[\beta]$, we set $N(G)[\beta] = G[\beta]$, the introduction rule is the same and the elimination rule is given as in the Yoneda lemma and the inverse property follows by the Yoneda isomorphism and the fact that I_G^{-1} is an inverse for I_G.
2. Given a negative type $N[\beta]$ we set $G(N)[\beta] = N[\beta]$, the introduction rule is the same and the inverse introduction rule is given as in the Yoneda lemma. We need to show that $N[\beta]$ has the structure of a functor and that I_N and I_N^{-1} are homomorphisms on the right with respect to that structure. Define N_1 as follows:
$$\frac{\beta_1^o : \mathbb{D}; \beta_2 : \mathbb{D} \vdash \psi \circ \epsilon : N[\beta_1/\beta] \to_Q \beta_2 \qquad \phi : N[\beta_1/\beta] \to_Q \beta_2 \vdash I_N[\phi] : N[\beta_1/\beta] \to_{\mathbb{C}} N[\beta_2/\beta]}{\beta_1^o : \mathbb{D}; \beta_2 : \mathbb{D} \mid \psi : \beta_1 \to_{\mathbb{D}} \beta_2 \vdash I_N[\psi \circ \epsilon_{\beta_1}/\phi] : N[\beta_1/\beta] \to_{\mathbb{C}} N[\beta_2/\beta]}$$

We need to show functoriality. For identity we have

$$
\begin{aligned}
I_N[\mathrm{id}_\beta \circ \epsilon_\beta/\phi] &= I_N[\epsilon_\beta] \\
&= I_N[\epsilon \circ \mathrm{id}_{N[\beta]}] \\
&= \mathrm{id}_{N[\beta]} && (\eta)
\end{aligned}
$$

For composition,

$$
\begin{aligned}
N_1[\theta] \circ N_1[\psi] &= I_N[\theta \circ \epsilon] \circ I_N[\psi \circ \epsilon] \\
&= I_N[(\theta \circ \epsilon) \circ I_N[\psi \circ \epsilon]] && \text{(left homomorphism)} \\
&= I_N[\theta \circ (\epsilon \circ I_N[\psi \circ \epsilon])] \\
&= I_N[\theta \circ (\psi \circ \epsilon)] && (\beta) \\
&= I_N[(\theta \circ \psi) \circ \epsilon] \\
&= N_1[\theta \circ \psi]
\end{aligned}
$$

Next we show I_N is a right homomorphism, the typing here is $\phi : \alpha \to_Q \beta_1, \theta : \beta_1 \to_{\mathbb{D}} \beta_2$.

$$
\begin{aligned}
N_1[\theta] \circ I_N[\phi] &= I_N[\theta \circ \epsilon] \circ I_N[\phi] \\
&= I_N[\theta \circ \epsilon \circ I_N[\phi]] && \text{(left homomorphism)} \\
&= I_N[\theta \circ \phi] && (\beta)
\end{aligned}
$$

and to show I_N^{-1} is a right homomorphism under $\phi : \alpha \to_{\mathbb{C}} N[\beta_1], \theta : \beta_1 \to_{\mathbb{D}} \beta_2$

$$
\begin{aligned}
I_N^{-1}[N_1[\theta] \circ \phi] &= \epsilon \circ (N_1[\theta] \circ \phi) && \text{(def)} \\
&= \epsilon \circ (I_N[\theta \circ \epsilon] \circ \phi) && \text{(def)} \\
&= (\epsilon \circ I_N[\theta \circ \epsilon]) \circ \phi && \text{(assoc)} \\
&= (\theta \circ \epsilon) \circ \phi && (\beta) \\
&= \theta \circ (\epsilon \circ \phi) && \text{(assoc)} \\
&= \theta \circ I_N^{-1}[\phi] && \text{(def)}
\end{aligned}
$$

Next we need to show that this is a *bijection.*

1. Starting with a negative type $N[\beta]$, we recover the original ϵ by the Yoneda lemma.
2. Starting with a right representable $G[\beta]$, we recover the original I_N^{-1} by the Yoneda lemma. We need to show that the action of G on arrows is the same as the one defined from G viewed as a negative type, which we will call G_1':

$$
\begin{aligned}
G_1'[\theta] &= I_G[\theta \circ \epsilon] \\
&= G_1[\theta] \circ I_G[\epsilon] && \text{(right homomorphism)} \\
&= G_1[\theta] \circ I_G[\epsilon \circ \mathrm{id}] \\
&= G_1[\theta] \circ \mathrm{id} \\
&= G_1[\theta]
\end{aligned}
$$

□

2 Specific Universal Properties: The Microcosm Principle

While we have now proved a beautiful internal theorem in great generality that has many useful instances in different models, we are hard pressed to actually *instantiate* the theorem in the syntax itself because we can't really come up with any useful universal properties! The reason is that Judgment Theory itself is too bare-bones.

$$\frac{}{1\,\mathrm{sort}} \qquad \frac{}{\alpha : C \vdash () : 1}\ \textsc{1-Ty-Intro} \qquad \frac{\alpha : C \vdash A : 1}{\alpha : C \vdash A = () : 1}\ \textsc{1-Ty-}\eta$$

$$\frac{}{\alpha^o : C; \beta : D \mid 1} \qquad \frac{}{\Phi \vdash () : 1}\ \textsc{1-Tm-Intro} \qquad \frac{\Phi \vdash t : 1}{\Phi \vdash t = () : 1}\ \textsc{1-Tm-}\eta$$

$$()[\gamma] = () \qquad 1[A/\alpha; B/\beta] = 1 \qquad ()[\phi] = ()$$

Figure 5: Terminal Sort, Judgment

For instance how would we say that a category $\mathbb{C}$ has a *terminal object*? Well, it would have to be in the first place a *single* object in $\mathbb{C}$, but so far our syntax only enables us to talk about functions and functors. Well an object of $\mathbb{C}$ is the same as a function from the unit sort, but so far our syntax only has base sorts.

So we see that defining the terminal object, *necessitates* some notion of terminal object in our "metatheory": Judgment Theory. This mysterious and pervasive aspect of higher category theory is called the *microcosm principle* because it says that in order to discuss a property of a single object in a single category (the microcosm), we need the same sort of object at the meta-level (the macrocosm). The adage here is "As above, so below".

That might all sound very spiritual, but we'll see that it's actually quite a nice heuristic for formalizing universal properties in internal category theory. In the spirit of this heuristic, instead of first giving a bunch of new type constructions for Judgment Theory and then applying them to define universal properties, we will consider them in pairs, first above, and then below.

2.1 Terminal Sort/Terminal Object/Unit Type

First we consider the terminal sort and then the idea of a terminal object in an internal category.

The rules say that there is a terminal sort 1 that has an element () and everything is equal to that element. Then we reproduce this at the level of judgments/terms: there is a terminal judgment 1 that has an element () and everything is equal to that element.

Next, the terminal sort trivially has a category structure using the trivial judgment as its hom set:

Definition 9 (Terminal Category)**.** *The terminal category* $\mathbb{1}$ *consists of*

1. *Its object sort is* 1
2. *Its arrow judgment is* $\alpha^o : 1; \alpha : 1 \mid 1$
3. *Its identity arrow is given by* $\alpha^o; \alpha \mid \cdot \vdash () : 1$
4. *Composition is defined by* $\phi : 1, \psi : 1 \vdash () : 1$
5. *Associativity and unitality are trivial by* 1-η

Definition 10 (Terminal Object Specification)**.** *In judgment type theory with a terminal sort/judgment, we can define for any category* $\mathbb{C}$ *a profunctor* $\alpha^o : \mathbb{C}; \beta : \mathbb{1} \mid 1$*, which specifies a terminal object.*

Now let's instantiate the definition for the terminal object defined as a negative type and as a representable and see how it reproduces the standard type-theoretic and category-theoretic definitions.

A representation of $\alpha^o : \mathbb{C}; \beta : \mathbb{1} \mid 1$ consists of a functor $\beta : \mathbb{1} \vdash 1_{\mathbb{C}} : \mathbb{C}$, an introduction rule

$$\phi : 1 \vdash I_1[\phi] : \alpha \to_{\mathbb{C}} 1_{\mathbb{C}}[\beta]$$

satisfying $I_1[\phi] \circ \gamma = I_1[\phi \circ \gamma]$ and an inverse $\psi : \alpha \to_{\mathbb{C}} 1_{\mathbb{C}}[\beta] \vdash I_1^{-1}[\psi] : 1$ which by 1-η is equal to () and therefore every $t : \alpha \to_{\mathbb{C}} 1_{\mathbb{C}}[\beta]$ is equal to $I_1[\phi]$. In light of this it would be appropriate to write $I_1[\phi]$ as () and then the left-homomorphism property is $() \circ \gamma = ()$, which is precisely the definition of substitution.

$$\frac{C\,\text{sort} \qquad D\,\text{sort}}{C \times D\,\text{sort}} \qquad \frac{\alpha : C \vdash A_1 : D_1 \qquad \alpha : C \vdash A_2 : D_2}{\alpha : C \vdash (A_1, A_2) : D_1 \times D_2} \qquad (A_1, A_2)[B/\alpha] = (A_1[B/\alpha], A_2[B/\alpha])$$

$$(\pi_i A)[B/\alpha] = \pi_i A[B/\alpha] \qquad \frac{\alpha : C \vdash A : D_1 \times D_2}{\alpha : C \vdash \pi_i A : D_i} \qquad \frac{}{\pi_i(A_1, A_2) = A_i}\times - \beta$$

$$\frac{\alpha : C \vdash A : D_1 \times D_2}{\alpha : C \vdash A = (\pi_1 A, \pi_2 A) : D_1 \times D_2}$$

$$\frac{\alpha^o : C; \beta : D \mid R_1 \qquad \alpha^o : C; \beta : D \mid R_2}{\alpha^o : C; \beta : D \mid R_1 \times R_2} \qquad (R_1 \times R_2)[A/\alpha; B/\beta] = (R_1[A/\alpha; B/\beta] \times R_2[A/\alpha; B/\beta])$$

$$\frac{\Phi \vdash t_1 : R_1 \qquad \Phi \vdash t_2 : R_2}{\Phi \vdash (t_1, t_2) : R_1 \times R_2} \qquad (t_1, t_2)[\phi] = (t_1[\phi], t_2[\phi]) \qquad \frac{\Phi \vdash u : R_1 \times R_2}{\Phi \vdash \pi_i u : R_i} \qquad (\pi_i u)[\phi] = \pi_i u[\phi]$$

$$\frac{}{\pi_i(t_1, t_2) = t_i} \qquad \frac{\Phi \vdash t : R_1 \times R_2}{t = (\pi_1 t, \pi_2 t)}$$

Figure 6: Product Sort, Judgment

This alternate syntax makes even more sense in light of the negative type presentation. A negative type satisfying the terminal object specification consists of a functor $1_{\mathbb{C}}$ as above and an introduction rule as above and an elimination form $\vdash \epsilon : 1$, which is trivial.

2.2 Products

Next the product

First the product category

Definition 11 (Product category)**.** *For any categories $\mathbb{C}_1$, $\mathbb{C}_2$ we can form the product category whose sort of objects is $\mathbb{C}_1 \times \mathbb{C}_2$, whose arrow judgment is $\alpha^o : \mathbb{C}_1 \times \mathbb{C}_2; \alpha \mid (\pi_1\alpha \to_{\mathbb{C}_1} \pi_1\alpha) \times (\pi_2\alpha \to_{\mathbb{C}_2} \pi_2\alpha)$. Identity is given by $\alpha^o : \mathbb{C}_1 \times \mathbb{C}_2; \alpha \mid \cdot \vdash (id_{\pi_1\alpha}, id_{\pi_2\alpha}) : (\pi_1\alpha \to_{\mathbb{C}_1} \pi_1\alpha) \times (\pi_2\alpha \to_{\mathbb{C}_2} \pi_2\alpha)$ and composition by*

$$\phi, \psi \vdash ((\pi_1\phi) \circ (\pi_1\psi), (\pi_2\psi) \circ (\pi_2\phi)) : (\pi_1\alpha_0 \to_{\mathbb{C}_1} \pi_1\alpha_2) \times (\pi_2\alpha_0 \to_{\mathbb{C}_2} \pi_2\alpha_2)$$

unitality, associativity follow from the same properties of $\mathbb{C}_1, \mathbb{C}_2$

Definition 12 (Product Functor Specification)**.** *For any category $\mathbb{C}$, we can specify the product functor by*

1. *The span is $\alpha^o : \mathbb{C}; \beta : \mathbb{C} \times \mathbb{C} \mid (\alpha \to_{\mathbb{C}} \pi_1\beta) \times (\alpha \to_{\mathbb{C}} \pi_2\beta)$*
2. *Left Composition is defined as*
$$\phi, \psi \vdash (\pi_1\phi \circ \psi, \pi_2\phi \circ \psi)$$
3. *Left unitality:*
$$\begin{aligned} (\pi_1\phi \circ id, \pi_2\phi \circ id) &= (\pi_1\phi, \pi_2\phi) \\ &= \phi \qquad (\times - eta) \end{aligned}$$

4. *Left associativity:*

$$\begin{aligned}
(\phi \circ \psi) \circ \psi' &= (\pi_1\phi \circ \psi, \pi_2\phi \circ \psi) \circ \psi' \\
&= (\pi_1(\pi_1\phi \circ \psi, \pi_2\phi \circ \psi) \circ \psi', \pi_2(\pi_1\phi \circ \psi, \pi_2\phi \circ \psi) \circ \psi') \\
&= ((\pi_1\phi \circ \psi) \circ \psi', (\pi_2\phi \circ \psi) \circ \psi') \\
&= (\pi_1\phi \circ (\psi \circ \psi'), \pi_2\phi \circ (\psi \circ \psi')) \\
&= \phi \circ (\psi \circ \psi')
\end{aligned}$$

5. *Right composition is defined as*

$$\phi, \theta \vdash (\pi_1\theta \circ \pi_1\phi, \pi_2\theta \circ \pi_2\phi)$$

6. *Right unitality:*

$$\begin{aligned}
(\pi_1(id, id) \circ \pi_1\phi, \pi_2(id, id) \circ \pi_2\phi) &= (id \circ \pi_1\phi, id \circ \pi_2\phi) && (\times - \beta) \\
&= (\pi_1\phi, id \circ \pi_2\phi) \\
&= \phi && (\times - \eta)
\end{aligned}$$

7. *Right associativity*

$$\begin{aligned}
\theta' \circ (\theta \circ \phi) &= \theta' \circ (\pi_1\theta \circ \pi_1\phi, \pi_2\theta \circ \pi_2\phi) \\
&= (\pi_1\theta' \circ \pi_1(\pi_1\theta \circ \pi_1\phi, \pi_2\theta \circ \pi_2\phi), \pi_2\theta' \circ \pi_2(\pi_1\theta \circ \pi_1\phi, \pi_2\theta \circ \pi_2\phi)) \\
&= (\pi_1\theta' \circ (\pi_1\theta \circ \pi_1\phi), \pi_2\theta' \circ (\pi_2\theta \circ \pi_2\phi)) \\
&= ((\pi_1\theta' \circ \pi_1\theta) \circ \pi_1\phi, (\pi_2\theta' \circ \pi_2\theta) \circ \pi_2\phi) \\
&= (\pi_1(\theta' \circ \theta) \circ \pi_1\phi, \pi_2(\theta' \circ \theta) \circ \pi_2\phi) \\
&= (\theta' \circ \theta) \circ \phi
\end{aligned}$$

3 Module System

Currently, the theorems presented in **??** were all metatheorems, we can give them a *formal* status by incorporating a simple module system on top of span theory. This serves as the basis for a *domain specific proof assistant* for category theory that we plan to use to teach category theory.

4 Equipment of Internal Categories

Next we show that internal categories provide us with our first *model* of judgment type theory by interpreting sorts as categories, functions as functors, spans as profunctors and 2-cells as natural transformations. Furthermore, this model can be seen as an *implementation* of Hom-sets, we translate Judgment type theory with hom sets into categories internal to judgmental type theory *without* hom sets: the home sets come from the category structure.

The interpretation of hom-formation and -introduction is obvious, but elim is more interesting...

5 Monads and Multicategories

Defining enriched multicategories requires an appropriate notion of *monad* on the enriching "category", so we next extend Judgment theory with a monad acting on sorts and judgments.

6 Semantics of Judgment Theory

$$\frac{C\,\mathrm{sort}}{TC\,\mathrm{sort}} \qquad \frac{\alpha : C \vdash A : D}{\alpha : C \vdash [A] : TD}\ \textsc{ret} \qquad \frac{\alpha : C \vdash \vec{A} : TD \qquad \beta : D \vdash \vec{B} : TE}{\alpha : C \vdash \mathrm{for}\,[\beta] \in \vec{A}.\ \vec{B} : TE}\ \textsc{kleisli}$$

$$\frac{}{\mathrm{for}\,[\beta] \in [A].\ \vec{A} = \vec{A}[A/\beta]} \qquad \frac{}{\mathrm{for}\,[\beta] \in \vec{A}.\ [\beta] = \vec{A}}$$

$$\frac{\alpha' : C' \vdash \vec{A} : TC \qquad \beta' : D' \vdash \vec{B} : TD \qquad \alpha^o : C; \beta : D \mid R}{\alpha'^o; \beta' \mid \mathrm{for}\,[\alpha; \beta] \in \vec{A} \wr \vec{B}.\ R}\ \textsc{Monad-Judg}$$

$$\frac{\alpha^o; \beta \mid \Phi \vdash t : R}{\Phi \vdash [t] : \mathrm{for}\,[\alpha; \beta] \in [\alpha] \wr [\beta].\ R} \qquad \frac{\Psi \vdash \vec{u} : \mathrm{for}\,[\alpha; \beta] \in \vec{A} \wr \vec{B}.\ R \qquad \Phi, \psi : R, \Phi' \vdash t : \mathrm{for}\,[\cdots] \in \cdots.\ S}{\Phi, \Psi, \Phi \vdash \mathrm{for}\,[\phi] \in u.\ t : \mathrm{for}\,[\cdots] \in \cdots.\ S}$$

Figure 7: Monadal Judgment Theory (additions to judgment theory)