Lecture 3: Initiality of IPL

Lecturer: Max S. New
Scribe: Ayan Chowdhury

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Definition 1 (BiHeyting Algebra). *A poset with finite meets, finite joins, and Heyting implication.*

Definition 2 (BiHeyting Pre-Algebra). *A preorder with specified finite meets, finite joins, and Heyting implication. A specified meet/join/implication is a binary function satisfying the universal property of meet/join/implication. There could be other elements satisfying the universal property chosen to be outputs of our specified functions. However, our functions are unique up to order equivalence.*

Theorem 1 (The Soundness Theorem). *Fix a signature* Σ*. Given an interpretation* i *of* Σ *in a biHeyting pre-algebra* P*, if* $\Gamma \vdash A$ *in* $\mathrm{IPL}(\Sigma)$ *then* $[[\Gamma]] \leq [[A]]$ *in* P*, where* $[[\cdot]]$ *is defined by*

$$[[X]] = i(x), \qquad [[A \wedge B]] = [[A]] \wedge [[B]], \qquad [[A \vee B]] = [[A]] \vee [[B]], \qquad \cdots$$

Proof. This is done by induction on deduction rules. □

Theorem 2 (The Completeness Theorem). *Fix a signature* Σ*. We have that* $\Gamma \vdash A$ *is provable in* $\mathrm{IPL}(\Sigma)$ *if* $[[\Gamma]] \leq [[A]]$ *in all biHeyting pre-algebras.*

Proof. Let us assume $[[\Gamma]] \leq [[A]]$ in all biHeyting pre-algebras. We have that the propositions of IPL form a biHeyting pre-algebra. It follows that $[[\Gamma]] \vdash A$. If $\Gamma = B_1, \cdots, B_n$, then we have that $[[B_1]], \cdots, [[B_n]] \vdash A$, which is equivalent to $B_1, \cdots, B_n \vdash A$, so we have $\Gamma \vdash A$. □

Definition 3 (Monotone function). *Let* P *and* Q *be pre-orders. A monotone function* $f : P \to Q$ *satisfies if* $x \leq y$ *then* $f(x) \leq f(y)$*.*

Our denotation function $[[\cdot]] : \mathrm{IPL}(\Sigma) \to P$, where P is a biHeyting pre-algebra, is an example of a monotone function.

Definition 4 (Isomorphism). *Let* P *and* Q *be pre-orders. An isomorphism* $f : P \to Q$ *is a monotone function with a monotone inverse.*

Functions fail to be isomorphisms if they are not bijective or if the inverse fails to preserve ordering.

Let us consider one particularly interesting non-example of an isomorphism. Let P denote the pre-order of all finite sets ordered by cardinality. Let $\mathbb{N}$ denote the natural numbers with its usual ordering. There exists a natural monotone function from P to $\mathbb{N}$ given by taking the cardinality of the set. There also exists a natural monotone function from $\mathbb{N}$ to P sending n to $\{a \in \mathbb{N} \mid a < n\}$. However, these fail to form an isomorphism as many sets in P get sent to the same set in $\mathbb{N}$. This motivates the definition of an equivalence of preorders.

Definition 5 (Equivalence of pre-orders)**.** *An equivalence between P and Q is a monotone function $f : P \to Q$ and a monotone function $f^{-1} : Q \to P$ such that $f^{-1}(f(p))$ is order equivalent to p and $f(f^{-1}(q))$ is order equivalent to q.*

We will denote the equivalence relation of order equivalence with $\geq\leq$.

Note that for any pre-order we have that P is equivalent to $P/\geq\leq$ (though we need the Axiom of Choice).

Definition 6 (biHeyting homomorphism)**.** *Let P and Q be biHeyting pre-algebras.*

A strict biHeyting homomorphism is a monotone function from P to Q which preserves finite specified meets, finite specified joins, and specified Heyting implication.

A weak biHeyting homomorphism is a monotone function from P to Q which up to isomorphism preserves finite meets, finite joins, and Heyting implication.

Theorem 3 (The Initiality Theorem)**.** *Fix a signature Σ. We have that* $\mathrm{IPL}(\Sigma)$ *is a biHeyting pre-algebra, with the natural interpretation. For any biHeyting pre-algebra P with interpretation i of Σ, there exists an essentially unique weak homomorphism that is also a unique strict homomorphism from* $\mathrm{IPL}(\Sigma)$ *to P that preserves the interpretation $[[X]]_i = i(X)$. This can be phrased as*

$$\begin{tikzcd}
IPL(\Sigma) \arrow[rr, dashed, "{[[\cdot]]_i}"] & & P \\
& \Sigma_0 \arrow[ul] \arrow[ur, "i"'] &
\end{tikzcd}$$

there exists such a $[[\cdot]]_i$ which is an essentially weak homomorphism and a unique strict homomorphism which makes this diagram commute.

Proof. We can define $[[\cdot]]_i$ inductively, as it is defined on Σ_0, and thus extends to a map on all of $\mathrm{IPL}(\Sigma)$.

Similarly we can show that any such map f is a unique strict homomorphism. For any $X \in \Sigma_0$ we have that $f(X) = i(X)$. By induction, as f preserves finite meets, finite joins, and Heyting implication, it follows that $f = [[\cdot]]_i$. A similar induction gives us that f is a an essentially weak homomorphism. $\square$

We say that $\mathrm{IPL}(\Sigma)$ is initial among all biHeyting pre-algebras with a choice of interpretation over Σ.

Theorem 4. *Consider if we have another initial object, $F(\Sigma)$. That is $F(\Sigma)$ satisfies our initiality theorem. Then we have that* $\mathrm{IPL}(\Sigma)$ *is uniquely equivalent to $F(\Sigma)$.*

Proof. We have the existence of a unique up to order equivalence weak homomorphism from $\mathrm{IPL}(\Sigma)$ to $F(\Sigma)$, denoted $[[\cdot]]_i$. We also have the existence unique up to order equivalence weak homomorphism from $\mathrm{IPL}(\Sigma)$ to $F(\Sigma)$, denoted $((\cdot))_i$.

We can show that weak biHeyting pre-algebra homomorphism compose. We can also show that the identity is a weak biHeyting pre-algebra homomorphism. It would then follow that $(([[\cdot]]_i))_i$ and $[[((\cdot))_i]]_i$ are biHeyting pre-algebra homomorphisms from $\mathrm{IPL}(\Sigma)$ to $\mathrm{IPL}(\Sigma)$ and $F(\Sigma)$ and $F(\Sigma)$ respectively. As the identity is a biHeyting pre-algebra homomorphism, and thus the unique such up to order-equivalence, we have that these maps are order-equivalent to the identity. Therefore $\mathrm{IPL}(\Sigma)$ and $F(\Sigma)$ are equivalent.

As our homomorphisms from the universal property are unique, it follows that such an equivalence is unique up to order equivalence. □

Note that the conditions that our homomorphisms compose and the identiy is a homomorphism are exactly the conditions needed of a category.

Theorem 5. *Booleans, denoted $\mathbb{B}$ are weakly initial for biHeyting pre-algebras on $\varnothing$.*

Proof. Let P be a biHeyting pre-algebra. We can construct a mapping $f_p : \mathbb{B} \to P$ where $f_p(0) = \bot_P$ and $f_p(1) = \top_P$. Let us show that such a mapping is a weak biHeyting homomorphism. We have that $\bot$ and $\top$ are preserved. For each operation, we can prove by cases on the values in $\mathbb{B}$ and the rules of boolean algebra that operations are preserved up to order equivalence under f_p. Thus f_p is indeed a weak biHeyting homomorphism, so we have that $\mathbb{B}$ is weakly initial. □

Note that in general our homomorphism will not be strict if P is a true pre-order and there are multiple $\top$ and $\bot$ elements. For example the map to IPL sends $0 \wedge 0 = 0$ to $\bot$ rather than $\bot \wedge \bot$.

Theorem 6. *In* $\mathrm{IPL}(\varnothing)$ *whether $\Gamma \vdash A$ is provable is decidable.*

Proof. By theorem 4 and 5 we have that $\mathrm{IPL}(\varnothing)$ is equivalent to $\mathbb{B}$. Thus for any $\Gamma \vdash A$ in $\mathrm{IPL}(\varnothing)$ we can map to $\mathbb{B}$, determine in finite time if $[[\Gamma]] \leq [[A]]$ using boolean rules. Mapping back with f_p then tells us that $f_p([[\Gamma]]) \vdash f_p([[A]])$ which by equivalence gives us $\Gamma \vdash A$. □

This shows that while IPL does not in general admit the law of excluded middle or double negation elimination, $\mathrm{IPL}(\varnothing)$ is equivalent to $\mathbb{B}$, giving us:

Corollary 1. *The law of excluded middle and double negation elimination are admissable in* $\mathrm{IPL}(\varnothing)$.