Problem Set 1: Logic and Order Theory

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Update: Fixed typo in problem 3 part 4.

Submit your solutions to this homework on Canvas alone or in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

If you haven't already, sign up to scribe and present homework solutions on the course github repo.

Problem 1 Distributivity

A lattice (poset with finite meets and joins) is *distributive* when binary meets and joins satisfy a distributive law:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

We say that propositions A and B of a logic are *equivalent* when the judgments $A \vdash B$ and $B \vdash A$ are both provable.

1. Show that IPL satisfies this distributive law in that for any propositions A, B, C, the propositions $A \wedge (B \vee C)$ and $(A \wedge B) \vee (A \wedge C)$ are equivalent.
2. Show that any biHeyting algebra (poset with finite meets, finite joins and an implication operation) is a distributive lattice.

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Problem 2 Intuitionistic and Classical Logic

The law of excluded middle or principle of omniscience is the following axiom scheme: for all propositions A the axiom

$$\frac{}{\Gamma \vdash A \vee \neg A}$$

The law of double negation elimination is the axiom scheme

$$\frac{}{\Gamma \vdash \neg(\neg A) \supset A}$$

Since the first non-boolean model of IPL that we saw is a 3 element Heyting algebra, it is easy to get the impression that intuitionistic logic is about "multi-valued logics" where there are some "intermediate" truth value other than just true and false. But this is misleading. Firstly, there are many boolean algebras with more than 2 elements (the powerset of any set) that are models of classical logic. Secondly, while there are more than 2 elements in a model, *within* the logic, we can never separate any proposition from true and false.

1. Show that in IPL extended with the law of excluded middle, the law of double negation elimination is admissible and vice-versa.

2. The following might be called the "intuitionistic law of excluded middle", for all Γ, A:

   $$\frac{}{\Gamma \vdash \neg(\neg(A \iff \top) \wedge \neg(A \iff \bot))}$$

   Intuitively this says "no proposition is not equivalent to true and not equivalent to false", where we are using the notations $\neg B = (B \supset \bot)$ and $B \iff C = (B \supset C) \wedge (C \supset B)$.

   Show that the intuitionistic law of excluded middle is derivable for all A in IPL. A full proof tree for this will be quite large, so I encourage you to develop intermediate reasoning principles to make this proof clearer.

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Problem 3 Conservativity Results

Fix a signature with propositional variables, but no axioms. In this problem our goal is to prove that IPL$(\top, \wedge, \supset)$ is a *conservative extension* of IPL$(\top, \wedge)$[1]. That is, we will show that for any judgment $\Gamma \vdash A$ where the propositions in Γ, A are generated using only $\top, \wedge$ and propositional variables, if there is a proof that uses the implication $\supset$ then there is a proof that doesn't use it. In other words, $\supset$ doesn't let us prove anything new about propositions that don't involve $\supset$. This means that we can use the richer logic of IPL$(\top, \wedge, \supset)$ to prove results that hold in any poset with finite meets, even those that don't support an implication structure. And while $\supset$ can't allow us to prove anything new, it might allow us to write a *shorter* or more *intuitive* proof.

[1] This conservativity result also holds when both have disjunction, but the proof is slightly more complex

To prove conservativity, we will show that the inclusion (a monotone function of posets) $i : \mathrm{IPL}(\top, \wedge) \to \mathrm{IPL}(\top, \wedge, \supset)$ is an *order embedding*: if $i(A) \leq i(B)$ then $A \leq B$. Recall that the ordering here is provability of the hypothetical judgment $\vdash$, so this means if $A \vdash B$ in $\mathrm{IPL}(\top, \wedge, \supset)$ then $A \vdash B$ in $\mathrm{IPL}(\top, \wedge)$.

Key to this proof is the *initiality* property of each variant of IPL:

- For any poset P with finite meets and an assignment $\sigma(X) \in P$ for each propositional variable, there is a *unique* monotone function $\overline{\sigma} : \mathrm{IPL}(\top, \wedge) \to P$ that preserves finite meets and respects the assignment of propositional variables $\overline{\sigma}(X) = \sigma(X)$.
- An analogous property holds for $\mathrm{IPL}(\top, \wedge, \supset)$ but the poset P must have an implication as well and $\overline{\sigma}$ is the unique monotone function preserving finite meets and implication that respects the assignment σ.

1. First, show that for any preorder Q, the set of downward-closed subsets $\mathcal{P}_{\downarrow}Q$ with subset inclusion as ordering is a poset and a Heyting algebra.
2. Next, for any preorder Q we can define a function $\downarrow : Q \to \mathcal{P}_{\downarrow}Q$ defined by $\downarrow(x) = \{y \mid y \leq x\}$.
   - Show that $\downarrow : Q \to \mathcal{P}_{\downarrow}Q$ is monotone.
   - Show that $\downarrow$ is an order embedding: if $\downarrow(x) \leq \downarrow(y)$ then $x \leq y$.
   - Show that $\downarrow$ preserves finite meets that exist[2].
3. Show that if $i : P \to Q$ and $j : Q \to R$ are monotone functions and $j \circ i : P \to R$ is an order embedding then i is an order embedding.
4. Use the initiality property of $\mathrm{IPL}(\top, \wedge, \supset)$ to construct a monotone function $f : \mathrm{IPL}(\top, \wedge, \supset) \to \mathcal{P}_{\downarrow}\mathrm{IPL}(\top, \wedge)$ that makes the following diagram commute:

$$\begin{tikzcd} & \mathcal{P}_{\downarrow}\mathrm{IPL}(\top, \wedge) & \\ \mathrm{IPL}(\top, \wedge) \arrow[ur, "\downarrow"] \arrow[rr, "i"'] & & \mathrm{IPL}(\top, \wedge, \supset) \arrow[ul, "f"'] \end{tikzcd}$$

   Note that you will need to use the initiality of $\mathrm{IPL}(\top, \wedge)$ to show the diagram commutes.

Then since $\downarrow$ is an embedding, we conclude that the inclusion i is an embedding.

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[2]though we do not need it for this problem, it also preserves any implications that exist