Problem Set 1

January 13, 2022

Homework is due midnight before next Thursday's class. Starred assignments will be presented in class. Turn in your homework on Canvas. You may work alone or in groups of two. I highly encourage you to write your solution in LaTeX. The LaTeX packages tikz and tikz-cd are useful for typesetting commutative diagrams, and there are WYSIWYG editors online that produce LaTeX code such as https://q.uiver.app/ and https://tikzcd.yichuanshen.de/ .

Problem 1

Give an example of a category not covered in class from an area of mathematics or computer science that interests you.

Next, give an example of an interesting functor whose domain is your category and another whose codomain is your category.

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Problem 2 (*) Cayley Representation

In class, we defined the Cayley representation of a category to show that all categories are subcategories of the category of sets and functions.

This was defined as follows. Given a small category C we can define a functor $\text{Cayley} : C \to \text{Set}$ by

1. $\text{Cayley}(A) = \{(B, f) | B \in C_0, f \in C_1(B, A)\}$
2. $\text{Cayley}(f : A \to B)(g) = f \circ g$

Then we can define a category $\overline{C}$ whose objects are sets of the form $\text{Cayley}(A)$ for some A and whose arrows are functions of the form $\text{Cayley}(f)$ for some f. This is a subcategory of Set and is isomorphic as a category to C.

Consider the following properties of functors and either (1) prove the functor Cayley always satisfies the given property or (2) give an counter-example: a category C where Cayley does not satisfy the given property.

1. A functor $F : C \to D$ is *faithful* if for every $A, B \in C$, $F_1 : C_1(A, B) \to C_1(F_0A, F_0B)$ is injective.

   Is Cayley always faithful?

2. A functor $F : C \to D$ is *full* if for every $A, B \in C$, $F_1 : C_1(A, B) \to C_1(F_0A, F_0B)$ is surjective.

   Is Cayley always full?

3. A functor $F : C \to D$ is *pseudomonic* if objects A, B are isomorphic in C if and only if F_0A and F_0B are isomorphic in D.

   Is Cayley always pseudomonic?

4. A functor $F : C \to D$ is *conservative* if a morphism $f : A \to B$ is an isomorphism if and only if $F_1f : F_0A \to F_0B$ is an isomorphism in D.

   Is Cayley always conservative?

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Problem 3 (*) Hoare Logic as a Functor

This problem is inspired by Melliès and Zeilberger [2015].

We can consider any monoid to be a model of a sequential untyped programming language, where we think of the elements of the monoid as (sequences of) statements and the operation as sequencing. In this case we would write the monoid operation as a semicolon: $m; n$ and the monoid identity as a "no-op" operation, which we write as *skip*.

Define a *Hoare logic* [Hoare, 1969] (P, T) over a monoid M as

1. A set $\mathcal{P}$ of "assertions"

2. A relation $T \subseteq \mathcal{P} \times M \times \mathcal{P}$ of "entailment". To match Hoare's notation, we can write $(P, s, Q) \in T$ as
$$\{P\}s\{Q\}$$

3. Satisfying the "skip rule": for every predicate $P \in \mathcal{P}$:
$$\{P\}skip\{P\}$$

4. And satisfying the "entailment rule": if $\{P\}s\{Q\}$ and $\{Q\}s'\{R\}$ then
$$\{P\}s; s'\{Q\}$$

Show that Hoare logics over a monoid M are in bijection with faithful functors with codomain M (viewed as a one-object category).

1. First, for any Hoare logic (P, T) over M define a category Triple(P, T) with a faithful functor command : Triple$(P, T) \to M$.

2. Next, for any category C and faithful functor $F : C \to M$, define a Hoare logic Analyse(C, F) over M.

3. Prove that Triple and Analyse are mutually inverse.

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References

C. A. R. Hoare. An axiomatic basis for computer programming. *Commun. ACM*, 12(10):576–580, oct 1969. ISSN 0001-0782. doi: 10.1145/363235.363259. URL `https://doi.org/10.1145/363235.363259`.

Paul-André Melliès and Noam Zeilberger. Functors are type refinement systems. In *Proceedings of the 42Nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '15, pages 3–16, New York, NY, USA, 2015. ACM. ISBN 978-1-4503-3300-9. doi: 10.1145/2676726.2676970. URL `http://doi.acm.org/10.1145/2676726.2676970`.