Problem Set 4

Released: October 9, 2025
Due: October 23, 2025, 11:59pm

Submit your solutions to this homework on Canvas alone or in a group of 2. Your solutions must be submitted in pdf produced using LaTeX.

Problem 1 Functor Comprehension Principle

Previously, you proved that if we have a specified product $A \times B$ for any two objects of $\mathcal{C}$ that there is a product functor $\mathcal{C}^2 \to \mathcal{C}$ that sends the pair A, B to $A \times B$. This property can be generalized to any universal construction, as long as the universal property itself (i.e., presheaf) is defined functorially.

1. Let $R : \mathcal{C} \to \mathcal{PD}$ be a functor to presheaves on $\mathcal{D}$ such that for every $A \in \mathcal{C}$, $R(A)$ is representable, i.e., we have a specified $F(A) : \mathcal{D}$ and natural isomorphism $i(A) : Y(F(A)) \cong R(A)$.

   Extend F to a functor $\mathcal{C} \to \mathcal{D}$, i.e., define a functorial action on morphisms that takes $f : \mathcal{C}(A, B)$ to $F(f) : \mathcal{D}(F(A), F(B))$ that preserves identity and composition.

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Definition 1. *A functor of SCwFs $F : \mathcal{S} \to \mathcal{T}$ consists of*

1. *A functor $F_c : \mathcal{S}_c \to \mathcal{T}_c$ of context/substitution categories.*
2. *A function $F_t : \mathcal{S}_t \to \mathcal{T}_t$ on types.*
3. *A natural transformation $F_{\mathrm{Tm}} : \mathrm{Tm}_{\mathcal{S}}(A) \to \mathrm{Tm}_{\mathcal{T}}(F_t A) \circ F_c^{op}$*

such that

1. *F_c preserves the terminal object up to isomorphism in that $F_c(\cdot)$ is terminal.*
2. *F_c preserves the context extension products in that the induced morphism $F_c(\Gamma \times A) \to F_c\Gamma \times F_t A$ is an isomorphism.*

Furthermore,

1. *We say F is* faithful *if for every Γ, A, the function $F_{\mathrm{Tm}} : \mathrm{Tm}_{\mathcal{S}}(A)(\Gamma) \to \mathrm{Tm}_{\mathcal{T}}(F_t A)(F_c \Gamma)$ is injective.*

2. *If S has a unit type, 1_S, we say F preserves the unit type if $\mathrm{Tm}_{\mathcal{T}}(F_t 1_S) \cong \mathrm{Tm}_{\mathcal{T}}(1_T)$.*

3. *If S has product types $A \times B$, we say F preserves product types if for every $A, B \in S_t$, that the induced natural transformation $\mathrm{Tm}_{\mathcal{T}}(F_t(A \times B)) \cong \mathrm{Tm}_{\mathcal{T}}(F_t A \times F_t B)$ is a natural isomorphism.*

4. *If S has function types $A \Rightarrow B$, we say F preserves function types if for every $A, B \in S_t$, that the induced natural transformation $\mathrm{Tm}_{\mathcal{T}}(F_t(A \Rightarrow B)) \to \mathrm{Tm}_{\mathcal{T}}(F_t A \Rightarrow F_t B)$ is an isomorphism.*

For the next problem, you will need to use the initiality of STLC semantics.

Theorem 1. *Initiality of STLC Semantics (Simplified) Let Σ be an $\mathrm{STLC}(1, \times)$ signature and let σ be an interpretation of Σ in a SCwF $\mathcal{S}$. Then*

1. *The semantics of $\mathrm{STLC}(1, \times)$ in $\mathcal{S}$ constitutes a SCwF functor $[\![\cdot]\!] : \mathrm{STLC}(1, \times)(\Sigma) \to \mathcal{S}$ that preserves product and unit types and agrees with σ on base types and function symbols.*

2. *If $F : \mathrm{STLC}(1, \times)(\Sigma) \to \mathcal{S}$ is also a SCwF functor that preserves product and unit types as well as agreeing with σ on base types and function symbols, then there is a[1] natural isomorphism $[\![\cdot]\!]_c \cong F_c$.*

An analogous theorem holds for $\mathrm{STLC}(1, \times, \Rightarrow)(\Sigma)$ but where the functors $[\![\cdot]\!], F$ additionally preserve function types .

Problem 2 Conservativity of Adding Function Types

Let $\Gamma \vdash M : A$ and $\Gamma \vdash N : B$ be terms in $\mathrm{STLC}(1, \times)(\Sigma)$. Then we can view the same terms as terms in the extended language $\mathrm{STLC}(1, \times, \Rightarrow)(\Sigma)$. Does the addition of new proof terms change whether or not we can prove $M = N$?

If $M = N$ is provable in the equational theory of $\mathrm{STLC}(1, \times)(\Sigma)$, then clearly it is also provable in the extended logic $\mathrm{STLC}(1, \times, \Rightarrow)(\Sigma)$ since the latter logic contains all of the proof rules of the former. However, the converse is non-trivial to establish, and is called the *conservativity* of the equational theory of $\mathrm{STLC}(1, \times, \Rightarrow)(\Sigma)$ over $\mathrm{STLC}(1, \times)(\Sigma)$. It says that adding function types to our language doesn't allow us to prove any new theorems that could be formulated in the original logic. Compare the conservativity theorem we proved for IPL in Problem Set 1.

Part 3 is quite difficult, so you may wish to finish the other parts of the construction before returning to it.

[1] this natural isomorphism is unique among natural isomorphisms that respect the type structure in an appropriate sense, but for this problem set we need only its existence

1. Show that if $F : \mathcal{S} \to \mathcal{T}$ and $G : \mathcal{S} \to \mathcal{T}$ are SCwF functors, and $G \circ F$ is faithful, then F is faithful.

2. Show that if $F, F' : \mathcal{S} \to \mathcal{T}$ are SCwF functors and $\alpha : F_c \cong F'_c$ is a natural isomorphism then F is faithful if and only if F' is faithful.

3. (CHALLENGING) Show that for any category $\mathcal{C}$, the category of presheaves $\mathcal{PC}$ is cartesian closed (HINT: the construction of exponentials is a direct generalization of the Heyting implication structure on downward closed sets in PS1).

4. Define for any SCwF $\mathcal{S}$, a SCwF functor $Y : \mathcal{S} \to \text{democratic}(\mathcal{PS}_c)$ that is faithful and preserves unit and product types.

5. Define a SCwF functor $i : \text{STLC}(1, \times)(\Sigma) \to \text{STLC}(1, \times, \Rightarrow)(\Sigma)$ that sends base types/function symbols to themselves and preserves product and unit types.

6. Define a SCwF functor $j : \text{STLC}(1, \times, \Rightarrow)(\Sigma) \to \text{democratic}(\mathcal{PS}_\rfloor)$ that agrees with $Y : \text{STLC}(1, \times)(\Sigma) \to \text{democratic}(\mathcal{PS}_\rfloor)$ on base types/function symbols and preserves product and unit types.

7. Conclude that i is faithful[2], and therefore that adding function types to STLC is a conservative extension of the equational theory.

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[2]through a more complex construction, it can be established that i is also full. See Crole chapter 4.10 for a similar proof.