Problem Set 5

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Due: Novemmber 13, 2025, 11:59pm

Submit your solutions to this homework on Canvas alone or in a group of 2. Your solutions must be submitted in pdf produced using LaTeX.

Definition 1. *Algebras, Initial Algebras Let $F : \mathcal{C} \to \mathcal{C}$ be a functor. An F-algebra is a morphism $\alpha : FX \to X$.*

A homomorphism from $\alpha : FX \to X$ to $\beta : FY \to Y$ is a morphism $\phi : X \to Y$ such that $\phi \circ \alpha = \beta \circ F$. Identity and composition of homomorphisms is given by identity and composition in $\mathcal{C}$. This defines a category F-Alg.

An initial F-algebra is an initial object in F-Alg. Given an initial algebra $i : F(\mu F) \to \mu F$, and an algebra $\alpha : FX \to X$, we write $rec^F \alpha : \mu F \to X$ to mean the unique homomorphism from i to α.

Definition 2. *Let $\mathcal{C}$ be a bicartesian category. A natural numbers object (NNO) in $\mathcal{C}$ is an initial algebra of the functor $F_{\mathbb{N}} X = 1 + X$.*

Problem 1 Programming with Peano

Let $\mathcal{C}$ be a cartesian closed category with a natural numbers object $i : F_{\mathbb{N}} N \to N$.

- Define a morphism $\mathrm{add} : N \times N \to N$ that when $\mathcal{C}$ is the category of sets is the usual addition operation on natural numbers.
- Prove that zero is a left and right unit of add. That is

  $$\mathrm{add} \circ (\mathrm{zero} \circ !, \mathrm{id}_N) = \mathrm{id}_N : N \to N$$

  and

  $$\mathrm{add} \circ (\mathrm{id}_N, \mathrm{zero} \circ !) = \mathrm{id}_N : N \to N$$

  where $\mathrm{zero} : 1 \to N$ is $i \circ \sigma_1$.

  HINT: depending on how you define add, one of these two will be easy and one will require the uniqueness property of an NNO.

. . . .

Problem 2 Generic Map-Fold Fusion

Let $\mathcal{C}$ be any category and $F, G : \mathcal{C} \to \mathcal{C}$ be functors. Assume $i : F(\mu F) \to \mu F$ is an initial F-algebra, and $j : G(\mu G) \to G$ is an initial G-algebra. Then for any natural transformation $\tau : F \Rightarrow G$ and G-algebra $\alpha : GX \to X$, we can construct a "pipeline" $\mathrm{rec}^G \alpha \circ \mathrm{rec}^F (j \circ \tau_{\mu G})$. This pipeline first "maps", transforming the F nodes of the μF-tree into G nodes, constructing a μG-tree. Then we perform a "fold", reducing the μG tree into a single X value.

Prove that any such pipeline can be fused into a single fold: construct an F-algebra $\beta : FX \to X$ such that

$$\mathrm{rec}^F \beta = \mathrm{rec}^G \alpha \circ \mathrm{rec}^F (j \circ \tau_{\mu G})$$

. . . .