Problem Set 6

Released: March 20, 2023
Due: March 31, 2023, 11:59pm

Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

Definition 1. *Let $\mathcal{C}$ be a cartesian category. A natural numbers object (NNO) in $\mathcal{C}$ consists of*

- *An object $N \in \mathcal{C}$*
- *Morphisms $zero : 1 \to N$ and $succ : N \to N$*
- *such that for any $z : 1 \to V$ and $s : V \to V$ there exists a unique morphism $rec(z, s) : N \to V$ that satisfies*
  - $rec(z, s) \circ zero = z$
  - $rec(z, s) \circ succ = s \circ rec(z, s)$

Diagrammatically,

$$\begin{tikzcd}
1 \arrow[r, "zero"] \arrow[dr, "z"'] & N \arrow[r, "succ"] \arrow[d, "rec(z,s)"] & N \arrow[d, "rec(z,s)"] \\
& V \arrow[r, "s"'] & V
\end{tikzcd}$$

Problem 1 Programming with Peano

Let $\mathcal{C}$ be a bicartesian closed category with a natural numbers object $(N, \mathrm{zero}, \mathrm{succ})$.

- Define a morphism $\mathrm{add} : N \times N \to N$ that when $\mathcal{C}$ is the category of sets is the usual addition operation on natural numbers.
- Prove that zero is a left and right unit of add. That is

  $$\mathrm{add} \circ (\mathrm{zero} \circ !, \mathrm{id}_N) = \mathrm{id}_N : N \to N$$

  and

  $$\mathrm{add} \circ (\mathrm{id}_N, \mathrm{zero} \circ !) = \mathrm{id}_N : N \to N$$

  HINT: depending on how you define add, one of these two will be easy and one will require the uniqueness property of an NNO.

- Prove that addition is commutative:

$$\text{add} \circ (\pi_2, \pi_1) = \text{add}$$

. . . .

Definition 2. *Let $\mathcal{C}$ be a cartesian category and $X \in \mathcal{C}_0$ an object of $\mathcal{C}$. An X-list object consists of*

- *An object $L_X \in \mathcal{C}_0$*
- *Morphisms $nil : X \to L_X$ and $cons : X \times L_X \to L_X$*
- *such that for any $n : X \to V$ and $c : X \times V \to V$ there exists a unique $fold(n, c) : L_X \to V$ satisfying*

$$fold(n, c) \circ nil = n : 1 \to V$$

and

$$fold(n, c) \circ cons = c \circ (\pi_1, fold(n, c)) : X \times L_X \to V$$

Problem 2 Functoriality of Lists

Assume that $\mathcal{C}$ is a cartesian category, and for each $X \in \mathcal{C}_0$ we have an X-list object $(L_X, \text{nil}_X, \text{cons}_X)$.

- Extend the list operation to a functor $L_- : \mathcal{C} \to \mathcal{C}$.
- Show that if $\mathcal{C}$ is a cartesian *closed* category, the action of the functor can be *internalized* as a family of morphisms

$$\text{map} : (X \Rightarrow Y) \Rightarrow L_X \Rightarrow L_Y$$

where $A \Rightarrow B$ is the exponential B^A.

. . . .