Problem Set 7

April 9, 2022

Homework is due the midnight before class on April 14.

If you haven't yet presented a homework in class, email me to setup eith an in-class presentation or a presentation outside of class.

Problem 1 Examples of Representables

The yoneda embedding $\mathrm{Y} : \mathbb{C} \to \mathrm{Set}^{\mathbb{C}^o}$ takes an object $A \in \mathbb{C}$ to the representable presheaf $\lambda X.\mathbb{C}_1(X, A)$. In a precise sense, all presheaves are constructed by gluing together the representable presheaves (Awodey Proposition 8.11).

Let's consider what the representable presheaves look like in some categories of presheaves that are familiar mathematical objects.

1. Let $\mathcal{G}$ be the category

   $$\begin{tikzcd} v \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & e \end{tikzcd}$$

   Then $\mathrm{Set}^{\mathcal{G}^o}$ is equivalent to the category of *multigraphs*, i.e., pairs of a set V of vertices, E of edges such that each edge has a source and target vertex.

   Describe for each object $a \in \mathcal{G}$ what the multigraph $\mathrm{Y}a$ is. Describe for each object $a \in \mathcal{G}$ and multigraph X what a homomorphism $\mathrm{Y}a \to X$ is.

2. Let $\mathcal{M}$ be a one-object category, i.e., a monoid. Then $\mathrm{Set}^{\mathcal{M}^o}$ is equivalent to the category of $\mathcal{M}$-*actions*[1]. We write the identity element of $\mathcal{M}$ as ϵ and the multiplication as $m \cdot m'$.

   An $\mathcal{M}$-action is a set X equipped with a "multiplication" operation $\times : \mathcal{M} \times X \to X$ satisfying (for any $m, m_1, m_2 \in \mathcal{M}$, and $x \in X$):

   $$\epsilon \cdot x = x$$

   $$m_1 \times (m_2 \times x) = (m_1 \cdot m_2) \times x$$

   An *equivariant map* $\phi : X \to Y$ between $\mathcal{M}$-actions is a function on the underlying sets satisfying

   $$\phi(m \times x) = m \times \phi(x)$$

[1] If $\mathcal{M}$ is a group this is equivalently a group action

Let $*$ be the object of $\mathcal{M}$ viewed as a one object category. Describe what the $\mathcal{M}$-action $\mathrm{Y}*$ is, and describe what an equivariant map $\mathrm{Y}* \to X$ is.

3. Let I be a set, viewed as a category with only identity arrows. Then $\mathrm{Set}^{I^o} = \mathrm{Set}^I$ is equivalent to the category of *families indexed by* I, i.e., $\{X_i \in \mathrm{Set}\}_{i \in I}$. A morphism of families $\{X_i\}_{i \in I} \to \{Y_i\}_{i \in I}$ is a family $\{f_i : X_i \to Y_i\}_{i \in I}$.

   For each $i \in I$, describe what family $\mathrm{Y}i$ is and what the morphisms $\mathrm{Y}i \to \{X_i\}_{i \in I}$.

Hint: to describe the morphisms, use the Yoneda lemma!

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Problem 2 A String of Adjoints

Awodey Chapter 9, Problem 7.

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