Problem Set 5

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Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

Definition 1. *Let $\mathcal{C}$ be a category with binary products.*

An initial object $0 \in \mathcal{C}$ is distributive *if for every $a \in \mathcal{C}$ the unique morphism*

$$0 \to a \times 0$$

is an isomorphism.

A binary coproduct $a_1 + a_2$ with injections $i_1 : a_1 \to a_1 + a_2$ and $i_2 : a_2 \to a_1 + a_2$ is distributive *if for every $b \in \mathcal{C}$, the morphism*

$$[id_b \times i_1, id_b \times i_2] : (b \times a_1) + (b \times a_2) \to b \times (a_1 + a_2)$$

is an isomorphism.

Problem 1 Sums and Distributive coproducts

Let $\mathcal{C}$ be a category with all binary products. In class we discussed that (almost tautologically) $\mathcal{C}$ has

- a terminal object if and only if self$\mathcal{C}$ has a unit type.
- all products if and only if self$\mathcal{C}$ has all product types.
- all exponentials if and only if self$\mathcal{C}$ has all function types.

Your task is to prove the following non-trivial correspondences:

1. $\mathcal{C}$ has a *distributive* initial object if and only if self$\mathcal{C}$ has an empty type.
2. For any $a, b \in \mathcal{C}$, $\mathcal{C}$ has a *distributive* coproduct of a and b if and only if self$\mathcal{C}$ has a sum type of a and b.

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Definition 2. *A CT structure $\mathcal{S}$ consists of*

1. *A category $\mathcal{S}_c$*
2. *A set $\mathcal{S}_T$.*
3. *For each type $A \in S_T$ a predicator $\mathrm{Tm}(A)$ on $\mathcal{S}_c$.*
4. *A terminal object $1 \in \mathcal{S}_c$*
5. *For each $\Gamma_1, \Gamma_2 \in \mathcal{S}_c$ a product structure $(\Gamma_1 \times \Gamma_2, \pi_1, \pi_2)$ for Γ_1, Γ_2, that is*
   - *An object $\Gamma_1 \times \Gamma_2 \in \mathcal{S}_c$*
   - *Morphisms ${\pi_1}^{\Gamma_1,\Gamma_2} : \Gamma_1 \times \Gamma_2 \to \Gamma_1$ and ${\pi_2}^{\Gamma_1,\Gamma_2} : \Gamma_1 \times \Gamma_2 \to \Gamma_2$.*
   - *Such that for any $\Delta \in \mathcal{S}_c$ and $f_1 : \Delta \to \Gamma_1$ and $f_2 : \Delta \to \Gamma_2$ there exists a unique $(f_1, f_2) : \Delta \to \Gamma_1 \times \Gamma_2$ such that ${\pi_1}^{\Gamma_1,\Gamma_2} \circ (f_1, f_2) = f_1$ and ${\pi_2}^{\Gamma_1,\Gamma_2} \circ (f_1, f_2) = f_2$.*
6. *For each $A \in \mathcal{S}_T$ a singleton context structure $(\mathrm{sole}A, \mathrm{var})$ for A, that is,*
   - *An object $\mathrm{sole}A \in \mathcal{S}_c$*
   - *An element $\mathrm{var}^A \in \mathrm{Tm}(A)(\mathrm{sole}A)$*
   - *Such that for any $\Gamma \in \mathcal{S}_c$ and $M \in \mathrm{Tm}(A)(\Gamma)$, there exists a unique $M/\mathrm{var}^A \in \Gamma \to \mathrm{sole}A$ such that $\mathrm{var}^A * M/\mathrm{var}^A = M$.*

Definition 3. *A CT structure homomorphism $F : \mathcal{S} \to \mathcal{T}$ consists of*

- *A functor $F_c : \mathcal{S}_c \to \mathcal{T}_c$ of context categories such that*
  - *If $1 \in \mathcal{S}_c$ is the chosen terminal object of $\mathcal{S}_c$ then $F_c 1$ is terminal in $\mathcal{T}_c$.*
  - *For every Γ_1, Γ_2, $F_c(\Gamma_1 \times \Gamma_2), F_c({\pi_1}^{\Gamma_1,\Gamma_2}), F_c({\pi_2}^{\Gamma_1,\Gamma_2})$ is a product structure for $F_c\Gamma_1, F_c\Gamma_2$ in $\mathcal{T}_c$.*
- *A function $F_T : \mathcal{S}_T \to \mathcal{T}_T$ of types and for each $A \in \mathcal{S}_T$, a natural transformation $F_{\mathrm{Tm}} : \mathrm{Tm}(A) \to \mathrm{Tm}(F_T A) \circ F_c^{op}$ such that*
  - *For each $A \in \mathcal{S}_T$, $(F_T(\mathrm{sole}A), F_{\mathrm{Tm}}(\mathrm{var}^A))$ is a singleton context structure for $F_T A$.*

Definition 4. *Let $\mathcal{S}$ be a CT structure.*

- *A unit type in $\mathcal{S}$ is a type $1 \in \mathcal{S}_T$ such that for every $\Gamma \in \mathcal{S}_c$ there exists a unique term $() \in \mathrm{Tm}(1)(\Gamma)$.*
- *A product of types $A_1, A_2 \in \mathcal{S}_T$ is a type $A_1 \times A_2 \in \mathcal{S}_T$ with terms $\pi_1 \in \mathrm{Tm}(A_1)(\mathrm{sole}(A_1 \times A_2))$ and $\pi_2 \in \mathrm{Tm}(A_2)(\mathrm{sole}(A_1 \times A_2))$ such that for any pair of terms $M_1 \in \mathrm{Tm}(A_1)(\Gamma)$ and $M_2 \in \mathrm{Tm}(A_2)(\Gamma)$ there exists a unique term $(M_1, M_2) \in \mathrm{Tm}(A_1 \times A_2)(\Gamma)$ satisfying $\pi_1 * (M_1, M_2) = M_1$ and $\pi_2 * (M_1, M_2) = M_2$.*

- *An exponential of types* $A, B \in \mathcal{S}_T$ *is a type* $A \Rightarrow B \in \mathcal{S}_T$ *with a term* $\mathrm{app} \in \mathrm{Tm}B(\mathrm{sole}(A \Rightarrow B) \times \mathrm{sole}A)$ *such that for any* $M \in \mathrm{Tm}B(\Gamma \times \mathrm{sole}A)$ *there exists a unique* $\lambda M \in \mathrm{Tm}(A \Rightarrow B)\Gamma$ *satisfying* $\mathrm{app} * (\lambda M * \pi_1^{\Gamma,\mathrm{sole}A}, \pi_2^{\Gamma,\mathrm{sole}A}) = M$

Definition 5. *A homomorphism of CT structures* $F : \mathcal{S} \to \mathcal{T}$ *is* faithful *if for each* $\Gamma \in \mathcal{S}_c$ *and* $A \in \mathcal{S}_T$*, the function* $F_{tm}^{A,\Gamma} : \mathrm{Tm}_{\mathcal{S}}(A)(\Gamma) \to \mathrm{Tm}_{\mathcal{T}}(FA)(F\Gamma)$ *is injective.*

Problem 2 Conservativity of Adding Function Types to STT

Fix a set of base types Σ_0. Let $\mathcal{L}(\times, 1)$ be the syntactic CT structure for STT generated from the base types and the $1, \times$ connectives, whereas $\mathcal{L}(\times, 1, \Rightarrow)$ is the analogous syntactic CT structure generated by $1, \times, \Rightarrow$.

Then we define a CT structure homomorphism $i : \mathcal{L}(\times, 1) \to \mathcal{L}(\times, 1, \Rightarrow)$, the inclusion of the smaller type theory into the larger one:

$$
\begin{aligned}
i_c(\Gamma) &= \Gamma \\
i_c(\gamma) &= \gamma \\
i_{ty}(A) &= A \\
i_{tm}([M]) &= [M]
\end{aligned}
$$

Observe that this preserves product types and the unit type.

Our goal is to prove that adding function types to STT with product types results in a *conservative extension* of the equational theory. That is, we want to show for any $\Gamma \in \mathcal{L}(\times, 1)_c$ and $A \in \mathcal{L}(\times, 1)_{ty}$, and $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$, if $\Gamma \vdash M = M' : A$ is provable in $STT(\times, 1, \Rightarrow)$, then in fact $\Gamma \vdash M = M' : A$ is already provable in $STT(\times, 1)$. Unraveling definitions, this says precisely that the homomorphism i is *faithful*.

We will prove this using a generalization of the method we used in problem set 1[1].

1. Show that if $F : \mathcal{S} \to \mathcal{T}$ and $G : \mathcal{T} \to \mathcal{U}$ are homomorphisms of CT structures and $G \circ F$ is faithful then F is faithful.

2. Show that if $F : \mathcal{S} \to \mathcal{T}$ and $F' : \mathcal{S} \to \mathcal{T}$ are homomorphisms of CT structures and $\alpha_c \in \mathcal{T}_c^{\mathcal{S}_c}(F, F')$ is a natural isomorphism and F is faithful then F' is faithful.

3. Show that for any category $\mathcal{C}$, the category of predicators $\mathscr{P}\mathcal{C}$ is cartesian closed (HINT: the cartesian closed structure is a direct generalization of the Heyting algebra structure you constructed in PS1). Therefore $\mathrm{self}(\mathscr{P}\mathcal{C})$ has unit, binary products and function types.

4. Define for every C-T structure $\mathcal{S}$, a homomorphism $Y : S \to \mathrm{self}(\mathscr{P}\mathcal{S}_c)$ (Hint: use the Yoneda embedding) that

[1] again, there is a more complex proof that proves conservativity when we additionally have sum types

- is faithful (Hint: use the Yoneda lemma)
- preserves product types

5. Define a homomorphism of CT structures $[\![\cdot]\!] : \mathcal{L}(\times, 1, \Rightarrow) \to \mathrm{self}(\mathscr{P}\mathcal{L}(\times, 1)_c)$ and a natural isomorphism between $[\![\cdot]\!] \circ i$ and Y. (Hint: use the soundness theorem for $\mathcal{L}(\times, 1, \Rightarrow)$ and completeness theorem for $\mathcal{L}(\times, 1)$).

6. Conclude that i is faithful.

In fact, this functor i satisfies an additional property: it is also *full*, meaning that $i_{tm}^{A,\Gamma}$ is not just injective but also *surjective*. That is, for any $\Gamma \in \mathcal{L}(\times, 1)_c$ and $A \in \mathcal{L}(\times, 1, \Rightarrow)_T$, if $\Gamma \vdash M : A$ is a term in $STT(\times, 1, \Rightarrow)$ then there exists a term $\Gamma \vdash M' : A$ in $STT(\times, 1)$ such that $\Gamma \vdash M = M' : A$ is provable. This can be proven using a more complex, but similar construction. See Crole chapter 4.10 for a variant of this argument.

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