Problem Set 2: Simple Type Theory

Released: January 23, 2023
Due: February 03, 2023, 11:59pm
Last modified: Jan 26, 2023, 1:00pm

Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

Corrections/Modifications: The following have been fixed from the initial released version:

1. The rule of substitution $()[\gamma] = ()$ has been corrected.

2. The notation $\gamma, M/x$ was previously incorrectly written as $\gamma[M/x]$.

3. Extra parentheses were added to help disambiguate the meaning of $(\gamma(x))[\delta]$: This means first apply γ to the variable x to get a term $\gamma(x)$ and then apply the substitution δ to that term.

4. The rule for weakening was reversed.

Problem 1   Exponential Isomorphisms

We say that $x : A \vdash M : B$ and $y : B \vdash N : A$ form an *isomorphism* if $x : A \vdash N[M/y] = x$ and $y : B \vdash M[N/x] = y : B$. In this case we say A and B are isomorphic, written $A \cong B$.

Construct the following isomorphisms (with proof):

1. $A \Rightarrow B \Rightarrow C \cong (A \times B) \Rightarrow C$

2. $A \Rightarrow (B \times C) \cong (A \Rightarrow B) \times (A \Rightarrow C)$

3. $(A \Rightarrow 1) \cong 1$

4. $(A + B) \Rightarrow C \cong (A \Rightarrow C) \times (B \Rightarrow C)$

5. $0 \Rightarrow C \cong 1$

This gives an idea of why $A \Rightarrow B$ is in category theory sometimes called the *exponential* and written B^A.

· · · · · · · · ·

Problem 2   Admissibile Rules in STT

When all variables are known to be distinct, substitution $M[N/x]$ can simply be defined as the replacement of x with N everywhere in the term M. This definition is the STT version of the admissibility of the substitution principle of IPL:

$$\frac{\Gamma \vdash M : A \qquad \Gamma, x : A \vdash N : B}{\Gamma \vdash N[M/x]} \text{ SUBST(*)}$$

The admissible principle of contraction can also be viewed as a textual substitution in the term:

$$\frac{\Gamma, x : A, y : A, \Delta \vdash M : C}{\Gamma, x : A, \Delta \vdash M[x/y] : C} \text{ CONTRACTION(*)}$$

On the other hand, the use of variables to stand for assumptions means that exchange and weakening have no effect on the proof term:

$$\frac{\Gamma, y : B, x : A, \Delta \vdash M : C}{\Gamma, x : A, y : B, \Delta \vdash M : C} \text{ EXCHANGE(*)} \qquad \frac{\Gamma, \Delta \vdash M : C}{\Gamma, x : A, \Delta \vdash M : C} \text{ WEAKENING(*)}$$

In this exercise you will show all of these principles are admissible and additionally prove some *equations* about substitution.

The simplest inductive proofs involve an auxiliary notion. We define a substitution from Δ to Γ to be a function γ that for each variable $x : A \in \Gamma$ produces a term $\Delta \vdash \gamma(x) : A$. We write $\gamma : \Delta \to \Gamma$ to mean a substitution from Δ to Γ. We can then define an admissible action of substitution:

$$\frac{\gamma : \Delta \to \Gamma \qquad \Gamma \vdash M : A}{\Delta \vdash M[\gamma] : A} \text{ GENSUBST}$$

Defined by induction on M:

$$x[\gamma] = \gamma(x)$$
$$f(M_1, \ldots,)[\gamma] = f(M_1[\gamma], \ldots)$$
$$(M, N)[\gamma] = (M[\gamma], N[\gamma])$$
$$(\pi_j M)[\gamma] = \pi_j M[\gamma]$$
$$()[\gamma] = ()$$
$$(i_j M)[\gamma] = i_j M[\gamma]$$
$$(\text{case}_+ M\{i_1 x_1 \to N_1 | i_2 x_2 \to N_2\})[\gamma] = (\text{case}_+ M[\gamma]\{i_1 x_1 \to N_1[\gamma, x_1/x_1] | i_2 x_2 \to N_2[\gamma, x_2/x_2]\})$$
$$(\text{case}_0 M\{\})[\gamma] = \text{case}_0 M[\gamma]\{\}$$
$$(\lambda x.M)[\gamma] = \lambda x.M[\gamma, x/x]$$
$$(MN)[\gamma] = M[\gamma] N[\gamma]$$

Where the notation $\gamma, M/x$ is the extension of the the function to map x to M:

$$(\gamma, M/x)(y) = M \qquad\qquad\qquad (\text{if } x = y)$$
$$(\gamma, M/x)(y) = \gamma(y) \qquad\qquad\qquad (\text{if } x \neq y)$$

Define the *identity* substitution $\mathrm{id}_\Gamma : \Gamma \to \Gamma$ to map each variable in Γ to itself: $\mathrm{id}(x) = x$.

Given $\gamma : \Delta \to \Gamma$ and $\delta : \Xi \to \Delta$, define the *composition* $\gamma \circ \delta : \Xi \to \Gamma$ as $(\gamma \circ \delta)(x) = (\gamma(x))[\delta]$.

Below assume $\gamma : \Delta \to \Gamma$, $\delta : \Xi \to \Delta$, $\xi : \Xi' \to \Xi$ and $\Gamma \vdash M : A$.

1. Show M that $\Delta \vdash M[\gamma] : A$, i.e., that the GenSubst typing rule is admissible.

2. Show that one-place substitution, weakening, exchange and contraction are all instances of GenSubst.

3. Show that $M[\mathrm{id}_\Gamma] = M$. Note that this and the following equalities are exact syntactic equalities, you will not need to use any $\beta\eta$ rules to prove it.

4. Show that $\gamma \circ \mathrm{id}_\Delta = \gamma$ and $\mathrm{id}_\Gamma \circ \gamma = \gamma$.

5. Show that $M[\gamma \circ \delta] = M[\gamma][\delta]$

6. Show as a corollary that if $x_2 : A_2 \vdash N_1 : A_1$ and $x_3 : A_3 \vdash N_2 : A_2$ and $x_4 : A_4 \vdash N_4 : A_4$ then $(N_1[N_2/x_2])[N_3/x_3] = (N_1[N_2/x_2])[N_3/x_3]$.

7. Show that $(\gamma \circ \delta) \circ \xi = \gamma \circ (\delta \circ \xi)$

· · · · · · · ·