

# Problem Set 2: Simple Type Theory

Released: January 23, 2023

Due: February 01, 2023, 11:59pm

Last modified: Jan 30, 2023, 4:30pm

Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

**Corrections/Modifications:** The following have been fixed from the initial released version:

1. The rule of substitution  $()[\gamma] = ()$  has been corrected.
2. The notation  $\gamma, M/x$  was previously incorrectly written as  $\gamma[M/x]$ .
3. Extra parentheses were added to help disambiguate the meaning of  $(\gamma(x))[\delta]$ : This means first apply  $\gamma$  to the variable  $x$  to get a term  $\gamma(x)$  and then apply the substitution  $\delta$  to that term.
4. The rule for weakening was reversed.
5. Corrected formulation of Problem 2.6.
6. Correct due date, matching Canvas and the course website.
7. Fix a typo in part 1 of problem 2.

## Problem 1 Exponential Isomorphisms

We say that  $x : A \vdash M : B$  and  $y : B \vdash N : A$  form an *isomorphism* if  $x : A \vdash N[M/y] = x$  and  $y : B \vdash M[N/x] = y : B$ . In this case we say  $A$  and  $B$  are isomorphic, written  $A \cong B$ .

Construct the following isomorphisms (with proof):

1.  $A \Rightarrow B \Rightarrow C \cong (A \times B) \Rightarrow C$
2.  $A \Rightarrow (B \times C) \cong (A \Rightarrow B) \times (A \Rightarrow C)$
3.  $(A \Rightarrow 1) \cong 1$
4.  $(A + B) \Rightarrow C \cong (A \Rightarrow C) \times (B \Rightarrow C)$

5.  $0 \Rightarrow C \cong 1$

This gives an idea of why  $A \Rightarrow B$  is in category theory sometimes called the *exponential* and written  $B^A$ .

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## Problem 2 Admissible Rules in STT

When all variables are known to be distinct, substitution  $M[N/x]$  can simply be defined as the replacement of  $x$  with  $N$  everywhere in the term  $M$ . This definition is the STT version of the admissibility of the substitution principle of IPL:

$$\frac{\Gamma \vdash M : A \quad \Gamma, x : A \vdash N : B}{\Gamma \vdash N[M/x]} \text{SUBST}(\ast)$$

The admissible principle of contraction can also be viewed as a textual substitution in the term:

$$\frac{\Gamma, x : A, y : A, \Delta \vdash M : C}{\Gamma, x : A, \Delta \vdash M[x/y] : C} \text{CONTRACTION}(\ast)$$

On the other hand, the use of variables to stand for assumptions means that exchange and weakening have no effect on the proof term:

$$\frac{\Gamma, y : B, x : A, \Delta \vdash M : C}{\Gamma, x : A, y : B, \Delta \vdash M : C} \text{EXCHANGE}(\ast) \quad \frac{\Gamma, \Delta \vdash M : C}{\Gamma, x : A, \Delta \vdash M : C} \text{WEAKENING}(\ast)$$

In this exercise you will show all of these principles are admissible and additionally prove some *equations* about substitution.

The simplest inductive proofs involve an auxiliary notion. We define a substitution from  $\Delta$  to  $\Gamma$  to be a function  $\gamma$  that for each variable  $x : A \in \Gamma$  produces a term  $\Delta \vdash \gamma(x) : A$ . We write  $\gamma : \Delta \rightarrow \Gamma$  to mean a substitution from  $\Delta$  to  $\Gamma$ . We can then define an admissible action of substitution:

$$\frac{\gamma : \Delta \rightarrow \Gamma \quad \Gamma \vdash M : A}{\Delta \vdash M[\gamma] : A} \text{GENSUBST}$$

Defined by induction on  $M$ :

$$\begin{aligned}
x[\gamma] &= \gamma(x) \\
f(M_1, \dots, )[\gamma] &= f(M_1[\gamma], \dots) \\
(M, N)[\gamma] &= (M[\gamma], N[\gamma]) \\
(\pi_j M)[\gamma] &= \pi_j M[\gamma] \\
()[\gamma] &= () \\
(i_j M)[\gamma] &= i_j M[\gamma] \\
(\text{case}_+ M\{i_1 x_1 \rightarrow N_1 \mid i_2 x_2 \rightarrow N_2\})[\gamma] &= (\text{case}_+ M[\gamma]\{i_1 x_1 \rightarrow N_1[\gamma, x_1/x_1] \mid i_2 x_2 \rightarrow N_2[\gamma, x_2/x_2]\}) \\
(\text{case}_0 M\{\})[\gamma] &= \text{case}_0 M[\gamma]\{\} \\
(\lambda x. M)[\gamma] &= \lambda x. M[\gamma, x/x] \\
(MN)[\gamma] &= M[\gamma] N[\gamma]
\end{aligned}$$

Where the notation  $\gamma, M/x$  is the extension of the the function to map  $x$  to  $M$ :

$$\begin{aligned}
(\gamma, M/x)(y) &= M & (\text{if } x = y) \\
(\gamma, M/x)(y) &= \gamma(y) & (\text{if } x \neq y)
\end{aligned}$$

Define the *identity* substitution  $\text{id}_\Gamma : \Gamma \rightarrow \Gamma$  to map each variable in  $\Gamma$  to itself:  $\text{id}(x) = x$ .

Given  $\gamma : \Delta \rightarrow \Gamma$  and  $\delta : \Xi \rightarrow \Delta$ , define the *composition*  $\gamma \circ \delta : \Xi \rightarrow \Gamma$  as  $(\gamma \circ \delta)(x) = (\gamma(x))[\delta]$ .

Below assume  $\gamma : \Delta \rightarrow \Gamma$ ,  $\delta : \Xi \rightarrow \Delta$ ,  $\xi : \Xi' \rightarrow \Xi$  and  $\Gamma \vdash M : A$ .

1. Show that  $\Delta \vdash M[\gamma] : A$ , i.e., that the GenSubst typing rule is admissible.
2. Show that one-place substitution, weakening, exchange and contraction are all instances of GenSubst.
3. Show that  $M[\text{id}_\Gamma] = M$ . Note that this and the following equalities are exact syntactic equalities, you will not need to use any  $\beta\eta$  rules to prove it.
4. Show that  $\gamma \circ \text{id}_\Delta = \gamma$  and  $\text{id}_\Gamma \circ \gamma = \gamma$ .
5. Show that  $M[\gamma \circ \delta] = M[\gamma][\delta]$
6. Show as a corollary that if  $x_2 : A_2 \vdash N_1 : A_1$  and  $x_3 : A_3 \vdash N_2 : A_2$  and  $x_4 : A_4 \vdash N_3 : A_3$  then  $(N_1[N_2/x_2])[N_3/x_3] = N_1[N_2[N_3/x_3]/x_2]$ .
7. Show that  $(\gamma \circ \delta) \circ \xi = \gamma \circ (\delta \circ \xi)$

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