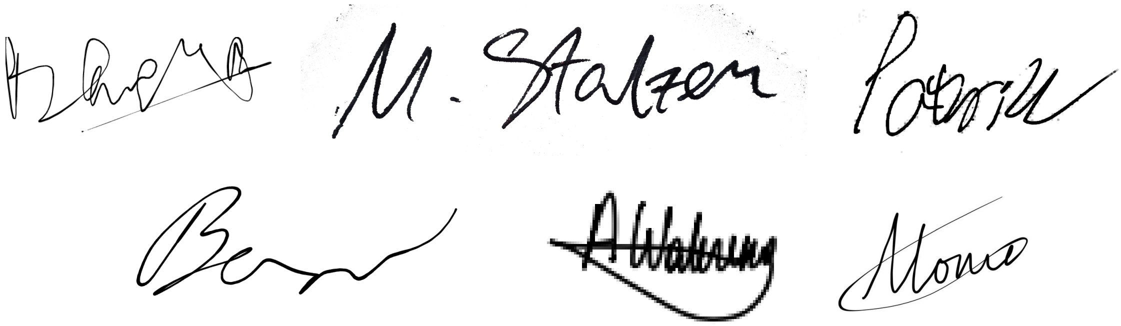
Complex Functions in Geometry

Authors

Blanca Morillo - s215247
Maximilian Stalzer - s215141
Akira-Miranda Adeyomi Adeniran-Lowe - s215170
Patrick Noergaard Skovlod - s211674
Toms Rudolfs Berzins - s215145
Anke Julia Van de Watering - s215231



April 29, 2022

Contents

1 Introduction

Through this project we have been exploring complex functions in geometry and their applications in the investigation of Constant Mean Curvature (CMC) Surfaces. CMC surfaces are used for representations of interfaces between two different types of fluid. For instance, if you overfill a glass of water the curved surface at the top can be represented as a CMC surface. This is created by surface tension, a force that pulls water molecules in which, in turn, stops the glass from overflowing. This is thus, the force that creates a tendency to minimize area.

A subset of CMCs surfaces that we will further investigate are minimal surfaces which are defined to be surfaces that locally minimize their area. This is the same as having a mean curvature of zero.

In this report we have included our solutions to the problems with asterisk signs in the Complex Geometry plan as well as to Problem A and B of section 5. Each problem is headed by its number which is then followed by a short description of the problem and an explanation of how we came to our solution. Initially, all methods used will be explained; however, in later questions the assumption will be made that the reader will be familiar with some the procedures that were used in earlier problems.

All plots are captioned with the name of the figure, the parameterization and a description of the image if necessary. A list of all figures in the report can be found on page 36.

It should be noted that, all calculations for the project were made using maple however in the body of the report only a description of the calculations and some equations will be shown. In order to see all commands used one should visit our Appendix

Lastly, the theory that we use to support our calculations comes from the project document: *Complex functions in Geometry* by David Brander.

1.1 Theory: Curvature

Understanding the concept of curvature, will help the reader with other concepts about complex on this paper. Thus,to dig deeper into these exercises we will explain what the curvature tells us about a surface. To learn more we read about the subject on this website Geometry with an Introduction to Cosmic Topology, Michael P. Hitchman

Curvature measures how the surface bends away from the tangent plane. The curvature of a surface can be negative, positive or zero.

If the curvature is positive at a point, the tangent plane will meet the surface at that point **see Figure 1, (a)**; if the curvature is negative at a point, the tangent plane will cut through the surface **see Figure 1, (b)**. Lastly, if the curvature is zero at a point the tangent plane meets with the surface at a line see **Figure 1, (c)**.

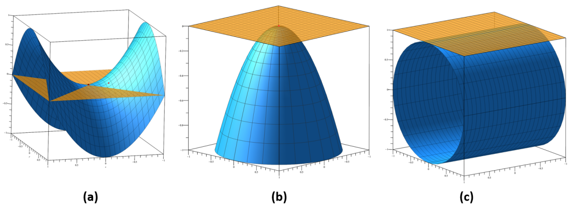


Figure 1: The curvature of a surface at a point can be (a) positive; (b) negative; or (c) zero.

2 Asterisk problems

2.1 Surfaces and their geometry

2.1.1 Problem 7

We are given the parametric form for the function $\boldsymbol{h}$:

$$\boldsymbol{h}(u,v) = \begin{bmatrix} u \\ v^2 \\ u\,v^3 \end{bmatrix} \text{ where } u \in [-1,1], v \in [-1,1] \tag{1}$$

This function is plotted in Figure 2.

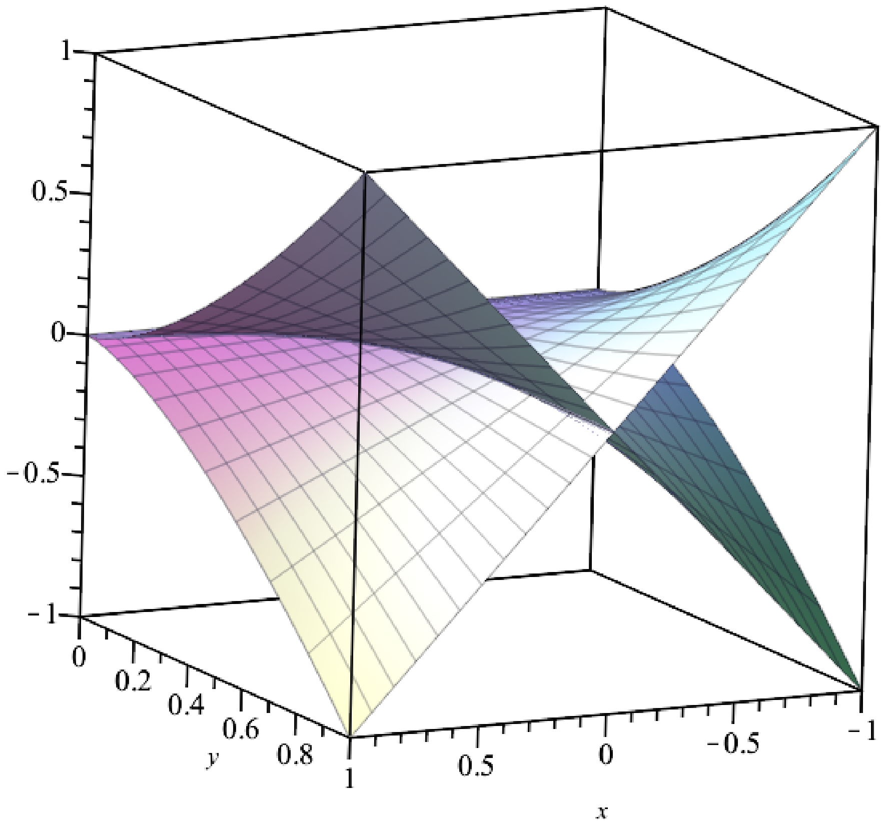


Figure 2: $h(u,v), u \in [-1,1], v \in [-1,1]$

A function is not a regular parameterized surface if the cross product of the tangent vectors equals 0. This means that the tangent vectors do not span a plane, i.e. the tangent vectors are not linearly independent.

Therefore, one must find the tangent vectors, compute the cross product between them, and check at which points, if any, the cross product equals zero.

The tangent vectors are found by taking the derivatives with respect to u and v of the parametric representation of h:

$$\boldsymbol{h}_u = \begin{bmatrix} 1 \\ 0 \\ v^3 \end{bmatrix} \qquad \boldsymbol{h}_v = \begin{bmatrix} 0 \\ 2\,v \\ 3\,u\,v^2 \end{bmatrix}$$

The cross product of these tangent vectors gives us:

$$\boldsymbol{h}_u \times \boldsymbol{h}_v = \begin{bmatrix} -2\,v^4 \\ -3\,u\,v^2 \\ 2\,v \end{bmatrix}$$

The cross product is equal to zero when $v = 0$:

$$\boldsymbol{h}_u \times \boldsymbol{h}_v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{, } when\; v = 0\; and\; u = u$$

Therefore, the parametric representation $\boldsymbol{h}(u, v)$ is regular except for when $v = 0$. Plugging this into the parametric representation (eq. 1) $\boldsymbol{h}$(u,v), we get the line $(u, 0, 0)$. We plot the line and the surface together in Figure 3.

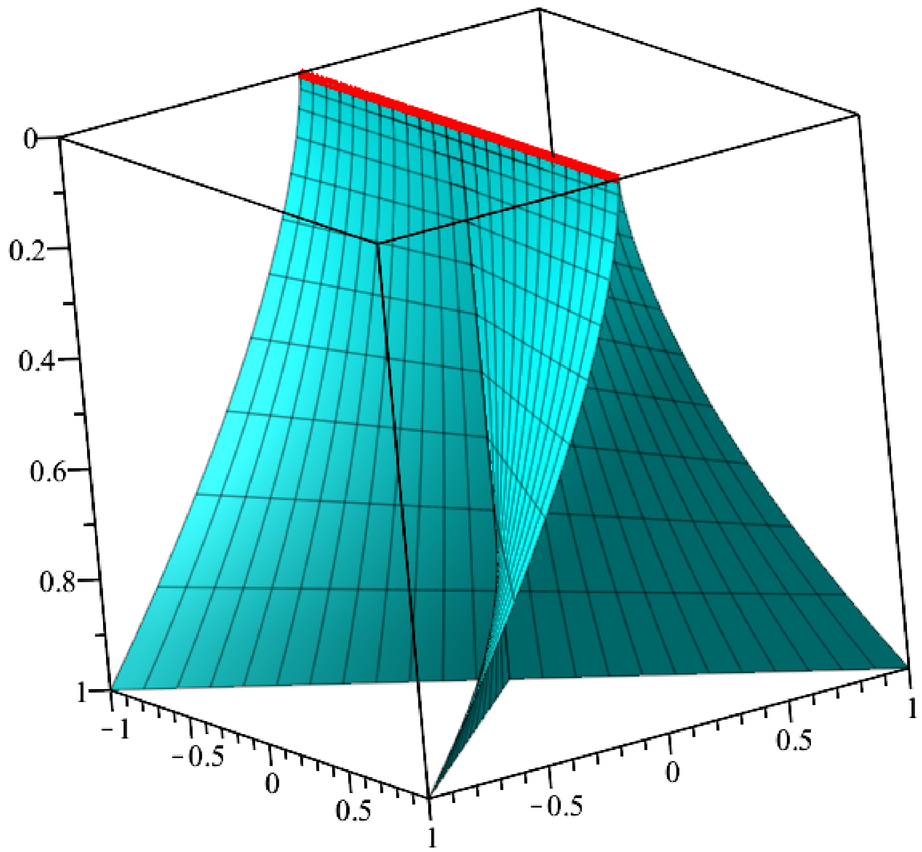


Figure 3: $h(u, v), u \in [-1, 1], v \in [-1, 1]$ and the line $(u, 0, 0), u \in [-1, 1]$

2.1.2 Problem 9

Part A We consider Enneper's surface:

$$\boldsymbol{E}(u,v) = \begin{bmatrix} u - \frac{u^3}{3} + uv^2 \\ -v + \frac{v^3}{3} - vu^2 \\ u^2 - v^2 \end{bmatrix} \text{ for } (u,v) \in \mathbb{R}^2, \tag{2}$$

We plot *Enneper's surface* in the given domain of $u \in [-2,2], v \in [-2,2]$ in Figure 4.

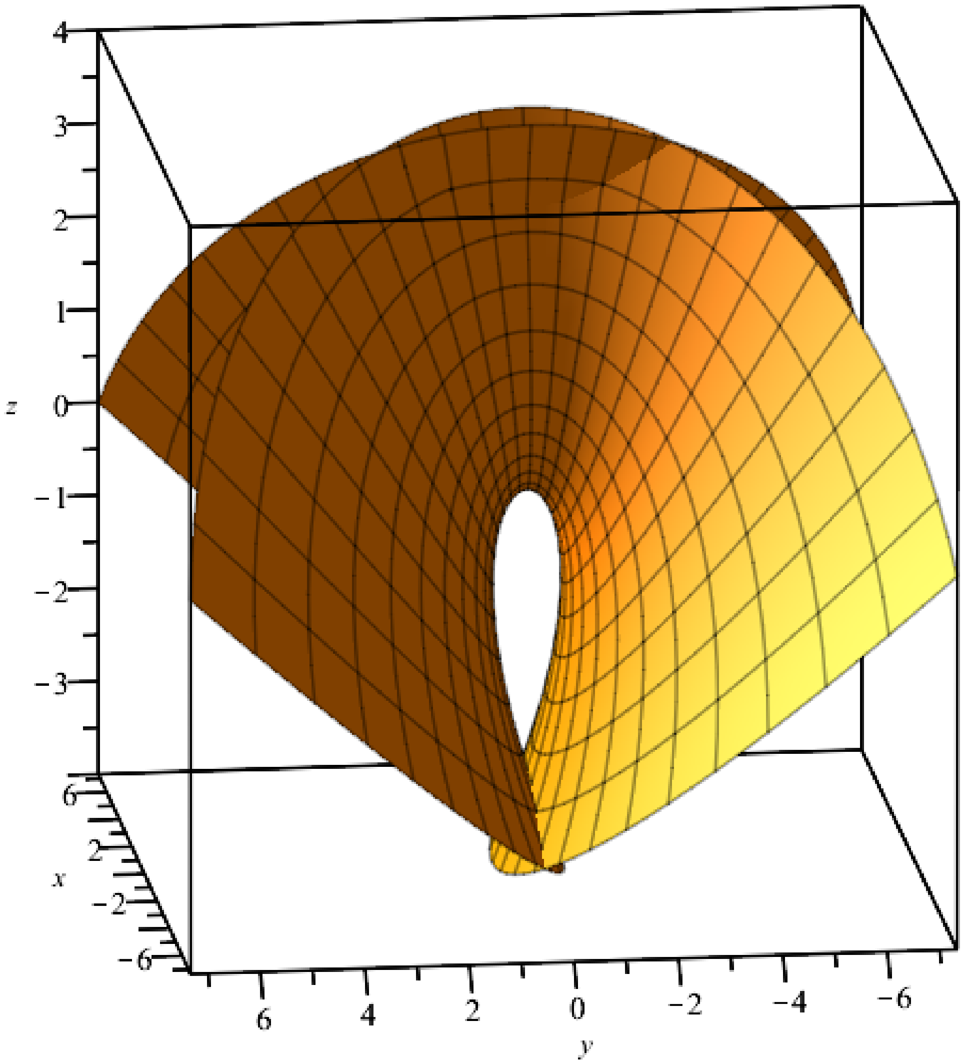


Figure 4: $E(u,v), u \in [-2,2], v \in [-2,2]$

Part B We can clearly see from the image that the surface has self-intersections. We now show this analytically:

By looking at the graph, we make the assumption that an intersection happens when $x = 0$. Thus, we can begin by solving for when the first component of $\boldsymbol{E}(u,v)$ (its x-coordinate) equals 0.

$$u - \frac{u^3}{3} + uv^2 = 0$$

which yields:

$$v_{1,2,3} = v \quad \text{and} \quad u_1 = 0 \quad u_2 = \sqrt{3v^2 + 3} \quad u_3 = -\sqrt{3v^2 + 3}$$

We proceed by substituting one of the solutions ($u_1 = 0$) into the parameterization of Enneper's surface $E(u, v)$:

$$\boldsymbol{E}(0, v) = \begin{bmatrix} 0 \\ -v + \frac{v^3}{3} \\ -v^2 \end{bmatrix}$$

We plot this function in Figure 5, which shows the y and z values in the plane where $x = 0$.

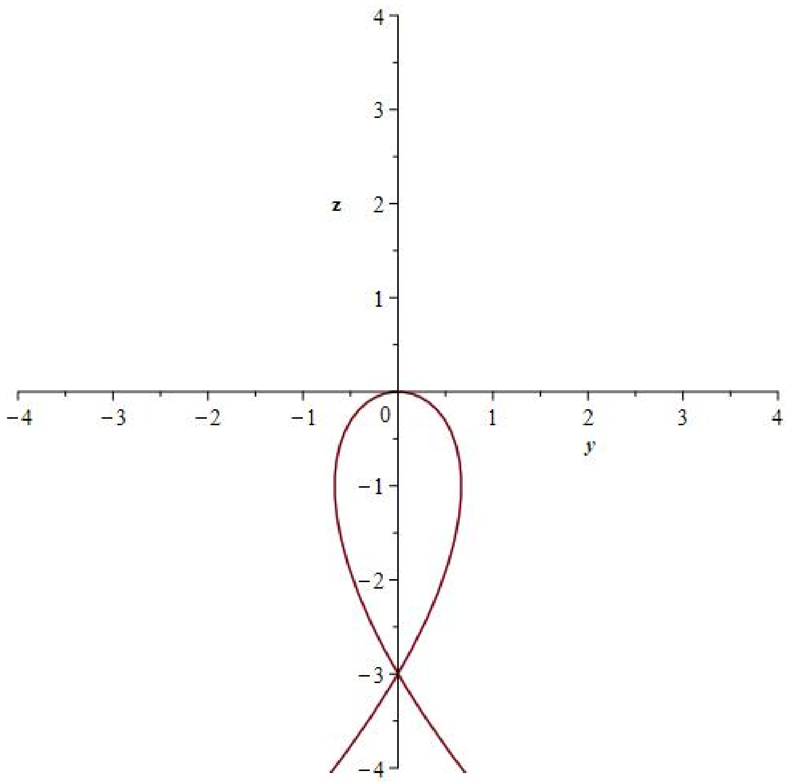


Figure 5: $E(0, v)$

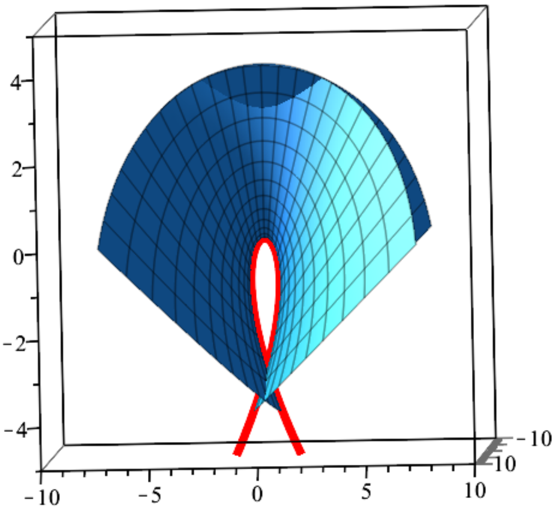


Figure 6: $E(0, v)$
$E(u, v), u \in [-2, 2], v \in [-2, 2]$

From Figure 5, we can discern that a 'self-intersection' in fact occurs when $y = 0$ and when $z = -3$.

Therefore, we can equate the y-coordinate of $E(u, v)$ to 0 and the z-coordinate of $E(u, v)$ to -3.

Solving for the y-coordinate:

$$-v + \frac{v^3}{3} = 0$$

which yields:

$$v_1 = 0, \quad v_2 = \sqrt{3}, \quad v_3 = -\sqrt{3}$$

and

$$-v^2 = -3$$

Solving for the z-coordinate:

$$-v^2 = -3$$

which yields:

$$v_1 = -\sqrt{3}, \quad v_2 = \sqrt{3}$$

We can see that we have two of the same v values when solving for y and z. This means that these different values of v ($v_1 = \sqrt{3}$ and $v_2 = -\sqrt{3}$)) are mapped to the same point in $\mathbb{R}^2$.

Thus:

$$\boldsymbol{E}(0,\sqrt{3}) = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \qquad \boldsymbol{E}(0,-\sqrt{3}) = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

Part C Nevertheless, $\boldsymbol{E}(u,v)$ is a regular parameterized surface.

We show this by using the procedure to check if a function has a regular parameterization explained in the *Complex functions in Geometry* document. This is done by checking if the coordinate tangent vectors are linearly independent. The function is regular if the cross product of the tangent coordinate vectors is non-zero.

Coordinate tangent vectors of $\boldsymbol{E}(u,v)$:

$$\boldsymbol{E}'_u = \begin{bmatrix} -u^2+v^2+1 \\ -2vu \\ 2u \end{bmatrix} \qquad \boldsymbol{E}'_v = \begin{bmatrix} 2vu \\ -u^2+v^2-1 \\ -2v \end{bmatrix}$$

The cross product:

$$\boldsymbol{E}'_u \times \boldsymbol{E}'_v = \begin{bmatrix} 2u^3+2uv^2+2u \\ 2vu^2+2v^3+2v \\ u^4+2v^2u^2+v^4-1 \end{bmatrix}$$

We solve for when the cross product of $\boldsymbol{E}$ yields 0 and we find that there are no real solutions for such. Therefore, Enneper's surface is a regular parameterized surface.

2.1.3 Problem 11

We now want find an expression for the unit normal of unit sphere in terms of u and v. The parameterization of a sphere:

$$\boldsymbol{f}(u,v) = \begin{bmatrix} cos(u)\,cos(v) \\ sin(u)\,cos(v) \\ sin(v) \end{bmatrix},\ u \in [0,2\pi], v \in [\frac{-\pi}{2}, \frac{\pi}{2}]. \tag{3}$$

The unit normal is a unit vector perpendicular to the tangent plane and can therefore be found by taking the cross product of the tangent vectors of the sphere and normalizing this. The tangent vectors are found to be:

$$\boldsymbol{f}'_u = \begin{bmatrix} -sin(u)\,cos(v) \\ cos(u)\,cos(v) \\ 0 \end{bmatrix} \qquad \boldsymbol{f}'_v = \begin{bmatrix} -cos(u)\,sin(v) \\ -sin(u)\,sin(v) \\ cos(v) \end{bmatrix}$$

The cross product of the tangent vectors:

$$\boldsymbol{f}'_u \times \boldsymbol{f}'_v = \begin{bmatrix} cos(u)\, cos(v)^2 \\ sin(u)\, cos(v)^2 \\ sin(v)\, cos(v) \end{bmatrix}$$

By normalizing the cross product the unit normal is found to be:

$$\boldsymbol{N} = \frac{\boldsymbol{f}'_u \times \boldsymbol{f}'_v}{|\boldsymbol{f}'_u \times \boldsymbol{f}'_v|} = \begin{bmatrix} cos(u)\, |cos(v)| \\ sin(u)\, |cos(v)| \\ \dfrac{cos(v)\, sin(v)}{|cos(v)|} \end{bmatrix}$$

2.1.4 Problem 12

To determine whether the unit normal of the sphere is pointing towards the center of the sphere, we pick a particular point on the sphere and find the unit normal at this point.

For the point $f(\pi, 0)$:

$$\boldsymbol{f}(\pi, 0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{N} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

The chosen point lies on the left side of the sphere and its unit normal also point to the left. This means the unit normal points out of the surface as shown in the figure below:

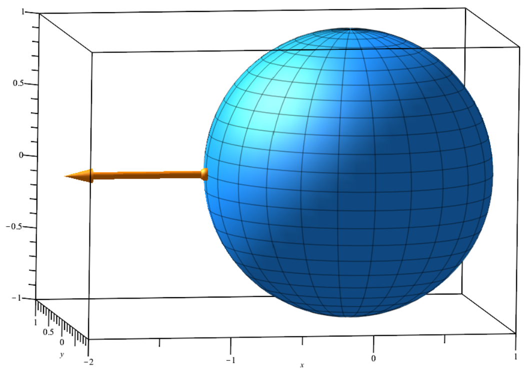


Figure 7: The outward pointing normal at $f(\pi, 0)$

We now want to find out whether the unit normal points outwards for all points on the sphere. To do this we can make a parameterization for the end points of the unit normal vector for all points on the sphere and add this to the sphere's parameterization. This can be written as:

$$\boldsymbol{g}(u,v) = \boldsymbol{f}(u,v) + \boldsymbol{N} = \begin{bmatrix} cos(u)\,(cos(v) + |cos(v)|) \\ sin(u)\,(cos(v) + |cos(v)|) \\ sin(v) + \dfrac{cos(v)\,sin(v)}{|cos(v)|} \end{bmatrix} \tag{4}$$

The above parameterization is plotted in Figure 7 together with the sphere. The parameterization of the end points of the unit normals is a bigger sphere with radius 2, and center at the origin. There are however two deviation from a perfect sphere, this being the lines on top and below the sphere. This corresponds to the points $\boldsymbol{f}(0,\pi/2)$ and $\boldsymbol{f}(0,-\pi/2)$. This is because at those points the cross product of the tangent vectors is zero, so there exists no unit normal at that point. Our parameterization of the sphere is not regular at $\boldsymbol{f}(0,\pi/2)$ and $\boldsymbol{f}(0,-\pi/2)$. Figure 8 shows also that the unit normal points out of the sphere for all other point on the sphere.

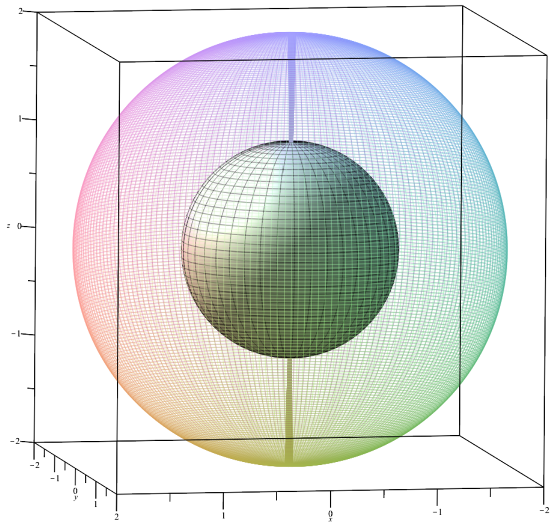


Figure 8: $f(u,v),\ g(u,v),\ u \in [0,2\pi],\ v \in [\frac{-\pi}{2},\frac{\pi}{2}]$

2.1.5 Problem 16

We want to find the principal curvatures of the catenoid. Principal curvatures are the minimum and maximum values of the curvature. The principal curvatures are the eigenvalues of the Weingarten matrix.

The Weingarten matrix is obtained by the first and second fundamental form.

The first fundamental form gives information about lengths, areas and angles on a surface.

$$\mathcal{F}_I = \begin{bmatrix} \boldsymbol{f}_u \cdot \boldsymbol{f}_u & \boldsymbol{f}_u \cdot \boldsymbol{f}_v \\ \boldsymbol{f}_u \cdot \boldsymbol{f}_v & \boldsymbol{f}_v \cdot \boldsymbol{f}_v \end{bmatrix} \tag{5}$$

The second fundamental form gives information about how the surface bends away from the tangent plane.

$$F_{II} = \begin{bmatrix} \boldsymbol{f}_{uu} \cdot \boldsymbol{N} & \boldsymbol{f}_{uv} \cdot \boldsymbol{N} \\ \boldsymbol{f}_{uv} \cdot \boldsymbol{N} & \boldsymbol{f}_{vv} \cdot \boldsymbol{N} \end{bmatrix} \tag{6}$$

where $\boldsymbol{N}$ is the unit normal.

The Weingarten matrix is then obtained by:

$$\boldsymbol{W} = F_I^{-1} \cdot F_{II} \tag{7}$$

To find the two principal curvatures of the Catenoid we must find the eigenvalues of the Weingarten matrix.

The parametric representation of the Catenoid is given by:

$$\boldsymbol{r}(u, v) = \begin{bmatrix} u \\ cosh(u)\, cos(v) \\ cosh(u)\, sin(v) \end{bmatrix} \tag{8}$$

We find the first fundamental form for $\boldsymbol{r}(u, v)$:

$$F_I(\boldsymbol{r}) = \begin{bmatrix} cosh(u)^2 & 0 \\ 0 & cosh(u)^2 \end{bmatrix}$$

Next, we find the unit normal from the normalised cross product of the tangent vectors of $\boldsymbol{r}(u, v)$:

$$\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \begin{bmatrix} \frac{sinh(u)}{cosh(u)} \\ -\frac{cos(v)}{cosh(u)} \\ -\frac{sin(v)}{cosh(u)} \end{bmatrix}$$

We find the second fundamental form from $\boldsymbol{r}(u, v)$ and $\boldsymbol{N}$:

$$F_{II}(r) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

From the first and second fundamental forms we can now find the Weingarten matrix:

$$\boldsymbol{W}_r = F_I^{-1}(\boldsymbol{r}) \cdot F_{II}(\boldsymbol{r}) = \begin{bmatrix} -\frac{1}{cosh(u)^2} & 0 \\ 0 & \frac{1}{cosh(u)^2} \end{bmatrix} \tag{9}$$

Since this is a diagonal matrix, the principal curvatures are the values in the diagonal.

$$\lambda_1 = -\frac{1}{cosh(v)^2} \qquad \lambda_2 = \frac{1}{cosh(v)^2}$$

The principal curvatures for the catenoid have the same magnitude but opposite signs. This means that the surface is saddle shaped at all points.

2.1.6 Problem 17

The principal directions are the two directions in the surface where the curvature is a minimum and a maximum.

The principal directions are the eigenvectors of the Weingarten matrix. The Weingarten matrix is the same as in Problem 16. Since it is a diagonal matrix, the eigenvectors are:

$$E_{\lambda_1} = \text{span}\{(1,0)\} \qquad E_{\lambda_2} = \text{span}\{(0,1)\}$$

Since the principal directions are given by the coordinate vectors the principal curves are the principals curvatures. This is since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the u-coordinate, whilst $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the v-coordinate. Thus, in this case principal curves and principal curvatures are the same.

We can show this by plotting the surface of the Catenoid together with its principal curvatures:

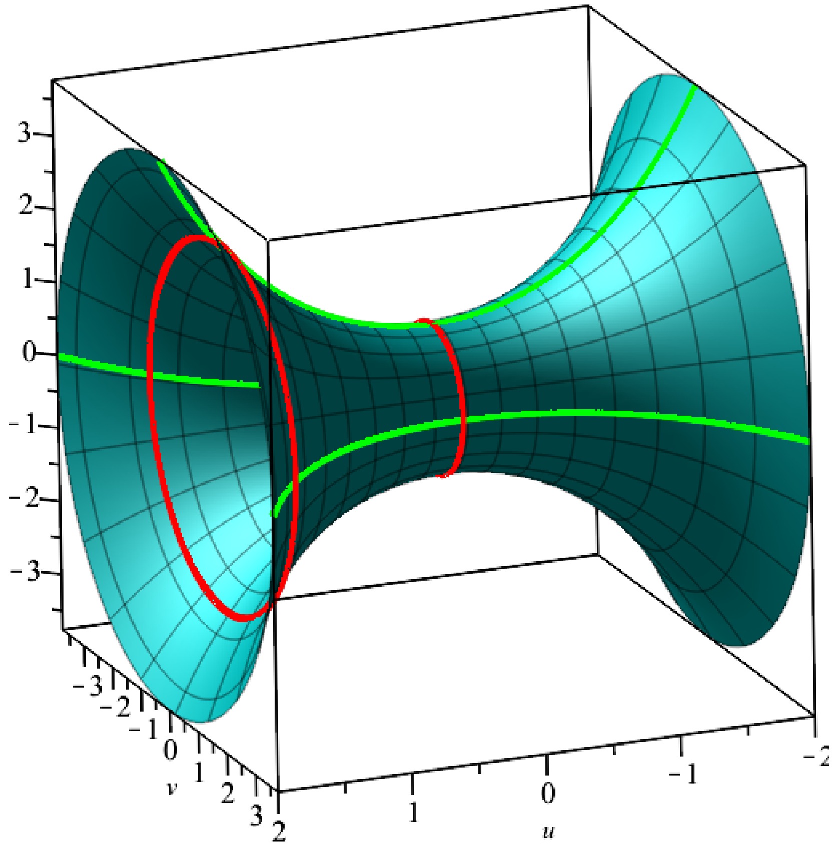


Figure 9: Principal curvatures of Catenoid

2.1.7 Problem 19

In this exercise we plot the catenoid colored by its Gauss curvature.

We compute Gauss Curvature for $\boldsymbol{r}(u, v)$:

The Gauss curvature of a surface is the product of the principal curvatures k_1 and k_2.

Formula for Gauss curvature is given by:

$$K = k_1 \cdot k_2 \tag{10}$$

We use the principal curvatures for the catenoid which we determined in Problem 16

Principal curvatures:

$$k_{1_r} = -\frac{1}{cosh(u)^2} \quad and \quad k_{2_r} = \frac{1}{cosh(u)^2}]$$

Gauss curvature K:

$$-\frac{1}{cosh(u)^2} \cdot \frac{1}{cosh(u)^2} = -\frac{1}{cosh(u)^4}$$

We want to color the surface of the Catenoid by the Gauss curvature. To color in maple we are given three variables that go from 0 to 1 for red, green and blue. We first see by how much the Gauss curvature varies in the range that we will be plotting the Catenoid.

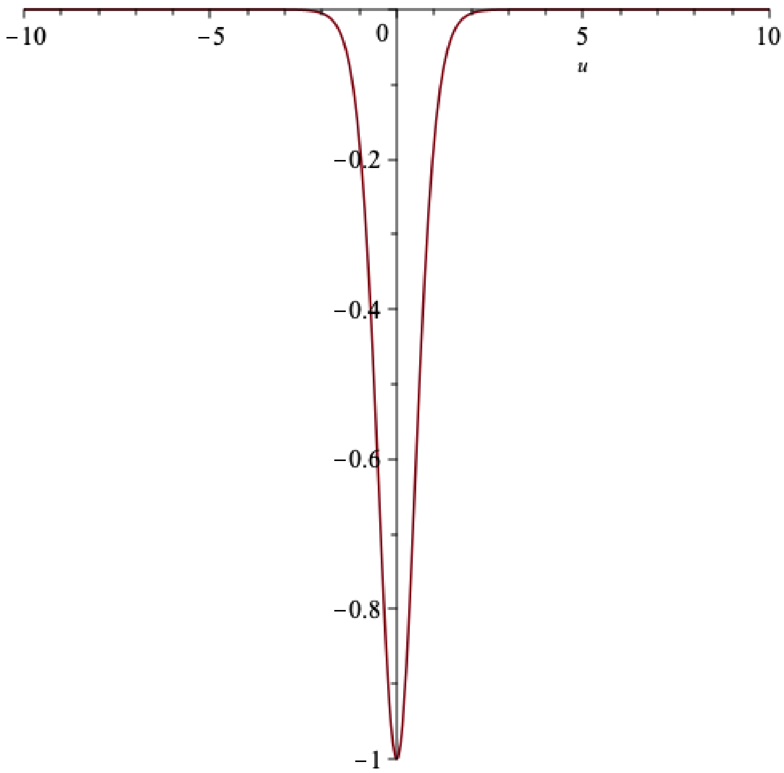


Figure 10: $K = -\frac{1}{cosh(u)^4}$

We can see that the Gauss Curvature varies from 0 to -1. We can now add 1 and set this to be our variable for blue (will vary from 0 to 1). Conversely, we can set our value for green to do the reverse. This way as the Gauss curvature on the surface changes, the color on the surface will also change.

To check what happens to the surface as u grows large we can compute the limit of the function of the Gauss Curvature as u approaches infinity.

$$\lim_{u\to\infty} K(u) = 0 \tag{11}$$

This means that as u grows larger, the surface becomes a plane.

Plot of catenoid colored by the Gauss Curvature:

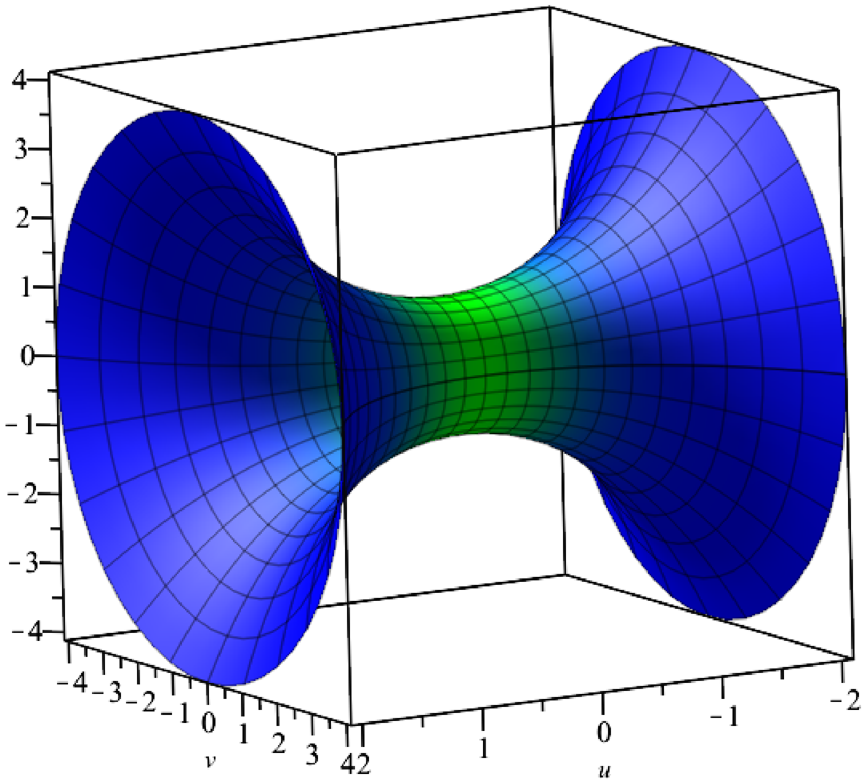


Figure 11: $r(u,v),\ u \in [0, 2\pi],\ v \in [\frac{-2\pi}{3}, \frac{2\pi}{3}]$

2.1.8 Problem 21

We calculate the Gauss and mean curvature for the surface $\boldsymbol{g}$ given by the parametric representation:

$$\boldsymbol{g}(u,v) = \begin{bmatrix} u^2 \\ u^3 \\ v \end{bmatrix},\ u \in [-1,1],\ v \in [-1,1] \tag{12}$$

We apply the first and second fundamental form to $\boldsymbol{g}(u,v)$ using the same methods as in Problem 16.

$$F_I(\boldsymbol{g}) = \begin{bmatrix} 9u^4 + 4u^2 & 0 \\ 0 & 1 \end{bmatrix} \qquad F_{II}(\boldsymbol{g}) = \begin{bmatrix} -\frac{6}{(9\,u^2+4)\sqrt{9\,u^4+4|u|^2}} & 0 \\ 0 & 0 \end{bmatrix}$$

The Weingarten matrix in terms of $\boldsymbol{u}$ is therefore:

$$\boldsymbol{W}_g(\boldsymbol{u}) = \begin{bmatrix} -\dfrac{6}{(9u^2+4)\sqrt{9u^4+4\,|u|^2}} & 0 \\ 0 & 0 \end{bmatrix}$$

The Weingarten matrix for $\boldsymbol{g}$ is a diagonal matrix, and therefore the principal curvatures are the values in the diagonal.

$$k_{1_g} = -\frac{6}{(9u^2+4)^{\frac{3}{2}}\,|u|} \qquad k_{2_g} = 0$$

The Gauss curvature is computed using eq.10.

$$K = 0 \cdot \left(-\frac{6}{(9u^2+4)^{\frac{3}{2}}\,|u|} \right) = 0$$

This means the surface $\boldsymbol{g}$ is a developable surface, which means the surface can be obtained by a plane which is bent or folded (add wikipedia source). The surface is plotted in Figure 12, where we can see that in the z direction there is no curvature, and $\boldsymbol{g}$ is therefore a developable surface.

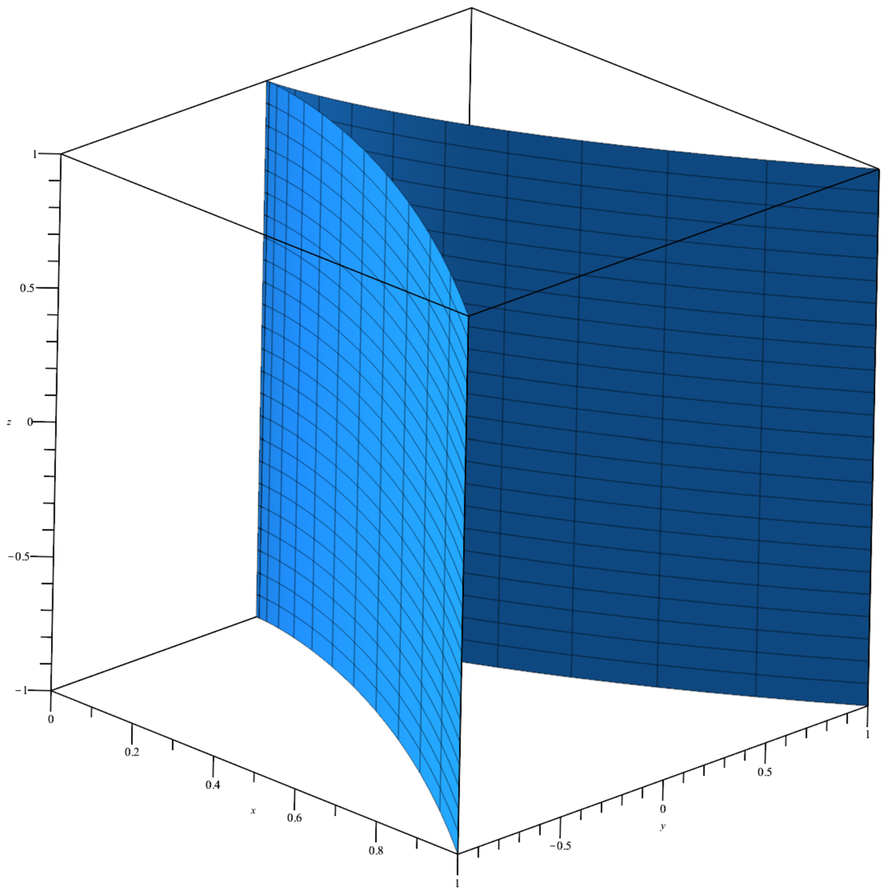


Figure 12: $g(u, v), u \in [-1, 1], v \in [-1, 1]$

Formula for the mean curvature is given by:

$$\boldsymbol{H} = \frac{k_1 + k_2}{2} \tag{13}$$

The mean curvature is:

$$\boldsymbol{H}(u) = \frac{1}{2}(0 - \frac{6}{(9u^2 + 4)^{3/2}|u|}) = -\frac{3}{(9u^2 + 4)^{3/2}|u|}$$

We now want to find the limit of the mean curvature as u tends to 0. We plot the function of the mean curvature:

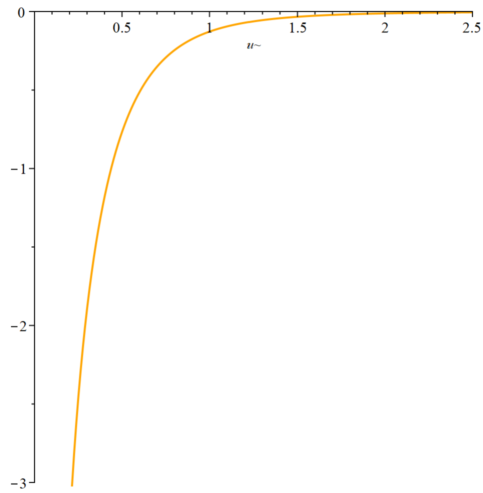


Figure 13: The mean curvature in terms of u

We observe that as the u tends to zero, the mean curvature tends to $-\infty$.

The surface $\boldsymbol{g}$ does not have constant mean curvature.

2.1.9 Problem 22

We are asked to give two surfaces that do not have constant mean curvature.

Example 1: The Torus

$$\boldsymbol{t}(u,v) = \begin{bmatrix} (2+cos(u))\,cos(v) \\ (2+cos(u))\,sin(v) \\ sin(u) \end{bmatrix} \quad u \in [-1,1],\ v \in [-1,1] \tag{14}$$

To prove that the Torus indeed does not have constant mean curvature, we must first compute the Weingarten matrix:

$$\boldsymbol{W}_t(u,v) = \begin{bmatrix} 1 & 0 \\ 0 & (2+cos(u))^3\,cos(u) \end{bmatrix} \tag{15}$$

Since the matrix is diagonal, the eigenvalues are easily recognizable.

$$\lambda_1 = 1 \qquad\qquad \lambda_2 = (2+cos(u))^3\,cos(u) \tag{16}$$

Following the mean curvature (eq.14) formula we get:

$$H_t(u,v) = \frac{1+(2+cos(u))^3\,cos(u)}{2} = \frac{1}{2} + \frac{(2+cos(u))^3\,cos(u)}{2}$$

We use the same method as in Problem 19 to colour the Torus by mean curvature. This is plotted in Figure 14.

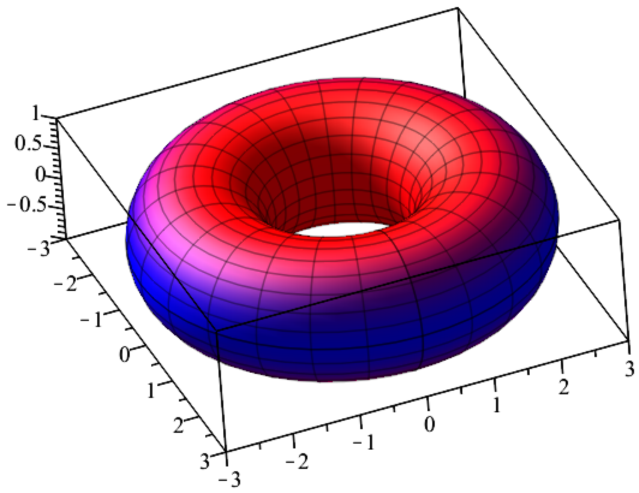


Figure 14: The Torus colored by mean curvature

As we can see from the mean curvature equation $H_t(u, v)$ and the graph colored by the mean curvature, the Torus does not have constant mean curvature. The graph of the Torus would be colored by one color if it had constant mean curvature.

Example 2:

$$\boldsymbol{f}(u,v) = \begin{bmatrix} u \\ u^2 \\ v \end{bmatrix},\ u \in [-1,1],\ v \in [-1,1]$$

We repeat the same process as we did previously in this exercise.

We find the Weingarten matrix:

$$\boldsymbol{W} = F_I^{-1} \cdot F_{II} = \begin{bmatrix} -\dfrac{-2}{(4\,u^2+1)^{\frac{3}{2}}} & 0 \\ \genfrac{}{}{0pt}{}{0}{v} & 0 \end{bmatrix}$$

Next, we compute the mean curvature (eq.14) with the matrix's eigenvalues (principal curvatures):

$$H_f(u,v) = \frac{\dfrac{-2\sqrt{1+4\,|u|^2}}{16\,|u|^2\,\bar{u}\,u + 4\,|u|^2 + 4\,\bar{u}\,u + 1}}{2} = -\frac{1}{(\,4u^2+1)^{3/2}}$$

Another way to clearly see that the function does not have constant mean curvature is plotting the mean curvature function on a graph:

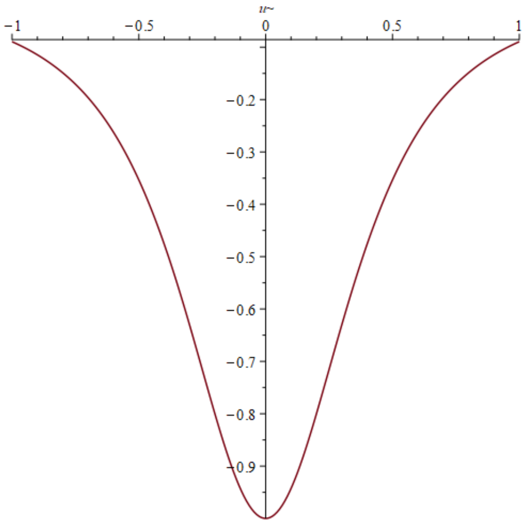


Figure 15: $H_f(u, v)$

As in the previous example with the Torus, we can appreciate from the graph and the equation of the mean curvature $H_f(u, v)$ that the function $\boldsymbol{f}(u, v)$ also does not have constant mean curvature.

Again we plot the function colored by the mean curvature:

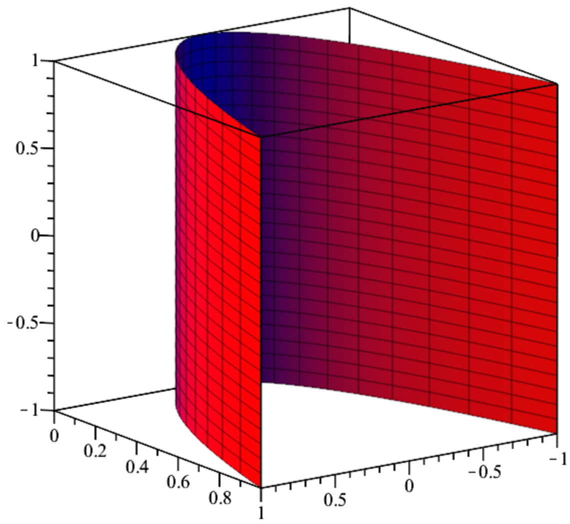


Figure 16: $f(u, v)$

2.2 Weierstrass-Enneper Representation

2.2.1 Problem 24

To check if a surface was given using conformal parameterization we check if the first fundamental form is of the form:

$$F_I = \begin{bmatrix} f'_u \cdot f'_u & 0 \\ 0 & f'_u \cdot f'_u \end{bmatrix} \tag{17}$$

To do this we calculate the first fundamental form and subtract the matrix with the square of the tangent vector with respect to u as its diagonals. The result of this subtraction should equal zero meaning that these matrices are the same:

$$\begin{bmatrix} f'_u \cdot f'_u & f'_u \cdot f'_v \\ f'_u \cdot f'_v & f'_v \cdot f'_v \end{bmatrix} - \begin{bmatrix} f'_u \cdot f'_u & 0 \\ 0 & f'_u \cdot f'_u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{18}$$

(a) Henneberg's surface

$$\boldsymbol{H}(u,v) = \begin{bmatrix} 2\,sinh(u)\,cos(v) - \frac{2}{3}\,sinh(3\,u)\,cos(3\,v) \\ 2\,sinh(u)\,sin(v) + \frac{2}{3}\,sinh(3\,u)\,sin(3\,v) \\ 2\,cosh(2\,u)\,cos(2\,v) \end{bmatrix} \tag{19}$$

First fundamental form of H:

$$\digamma_I(\boldsymbol{H}) = \begin{bmatrix} \boldsymbol{H}'_u \cdot \boldsymbol{H}'_u & \boldsymbol{H}'_u \cdot \boldsymbol{H}'_v \\ \boldsymbol{H}'_u \cdot \boldsymbol{H}'_v & \boldsymbol{H}'_v \cdot \boldsymbol{H}'_v \end{bmatrix}$$

Where:

$$\boldsymbol{H}'_u = \begin{bmatrix} 2\,cosh(u)\,cos(v) - 2\,cosh(3\,u)\,cos(3\,v) \\ 2\,cosh(u)\,sin(v) + 2\,cosh(3\,u)\,sin(3\,v) \\ 4\,sinh(2\,u)\,cos(2\,v) \end{bmatrix}$$

$$\boldsymbol{H}'_v = \begin{bmatrix} -2\,sinh(u)\,sin(v) + 2\,sinh(3u)\,sin(3\,v) \\ 2\,sinh(u)\,cos(v) + 2\,sinh(3\,u)\,cos(3\,v) \\ -4\,cosh(2\,u)\,sin(2\,v) \end{bmatrix}$$

When we subtract the matrix that would represent conformal parameterization we obtain the zero matrix:

$$\begin{bmatrix} \boldsymbol{H}'_u \cdot \boldsymbol{H}'_u & \boldsymbol{H}'_u \cdot \boldsymbol{H}'_v \\ \boldsymbol{H}'_u \cdot \boldsymbol{H}'_v & \boldsymbol{H}'_v \cdot \boldsymbol{H}'_v \end{bmatrix} - \begin{bmatrix} \boldsymbol{H}'_u \cdot \boldsymbol{H}'_u & 0 \\ 0 & \boldsymbol{H}'_u \cdot \boldsymbol{H}'_u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The parameterization of Henneberg's surface was given in conformal form.

(b) Catalan's surface

$$\boldsymbol{C}(u,v) = \begin{bmatrix} u - sin(u) \cdot cosh(v) \\ 1 - cos(u) \cdot cosh(v) \\ 4sin(\frac{u}{2}) \cdot sinh(\frac{v}{2}) \end{bmatrix} \tag{20}$$

The first fundamental form of this parameterization of Catalan's surface is given by:

$$\digamma_I(\boldsymbol{C}) = \begin{bmatrix} \boldsymbol{C}'_u \cdot \boldsymbol{C}'_u & \boldsymbol{C}'_u \cdot \boldsymbol{C}'_v \\ \boldsymbol{C}'_u \cdot \boldsymbol{C}'_v & \boldsymbol{C}'_v \cdot \boldsymbol{C}'_u \end{bmatrix}$$

Where:

$$\boldsymbol{C}'_u = \begin{bmatrix} 1 - cos(u)\,cosh(v) \\ sin(u)\,cosh(v) 2\,cos(\frac{u}{2})\,sinh(\frac{v}{2}) \end{bmatrix}$$

$$\boldsymbol{C}'_v = \begin{bmatrix} -sin(u)\, sinh(v) \\ -cos(u)\, sinh(v) \\ 2\, sin(\frac{u}{2})\, cosh(\frac{v}{2}) \end{bmatrix}$$

When subtracting the matrices we again obtain the zero matrix.

$$\begin{bmatrix} \boldsymbol{C}'_u \cdot \boldsymbol{C}'_u & \boldsymbol{C}'_u \cdot \boldsymbol{C}'_v \\ \boldsymbol{C}'_u \cdot \boldsymbol{C}'_v & \boldsymbol{C}'_v \cdot \boldsymbol{C}'_v \end{bmatrix} - \begin{bmatrix} \boldsymbol{C}'_u \cdot \boldsymbol{C}'_u & 0 \\ 0 & \boldsymbol{C}'_u \cdot \boldsymbol{C}'_u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This shows that the parameterization of Catalan's surface was also given in conformal form.

(c) Scherk's fifth surface

$$\boldsymbol{S}(u,v) = \begin{bmatrix} arcsinh(u) \\ arcsinh(v) \\ arcsin(v\, u) \end{bmatrix} \tag{21}$$

The first fundamental form for Scherk's fifth surface is given by:

$$F_I(\boldsymbol{S}) = \begin{bmatrix} \frac{v^2\, u^2 + v^2 |v^2\, u^2 - 1|}{(u^2+1)\, |v^2\, u^2 - 1|} & \frac{v\, u}{|v^2\, u^2 - 1|} \\ \frac{v\, u}{|v^2\, u^2 - 1|} & \frac{|v^2\, u^2 - 1| + u^2\, (v^2+1)}{(v^2+1)\, |v^2\, u^2 - 1|} \end{bmatrix} \tag{22}$$

We can see immediately that it was not given with conformal parameterization since we did not obtain a diagonal matrix. In other words, the bottom left corner and top right corner corresponding to $S'_u \cdot S'_v$ does not equal zero.

2.2.2 Problem 26

The function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = z^3 \tag{23}$$

is not one-to-one.

We can show this with an example using polar coordinates:

$$f(r,\theta) = r^3\, (cos(3\,\theta) + I\, sin(3\,\theta))$$
$$f(r,\pi) = -r^3$$
$$f(r,-\pi) = -r^3$$

As we can see, the two different input angles yield the same output. This shows that function $f(z) = z^3$ is many-to-one.

2.2.3 Problem 27(c)

To find the the real and imaginary part of the function $f(z) = e^z$ where $z = u + iv$, in terms of u and v, we "separate" the function in two parts —one without I (the real part) and one with I (the imaginary part) —.

This is done by changing the function to trigonometric form:

$$f(u + I\,v) = e^{u+I\,v} = e^{u}\,cos(v) + I\,e^{u}\,sin(v)$$

Now, it is easier to see which is the real and imaginary part of the function:

$$\mathrm{Re}(e^{u+I\,v}) = e^{u}\,cos(v)$$

$$\mathrm{Im}(e^{u+I\,v}) = e^{u}\,sin(v)$$

2.2.4 Problem 28

We now want to calculate the minimal surfaces corresponding to the given Weierstrass data. We do this following the Weierstrass-Enneper representation, which determines a conformally parameterized minimal surface from two complex analytic functions.

The Weierstrass-Enneper representation from Theorem 4.1 from the *Complex functions in Geometry* problem paper is given below.

$$\boldsymbol{WE} = \begin{bmatrix} Re\int_0^z f(w)(1-g(w)^2)\,dw \\ Re\int_0^z i\,f(w)(1+g(w)^2)\,dw \\ 2\,Re\int_0^z f(w)\,g(w)\,dw \end{bmatrix} \tag{24}$$

This was the Weierstrass data given in the problem:

$$f(z) = 1 \qquad g(z) = z. \tag{25}$$

$$(f, g) = (-\frac{e^{-z}}{2}, -e^{z}) \tag{26}$$

$$(f, g) = (-i\frac{e^{-z}}{2}, -e^{z}) \tag{27}$$

We use a Maple procedure to compute the paramaterizations. We then identify which functions they are, by plotting the computed parameterizations.

Weierstrass data (a): identified as Enneper's surface

$$\boldsymbol{WE}_{f(z),g(z)}(u,v) = \begin{bmatrix} -u^3 + 3\,u\,v^2 + u \\ -3\,u^2\,v + v^3 - v \\ 2\,u^2 - 2\,v^2 \end{bmatrix}$$

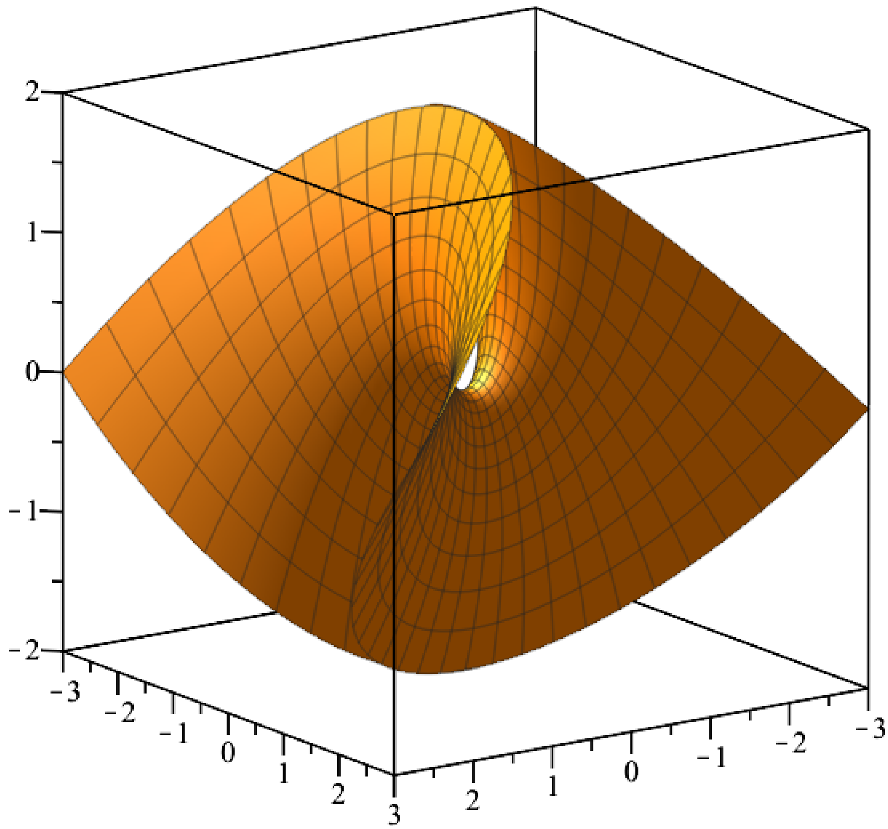


Figure 17: Weierstrass-Enneper Representation of Enneper's surface

Weierstrass data (b): identified as a Catenoid

$$\boldsymbol{WE}_{(f,g)}(u,v) = \begin{bmatrix} \dfrac{cos(v)\,(e^{-u} - e^{u})}{2} \\ -\dfrac{sin(v)\,(e^{-u} - e^{u})}{2} \\ u \end{bmatrix}$$

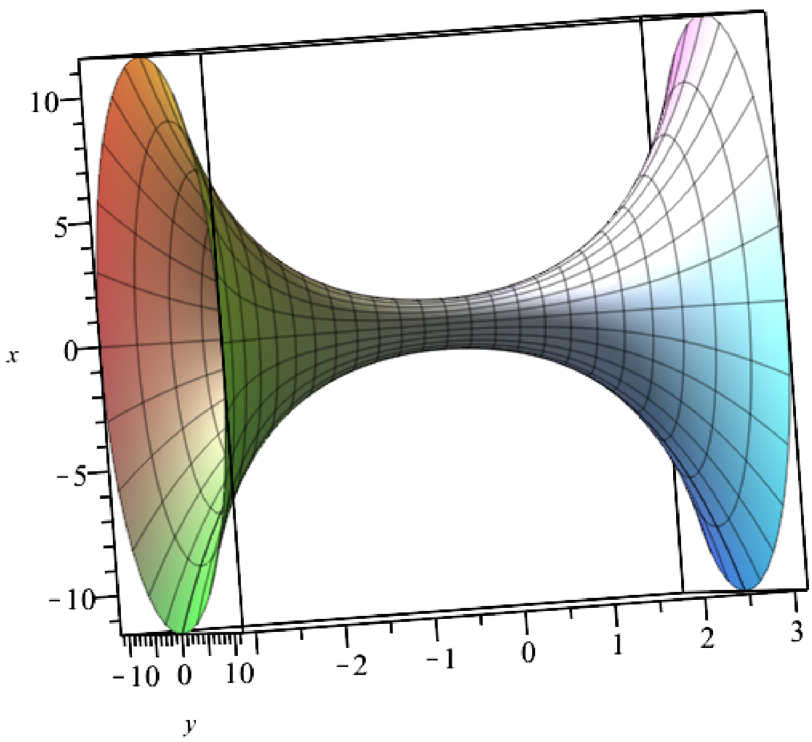


Figure 18: Weierstrass-Enneper Representation of a catenoid

Weierstrass data (c): identified as a helicoid

$$\boldsymbol{WE}_{(f,g)}(u,v) = \begin{bmatrix} \dfrac{sin(v)\,(e^{-u} - e^{u})}{2} \\ -\dfrac{cos(v)\,(e^{-u} - e^{u})}{2} \\ -v \end{bmatrix}$$

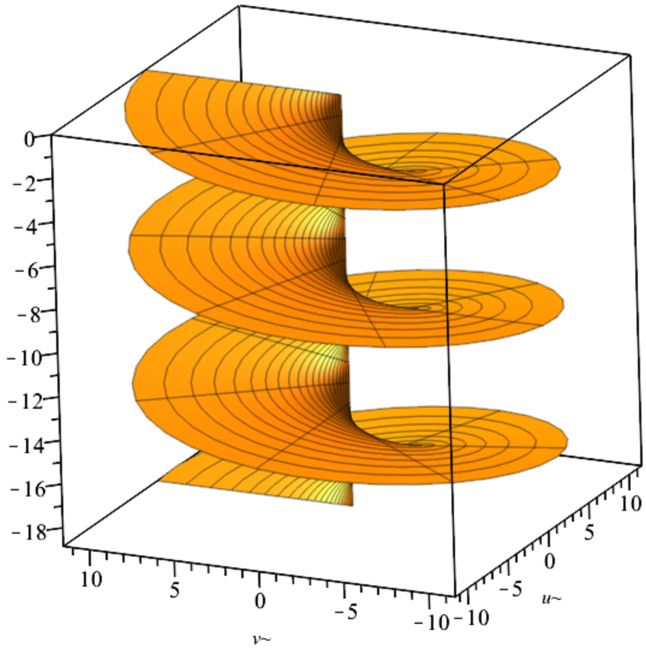


Figure 19: Weierstrass-Enneper Representation of a helicoid

2.2.5 Problem 30

With Weierstrass data it is easy to compute the Gauss Curvature. The following formula was used to do so:

$$K(f(w), g(w)) = -\frac{4\,|g'(w)|^2}{|f(w)|^2\,(1+|g(w)|^2)^4} \; where \; w = u + I\,v \tag{28}$$

We will compute the Gauss Curvature with the equation above for the Weierstrass data given in the previous equation 25 and 26.

(a) Enneper:

$$K(f(w), g(w)) = -\frac{4}{(1+u^2+v^2)^4} \tag{29}$$

Gauss curvature will always be negative for Enneper's surface.

(b) Helicoid:

$$K(f(w), g(w)) = -\frac{16\,e^{4\,u}}{(1+e^{2\,u})^4} \tag{30}$$

Gauss curvature will also always be negative for the Helicoid.

In fact, for any Weierstrass data the Gauss curvature will be non-positive. When we use the Weierstrass-Enneper representation, the surface we obtain is always a minimal surface. And since minimal surfaces have zero mean curvature, the principal curvatures are equal and opposite. This leads to the Gauss curvature always being either negative or zero for minimal surfaces, and all Weierstrass data.

3 Problem 5A

3.1 Enneper's surface is not area minimizing

In this section we will show that Enneper's surface is not area minimizing over large regions, even if we restrict the surface to a subset where it has no self-intersections.

First we confirm that Enneper's surface is a minimal surface by finding its mean curvature:

$$F_I(\boldsymbol{E}(u,v)) = \begin{bmatrix} (u^2+v^2+1)^2 & 0 \\ 0 & (u^2+v^2+1)^2 \end{bmatrix} \qquad F_{II}(\boldsymbol{E}(u,v)) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

The Weingarten matrix:

$$\boldsymbol{W}_{\boldsymbol{E}}(u,v) = F_I(\boldsymbol{E})^{-1} \cdot F_{II}(\boldsymbol{E}) = \begin{bmatrix} \dfrac{-2}{(u^2+v^2+1)^2} & 0 \\ 0 & \dfrac{2}{(u^2+v^2+1)^2} \end{bmatrix} \tag{31}$$

Since it is a diagonal matrix; the expressions in the diagonal are the eigenvalues, thus we can further deduce that these are the principal curvatures. The mean curvature is then computed to be:

$$H = \frac{1}{2} \cdot \left(\frac{-2}{(u^2+v^2+1)^2} + \frac{2}{(u^2+v^2+1)^2} \right) = 0$$

Therefore as the mean curvature is zero, we can conclude that Enneper's surface is a minimal surface.

We start by substituting our parameters by polar coordinates in the parameterization of Enneper's surface (eq.2).

$$\boldsymbol{E}_z(r,\theta) = \begin{bmatrix} -\dfrac{4\,r^3\,cos(\theta)^3}{3} + (r^2+1)\,r\,cos(\theta) \\ -\dfrac{(4\,r^2\,cos(\theta)^2 - r^2 + 3)r\,sin(\theta)}{3} \\ r^2(2\,cos(\theta)^2 - 1) \end{bmatrix} \tag{32}$$

We want to restrict the parameterization to a subset where there are no self-intersections. This is for the surfaces where $r < \sqrt{3}$. We can clearly see that in Figure 20 , where the surface for $r = \sqrt{3}$ is plotted, that there are no self intersections for smaller radii.

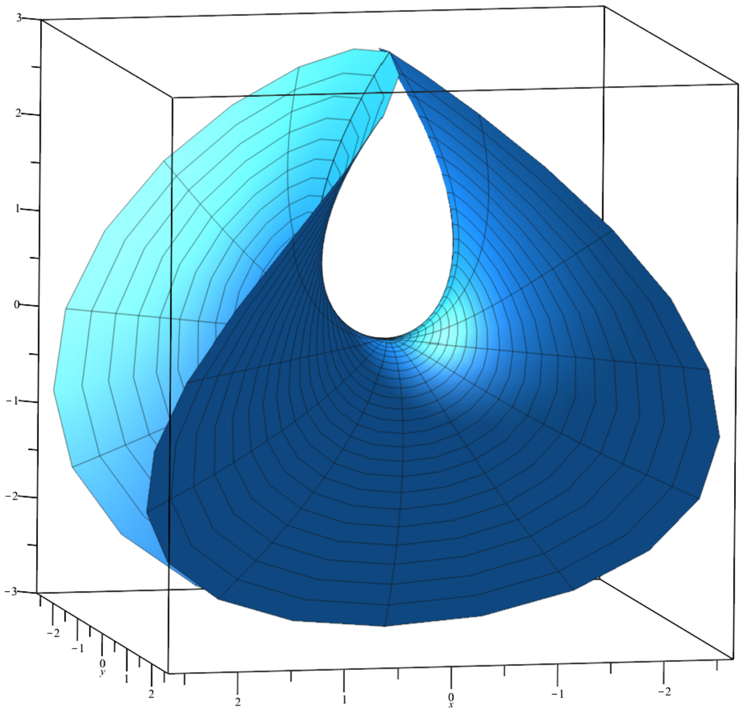


Figure 20: Enneper's surface for $r = \sqrt{3}$

We now consider the curve on Enneper's surface where $r = 1.5$, which has the parameterization:

$$\boldsymbol{p}(\theta) = \begin{bmatrix} 1.5\,cos(\theta) - 1.125\,cos(\theta)^3 + 3.375\,cos(\theta)\,sin(\theta)^2 \\ -1.5\,sin(\theta) + 1.125\,sin(\theta)^3 - 3.375\,sin(\theta)\,cos(\theta)^2 \\ 2.25\,cos(\theta)^2 - 2.25\,sin(\theta)^2 \end{bmatrix}$$

The curve is plotted together with Enneper's surface in Figure 21.

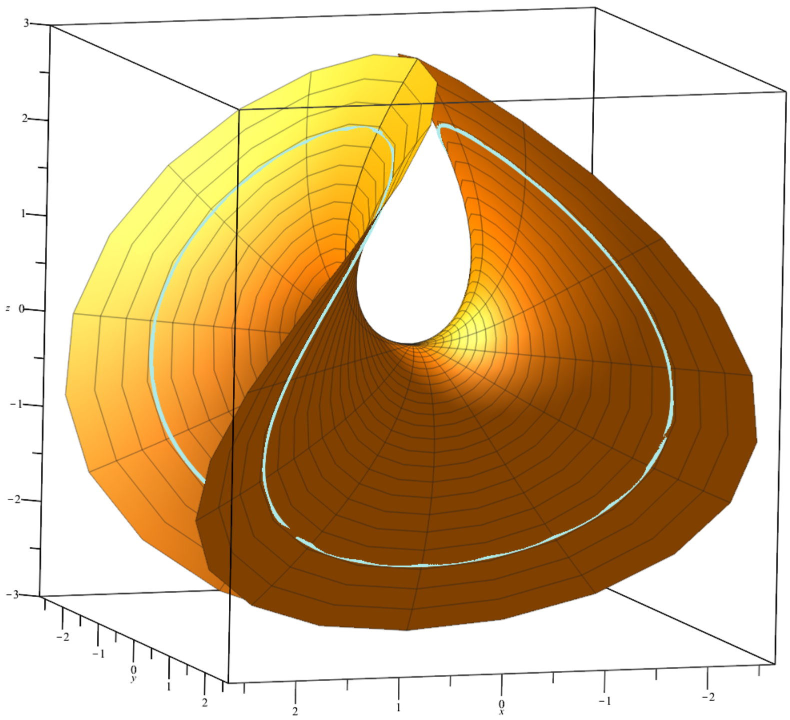


Figure 21: $E_z(r,\theta),\ r \in [0,\sqrt{3}], \theta \in [0, 2\,\pi]$
$p(\theta),\ \theta \in [0, 2\,\pi]$

This curve does not only lie on Enneper's surface, but also on a cylindrical surface. A cylindrical surface is a curve in a plane, which is extended orthogonally into the third dimension.

We create the parametric representation of the cylindrical surface by projecting the curve to the xz-plane, and adding another parameter w to extend the curve into the the y-direction.

$$\boldsymbol{C}(\theta, w) = \begin{bmatrix} 1.5 \cdot cos(\theta) - \dfrac{(1.5 \cdot cos(\theta))^3}{3} + (1.5 \cdot cos(\theta))\,(1.5 \cdot sin(\theta))^2 \\ w \\ (1.5 \cdot cos(\theta))^2 - (1.5 \cdot sin(\theta))^2 \end{bmatrix} \tag{33}$$

The cylinder and the curve are plotted together in Figure 22.

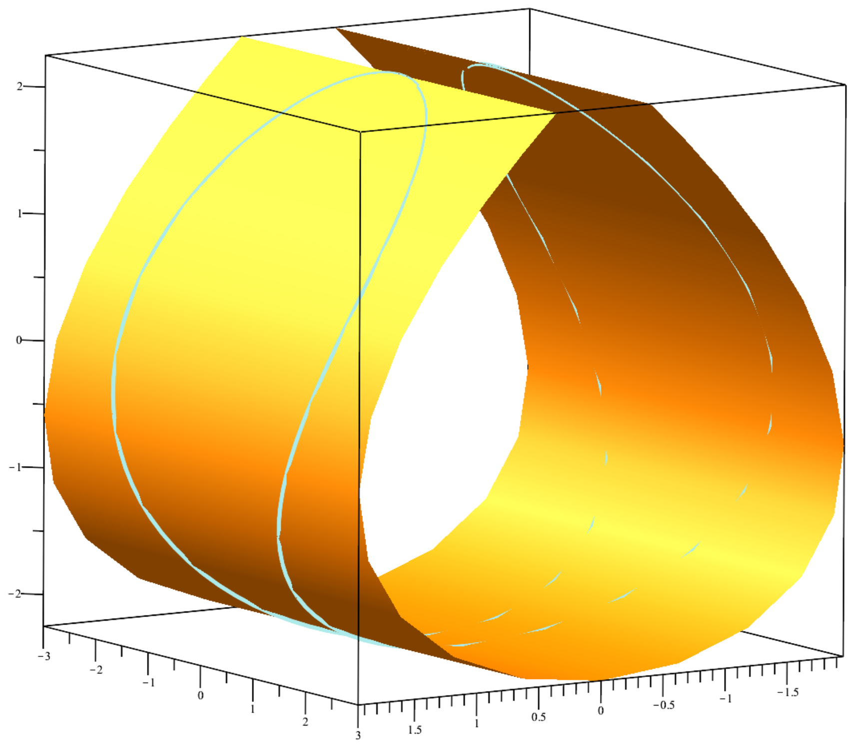


Figure 22: $C(\theta, w), \in [0, 2Pi], \in [-3, 3]$
$p(\theta),\ \theta \in [0, 2\,\pi]$

We now make a parameterizaton of the cylinder which is bounded by the curve $p(\theta)$. The new parameterization is:

$$\boldsymbol{S}(\theta, w) = \begin{bmatrix} 1.5 \cdot cos(\theta) - 1.125 \cdot cos(\theta)^3 + 3.375 \cdot cos(\theta)\ \cdot sin(\theta)^2 \\ w \cdot (-1.5 \cdot sin(\theta) - 1.125 \cdot sin(\theta)^3 - 3.375 \cdot sin(\theta) \cdot cos(\theta)^2) \\ 2.25 \cdot cos(\theta)^2 - 2.25 \cdot sin(\theta)^2 \end{bmatrix} \tag{34}$$

This surface is shown in Figure 23.

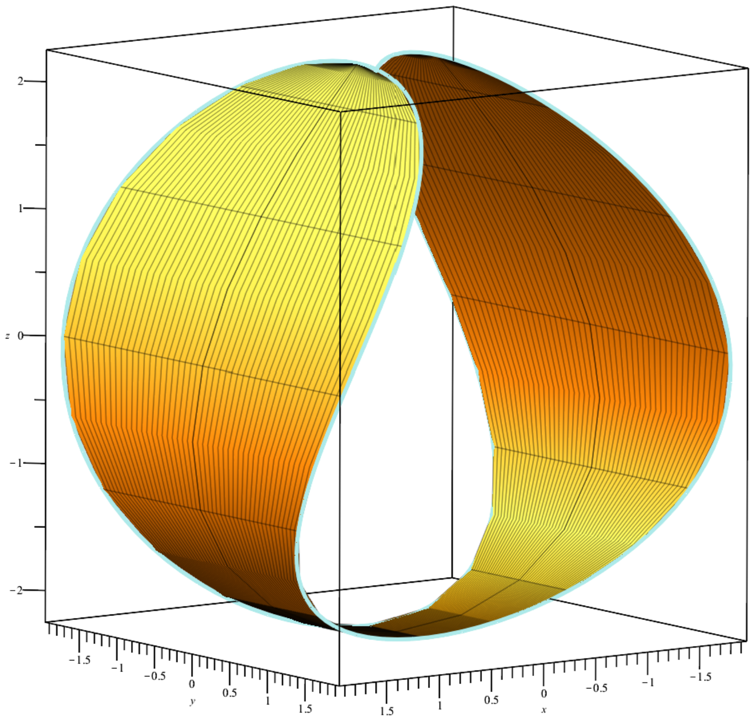


Figure 23: $S(\theta, w),\ \theta \in [0, 2\pi], w \in [0, 1]$
$p(\theta),\ \theta \in [0, 2\,\pi]$

We compare the surface area of Enneper's surface bounded by the curve (eq.33) to the surface area of the cylinder bounded by the same curve (eq.34). To determine the surface area we take the double integral of the Jacobian function. The Jacobian function for a surface is the normalized cross product of the tangent vectors.

1. **Surface area of Enneper's surface**

   The two tangent vectors:

$$\boldsymbol{E}'_{z_r} = \begin{bmatrix} cos(\theta) - r^2 cos(\theta)^3 + 3r^2 cos(\theta) sin(\theta)^2 \\ -sin(\theta) + r^2 sin(\theta)^3 - 3r^2 sin(\theta) cos(\theta)^2 \\ 2r cos(\theta)^2 - 2r sin(\theta)^2 \end{bmatrix}$$

$$\boldsymbol{E}'_{z_\theta} = \begin{bmatrix} -r sin(\theta) + 3r^3 sin(\theta) cos(\theta)^2 - r^3 sin(\theta)^3 \\ -r cos(\theta) + 3r^3 cos(\theta) sin(\theta)^2 - r^3 cos(\theta)^3 \\ -4r^2 cos(\theta) sin(\theta) \end{bmatrix}$$

   The cross product of the tangent vectors:

$$\boldsymbol{E}'_{z_r} \times \boldsymbol{E}'_{z_\theta} = \begin{bmatrix} 2\,r^2 cos(\theta)(r^2 + 1 \\ 2\,r^2\ sin(\theta)(r^2 + 1) \\ r^5 - r \end{bmatrix}$$

   The Jacobian function:

$$Jacobi_{\boldsymbol{E}_z} = \frac{\boldsymbol{E}'_{z_r} \times \boldsymbol{E}'_{z_\theta}}{|\boldsymbol{E}'_{z_r} \times \boldsymbol{E}'_{z_\theta}|} = \sqrt{|r^2(r^2+1)^4|}$$

The surface area of Enneper's surface for $r = 1.5$.

$$\int_0^{1.5} \int_0^{2\,\pi} \sqrt{r^2(r^2+1)^4}\, d\theta\, dr = 34.9011309$$

2. **Surface area of cylinder**

Tangent vectors:

$$\boldsymbol{S}'_w = \begin{bmatrix} 0 \\ -1.5 \cdot sin(\theta) + 1.125 \cdot sin(\theta)^3 - 3.375 \cdot sin(\theta) \cdot cos(\theta)^2 \\ 0 \end{bmatrix}$$

$$\boldsymbol{S}'_\theta = \begin{bmatrix} -1.5 \cdot sin(\theta) + 10.125 \cdot sin(\theta)cos(\theta)^2 - 3.375 \cdot sin(\theta)^3 \\ w(-1.5 \cdot cos(\theta) + 10.125 \cdot cos(\theta)sin(\theta)^2 - 3.375 \cdot cos(\theta)^3) \\ -9 \cdot cos(\theta)sin(\theta) \end{bmatrix}$$

Cross product of the tangent vectors:

Note that the numbers are rounded off to three significant figures, in order to fit the equations on the page.

$$\boldsymbol{S}'_\theta \times \boldsymbol{S}'_w = \begin{bmatrix} (-40.5 \cdot cos(\theta)^2 - 3.38) \cdot cos(\theta) \cdot sin(\theta)^2 \\ 0 \\ -60.8 \cdot (cos(\theta) + 0.601)(cos(\theta) - 0.601) \end{bmatrix}$$

Jacobian function for the cylinder:

$$Jacobi_{\boldsymbol{S}} = \frac{\boldsymbol{S}'_\theta \times \boldsymbol{S}'_w}{|\boldsymbol{S}'_\theta \times \boldsymbol{S}'_w|} = sin(\theta)^2 \sqrt{(3690\, cos(\theta)^8 - 410\, cos(\theta)^6 + 336\, cos(\theta)^4 + 73.1\, cos(\theta)^2 + 3.34)} \tag{35}$$

The surface area of the cylinder bounded by the curve on Enneper's surface at $r = 1.5$:

$$\int_0^{2\,\pi} \int_0^1 Jacobi_S\, dw\, d\theta = 31.66323514$$

From the two computations we can clearly see that the area of the cylinder is smaller than the area of Enneper's surface. Here we have shown that Enneper's surface is not always area minimizing.

Let's now restrict our domain to a smaller subset, and calculate the area of both surfaces again.

The new domain is: $r \in [0, 1]$ and $\theta \in [0, 2\pi]$. We use the same method as above and find the following areas:

Area of Enneper's surface: 7.330382858

Area of cylinder: 8.060149286

For $r = 1$ Enneper's surface is in fact area minimizing. We can conclude that Enneper's surface is locally area minimizing - nearby surfaces with the same boundary curve are larger, if we restrict to a sufficiently small subset.

4 Problem 5B

4.1 Associated families of minimal surfaces

Given a minimal surface X, there is an associated family X_t of minimal surfaces with $t \in [0, 2\pi]$ given by the Weierstrass data $(e^{it}f, g)$. Families of associated minimal surfaces are related by the same first fundamental form and Gauss curvature for any t.

Since Weierstrass data comes in a conformal parameterization, the first fundamental form looks like:

$$F_I = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$$

where E can be calculated by the following formula:

$$E = |f|^2 \left(1 + |g|^2\right)^2 \tag{36}$$

In a similar way, the Gaussian curvature can be easily computed with:

$$K = -\frac{4|g'|^2}{|f|\left(1 + |g|^2\right)^4} \tag{37}$$

Thus with $(e^{it}f, g)$ the first fundamental form can be computed in the following way:

First, by calculating the E component with equation 36:

$$E = |\mathrm{e}^{\mathrm{I}\cdot t} \cdot f|^2 \cdot \left(1 + |g|^2\right)$$

And then by placing it in the diagonal of the first fundamental form matrix :

$$\begin{bmatrix} |f \cdot e^{It}|^2 \cdot \left(1 + |g|^2\right)^2 & 0 \\ 0 & |f \cdot e^{It}|^2 \cdot \left(1 + |g|^2\right)^2 \end{bmatrix}$$

Using equation 37, Gaussian curvature is simply:

$$K = -\frac{4|g'|^2}{|f \cdot e^{It}|\left(1 + |g|^2\right)^4}$$

Due to these properties of the Weierstrass data,—the ease by which geometric properties are calculated—one can easily see how these surfaces are in fact related.

For instance, the surfaces X_0 X_π and $X_{2\pi}$ are associated by a surface family X_t. We can show this by computing their respective first fundamental forms:

We can use the formula of E that we previously calculated for the given Weierstrass data $(e^{it}f, g)$ $E(t) = |\mathrm{e}^{\mathrm{I}\cdot t} \cdot f|^2 \cdot \left(1 + |g|^2\right)$ as well as the Gauss Curvature expression $K = -\frac{4|g'|^2}{|f \cdot e^{It}|\left(1+|g|^2\right)^4}$

X_0 If $t = 0$ then, the first fundamental form is:

$$E(0) = |f \cdot e^{0t}|^2 \left(1 + |g|^2\right)$$

$$E(0) = |f|^2 \left(1 + |g|^2\right)$$

$$F_I = \begin{bmatrix} |f|^2 \left(1 + |g|^2\right) & 0 \\ 0 & |f|^2 \left(1 + |g|^2\right) \end{bmatrix}$$

and the Gaussian Curvature:

$$K(0) = -\frac{4|g|^2}{|f|\left(1 + |g|^2\right)^4}$$

X_π If $t = \pi$ then, the first fundamental form is:

$$E(\pi) = |f \cdot \mathrm{e}^{\mathrm{I}\pi}|^2 \left(1 + |g|^2\right)$$

$$E(\pi) = |f \cdot -1|^2 \left(1 + |g|^2\right)$$

Because the -1 is inside the absolute value it essentially becomes:

$$E(\pi) = |f|^2 \left(1 + |g|^2\right)$$

$$F_I = \begin{bmatrix} |f|^2 \left(1 + |g|^2\right) & 0 \\ 0 & |f|^2 \left(1 + |g|^2\right) \end{bmatrix}$$

and the Gaussian Curvature:

$$K(\pi) = -\frac{4|g|^2}{|f \cdot \mathrm{e}^{\mathrm{I}\pi}|\left(1 + |g|^2\right)^4}$$

$$K(\pi) = -\frac{4|g|^2}{|f \cdot -1|\left(1 + |g|^2\right)^4}$$

$$K(\pi) = -\frac{4|g|^2}{|f|\left(1 + |g|^2\right)^4}$$

$X_{2\pi}$ If $t = 2\pi$ then, the first fundamental form is:

$$E(2\pi) = |f \cdot \mathrm{e}^{\mathrm{I}2\pi}|^2 \left(1 + |g|^2\right)$$

$$E(2\pi) = |f \cdot 1|^2 \left(1 + |g|^2\right)$$

$$E(2\pi) = |f|^2 \left(1 + |g|^2\right)$$

$$F_I = \begin{bmatrix} |f|^2 \left(1 + |g|^2\right) & 0 \\ 0 & |f|^2 \left(1 + |g|^2\right) \end{bmatrix}$$

and the Gaussian Curvature:

$$K(2\pi) = -\frac{4|g|^2}{|f \cdot \mathrm{e}^{\mathrm{I}2\pi}|\left(1+|g|^2\right)^4}$$

$$K(2\pi) = -\frac{4|g|^2}{|f \cdot 1|\left(1+|g|^2\right)^4}$$

$$K(2\pi) = -\frac{4|g|^2}{|f|\left(1+|g|^2\right)^4}$$

As we can see, the Gauss Curvature and first fundamental forms are the same. Thus, these surfaces X_t form a family of minimal surfaces.

We can show how these families are related in space by plotting some example of families of surfaces on Maple with the aid of the Maple starter sheet:

Figure 24 shows Enneper's surface using Weierstrass data. Note that it is colored by its Gauss curvature:

$$f(z) = e^{I\,t}, g(z) = z$$

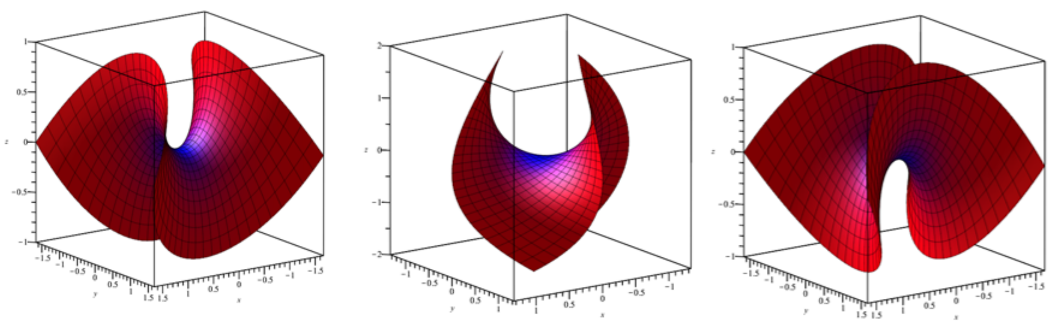


Figure 24: From left to right: when $t = 0$, $t = \frac{\pi}{2}$ and $t = \pi$

To view the animation of the above figure please follow this link.

Additionally, Figure 25 shows Bour's surface also given by Weierstrass data found on Wolfram Mathworld and colored by Gauss curvature:

$$f(z) = e^{I\,t}, g(z) = \sqrt{z}$$

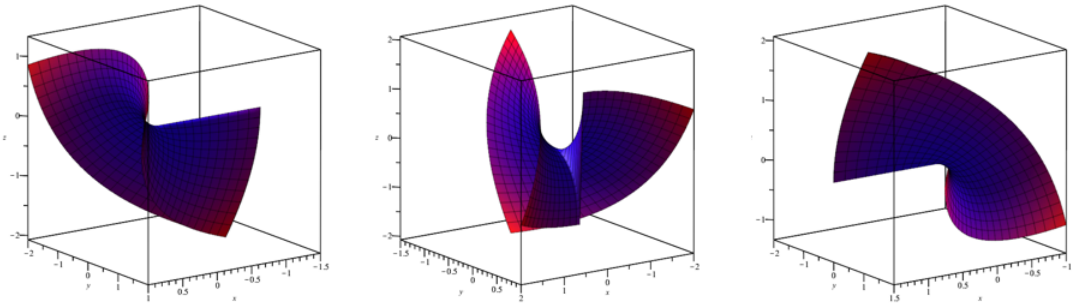


Figure 25: From left to right: when $t = 0$, $t = \frac{\pi}{2}$ and $t = \pi$

To view the animation of the above figure please follow this link.

List of Figures

5 Appendix A

7 *

$restart; with(plots):$

$h(u, v) := \langle u, v^2, u \cdot v^3 \rangle$

$$h := (u, v) \mapsto \langle u, v^2, u \cdot v^3 \rangle \tag{1.1}$$

$plot3d([u, v^2, u \cdot v^3], u = -1..1, v = -1..1):$

$\frac{\partial}{\partial u}(\underline{h}(\underline{u, v}))$

$$\begin{bmatrix} 1 \\ 0 \\ v^3 \end{bmatrix} \tag{1.2}$$

$\frac{\partial}{\partial v}(\underline{h}(\underline{u, v}))$

$$\begin{bmatrix} 0 \\ 2\,v \\ 3\,u\,v^2 \end{bmatrix} \tag{1.3}$$

$cp := (u, v) \rightarrow CrossProduct(\langle 1, 0, v^3 \rangle, \langle 0, 2 \cdot v, 3 \cdot u \cdot v^2 \rangle)$

$$cp := (u, v) \mapsto CrossProduct(\langle 1, 0, v^3 \rangle, \langle 0, 2 \cdot v, 3 \cdot u \cdot v^2 \rangle) \tag{1.4}$$

$cp(u, v)$

$$\begin{bmatrix} -2\,v^4 \\ -3\,u\,v^2 \\ 2\,v \end{bmatrix} \tag{1.5}$$

$cp(u, 0)$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.6}$$

$plt1 := plot3d([u, v^2, u \cdot v^3], u = -1..1, v = -1..1):$
$plt2 := plot3d([u, 0, 0], u = -1..1, thickness = 5, color = blue):$
$display(plt1, plt2):$

9a *

$restart; with(plots):$

$E(u, v) := \left\langle u - \frac{u^3}{3} + u \cdot v^2, -v + \frac{v^3}{3} - v \cdot u^2, u^2 - v^2 \right\rangle$

$$E := (u, v) \mapsto \left\langle u - \frac{1}{3} \cdot u^3 + u \cdot v^2, -v + \frac{1}{3} \cdot v^3 - v \cdot u^2, u^2 - v^2 \right\rangle \quad \textbf{(2.1)}$$

$plot3d([E(u, v)], u = -2..2, v = -2..2):$

9b *

$restart; with(LinearAlgebra):$

$E(u, v) := \left\langle u - \frac{u^3}{3} + u \cdot v^2, -v + \frac{v^3}{3} - v \cdot u^2, u^2 - v^2 \right\rangle$

$$E := (u, v) \mapsto \left\langle u - \frac{1}{3} \cdot u^3 + u \cdot v^2, -v + \frac{1}{3} \cdot v^3 - v \cdot u^2, u^2 - v^2 \right\rangle \quad \textbf{(3.1)}$$

$solve\left(u - \frac{u^3}{3} + u \cdot v^2 = 0, \{u, v\}\right)$

$$\{u = 0, v = v\}, \left\{u = \sqrt{3\,v^2 + 3}, v = v\right\}, \left\{u = -\sqrt{3\,v^2 + 3}, v = v\right\} \quad \textbf{(3.2)}$$

$E(0, v)$

$$\begin{bmatrix} 0 \\ -v + \frac{1}{3}\,v^3 \\ -v^2 \end{bmatrix} \quad \textbf{(3.3)}$$

$display\left(plot\left(\left[-v + \frac{1}{3}\,v^3, -v^2, v = -10..10\right]\right), view = [-4..4, -4..4]\right):$

$solve\left(-v + \frac{1}{3}\,v^3 = 0\right)$

$$0, \sqrt{3}, -\sqrt{3} \quad \textbf{(3.4)}$$

$solve(-v^2 = -3)$

$$-\sqrt{3}, \sqrt{3} \quad \textbf{(3.5)}$$

$E(0, -\mathrm{sqrt}(3))$

$$\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \quad \textbf{(3.6)}$$

$E(0, \mathrm{sqrt}(3))$

$$\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \quad \textbf{(3.7)}$$

9c *

$restart; with(LinearAlgebra):$

$$E(u, v) := \left\langle u - \frac{u^3}{3} + u \cdot v^2, -v + \frac{v^3}{3} - v \cdot u^2, u^2 - v^2 \right\rangle$$

$$E := (u, v) \mapsto \left\langle u - \frac{1}{3} \cdot u^3 + u \cdot v^2, -v + \frac{1}{3} \cdot v^3 - v \cdot u^2, u^2 - v^2 \right\rangle \tag{4.1}$$

$$\frac{\partial}{\partial u} (E(u, v))$$

$$\begin{bmatrix} -u^2 + v^2 + 1 \\ -2\, v\, u \\ 2\, u \end{bmatrix} \tag{4.2}$$

$$\frac{\partial}{\partial v} (E(u, v))$$

$$\begin{bmatrix} 2\, v\, u \\ -u^2 + v^2 - 1 \\ -2\, v \end{bmatrix} \tag{4.3}$$

$$cp := (u, v) \rightarrow simplify\left(CrossProduct\left(\frac{\partial}{\partial u} (E(u, v)), \frac{\partial}{\partial v} (E(u, v)) \right) \right)$$

$$cp := (u, v) \rightarrow simplify\left(LinearAlgebra\text{:-}CrossProduct\left(\frac{\partial}{\partial u} E(u, v), \frac{\partial}{\partial v} E(u, v) \right) \right) \tag{4.4}$$

$$solve\left(\{2\, u^3 + 2\, u\, v^2 + 2\, u = 0, 2\, v\, u^2 + 2\, v^3 + 2\, v = 0, u^4 + 2\, v^2\, u^2 + v^4 - 1 = 0\}, \{u, v\} \right)$$

$$\{u = u, v = RootOf(\_Z^2 + u^2 + 1)\}, \{u = RootOf(\_Z^2 + 1), v = 0\} \tag{4.5}$$

11 *

$restart;\ with(plots):$

$$f(u, v) := \langle \cos(u) \cdot \cos(v), \sin(u) \cdot \cos(v), \sin(v) \rangle$$

$$f := (u, v) \mapsto \langle \cos(u) \cdot \cos(v), \sin(u) \cdot \cos(v), \sin(v) \rangle \tag{5.1}$$

$$\frac{\partial}{\partial u} (f(u, v))$$

$$\begin{bmatrix} -\sin(u)\, \cos(v) \\ \cos(u)\, \cos(v) \\ 0 \end{bmatrix} \tag{5.2}$$

$$\frac{\partial}{\partial v} (f(u, v))$$

$$\begin{bmatrix} -\cos(u)\, \sin(v) \\ -\sin(u)\, \sin(v) \\ \cos(v) \end{bmatrix} \tag{5.3}$$

$with(VectorCalculus):$

$cp := (u, v) \rightarrow CrossProduct(\langle -\sin(u)\cdot\cos(v), \cos(u)\cdot\cos(v), 0\rangle, \langle -\cos(u)\cdot\sin(v), -\sin(u)\cdot\sin(v), \cos(v)\rangle) :$

$cp(u, v)$ assuming *real*

$$\left(\cos(u)\cos(v)^2\right)e_x + \left(\sin(u)\cos(v)^2\right)e_y + \left(\sin(u)^2\cos(v)\sin(v) + \cos(u)^2\cos(v)\sin(v)\right)e_z \quad \mathbf{(5.4)}$$

$cp(u, v) := simplify(cp(u, v))$

$$cp(u, v) := \left(\cos(u)\cos(v)^2\right)e_x + \left(\sin(u)\cos(v)^2\right)e_y + \left(\sin(v)\cos(v)\right)e_z \quad \mathbf{(5.5)}$$

$un := (u, v) \rightarrow simplify\left(\frac{cp(u, v)}{Norm(cp(u, v), 2)}\right)$

$$un := (u, v) \mapsto simplify\left(cp(u, v)\cdot Norm(cp(u, v), 2)^{-1}\right) \quad \mathbf{(5.6)}$$

$un(u, v)$ assuming *real*

$$\left(\cos(u)\,|\cos(v)|\right)e_x + \left(\sin(u)\,|\cos(v)|\right)e_y + \left(\frac{\cos(v)\sin(v)}{|\cos(v)|}\right)e_z \quad \mathbf{(5.7)}$$

$f(u, v) + cp(u, v)$

$$\left(\cos(u)\cos(v) + \cos(u)\cos(v)^2\right)e_x + \left(\sin(u)\cos(v) + \sin(u)\cos(v)^2\right)e_y + \left(\sin(v) + \sin(v)\cos(v)\right)e_z \quad \mathbf{(5.8)}$$

$simplify(\%)$

$$\left(\cos(u)\cos(v)\left(\cos(v) + 1\right)\right)e_x + \left(\sin(u)\cos(v)\left(\cos(v) + 1\right)\right)e_y + \left(\sin(v)\left(\cos(v) + 1\right)\right)e_z \quad \mathbf{(5.9)}$$

12 *

$f(u, v) := \langle\cos(u)\cdot\cos(v), \sin(u)\cdot\cos(v), \sin(v)\rangle$

$$f := (u, v) \mapsto \langle\cos(u)\cdot\cos(v), \sin(u)\cdot\cos(v), \sin(v)\rangle \quad \mathbf{(6.1)}$$

$f(\mathrm{Pi}, \mathrm{Pi})$

$$(1)e_x + (0)e_y + (0)e_z \quad \mathbf{(6.2)}$$

$un(\mathrm{Pi}, \mathrm{Pi})$

$$(-1)e_x + (0)e_y + (0)e_z \quad \mathbf{(6.3)}$$

$f(\mathrm{Pi}, \mathrm{Pi}) + un(\mathrm{Pi}, \mathrm{Pi})$

$$(0)e_x + (0)e_y + (0)e_z \quad \mathbf{(6.4)}$$

$f(u, v)$

$$\left(\cos(u)\cos(v)\right)e_x + \left(\sin(u)\cos(v)\right)e_y + \left(\sin(v)\right)e_z \quad \mathbf{(6.5)}$$

$un(u, v)$

$$\left(\mathrm{csgn}(\cos(v))\cos(v)\cos(u)\right)e_x + \left(\mathrm{csgn}(\cos(v))\cos(v)\sin(u)\right)e_y + \left(\mathrm{csgn}(\cos(v))\sin(v)\right)e_z \quad \mathbf{(6.6)}$$

$f(0, 0)$

$$(1)e_x + (0)e_y + (0)e_z \quad \mathbf{(6.7)}$$

$ad := (u, v) \rightarrow f(u, v) + un(u, v)$

$$ad := (u, v) \mapsto f(u, v) + un(u, v) \tag{6.8}$$

$ad(u, v)$ assuming *real*

$$(\cos(u)\cos(v) + \cos(u)\,|\cos(v)|)e_x + (\sin(u)\cos(v) + \sin(u)\,|\cos(v)|)e_y + \left(\sin(v) + \frac{\cos(v)\sin(v)}{|\cos(v)|}\right)e_z \tag{6.9}$$

$ad\left(u, \frac{\mathrm{Pi}}{2}\right)$

Error, (in un) numeric exception: division by zero

$un\left(u, -\frac{\mathrm{Pi}}{2}\right)$

Error, (in un) numeric exception: division by zero

$sphere := plot3d\left([f(u, v)], u = 0 .. 2\cdot \mathrm{Pi}, v = -\frac{\mathrm{Pi}}{2} .. \frac{\mathrm{Pi}}{2}, numpoints = 40000\right):$

$unitnormal := plot3d\left([ad(u, v)], u = 0 .. 2\cdot \mathrm{Pi}, v = -\frac{\mathrm{Pi}}{2} .. \frac{\mathrm{Pi}}{2}, style = wireframe, numpoints = 40000\right):$

$display(sphere, unitnormal):$

16 *

Initialisation

restart; *with*(*plots*) :; *with*(*LinearAlgebra*) :
assume(*u*, *real*);
assume(*v*, *real*);
first := **proc**(*f*) **description** "calculate the first fundamental form of the paramaterized surface f(u,v)"; *Matrix*([[*DotProduct*(*Vector*(*diff*(*f*, *u*)), *Vector*(*diff*(*f*, *u*))), *DotProduct*(*Vector*(*diff*(*f*, *u*)), *Vector*(*diff*(*f*, *v*)))], [*DotProduct*(*Vector*(*diff*(*f*, *u*)), *Vector*(*diff*(*f*, *v*))), *DotProduct*(*Vector*(*diff*(*f*, *v*)), *Vector*(*diff*(*f*, *v*)))]]); **end proc**:
second := **proc**(*f*, *N*)
description "calculate the second fundamental form of the paramaterized surface f(u,v), with unit normal N(u,v)"; *Matrix*([[*DotProduct*(*Vector*(3, *diff*(*f*, *u*, *u*)), *N*), *DotProduct*(*Vector*(3, *diff*(*f*, *u*, *v*)), *N*)], [*DotProduct*(*Vector*(3, *diff*(*f*, *u*, *v*)), *N*), *DotProduct*(*Vector*(3, *diff*(*f*, *v*, *v*)), *N*)]]); **end proc**:
calcnormal := **proc**(*f*) **description** "calculate the unit normal of the paramaterized surface f(u,v)"; *CrossProduct*(*Vector*(3, *diff*(*f*, *u*)), *Vector*(3, *diff*(*f*, *v*))) / *Norm*(*CrossProduct*(*Vector*(3, *diff*(*f*, *u*)), *Vector*(3, *diff*(*f*, *v*))), 2); **end proc**:

$r := (u, v) \rightarrow \langle u, \cosh(u)\cdot\cos(v), \cosh(u)\cdot\sin(v)\rangle$

$$r := (u, v) \mapsto \langle u, \cosh(u)\cdot\cos(v), \cosh(u)\cdot\sin(v)\rangle \tag{7.1}$$

$rfirst := (u, v) \rightarrow simplify(first(r(u, v)))$

$$rfirst := (u, v) \mapsto simplify(first(r(u, v))) \tag{7.2}$$

$rfirst(u, v)$

$$\begin{bmatrix} \cosh(u\sim)^2 & 0 \\ 0 & \cosh(u\sim)^2 \end{bmatrix} \tag{7.3}$$

$un := (u, v) \rightarrow calcnormal(r(u, v))$

$$un := (u, v) \mapsto calcnormal(r(u, v)) \tag{7.4}$$

$simplify(un(u, v))$

$$\begin{bmatrix} \frac{\sinh(u\sim)}{\cosh(u\sim)} \\ -\frac{\cos(v\sim)}{\cosh(u\sim)} \\ -\frac{\sin(v\sim)}{\cosh(u\sim)} \end{bmatrix} \tag{7.5}$$

$rsecond := (u, v) \rightarrow simplify(second(r(u, v), un(u, v)))$

$$rsecond := (u, v) \mapsto simplify(second(r(u, v), un(u, v))) \tag{7.6}$$

$rsecond(u, v)$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.7}$$

$Weinr := (u, v) \rightarrow simplify(MatrixInverse(rfirst(u, v))).(rsecond(u, v)):$
$Weinr(u, v)$

$$\begin{bmatrix} -\frac{1}{\cosh(u\sim)^2} & 0 \\ 0 & \frac{1}{\cosh(u\sim)^2} \end{bmatrix} \tag{7.8}$$

$restart; with(LinearAlgebra):$

$Weinr(u, v) := \left\langle -\frac{1}{\cosh(v)^2}, 0; 0, \frac{1}{\cosh(v)^2} \right\rangle$

$$Weinr := (u, v) \mapsto \left\langle \left\langle -\frac{1}{\cosh(v)^2} \middle| 0 \right\rangle, \left\langle 0 \middle| \frac{1}{\cosh(v)^2} \right\rangle \right\rangle \tag{7.9}$$

$Eigenvalues(Weinr(u, v), output = list)$ assuming $real$

$$\left[-\frac{4\left(e^v\right)^2}{\left(e^v\right)^4 + 2\left(e^v\right)^2 + 1}, \frac{4\left(e^v\right)^2}{\left(e^v\right)^4 + 2\left(e^v\right)^2 + 1} \right] \tag{7.10}$$

$with(plots):$

$c := (u, v) \rightarrow \langle u, \cosh(u) \cdot \cos(v), \cosh(u) \cdot \sin(v) \rangle$

$$c := (u, v) \mapsto \langle u, \cosh(u)\cdot\cos(v), \cosh(u)\cdot\sin(v)\rangle \tag{7.11}$$

17 *

$\langle 1, 0\rangle$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.1}$$

$\langle 0, 1\rangle$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8.2}$$

```
u1 := -2 :
u2 := 2 :
v1 := 0 :
v2 := 2·Pi :
plottorus := plot3d(c(u, v), u = u1 .. u2, v = v1 .. v2, grid = [50, 100], scaling = constrained, colour
    = magenta) :
ucurve1 := spacecurve(c(u, 0), u = u1 .. u2, color = green, thickness = 5) :
ucurve2 := spacecurve(c(u, Pi/2), u = u1 .. u2, color = green, thickness = 5) :
ucurve3 := spacecurve(c(u, Pi), u = u1 .. u2, color = green, thickness = 5) :
vcurve1 := spacecurve(c(0, v), v = v1 .. v2, color = red, thickness = 5) :
vcurve2 := spacecurve(c(Pi/2, v), v = v1 .. v2, color = red, thickness = 5) :
vcurve3 := spacecurve(c(Pi, v), v = v1 .. v2, color = red, thickness = 5) :
display(plottorus, ucurve1, ucurve3, vcurve1, vcurve2) :
```

19 *

```
restart; with(plots) :
r := (u, v) → ⟨v, cosh(v) * cos(u), cosh(v) * sin(u)⟩
```

$$r := (u, v) \mapsto \langle v, \cosh(v)\cdot\cos(u), \cosh(v)\cdot\sin(u)\rangle \tag{9.1}$$

$K := -\dfrac{1}{\cosh(v)^4}$

$$K := -\frac{1}{\cosh(v)^4} \tag{9.2}$$

```
plot(K) :
plot3d(r(u, v), u = 0 ..2 * Pi, v =-2 * Pi/3 ..2 * Pi/3, color = [K + 1, 0, -K]) :
```

21 *

$g(u, v) := \langle u^2, u^3, v\rangle$

$$g := (u, v) \mapsto \langle u^2, u^3, v\rangle \tag{10.1}$$

$first(g(u, v))$

$$\begin{bmatrix} 9\,u\sim^4 + 4\,u\sim^2 & 0 \\ 0 & 1 \end{bmatrix} \tag{10.2}$$

$simplify(calcnormal(g(u, v)))$

$$\begin{bmatrix} \dfrac{3\,|u\sim|}{\sqrt{9\,u\sim^2 + 4}} \\ -\dfrac{2\,\mathrm{signum}(u\sim)}{\sqrt{9\,u\sim^2 + 4}} \\ 0 \end{bmatrix} \tag{10.3}$$

$simplify(second(g(u, v), simplify(calcnormal(g(u, v)))))$

$$\begin{bmatrix} -\dfrac{6\,|u\sim|}{\sqrt{9\,u\sim^2 + 4}} & 0 \\ 0 & 0 \end{bmatrix} \tag{10.4}$$

$Weing := (u, v) \rightarrow MatrixInverse(\,first(g(u, v))) \cdot second(g(u, v), calcnormal(g(u, v)))$

$$Weing := (u, v) \rightarrow Typesetting\text{:-}delayDotProduct(MatrixInverse(\,first(g(u, v))), second(g(u, v), calcnormal(g(u, v)))) \tag{10.5}$$

$Weing(u, v)$

$$\begin{bmatrix} -\dfrac{6}{(9\,u\sim^2 + 4)\,\sqrt{9\,u\sim^4 + 4\,|u\sim|^2}} & 0 \\ 0 & 0 \end{bmatrix} \tag{10.6}$$

$simplify(Eigenvalues(Weing(u, v)))$

$$\begin{bmatrix} -\dfrac{6}{(9\,u\sim^2 + 4)^{3/2}\,|u\sim|} \\ 0 \end{bmatrix} \tag{10.7}$$

$k1 := 0$

$$k1 := 0 \tag{10.8}$$

$k2 := -\dfrac{6}{(9\,u\sim^2 + 4)^{3/2}\,|u\sim|}$

$$k2 := -\frac{6}{(9\,u\sim^2 + 4)^{3/2}\,|u\sim|} \tag{10.9}$$

$K := k1 \cdot k2$

$$K := 0 \tag{10.10}$$

$H := u \rightarrow \dfrac{(k1 + k2)}{2}$

$$H := u \mapsto \frac{k1}{2} + \frac{k2}{2} \qquad \textbf{(10.11)}$$

simplify$(H(u))$

$$-\frac{3}{\left(9\,u\sim^{2}+4\right)^{3/2}\,|u\sim|} \qquad \textbf{(10.12)}$$

limit$(H(u), u=0)$

$$-\infty \qquad \textbf{(10.13)}$$

22 *

$t := (u, v) \rightarrow \langle (2+\cos(u))\cdot\cos(v), (2+\cos(u))\cdot\sin(v), \sin(u)\rangle$

$$t := (u, v) \mapsto \langle (2+\cos(u))\cdot\cos(v), (2+\cos(u))\cdot\sin(v), \sin(u)\rangle \qquad \textbf{(11.1)}$$

$\mathit{Weint} := (u, v)\rightarrow \mathit{first}(t(u, v)) \bullet \mathit{second}(t(u, v), \mathit{calcnormal}(t(u, v)))$

$\mathit{Weint} := (u, v)\rightarrow \mathit{Typesetting{:}\text{-}delayDotProduct}(\mathit{first}(t(u, v)), \mathit{second}(t(u, v), \mathit{calcnormal}(t(u, v))))$ **(11.2)**

simplify$(\mathit{Weint}(u, v))$

$$\begin{bmatrix} 1 & 0 \\ 0 & (2+\cos(u\sim))^{3}\cos(u\sim) \end{bmatrix} \qquad \textbf{(11.3)}$$

$k1 := 1$

$$k1 := 1 \qquad \textbf{(11.4)}$$

$k2 := (2+\cos(u\sim))^{3}\cos(u\sim)$

$$k2 := (2+\cos(u\sim))^{3}\cos(u\sim) \qquad \textbf{(11.5)}$$

$H := \frac{(k1+k2)}{2}$

$$H := \frac{1}{2} + \frac{(2+\cos(u\sim))^{3}\cos(u\sim)}{2} \qquad \textbf{(11.6)}$$

plot3d$([(2+\cos(u))\cdot\cos(v), (2+\cos(u))\cdot\sin(v), \sin(u)], u=0..2\cdot\mathrm{Pi}, v=0..2\cdot\mathrm{Pi}, \mathit{scaling} = \mathit{constrained})$:

plot3d$(\langle u, u\hat{}2, v\rangle, u=-1..1, v=-1..1)$:

$f := (u, v)\rightarrow\langle u, u\hat{}2, v\rangle$;

$N := (u, v) \rightarrow \mathit{calcnormal}(f(u, v))$;
simplify$(N(u, v))$;

$\mathit{first2} := (u, v) \rightarrow \mathit{first}(f(u, v))$;

simplify$(\mathit{first2}(u, v))$;

$second2 := (u, v) \to second(f(u, v), N(u, v));$

$simplify(second2(u, v));$

$Weinf2 := (u, v) \to (MatrixInverse(first2(u, v))) . (second2(u, v));$

$simplify(Weinf2(u, v));$

$Eigenvalues(Weinf2(u, v), output = list);$

$H1 := (-2 * \mathrm{sqrt}(1 + 4 * \mathrm{abs}(u)\text{^}2) / (16 * \mathrm{abs}(u)\text{^}2 * conjugate(u) * u + 4 * \mathrm{abs}(u)\text{^}2 + 4 * conjugate(u) * u + 1)) * (1/2);$

$simplify(\%) :$

$plot(H1(u), u = -1 ..1) :$

$$f := (u, v) \mapsto \langle u, u^2, v \rangle$$

$$N := (u, v) \mapsto calcnormal(f(u, v))$$

$$\begin{bmatrix} \dfrac{2\, u\sim}{\sqrt{4\, u\sim^2 + 1}} \\ -\dfrac{1}{\sqrt{4\, u\sim^2 + 1}} \\ 0 \end{bmatrix}$$

$$first2 := (u, v) \mapsto first(f(u, v))$$

$$\begin{bmatrix} 4\, u\sim^2 + 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$second2 := (u, v) \mapsto second(f(u, v), N(u, v))$$

$$\begin{bmatrix} -\dfrac{2}{\sqrt{4\, u\sim^2 + 1}} & 0 \\ 0 & 0 \end{bmatrix}$$

$$Weinf2 := (u, v) \to Typesetting\text{:-}delayDotProduct(MatrixInverse(first2(u, v)), second2(u, v))$$

$$\begin{bmatrix} -\dfrac{2}{\left(4\, u\sim^2 + 1\right)^{3/2}} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[0, -\frac{2\sqrt{1+4|u\sim|^2}}{16|u\sim|^2 u\sim^2 + 4|u\sim|^2 + 4u\sim^2 + 1}\right]$$

$$H1 := -\frac{\sqrt{1+4|u\sim|^2}}{16|u\sim|^2 u\sim^2 + 4|u\sim|^2 + 4u\sim^2 + 1} \tag{11.7}$$

24 *

(a) Henneberg's surface

$$H(u, v) := \left\langle 2\cdot\sinh(u)\cdot\cos(v) - \frac{2}{3}\cdot\sinh(3\cdot u)\cdot\cos(3\cdot v), 2\cdot\sinh(u)\cdot\sin(v) + \frac{2}{3}\cdot\sinh(3\cdot u)\cdot\sin(3\cdot v), 2\cdot\cosh(2\cdot u)\cdot\cos(2\cdot v)\right\rangle$$

$$H := (u, v) \mapsto \left\langle 2\cdot\sinh(u)\cdot\cos(v) - \frac{2\cdot\sinh(3\cdot u)\cdot\cos(3\cdot v)}{3}, 2\cdot\sinh(u)\cdot\sin(v) + \frac{2\cdot\sinh(3\cdot u)\cdot\sin(3\cdot v)}{3}, 2\cdot\cosh(2\cdot u)\cdot\cos(2\cdot v)\right\rangle \tag{12.1}$$

$simplify(first(H(u, v)))$

$$[[4\cosh(3u\sim)^2 - 8\cosh(u\sim)(\cos(3v\sim)\cos(v\sim) - \sin(3v\sim)\sin(v\sim))\cosh(3u\sim) + (16\cosh(2u\sim)^2 - 16)\cos(2v\sim)^2 + 4\cosh(u\sim)^2, 0], [0, (-16\cos(2v\sim)^2 + 16)\cosh(2u\sim)^2 + 8\sinh(u\sim)(\cos(3v\sim)\cos(v\sim) - \sin(3v\sim)\sin(v\sim))\sinh(3u\sim) + 4\cosh(3u\sim)^2 + 4\cosh(u\sim)^2 - 8]] \tag{12.2}$$

$$E := \frac{\partial}{\partial u}(H(u, v)).\frac{\partial}{\partial u}(H(u, v))$$

$$E := (2\cosh(u\sim)\cos(v\sim) - 2\cosh(3u\sim)\cos(3v\sim))^2 + (2\cosh(u\sim)\sin(v\sim) + 2\cosh(3u\sim)\sin(3v\sim))^2 + 16\sinh(2u\sim)^2\cos(2v\sim)^2 \tag{12.3}$$

$$simplify\left(\frac{\partial}{\partial u}(H(u, v))\right)$$

$$\begin{bmatrix} 2\cosh(u\sim)\cos(v\sim) - 2\cosh(3u\sim)\cos(3v\sim) \\ 2\cosh(u\sim)\sin(v\sim) + 2\cosh(3u\sim)\sin(3v\sim) \\ 4\sinh(2u\sim)\cos(2v\sim) \end{bmatrix} \tag{12.4}$$

$$simplify\left(\frac{\partial}{\partial v}(H(u, v))\right)$$

$$\begin{bmatrix} -2\sinh(u\sim)\sin(v\sim) + 2\sinh(3u\sim)\sin(3v\sim) \\ 2\sinh(u\sim)\cos(v\sim) + 2\sinh(3u\sim)\cos(3v\sim) \\ -4\cosh(2u\sim)\sin(2v\sim) \end{bmatrix} \tag{12.5}$$

$$conformalform := \left\langle \frac{\partial}{\partial u}(H(u,v)).\frac{\partial}{\partial u}(H(u,v)), 0; 0, \frac{\partial}{\partial u}(H(u,v)).\frac{\partial}{\partial u}(H(u,v))\right\rangle$$

$$conformalform := [[(2\cosh(u\sim)\cos(v\sim) - 2\cosh(3u\sim)\cos(3v\sim))^2 + (2\cosh(u\sim)\sin(v\sim) + 2\cosh(3u\sim)\sin(3v\sim))^2 + 16\sinh(2u\sim)^2\cos(2v\sim)^2, 0], [0, (2\cosh(u\sim)\cos(v\sim) - 2\cosh(3u\sim)\cos(3v\sim))^2 + (2\cosh(u\sim)\sin(v\sim) + 2\cosh(3u\sim)\sin(3v\sim))^2 + 16\sinh(2u\sim)^2\cos(2v\sim)^2]] \quad \textbf{(12.6)}$$

$simplify(conformalform - first(H(u,v)))$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(12.7)}$$

<u>(b) Catalan's surface</u>

$$C := (u,v) \to \left\langle u - \sin(u)\cdot\cosh(v), 1 - \cos(u)\cdot\cosh(v), 4\cdot\sin\left(\frac{u}{2}\right)\cdot\sinh\left(\frac{v}{2}\right)\right\rangle$$

$$C := (u,v) \mapsto \left\langle u - \sin(u)\cdot\cosh(v), 1 - \cos(u)\cdot\cosh(v), 4\cdot\sin\left(\frac{u}{2}\right)\cdot\sinh\left(\frac{v}{2}\right)\right\rangle \quad \textbf{(12.8)}$$

$simplify(first(C(u,v)))$

$$\left[\left[\left(4\cosh\left(\frac{v\sim}{2}\right)^2 - 4\right)\cos\left(\frac{u\sim}{2}\right)^2 - 2\cos(u\sim)\cosh(v\sim) + \cosh(v\sim)^2 + 1, 0\right], \left[0, \left(-4\cos\left(\frac{u\sim}{2}\right)^2 + 4\right)\cosh\left(\frac{v\sim}{2}\right)^2 + \cosh(v\sim)^2 - 1\right]\right] \quad \textbf{(12.9)}$$

$$E = simplify\left(\frac{\partial}{\partial u}(C(u,v)).\frac{\partial}{\partial u}(C(u,v))\right)$$

$$E = \left(4\cosh\left(\frac{v\sim}{2}\right)^2 - 4\right)\cos\left(\frac{u\sim}{2}\right)^2 - 2\cos(u\sim)\cosh(v\sim) + \cosh(v\sim)^2 + 1 \quad \textbf{(12.10)}$$

$$simplify\left(\frac{\partial}{\partial u}(C(u,v))\right)$$

$$\begin{bmatrix} 1 - \cos(u\sim)\cosh(v\sim) \\ \sin(u\sim)\cosh(v\sim) \\ 2\cos\left(\frac{u\sim}{2}\right)\sinh\left(\frac{v\sim}{2}\right) \end{bmatrix} \quad \textbf{(12.11)}$$

$$simplify\left(\frac{\partial}{\partial v}(C(u,v))\right)$$

$$\begin{bmatrix} -\sin(u\sim)\sinh(v\sim) \\ -\cos(u\sim)\sinh(v\sim) \\ 2\sin\left(\frac{u\sim}{2}\right)\cosh\left(\frac{v\sim}{2}\right) \end{bmatrix} \quad \textbf{(12.12)}$$

$$simplify\left(IdentityMatrix(2)\cdot\frac{\partial}{\partial u}(C(u,v)).\frac{\partial}{\partial u}(C(u,v)) - first(C(u,v))\right)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{12.13}$$

(c) Scherk

$S := (u, v) \rightarrow \langle \operatorname{arcsinh}(u), \operatorname{arcsinh}(v), \arcsin(u \cdot v) \rangle$

$$S := (u, v) \mapsto \langle \operatorname{arcsinh}(u), \operatorname{arcsinh}(v), \arcsin(v \cdot u) \rangle \tag{12.14}$$

$simplify(first(S(u, v)))$

$$\begin{bmatrix} \dfrac{v\sim^2 u\sim^2 + v\sim^2 + |v\sim^2 u\sim^2 - 1|}{(u\sim^2 + 1)\,|v\sim^2 u\sim^2 - 1|} & \dfrac{v\sim u\sim}{|v\sim^2 u\sim^2 - 1|} \\ \dfrac{v\sim u\sim}{|v\sim^2 u\sim^2 - 1|} & \dfrac{|v\sim^2 u\sim^2 - 1| + u\sim^2 (v\sim^2 + 1)}{(v\sim^2 + 1)\,|v\sim^2 u\sim^2 - 1|} \end{bmatrix} \tag{12.15}$$

$E = simplify\left(\dfrac{\partial}{\partial u} (S(u, v)) . \dfrac{\partial}{\partial u} (S(u, v)) \right)$

$$E = \frac{v\sim^2 u\sim^2 + v\sim^2 + |v\sim^2 u\sim^2 - 1|}{(u\sim^2 + 1)\,|v\sim^2 u\sim^2 - 1|} \tag{12.16}$$

$simplify\left(\dfrac{\partial}{\partial u} (S(u, v)) \right)$

$$\begin{bmatrix} \dfrac{1}{\sqrt{u\sim^2 + 1}} \\ 0 \\ \dfrac{v\sim}{\sqrt{-v\sim^2 u\sim^2 + 1}} \end{bmatrix} \tag{12.17}$$

$simplify\left(\dfrac{\partial}{\partial v} (S(u, v)) \right)$

$$\begin{bmatrix} 0 \\ \dfrac{1}{\sqrt{v\sim^2 + 1}} \\ \dfrac{u\sim}{\sqrt{-v\sim^2 u\sim^2 + 1}} \end{bmatrix} \tag{12.18}$$

26 *

restart

$f := (\text{theta}) \rightarrow r^3 \cdot (\cos(3 \cdot \text{theta}) + I \cdot \sin(3 \cdot \text{theta}))$

$$f := \theta \mapsto r^3 \cdot (\cos(3 \cdot \theta) + \mathrm{I} \cdot \sin(3 \cdot \theta)) \tag{13.1}$$

$f(\text{Pi})$

$$-r^3 \tag{13.2}$$

$f(-\mathrm{Pi})$

$$-r^3 \tag{13.3}$$

27 *

(c)

$\exp(u + I\cdot v)$

$$\mathrm{e}^{u + \mathrm{I}\,v} \tag{14.1}$$

$evalc(\exp(u + I\cdot v))$

$$\mathrm{e}^{u}\cos(v) + \mathrm{I}\,\mathrm{e}^{u}\sin(v) \tag{14.2}$$

$evalc(\mathrm{Re}(\exp(u + I\cdot v)))$

$$\mathrm{e}^{u}\cos(v) \tag{14.3}$$

$evalc(\mathrm{Im}(\exp(u + I\cdot v)))$

$$\mathrm{e}^{u}\sin(v) \tag{14.4}$$

28 *

(a) Enneper

$f(z) := 1$

$$f := z \mapsto 1 \tag{15.1}$$

$g(z) := u + I\cdot v$

$$g := z \mapsto u + \mathrm{I}\cdot v \tag{15.2}$$

$plt1 := WE1(f(z), g(z))$

$$plt1 := \begin{bmatrix} -u\sim^3 + 3\,u\sim v\sim^2 + u\sim \\ -3\,u\sim^2 v\sim + v\sim^3 - v\sim \\ 2\,u\sim^2 - 2\,v\sim^2 \end{bmatrix} \tag{15.3}$$

$plot3d(plt1, u = -1..1, v = -1..1):$

(b) Catenoid

$f2(w) := -\dfrac{\exp(-(w))}{2}$

$$f2 := w \mapsto -\frac{\mathrm{e}^{-w}}{2} \tag{15.4}$$

$g2(w) := -\exp(w)$

$$g2 := w \mapsto -\mathrm{e}^{w} \tag{15.5}$$

$plt2 := simplify(WE1(f2(w), g2(w)))$

$$plt2 := \begin{bmatrix} \frac{\cos(v\sim)\left(e^{-u\sim}+e^{u\sim}\right)}{2} \\ \frac{\sin(v\sim)\left(e^{-u\sim}+e^{u\sim}\right)}{2} \\ u\sim \end{bmatrix} \quad \textbf{(15.6)}$$

$plot3d\left(\left[\frac{\cos(v)\left(e^{-u}+e^{u}\right)}{2}, \frac{\sin(v)\left(e^{-u}+e^{u}\right)}{2}, u\right], u=-\mathrm{Pi}..\mathrm{Pi}, v=-\mathrm{Pi}..\mathrm{Pi}\right):$

(c) Helicoid

$f3(w) := -\frac{I\cdot\exp(-w)}{2}$

$$f3 := w \mapsto -\frac{\mathrm{I}}{2}\cdot e^{-w} \quad \textbf{(15.7)}$$

$g3(w) := -\exp(w)$

$$g3 := w \mapsto -e^{w} \quad \textbf{(15.8)}$$

$plt3 := WE1(f3(w), g3(w))$

$$plt3 := \begin{bmatrix} \frac{\sin(v\sim)\left(e^{-u\sim}-e^{u\sim}\right)}{2} \\ -\frac{\cos(v\sim)\left(e^{-u\sim}-e^{u\sim}\right)}{2} \\ -v\sim \end{bmatrix} \quad \textbf{(15.9)}$$

$plot3d(plt3, u=0..\mathrm{Pi}, v=0..2\cdot\mathrm{Pi}):$

30

$restart$
$with(LinearAlgebra):$
$with(plots):$

$K := (f, g) \rightarrow \frac{-4\cdot \mathrm{abs}(diff(g, w))^2}{\mathrm{abs}(f)^2\cdot\left(1+\mathrm{abs}(g)^2\right)^4}$

$$K := (f, g) \rightarrow -\frac{4\left|\frac{\partial}{\partial w} g\right|^2}{|f|^2\left(1+|g|^2\right)^4} \quad \textbf{(16.1)}$$

(a) Enneper

$f1(w) := 1$

$$f1 := w \mapsto 1 \quad \textbf{(16.2)}$$

$g1(w) := w$

$$g1 := w \mapsto w \quad \textbf{(16.3)}$$

$simplify(K(f1(w), g1(w)))$

$$-\frac{4}{\left(1+|w|^2\right)^4} \tag{16.4}$$

(b) Helicoid

$f2(w) := -\frac{I \cdot \exp(-w)}{2}$

$$f2 := w \mapsto -\frac{\mathrm{I}}{2} \cdot \mathrm{e}^{-w} \tag{16.5}$$

$g2(w) := -\exp(w)$

$$g2 := w \mapsto -\mathrm{e}^{w} \tag{16.6}$$

$simplify(K(f2(w), g2(w)))$

$$-\frac{16\,\mathrm{e}^{4\,\Re(w)}}{\left(1+\mathrm{e}^{2\,\Re(w)}\right)^4} \tag{16.7}$$

A

restart;
with(*LinearAlgebra*) :
with(*plots*) :

$$E := (u, v) \rightarrow \left\langle u - \frac{u^3}{3} + u \cdot v^2, -v + \frac{v^3}{3} - v \cdot u^2, u^2 - v^2 \right\rangle$$

$$E := (u, v) \mapsto \left\langle u - \frac{1}{3} \cdot u^3 + u \cdot v^2, -v + \frac{1}{3} \cdot v^3 - v \cdot u^2, u^2 - v^2 \right\rangle \quad \textbf{(1.1)}$$

$$EZ := (r, \text{theta}) \rightarrow E(r \cdot \cos(\text{theta}), r \cdot \sin(\text{theta}))$$

$$EZ := (r, \theta) \mapsto E(r \cdot \cos(\theta), r \cdot \sin(\theta)) \quad \textbf{(1.2)}$$

$$EZ(r, \text{theta})$$

$$\begin{bmatrix} r\cos(\theta) - \dfrac{r^3 \cos(\theta)^3}{3} + r^3 \cos(\theta)\sin(\theta)^2 \\ -r\sin(\theta) + \dfrac{r^3 \sin(\theta)^3}{3} - r^3 \sin(\theta)\cos(\theta)^2 \\ r^2\cos(\theta)^2 - r^2 \sin(\theta)^2 \end{bmatrix} \quad \textbf{(1.3)}$$

$r1 := 0$:
$r2 := \text{sqrt}(3)$:
$t1 := 0$:
$t2 := 2 \cdot \text{Pi}$:
$E(1.5 \cdot \cos(\text{theta}), 1.5 \cdot \sin(\text{theta}))$

$$\begin{bmatrix} 1.5\cos(\theta) - 1.125000000\cos(\theta)^3 + 3.375\cos(\theta)\sin(\theta)^2 \\ -1.5\sin(\theta) + 1.125000000\sin(\theta)^3 - 3.375\sin(\theta)\cos(\theta)^2 \\ 2.25\cos(\theta)^2 - 2.25\sin(\theta)^2 \end{bmatrix} \quad \textbf{(1.4)}$$

Enneper := *plot3d*(*EZ*(*r*, theta), *r* = *r1* ..*r2*, theta = *t1* ..*t2*, *colour* = *coral*) :
curve := *spacecurve*(*E*(1.5·cos(theta), 1.5·sin(theta)), theta = *t1* ..*t2*, *thickness* = 5, *colour* = *turquoise*) :
display(*Enneper*, *curve*) :
display(*curve*) :

$$c := (u, v, w) \rightarrow \left\langle u - \frac{u^3}{3} + u \cdot v^2, w \cdot \left(-v + \frac{v^3}{3} - v \cdot u^2\right), u^2 - v^2 \right\rangle$$

$$c := (u, v, w) \mapsto \left\langle u - \frac{1}{3} \cdot u^3 + u \cdot v^2, w \cdot \left(-v + \frac{1}{3} \cdot v^3 - v \cdot u^2\right), u^2 - v^2 \right\rangle \quad \textbf{(1.5)}$$

$$cZ := (\text{theta}, w) \rightarrow c(1.5 \cdot \cos(\text{theta}), 1.5 \cdot \sin(\text{theta}), w)$$

$$cZ := (\theta, w) \mapsto c(1.5 \cdot \cos(\theta), 1.5 \cdot \sin(\theta), w) \quad \textbf{(1.6)}$$

$cZ(\text{theta}, w)$

$$\begin{bmatrix} 1.5\cos(\theta) - 1.125000000\cos(\theta)^3 + 3.375\cos(\theta)\sin(\theta)^2 \\ w\left(-1.5\sin(\theta) + 1.125000000\sin(\theta)^3 - 3.375\sin(\theta)\cos(\theta)^2\right) \\ 2.25\cos(\theta)^2 - 2.25\sin(\theta)^2 \end{bmatrix} \quad \textbf{(1.7)}$$

$$e := (t, w) \to \left\langle 1.5\cdot\cos(t) - \frac{(1.5\cdot\cos(t))^3}{3} + 1.5\cdot\cos(t)\cdot(1.5\cdot\sin(t))^2, w, (1.5\cdot\cos(t))^2 - (1.5\cdot\sin(t))^2 \right\rangle$$

$$e := (t, w) \mapsto \langle 1.5\cdot\cos(t) - 1.125000000\cdot\cos(t)^3 + 3.375\cdot\cos(t)\cdot\sin(t)^2, w, 1.5^2\cdot\cos(t)^2 - 2.25\cdot\sin(t)^2\rangle \quad \textbf{(1.8)}$$

$cylinderrr := plot3d(e(t, w), t = 0..2\cdot\text{Pi}, w = -3..3, thickness = 5, colour = coral, style = patchnogrid) :$
$display(cylinderrrr, curve) :$
$t1 := 0 :$
$t2 := 2\cdot\text{Pi} :$
$w1 := 0 :$
$w2 := 1 :$
$cylinder := plot3d(cZ(\text{theta}, w), \text{theta} = t1..t2, w = w1..w2, scaling = constrained, colour = coral) :$
$planecurve := spacecurve(c(1.5\cdot\cos(\text{theta}), 1.5\cdot\sin(\text{theta}), 0), \text{theta} = t1..t2, thickness = 3, colour = black) :$

$display(curve, cylinder) :$
$with(VectorCalculus) :$
$m := Jacobian(E(r\cdot\cos(\text{theta}), r\cdot\sin(\text{theta})), [r, \text{theta}])$

$$m := \quad \textbf{(1.9)}$$

$$[[\cos(\theta) - r^2\cos(\theta)^3 + 3r^2\cos(\theta)\sin(\theta)^2, -r\sin(\theta) + 3r^3\sin(\theta)\cos(\theta)^2 - r^3\sin(\theta)^3],$$
$$[-\sin(\theta) + r^2\sin(\theta)^3 - 3r^2\sin(\theta)\cos(\theta)^2, -r\cos(\theta) + 3r^3\cos(\theta)\sin(\theta)^2 - r^3\cos(\theta)^3],$$
$$[2r\cos(\theta)^2 - 2r\sin(\theta)^2, -4r^2\cos(\theta)\sin(\theta)]]$$

$diff(E(r\cdot\cos(\text{theta}), r\cdot\sin(\text{theta})), r)$

$$\left(\cos(\theta) - r^2\cos(\theta)^3 + 3r^2\cos(\theta)\sin(\theta)^2\right)e_x + \left(-\sin(\theta) + r^2\sin(\theta)^3 - 3r^2\sin(\theta)\cos(\theta)^2\right)e_y + \left(2r\cos(\theta)^2 - 2r\sin(\theta)^2\right)e_z \quad \textbf{(1.10)}$$

$Jac := simplify(CrossProduct(Column(m, 1), Column(m, 2)))$

$$Jac := \left(2r^2\cos(\theta)(r^2+1)\right)e_x + \left(2r^2\sin(\theta)(r^2+1)\right)e_y + (r^5 - r)e_z \quad \textbf{(1.11)}$$

$Norm(Jac)$

$$\sqrt{r^2 (r^2+1)^4} \tag{1.12}$$

$J := \%$

$$J := \sqrt{r^2 (r^2+1)^4} \tag{1.13}$$

$\int_0^{1.5} \int_0^{2\cdot \mathrm{Pi}} J \,\mathrm{dtheta}\, \mathrm{d}r$

$$34.90113090 \tag{1.14}$$

$\int_0^{1} \int_0^{2\cdot \mathrm{Pi}} J \,\mathrm{dtheta}\, \mathrm{d}r$

$$\frac{7\pi}{3} \tag{1.15}$$

$evalf(\%)$

$$7.330382858 \tag{1.16}$$

$with(VectorCalculus):$

$cZ := (\mathrm{theta}, w) \rightarrow c(1.5\cdot\cos(\mathrm{theta}), 1.5\cdot\sin(\mathrm{theta}), w)$

$$cZ := (\theta, w) \mapsto c(1.5\cdot\cos(\theta), 1.5\cdot\sin(\theta), w) \tag{1.17}$$

$n := Jacobian(cZ(\mathrm{theta}, w), [\mathrm{theta}, w])$

$$n := [[-1.5\sin(\theta) + 10.12500000\sin(\theta)\cos(\theta)^2 - 3.375\sin(\theta)^3, 0], [w(-1.5\cos(\theta) + 10.12500000\cos(\theta)\sin(\theta)^2 - 3.375\cos(\theta)^3), -1.5\sin(\theta) + 1.125000000\sin(\theta)^3 - 3.375\sin(\theta)\cos(\theta)^2], [-9.00\cos(\theta)\sin(\theta), 0]] \tag{1.18}$$

$Jac2 := simplify(CrossProduct(Column(n, 1), Column(n, 2)))$

$$Jac2 := ((-40.5\cos(\theta)^2 - 3.375)\cos(\theta)\sin(\theta)^2)e_x + (0)e_y + (-60.75(\cos(\theta) + 0.600925212577331)(\cos(\theta) - 0.600925212577331)(\cos(\theta)^2 + 0.0833333333333321)\sin(\theta)^2)e_z \tag{1.19}$$

$diff(cZ(\mathrm{theta}, w), w)$

$$(0)e_x + (-1.5\sin(\theta) + 1.125000000\sin(\theta)^3 - 3.375\sin(\theta)\cos(\theta)^2)e_y + (0)e_z \tag{1.20}$$

$diff(cZ(\mathrm{theta}, w), \mathrm{theta})$

$$(-1.5\sin(\theta) + 10.12500000\sin(\theta)\cos(\theta)^2 - 3.375\sin(\theta)^3)e_x + (w(-1.5\cos(\theta) + 10.12500000\cos(\theta)\sin(\theta)^2 - 3.375\cos(\theta)^3))e_y + (-9.00\cos(\theta)\sin(\theta))e_z \tag{1.21}$$

$Norm(Jac2), assume(\mathrm{theta}, real)$

$$\left(\sin(\theta)^4\left(3690.5625\cos(\theta)^8-410.062500000352\cos(\theta)^6+336.023437499690\cos(\theta)^4+73.0898437496420\cos(\theta)^2+3.34204101524028\right)\right)^{1/2} \tag{1.22}$$

$J2 := \%$

$$J2 := \left(\sin(\theta\sim)^4\left(3690.5625\cos(\theta\sim)^8-410.062500000352\cos(\theta\sim)^6+336.023437499690\cos(\theta\sim)^4+73.0898437496420\cos(\theta\sim)^2+3.34204101524028\right)\right)^{1/2} \tag{1.23}$$

$\int_0^{2\cdot \mathrm{Pi}}\int_0^1 J2\, \mathrm{d}w\, \mathrm{dtheta}$

$$\int_0^{2\pi}\left(\sin(\theta\sim)^4\left(3690.5625\cos(\theta\sim)^8-410.062500000352\cos(\theta\sim)^6+336.023437499690\cos(\theta\sim)^4+73.0898437496420\cos(\theta\sim)^2+3.34204101524028\right)\right)^{1/2}\mathrm{d}\theta\sim \tag{1.24}$$

$evalf(\%)$

$$31.66323514 \tag{1.25}$$

B

$WE1(\exp(I\cdot 0)\cdot f, g)$

$$\begin{bmatrix} (-g^2+1)\, f u\sim \\ -f v\sim (g^2+1) \\ 2\, g\, f u\sim \end{bmatrix} \tag{1}$$

$WE1(\exp(I\cdot \mathrm{Pi})\cdot f, g)$

$$\begin{bmatrix} f u\sim (g^2-1) \\ f v\sim (g^2+1) \\ -2\, g\, f u\sim \end{bmatrix} \tag{2}$$

$WE1(\exp(I\cdot 2\cdot \mathrm{Pi})\cdot f, g)$

$$\begin{bmatrix} (-g^2+1)\, f u\sim \\ -f v\sim (g^2+1) \\ 2\, g\, f u\sim \end{bmatrix} \tag{3}$$

$E := (t)\to \mathrm{abs}(\exp(I\cdot t)\cdot f)^2\cdot(1+\mathrm{abs}(g)^2)$

$$E := t \mapsto \left|\mathrm{e}^{\mathrm{I}\,t}\cdot f\right|^2\cdot\left(1+|g|^2\right) \tag{4}$$

$E(0)$

$$|f|^2\left(1+|g|^2\right) \tag{5}$$

$E(\mathrm{Pi})$

$$|f|^2\left(1+|g|^2\right) \tag{6}$$

$E(2\cdot \mathrm{Pi})$

$$|f|^2\left(1+|g|^2\right) \tag{7}$$

$f := w\to 1$

$$f := w \mapsto 1 \tag{8}$$

$g := w\to w$

$$g := w \mapsto w \tag{9}$$

$weinf := simplify\Big(first(simplify(WE1(\exp(I\cdot t)\cdot f(w), g(w))))^{-1} \bullet second((WE1(\exp(I\cdot t)\cdot f(w), g(w)), calcnormal((WE1(\exp(I\cdot t)\cdot f(w), g(w))))))\Big)$

$$weinf := \begin{bmatrix} -\dfrac{2\cos(t\sim)}{(u\sim^2+v\sim^2+1)^2} & \dfrac{2\sin(t\sim)}{(u\sim^2+v\sim^2+1)^2} \\ \dfrac{2\sin(t\sim)}{(u\sim^2+v\sim^2+1)^2} & \dfrac{2\cos(t\sim)}{(u\sim^2+v\sim^2+1)^2} \end{bmatrix} \tag{10}$$

$simplify(Eigenvalues(weinf))$

$$\left[\begin{array}{c} -\frac{2}{(u\sim^2+v\sim^2+1)^2} \\ \frac{2}{(u\sim^2+v\sim^2+1)^2} \end{array}\right] \quad \mathbf{(11)}$$

$$\left(\left(\frac{2}{(u\sim^2+v\sim^2+1)^2}\right)\cdot\left(-\frac{2}{(u\sim^2+v\sim^2+1)^2}\right)\right)$$

$$-\frac{4}{(u\sim^2+v\sim^2+1)^4} \quad \mathbf{(12)}$$

$$plot3d\left(-\frac{4}{(u\sim^2+v\sim^2+1)^4}\right):$$

$$plot3d\left(-\frac{\left(-\frac{4}{(u\sim^2+v\sim^2+1)^4}\right)}{4}\right):$$

$$gausscolor := \left[\frac{\left(-\frac{4}{(u\sim^2+v\sim^2+1)^4}\right)}{4}+1, 0, -\frac{\left(-\frac{4}{(u\sim^2+v\sim^2+1)^4}\right)}{4}\right]$$

$$gausscolor := \left[-\frac{1}{(u\sim^2+v\sim^2+1)^4}+1, 0, \frac{1}{(u\sim^2+v\sim^2+1)^4}\right] \quad \mathbf{(13)}$$

$plt1 := simplify(WE1(\exp(I\cdot t)\cdot f(w), g(w)))$

$$plt1 := \left[\begin{array}{c} \frac{(-u\sim^3+3\,u\sim\,v\sim^2+3\,u\sim)\cos(t\sim)}{3}+\left(u\sim^2-\frac{v\sim^2}{3}-1\right)v\sim\sin(t\sim) \\ \frac{(-3\,u\sim^2\,v\sim+v\sim^3-3\,v\sim)\cos(t\sim)}{3}-\frac{\sin(t\sim)\,u\sim\,(u\sim^2-3\,v\sim^2+3)}{3} \\ (u\sim^2-v\sim^2)\cos(t\sim)-2\,u\sim\,v\sim\sin(t\sim) \end{array}\right] \quad \mathbf{(14)}$$

$plot3d(simplify(WE1(\exp(I\cdot 0)\cdot f(w), g(w))), u=-1..1, v=-1..1, color=gausscolor):$

$plot3d\left(simplify\left(WE1\left(\exp\left(\frac{I\cdot \mathrm{Pi}}{2}\right)\cdot f(w), g(w)\right)\right), u=-1..1, v=-1..1, color=gausscolor\right):$

$plot3d(simplify(WE1(\exp(I\cdot \mathrm{Pi})\cdot f(w), g(w))), u=-1..1, v=-1..1, color=gausscolor):$

$animate3d(plt1, u=-1..1, v=-1..1, t=0..2*\mathrm{Pi}, frames=50, color=gausscolor, symbol=point, symbolsize=5):$

$f := w \rightarrow 1$

$$f := w \mapsto 1 \quad \mathbf{(15)}$$

$g := w \rightarrow \mathrm{sqrt}(w)$

$$g := w \mapsto \sqrt{w} \quad \mathbf{(16)}$$

$weinf := simplify\left(first(simplify(WE1(\exp(I \cdot t) \cdot f(w), g(w))))^{-1} \cdot second((WE1(\exp(I \cdot t) \cdot f(w), g(w)), calcnormal((WE1(\exp(I \cdot t) \cdot f(w), g(w))))))\right)$

$plt1 := simplify(WE1(\exp(I \cdot t) \cdot f(w), g(w)))$

$$plt1 := \left[\left[\frac{(-u\sim^2 + v\sim^2 + 2\, u\sim)\cos(t\sim)}{2} + \sin(t\sim)\, v\sim\,(u\sim - 1)\right], \left[\frac{(-u\sim^2 + v\sim^2 - 2\, u\sim)\sin(t\sim)}{2} - \cos(t\sim)\, v\sim\,(u\sim + 1)\right], \left[\frac{4\sin(t\sim)\,\mathrm{csgn}(-v\sim + \mathrm{I}\, u\sim)\left(u\sim + \frac{\sqrt{u\sim^2 + v\sim^2}}{2}\right)\sqrt{2\sqrt{u\sim^2 + v\sim^2} - 2\, u\sim}}{3} + \frac{4\left(u\sim - \frac{\sqrt{u\sim^2 + v\sim^2}}{2}\right)\cos(t\sim)\sqrt{2\sqrt{u\sim^2 + v\sim^2} + 2\, u\sim}}{3}\right]\right] \tag{17}$$

$plot3d(simplify(WE1(\exp(I \cdot 0) \cdot f(w), g(w))), u = -1..1, v = -1..1, color = gausscolor) :$

$plot3d\left(simplify\left(WE1\left(\exp\left(\frac{I \cdot \mathrm{Pi}}{2}\right) \cdot f(w), g(w)\right)\right), u = -1..1, v = -1..1, color = gausscolor\right) :$

$plot3d(simplify(WE1(\exp(I \cdot \mathrm{Pi}) \cdot f(w), g(w))), u = -1..1, v = -1..1, color = gausscolor) :$

$animate3d(plt1, u = -1..1, v = -1..1, t = 0..2 * \mathrm{Pi}, frames = 50, color = gausscolor, symbol = point, symbolsize = 5) :$