Exercise 5 - Theory

Friendship paradox: Your friends have more friends than you do.

$$\langle k_{\rm nn}\rangle = \frac{\langle k^2\rangle}{\langle k\rangle}.$$

To calculate the average number of friends of friends one could start by choosing an endpoint of a random connection. The probability of choosing a certain node is $\frac{1}{2L} = \frac{1}{\sum_{j=1}^N k_j}$, where L is the number of links in the network. The probability that this is connected to a node i is therefore $p_i = \frac{k_i}{\sum_j k_j}$. Now we can calculate the expected value by noting the number of friends of this node i, i.e. its degree, and summing that up with the probability of it being chosen from above.

$$\langle k_{\rm nn}\rangle = \sum_{i=1}^N k_i p_i = \sum_i \frac{k_i^2}{\sum_j k_j} = \frac{\frac{1}{N}\sum_i k_i^2}{\frac{1}{N}\sum_j k_j} = \frac{\langle k^2\rangle}{\langle k\rangle}.$$

From the lecture we know that for a power law distribution $P(k) = (\gamma-1)k_{\rm min}^{\gamma-1}k^{-\gamma}$ the moments are $\langle k^m\rangle = \frac{\gamma-1}{m-\gamma+1}k_{\rm min}^{\gamma-1}(k_{\rm max}^{m-\gamma+1} - k_{\rm min}^{m-\gamma+1})$. Thus for the average number of next-nearest connections we have

$$\langle k_{\rm nn}\rangle = \frac{\langle k^2\rangle}{\langle k\rangle} = \frac{2-\gamma}{3-\gamma}\,\frac{k_{\rm max}^{3-\gamma} - k_{\rm min}^{3-\gamma}}{k_{\rm max}^{2-\gamma} - k_{\rm min}^{2-\gamma}}.$$

By using L'Hôpital's rule we get the following limits for $\gamma \to 2$

$$\begin{aligned}\langle k_{\rm nn}\rangle &= \lim_{\gamma\to 2}\frac{(2-\gamma)(k_{\rm max}^{3-\gamma} - k_{\rm min}^{3-\gamma})}{(3-\gamma)(k_{\rm max}^{2-\gamma} - k_{\rm min}^{2-\gamma})} = \lim_{\gamma\to 2}\frac{k_{\rm max}^{3-\gamma} - k_{\rm min}^{3-\gamma}}{3-\gamma}\lim_{\gamma\to 2}\frac{2-\gamma}{k_{\rm max}^{2-\gamma} - k_{\rm min}^{2-\gamma}} \\ &= (k_{\rm max} - k_{\rm min})\lim_{\gamma\to 2}\frac{-1}{-k_{\rm max}^{2-\gamma}\log(k_{\rm max}) + k_{\rm min}^{2-\gamma}\log(k_{\rm min})} = \frac{k_{\rm max} - k_{\rm min}}{\log(k_{\rm max}) - \log(k_{\rm min})}\end{aligned}$$

$$\langle k\rangle = \lim_{\gamma\to 2}\frac{\gamma-1}{2-\gamma}k_{\rm min}^{\gamma-1}(k_{\rm max}^{2-\gamma} - k_{\rm min}^{2-\gamma}) = k_{\rm min}\big(\log(k_{\rm max}) - \log(k_{\rm min})\big)$$

It is easy to see that since $k_{\rm max} > k_{\rm min} > 0$ we have that $\langle k_{\rm nn}\rangle > \langle k\rangle$.

Analogously for $\gamma \to 3$ we have that $\langle k_{\rm nn}\rangle > \langle k\rangle$ since

$$\langle k_{\rm nn}\rangle = \lim_{\gamma\to 3}\frac{(2-\gamma)(k_{\rm max}^{3-\gamma} - k_{\rm min}^{3-\gamma})}{(3-\gamma)(k_{\rm max}^{2-\gamma} - k_{\rm min}^{2-\gamma})} = k_{\rm max}k_{\rm min}\frac{\log(k_{\rm max}) - \log(k_{\rm min})}{k_{\rm max} - k_{\rm min}},$$

$$\langle k\rangle = 2\,k_{\rm min}\Big(1 - \frac{k_{\rm min}}{k_{\rm max}}\Big).$$

For a Poisson distribution $P(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ the second moments is $\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1)$. Thus the average number of friends of friends is $\langle k_{\mathrm{nn}} \rangle = \frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle + 1$ which also confirms the friendship paradox for random networks but it is much weaker than in scale-free networks.

By substituting $k_{\max} = k_{\min} N^{\frac{1}{\gamma - 1}}$ we get the following formula for the average next-nearest neighbor degree for $2 < \gamma < 3$.

$$\langle k_{\mathrm{nn}} \rangle = \frac{2-\gamma}{3-\gamma} k_{\min} \frac{N^{\frac{3-\gamma}{\gamma-1}} - 1}{N^{\frac{2-\gamma}{\gamma-1}} - 1} = \mathcal{O}(N)$$

In the limiting cases we have $\langle k_{\mathrm{nn}} \rangle = k_{\min} \frac{N-1}{\log(N)}$ for $\gamma = 2$ and $\langle k_{\mathrm{nn}} \rangle = \frac{k_{\min}}{2} \frac{\sqrt{N}}{\sqrt{N} - 1} \log(N)$ for $\gamma = 3$.

So the average next-nearest neighbor degree grows monotonely in N for all $2 \leq \gamma \leq 3$ but the slope of growth decreases with increasing γ. For $2 \leq \gamma < 3$ $\langle k_{\mathrm{nn}} \rangle$ grows linearly in N but for $\gamma = 3$ it only grows logarithmically in N.