Exercise 3 - Theory

In [ ]:

```
import networkx as nx
from matplotlib import pyplot as plt
```

Problem 1

Consider a balanced (or Cayley) tree which is a symmetric tree, constructed starting from central node with $k \geq 2$ neighbors. Each neighboring node again has k neighbors until nodes are reached of distance $h \geq 0$ from the central node. These nodes only have degree 1 (leaf nodes).

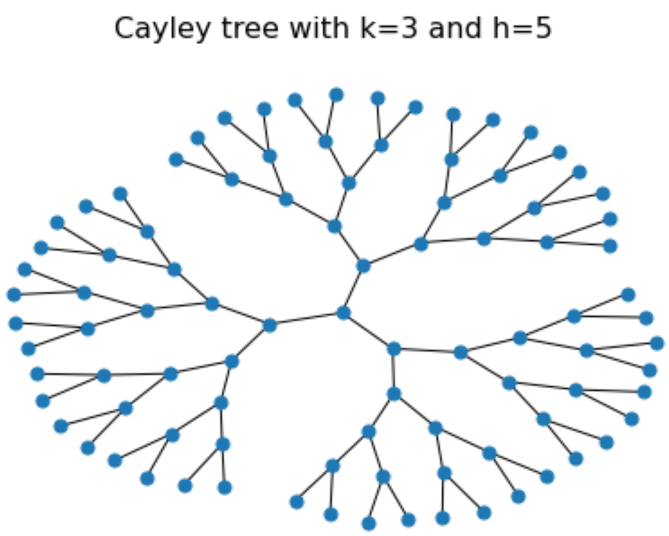
In [ ]:

```
def cayley_tree(k, h):
    if h == 0:
        return nx.trivial_graph()
    elif h == 1:
        return nx.star_graph(k)
    else:
        G = nx.star_graph(k)
        H = nx.balanced_tree(k-1, h-1)
        N = len(H)
        for i in range(k):
            n = (i+1)*N
            H = nx.convert_node_labels_to_integers(H, first_label=n)
            H = nx.relabel_nodes(H,{n:i+1})
            G = nx.compose(G, H)
        return G
```

In [ ]:

```
k = 3
h = 5
G = cayley_tree(k, h)

nx.draw_kamada_kawai(G, node_size=50)
ax = plt.gca()
ax.set_title(f"Cayley tree with k={k} and h={h}", fontsize=16)
plt.show()
```



a) Calculate the total number of nodes such a tree possesses given k and h.

The first layer grows k nodes and after that $(k-1)N_i$ nodes, where N_i is the number of nodes in the previous layer. Thus we get the recursion for the number of nodes in each layer:

$N_0 = 1,\ N_1 = k,\ N_{i+1} = (k-1)N_i$. Resolving this recursion yields $N_i = k(k-1)^{i-1}$ for $i \geq 1$ and thus in total for the whole Cayley tree

$$N = N_0 + \sum_{i=1}^{h} N_i = 1 + k\sum_{i=0}^{h-1}(k-1)^i = 1 + k\frac{(k-1)^h - 1}{(k-1)-1} = \frac{k(k-1)^h - 2}{k-2}.$$

b) Calculate the number of leaf nodes and compare it to the number of inner nodes. What can you say about its ratio? (For which values of k and h the number of inner nodes exceeds the number of leaf nodes?)

$$N_{\text{inner}}^k(h) = 1 + k\sum_{i=0}^{h-2}(k-1)^i = \frac{k(k-1)^h - 2k + 2}{(k-2)(k-1)},\ N_{\text{outer}}^k(h) = k(k-1)^{h-1}$$

For $k = 2$ we get the inequality $N_{\text{inner}}^2(h) = 2h - 1 \geq 2 = N_{\text{outer}}^2(h)$ which is satisfied if $h \geq \frac{3}{2}$. If $k = 3$ then $N_{\text{inner}}^3(h) = 3 \cdot 2^{h-1} - 2$ is obviously never greater than $N_{\text{outer}}^3(h) = 3 \cdot 2^{h-1}$. For $4 \leq k \to \infty$ the number of inner nodes $N_{\text{inner}}^k(h)$ grows less quickly than $N_{\text{outer}}^k(h)$ does which can be shown via induction and thus there are always more outer nodes than inner nodes.

c) Find an expression for the diameter of a Cayley tree in terms of the total number of nodes.

No shortest path can be longer than $2h$ since every node is connected via the central node. Two leaves from different main branches have distance $2h$. Thus the diameter is $2h$.

d) Calculate the clustering coefficient.

No nodes have any connected neighbors thus the clustering coefficient is 0.

Problem 2

Consider an Erdős–Rényi (ER) network with $N = 5000$ nodes and connection probability of node pairs of $p = 0.002$.

a) What is the expected number of edges and the average degree?

$$\langle L \rangle = p\,\frac{N(N-1)}{2} = 24995 \approx 25000$$

$$\langle k \rangle = p\,(N-1) = 9.998 \approx 10$$

b) In which regime is the network and what is the critical value of $p = p_c$?

The ER network is in connected regime because $\langle k \rangle > \log(N) \approx 8.5$. The critical value is $p_c = \frac{1}{N-1} \approx 0.0002$.

c) What is the expected size of the giant component?

Since the ER network is in its connected regime we expect the number of nodes in the giant component to be N.

d) Suppose you are making a random ER network with 100 (or 1000) nodes and you want the average degree to be 5 (or 20). What approximate value for p would you choose?

Rearranging the formula from above we get $p = \frac{\langle k \rangle}{N-1}$. Thus for $N = 100$ and $\langle k \rangle = 5$ one could choose $p = 0.05$ or $p = 0.02$ for $N = 1000$ and $\langle k \rangle = 20$.

Problem 3

Find an approximate expression for the minimal number of trials you must throw a die to estimate whether it is biased or not.