PHYS435 Assignment 4

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1. Find the radii of convergence of the following series using both Cauchy's root test and ratio tests:

$$\text{a)}\ \sum_{n=2}^{\infty} \frac{z^n}{\ln(n)}$$

$$\text{b)}\ \sum_{n=2}^{\infty} z^n \ln(n)$$

Solution:

a)

Radius of convergence is defined as

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$

For the first series, we have

$$a_n = \frac{1}{\ln(n)}$$

and, putting this into the root test above and rearranging it into an indeterminate form that we can use l'Hopital's rule on, we find

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{1}{\ln(n)}\right|^{1/n} = \lim_{n\to\infty} |\ln(n)|^{-1/n} = 1$$

and the radius of convergence is $R = 1$.

Using the ratio test for the same series we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\ln(n+1)}}{\frac{1}{\ln(n)}} = \frac{\ln(n)}{\ln(n+1)}$$

Taking the limit and simplifying we have

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{\ln(n+1)}\right) = 1$$

...as the difference between n and $n+1$ becomes negligible as n grows very large, confirming what we found with the root test. Thus the series will converge if $|z| < 1$.

b)

$$\frac{1}{R} = \lim_{n\to\infty} |\ln(n)|^{1/n}$$

we can again manipulate this into an indeterminate form

$$= \lim_{n\to\infty} \frac{1}{n}\ln(\ln(n)) = \frac{0}{\infty}$$

which yields 0 when applying l'Hopital's rule. Exponentiating both sides as before we then have

$$\lim_{n\to\infty} e^{\ln(\ln(n))/n} = e^0 = 1$$

and so the radius of convergence here is 1 as well.

Ratio test:

We now have

$$\lim_{n\to\infty}\left(\frac{\ln(n+1)}{\ln(n)}\right)$$

and again the $+1$ becomes negligible as n grows very large, so the radius of convergence is just 1 once again.

2. Find and plot all values of $z = \sqrt[6]{-2-i}$.

Solution:

We can use DeMoivre's Theorem* to find these sixth roots. De Moivre's Theorem is a formula used to compute powers and roots of complex numbers. It states that for any complex number $z = r(\cos\theta + i\sin\theta)$ (where r is the magnitude of z and θ is the argument of z), and any integer n, the nth power of z is given by:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Four our purposes we have $n = 1/6$.

First, we convert $-2-i$ to polar form, $r(\cos\theta + i\sin\theta)$, where r is the magnitude of the complex number and θ is the argument (or angle).

First the magnitude r:

$$r = \sqrt{(-2)^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$$

Calculate the argument θ. Since the complex number is in the third quadrant, we'll find the angle with respect to the positive x-axis and then add π to get the correct quadrant:

$$\theta = \arctan\left(\frac{-1}{-2}\right) + \pi = \arctan\left(\frac{1}{2}\right) + \pi$$

Now we can express $-2-i$ in polar form:

$$-2-i = \sqrt{5}\left(\cos\left(\arctan\left(\frac{1}{2}\right)+\pi\right) + i\sin\left(\arctan\left(\frac{1}{2}\right)+\pi\right)\right)$$

Apply De Moivre's Theorem to find the sixth roots. We want to find z such that $z^6 = -2-i$. According to De Moivre's Theorem, the nth roots are given by:

$$z_k = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$$

where $k = 0, 1, \ldots, n-1$.

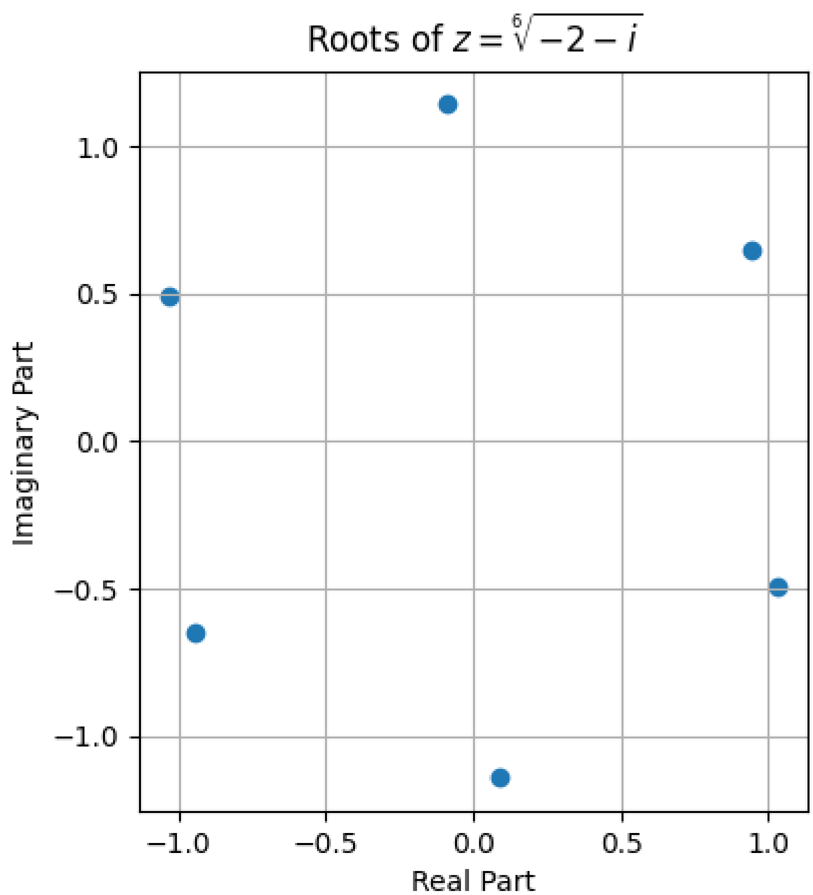
For the sixth roots, $n = 6$, and $r = \sqrt{5}$, $\theta = \arctan\left(\frac{1}{2}\right) + \pi$. So we have:

$$z_k = \sqrt[6]{5}\left(\cos\left(\frac{\arctan\left(\frac{1}{2}\right) + \pi + 2k\pi}{6}\right) + i\sin\left(\frac{\arctan\left(\frac{1}{2}\right) + \pi + 2k\pi}{6}\right)\right)$$

for $k = 0, 1, 2, 3, 4, 5$. The roots are then described by

$$\sum_{k=0}^{5} \sqrt[6]{5}\left(\cos\left(\frac{\arctan\left(\frac{1}{2}\right) + \pi + 2k\pi}{6}\right) + i\sin\left(\frac{\arctan\left(\frac{1}{2}\right) + \pi + 2k\pi}{6}\right)\right)$$

Each value of k will give you a different root, equally spaced around the circle in the complex plane, each 60° (or $\pi/3$ radians) apart.



- [https://math.libretexts.org/Bookshelves/Precalculus/Book%3A*Trigonometry*(Sundstrom_and_Schlicker)/05%3A_Complex_Numbers_and_Polar_Coordinates/5.03%3A_De Moivres_Theorem_and_Powers_of_Complex_Numbers]

Code to produce the above plot:

```
import numpy as np
import matplotlib.pyplot as plt

# Define the complex number
c = -2 - 1j

# Calculate the roots
roots = [np.power(c, 1/6) * np.exp(1j * 2 * np.pi * k / 6) for k in range(6)]

# Extract real and imaginary parts
real_parts = [root.real for root in roots]
imag_parts = [root.imag for root in roots]

# Plot the roots
plt.scatter(real_parts, imag_parts)
plt.xlabel('Real Part')
plt.ylabel('Imaginary Part')
plt.title('Roots of $z = \sqrt[6]{ -2-i }$')
plt.grid(True)
plt.gca().set_aspect('equal', adjustable='box')
plt.show()
```

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3. For each of the following functions, locate and identify the singularities:

$$\text{a) } f(z) = \frac{\sin(\sqrt{z})}{\sqrt{z}}$$

$$\text{b) } f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$$

$$\text{c) } f(z) = \frac{z-2}{z^2}\sin\left(\frac{1}{1-z}\right)$$

Solution:

To find the singularities we need to identify places where the functions are not analytic.

The first function is analytic everywhere except maybe at $z = 0$ - however, we can remove this singularity by noting that we can write $\sin$ as a Taylor series:

$$\sin(\sqrt{z}) = \sqrt{z} - \frac{z^{3/2}}{3!} + \frac{z^{5/2}}{5!} + \dots$$

and so

$$\frac{\sin(\sqrt{z})}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} \dots$$

and the singularity is removable.

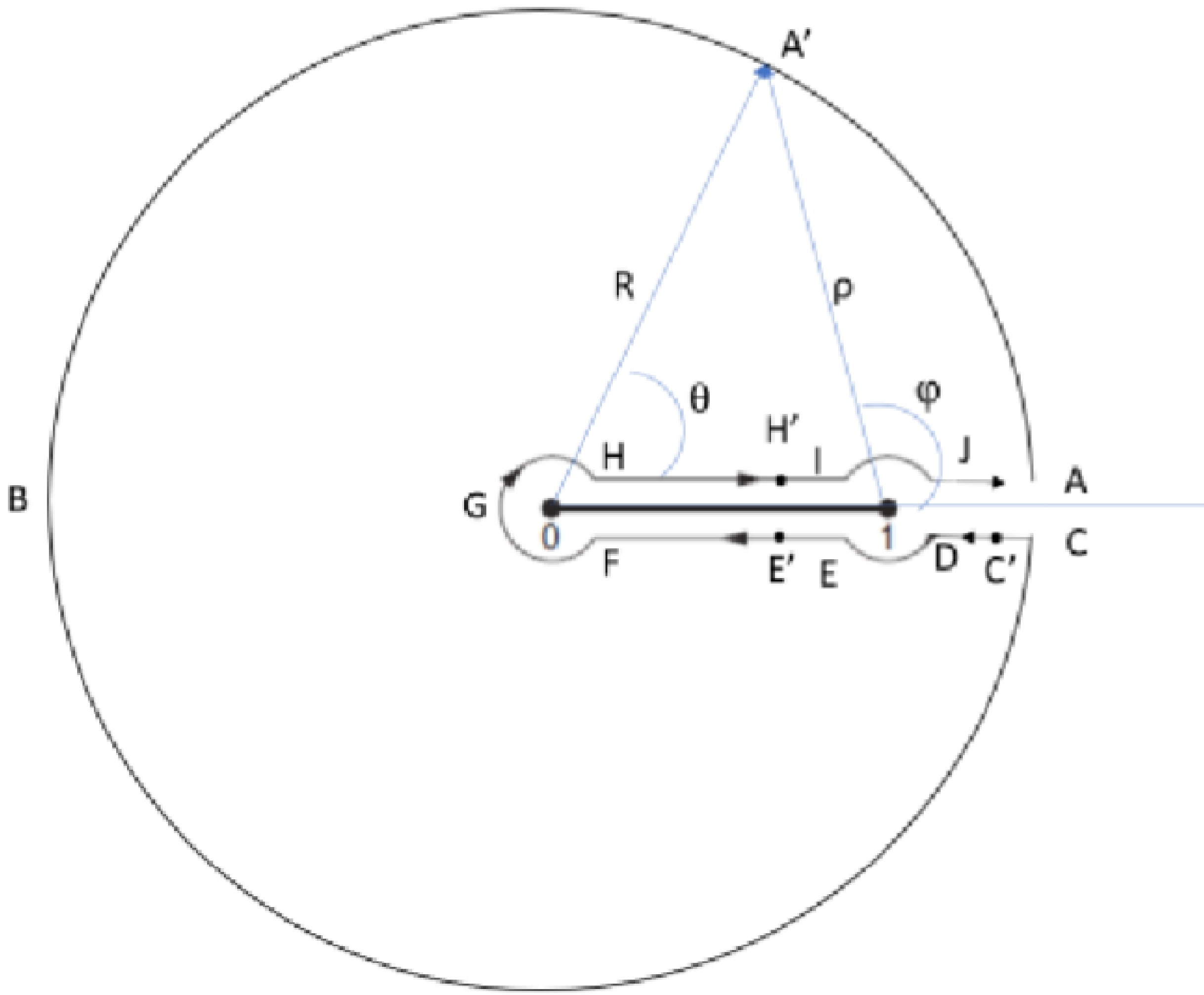
For the second function, we can identify from the denominator that there is a pole of order 3 at $z = 1$ and a pole of order 2 at $z = \frac{-2}{3}$. We can also note that as $z \to \infty$, the numerator will grow faster than the denominator, so the limit will go to infinity - the limit of the reciprocal will then go to zero, and so $z = \infty$ is a pole as well.

For the third function, there is a pole of order 2 due to the $\frac{1}{z^2}$ term. At $z = 1$, there is an essential singularity introduced by $\sin\left(\frac{1}{1-z}\right)$, since it goes to ∞ as $z \to 1$, and the sine function oscillates infinitely often near an essential singularity.

4. Consider $f(z) = \frac{1}{(z^2 - s^3)^{1/3}}$.

   a) Locate and name the singularities and branch points of $f(z)$.

A possible closed path for which $f(z)$ is single-valued (starting from A' and moving along the path $A'BCDEFGHIJAA'$) on the complex plane is plotted below:



Solution:

We find singularities where the denominator is equal to zero - i.e. we need to solve

$$z^2 - s^3 = 0$$

and so we find singularities at

$$z = \pm s^{3/2}$$

These singularities are not removable, essential, or poles, so they are branch points, which occur when functions are multivalued. The cube root function is one such function that is multivalued.

The value of the denominator would also shrink to zero as $z \to \infty$, so this is also a branch point.

5.

a) Determine if the functions $f_1(z) = |z|^2$ and $f_2(z) = \frac{1}{z^2}$ are analytic.

b) Evaluate the path integral $\int_c f_1(z)\,dz$ where the contour C

1. the straight line segment with the initial point $z = -1$ and final point $z = i$.
2. the 90° arc of the unit circle with initial point $z = -1$ and final point $z = i$.

c) Evaluate the path integral $\int_c f_2(z)\,dz$ for the above two contours.

d) Do the values of the two path integrals for $f_1(z)$ agree? What about the values of the path integrals for $f_2(z)$? Can you give a reason for these observations?

Solution:

To determine if a function is analytic, we need to check if it satisfies the Cauchy-Riemann equations:

The first function, $f_1(z) = |z^2|$ can be written in terms of x and y as:

$$f_1(z) = |x + iy|^2 = x^2 + y^2$$

This is a real-valued function, and its partial derivatives with respect to x and y

$$\frac{\partial f_1}{\partial x} = 2x, \quad \frac{\partial f_1}{\partial y} = 2y$$

The Cauchy-Riemann equations state that for a function with real and imaginary components u and v, respectively, the following must hold

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $f_1(z)$ is purely real (i.e., $v(x,y)=0$), its partial derivative w.r.t y would need to be zero for it to satisfy the Cauchy-Riemann equations - since it is not, it is not analytic.

The function $f_2(z)=\frac{1}{z^2}$ can be written as:

$$f_2(z)=\frac{1}{(x+iy)^2}=\frac{1}{x^2-y^2+2ixy}$$

To express this in terms of real and imaginary parts, we can multiply the numerator and denominator by the complex conjugate of the denominator:

$$f_2(z)=\frac{x^2-y^2-2ixy}{(x^2-y^2)^2+(2xy)^2}$$

This gives us

$$u(x,y)=\frac{x^2-y^2}{(x^2-y^2)^2+(2xy)^2},\quad v(x,y)=\frac{-2xy}{(x^2-y^2)^2+(2xy)^2}$$

visual examination of these functions would reveal that this function is not analytic either, though my confidence interval is much lower than in the case of f_1.