PHYS455 Assignment II - Potential

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1. Using previously derived electric fields and setting the potential to zero at infinity:

a). What is the potential as a function of position **outside** of a uniformly charged, infinitesimally thin, spherical shell of radius R (i.e., for $r > R$)?

Solution:

Via Gauss' Law, the electric field outside the shell is the same as that of a point charge at the origin as all of the charge is enclosed and the charge distribution is spherically symmetric:

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

...for $r > R$. We can then integrate this to find the potential:

$$V(r) - \cancel{V(\infty)} = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}$$

$$V(r) = -\frac{Q}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{1}{r'^2}\, dr'$$

where the prime is to separate the r dependence of the electric field from the point $r > R$ we are integrating to.

$$V(r) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r'}\right]_{\infty}^{r} = \frac{Q}{4\pi\epsilon_0 r}$$

which is the potential due to a point charge as expected.

b). What is the potential as a function of position **inside** of a uniformly charged, infinitesimally thin, spherical shell of radius R (i.e., for $r < R$)?

Solution:

We can use a similar procedure to find the potential inside the shell, again working our way in from infinity. We know the field outside the shell is that of a point charge, and the field inside the shell is zero from Gauss's law because no charge is enclosed. So:

$$V(r) = \frac{-Q}{4\pi\epsilon_0}\int_{\infty}^{R}\frac{1}{r'^2}\,dr' - \int_{R}^{r}(0)\,dr'$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r'}\right]_{\infty}^{R} = \frac{Q}{4\pi\epsilon_0 R}$$

...which is a constant as expected.

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2. There are N concentric thin spherical shells with radii r_i, with i ranging from 1 through N, and $r_i < r_{i+1}$. Each shell is spherically symmetric with the i-th shell having charge Q_i . Show that the scalar potential for $r_i < r < r_{i+1}$ is

$$\phi(r) = \frac{1}{4\pi\epsilon_0 r}\sum_{j=1}^{i} Q_j + \frac{1}{4\pi\epsilon_0}\sum_{j=i+1}^{N}\frac{Q_j}{r_j}$$

Solution:

As in the previous question, we can note that for $r_i < r < r_{i+1}$, a Gaussian surface of radius r would enclose all of the shells r_0 to r_i, (i.e. all the charge up to Q_i) and the electric field would look like that of a point charge at the origin with the total charge being the sum of all those charges. The same is true of the potential and the first half of the problem is very similar to the previous problem: the potential due to the shells interior to r is given by

$$\phi_{\text{inside}}(r) = \frac{1}{4\pi\epsilon_0 r}\sum_{j=1}^{i} Q_j$$

We then need to account for all the shells outside of r - these cannot be treated the same way, and instead will need to be considered individually, as each shell has a different radius. Because potential obeys the principle of superposition, the potential at a point r is the sum of the potentials due to all the shells, so long as the same reference point is chosen (∞, in our case).

For the shell of radius r_j with charge Q_j, we can find the potential due to that shell at some point $r < r_j$ by using the same formulation as above:

$$\phi_j(r) = -\frac{Q_j}{4\pi\epsilon_0}\int_{\infty}^{r_j}\frac{1}{r'^2}\,dr'$$

$$= \frac{Q_j}{4\pi\epsilon_0}\left[\frac{1}{r'}\right]_{\infty}^{r_j}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q_j}{r_j}$$

and the sum of all such shells, starting at r_{i+1}, would give the total potential for all shells up to N outside the point of interest at r:

$$\phi_{\text{outside}}(r) = \frac{1}{4\pi\epsilon_0} \sum_{j=i+1}^{N} \frac{Q_j}{r_j}$$

...where the constant $\frac{1}{4\pi\epsilon_0}$ has been brought outside the sum.

Thus the total potential at r due to all the shells, interior and exterior to r, is

$$\phi(r) = \phi_{\text{inside}} + \phi_{\text{outside}} = \frac{1}{4\pi\epsilon_0 r} \sum_{j=1}^{i} Q_j + \frac{1}{4\pi\epsilon_0} \sum_{j=i+1}^{N} \frac{Q_j}{r_j}$$

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3. There is a rectangular cavity, within which the volume charge density is zero. The boundary conditions on the scalar potential ϕ are such that there is no dependence of ϕ on z. We specify the cavity by its four vertices in a plane perpendicular to $\hat{z}$. Its vertices in that plane are $(0,0), (a,0), (0,b)$, and (a,b). The boundary condition is $\phi(0,y) = 0, \phi(x,0) = 0, \phi(a,y) = 0$, and $\phi(x,b) = \alpha x b$.

Show that

$$\phi = \sum_{n=1}^{\infty} A_n \sin(k_n x) \left[e^{k_n x} - e^{-k_n y}\right]$$

where

$$A_n = \frac{2\alpha b}{a\left(e^{k_n b} - e^{-k_n b}\right)} \int_0^a x \sin(k_n x)\, dx$$

and

$$k_n = \frac{n\pi}{a}.$$

Solution:

Since there is no charge density within the region, the scalar potential in the region satisfies Laplace's equation:

$$\nabla^2 \phi = 0$$

and, since there is no dependence on z (i.e. $\frac{\partial \phi}{\partial z} = 0$), this reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which is the differential equation we need to find a solution for, subject to the boundary conditions given in the question. Via the first uniqueness theorem, if the potential is known over the entire boundary of a region, the solution to Laplace's equation is uniquely determined, so any function that fulfills the boundary conditions is indeed *the* solution.

We are going to look for a solution in the form of a product of functions of the separate variables at play, i.e. in the form

$$\phi(x, y) = X(x)Y(y)$$

Putting this into the above, we have

$$Y(y)\frac{\partial^2 X}{\partial x^2} + X(x)\frac{\partial^2 Y}{\partial y^2} = 0$$

(since X is a function of x only, and vice versa). Rearranging this to separate the variables, we can obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

where each term is dependent only on one variable. The only way that this can be true is if both of these terms are constant - if one was held constant and the other allowed to vary, then the above equation could no longer be true. They could also both be zero but that would not be particularly illuminating. We can then say that

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1, \ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2$$

and

$$C_1 - C_2 = 0$$

so one of these constants must be the negative of the other. We can then put the partial differential equation we started with in terms of two ordinary differential equations - since the constants are undetermined, we can rearrange them into the square of some other undetermined constant, and have

$$\frac{\partial^2 X}{\partial x^2} = -k^2 X \text{ and } \frac{\partial^2 Y}{\partial y^2} = k^2 Y$$

the general solutions to which are

$$X(x) = A\sin(kx) + B\cos(kx)$$

$$Y(y) = Ce^{ky} + De^{-ky}$$

and we can express ϕ as

$$\phi(x, y) = (A\sin(kx) + B\cos(kx))\left(Ce^{ky} + De^{-ky}\right)$$

This is the separable solution to Laplace's equation - we now need to find the coefficients that will cause this to satisfy the given boundary conditions. Note that immediately we can say that $B = 0$ because the cosine function cannot take on the value of zero when its argument is zero and therefore it is impossible to satisfy the boundary condition $\phi(0, y) = 0$

with the cosine term there (though some will point out they never needed it anyways, as any function of interest in this context will be expressible as a Fourier sine series).

In order to satisfy $\phi(0, y) = \phi(a, y)$ always, we must have

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \sin(k_n x) \left(C_n e^{k_n y} + D_n e^{-k_n y}\right)$$

where $k_n = \frac{n\pi}{a}$.

In order for $\phi(x, 0) = 0$ to hold we need $C_n = -D_n$ (as the exponential terms will become 0 when $y = 0$), and this must hold for all n - therefore we can replace D_n by $-C_n$ and factor that out, combining it with A_n, which is allowed as the constants are still undetermined:

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \sin(k_n x) \left(e^{k_n y} - e^{-k_n y}\right)$$

We have one final boundary condition to match, which is $\phi(x, b) = \alpha x b$. We can recognize that the above is a Fourier sine series expansion, where the coefficient A_n is given by the formula

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

where L is the interval under consideration, here equal to a. The above formula is reliant on the orthogonality properties of We know that the value of the function $f(x)$ (in this case, $\phi(x, b)$) is $\alpha x b$, so we can write

$$A_n = \frac{2\alpha b}{a} \int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx$$

but if we set $y = b$ in the solution above (before looking at A_n) to check the boundary condition, we find

$$\alpha x b = \sum_{n=1}^{\infty} A_n \sin(k_n x) \left(e^{k_n b} - e^{-k_n b}\right)$$

and so we realize we need the factor of $\left(e^{k_n b} - e^{-k_n b}\right)$ to 'normalize' the coefficients A_n and ensure they match the boundary conditions without becoming undefined:

$$A_n = \frac{2\alpha b}{a\left(e^{k_n b} - e^{-k_n b}\right)} \int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx$$

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4. There is a rectangular cavity, within which the volume charge density is zero. The boundary conditions on the scalar potential ϕ are such that there is no dependence of ϕ on z. We specify the cavity by its four vertices in a plane perpendicular to $\hat{z}$. Its vertices

in that plane are $(0,0), (a,0), (0,b)$, and (a,b). The boundary condition is $\phi(0,y)=0, \phi(x,0)=2\alpha\sin\left(\frac{7\pi x}{a}\right), \phi(a,y)=0$, and $\phi(x,b)=0$. What is $\phi(x,y)$?

Solution:

The nonzero boundary condition is a function of x, so the solution will take the form

$$\phi(x,y)=\sum_{n=1}^{\infty} A_n \sin(k_n x)\left(C_n e^{k_n y}+D_n e^{-k_n y}\right)$$

where k_n will again be equal to $\frac{n\pi}{a}$. We will again need $C=-D$ and so we have

$$\phi(x,y)=\sum_{n=1}^{\infty} A_n \sin(k_n x)\left(e^{k_n y}-e^{-k_n y}\right)$$

If we examine the form of the coefficients A_n, however, we note that the boundary condition is a sine function of the form $k_n x$, where $k_n=\frac{7\pi}{a}=k_7$. Therefore via orthogonality the only term that will contribute is for A_7, given by

$$A_7=\frac{4\alpha}{a}\int_0^a \sin^2\left(\frac{7\pi x}{a}\right)dx=2\alpha$$

and so the sum is unnecessary and the potential is given by

$$\phi(x,y)=2\alpha\sin\left(\frac{7\pi x}{a}\right)\left(e^{7\pi y/a}-e^{-7\pi y/a}\right)$$
