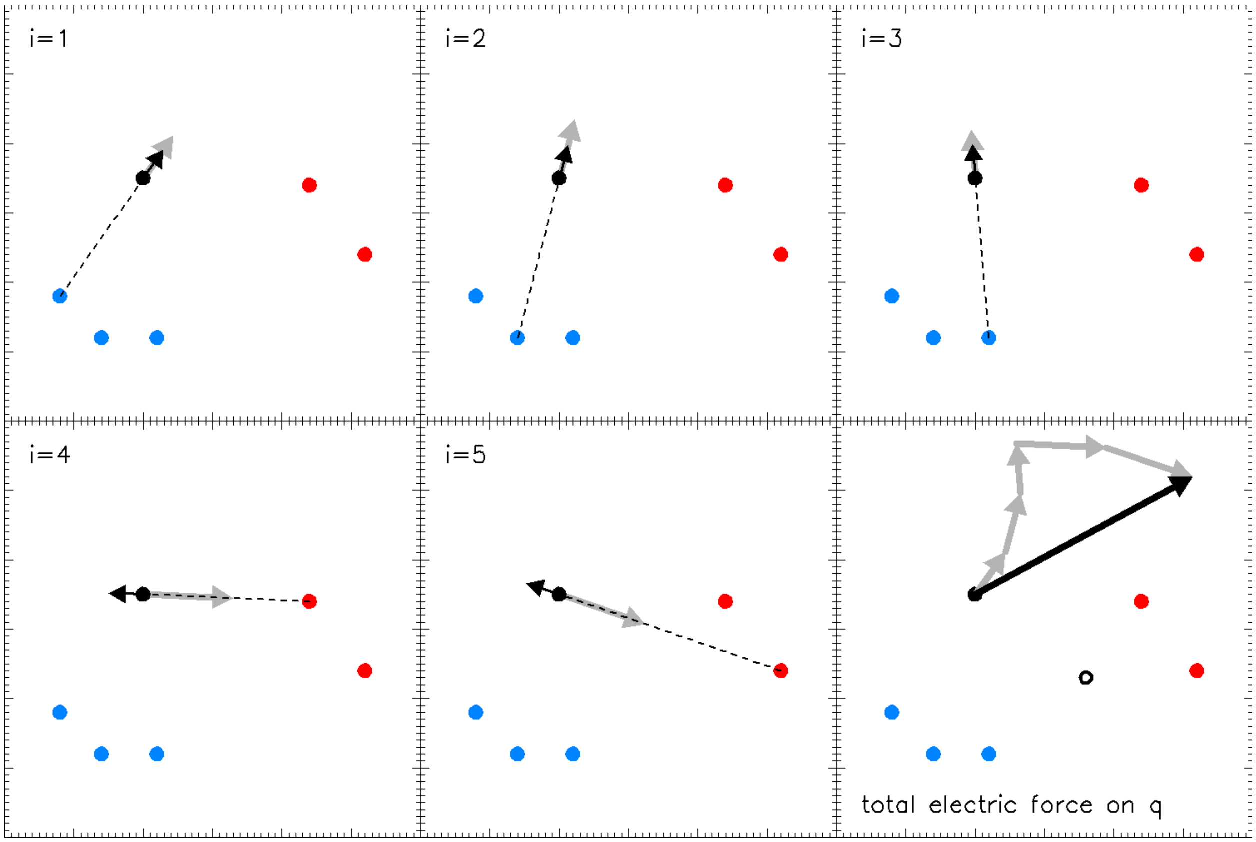
P455 - Assignment 1 - Electric Fields

Released in Class - September 15th, 2023

Due - 5:00pm September 20th, 2023

EFA-1: The figure below is a panel plot illustrating how using $\vec{E} = \sum_{i=1}^{i=N} \Delta\vec{E}_i$, where $\Delta\vec{E}_i = qq_i(\vec{r} - \vec{r}_i)/4\pi\epsilon_\circ|\vec{r} - \vec{r}_i|^3$, to determine the force on a point charge q due to N other point charges (q_1 through q_N) works in practice. Here, $N = 5$, and the location of q is indicated by the black solid circle, and of the five other charges by the red and blue solid circles. Shown are the unit vectors $(\vec{r} - \vec{r}_i)/|\vec{r} - \vec{r}_i|$, the forces $\Delta\vec{F}_i$, the radial separations $|\vec{r} - \vec{r}_i|$, and the total force on the charge q.



a) Looking at charges q_1 through q_5, Which color indicates a positive charge, and which a negative one?

b) Which vectors indicate the forces $\Delta\vec{F}_i$?

c) Which vectors indicate the unit vectors $(\vec{r} - \vec{r}_i)/|\vec{r} - \vec{r}_i|$?

d) Which vector indicates the net force on q?

e) How are the distances $|\vec{r} - \vec{r}_i|$ indicated?

a. The blue solid circles have forces directed radially away from the probe charge q, and the red circles have a force directed towards q, so blue corresponds to positive charges and red to negative per the definition of the electric field.

b. The forces ΔF_i are indicated by the grey arrows.

c. The unit vectors are indicated by the small thin black arrows - this is easy to conclude as they are the only vectors that are all the same length

d. The net force on q is indicated by the thicker black arrow in the sixth panel, the vector sum of all the grey arrows.

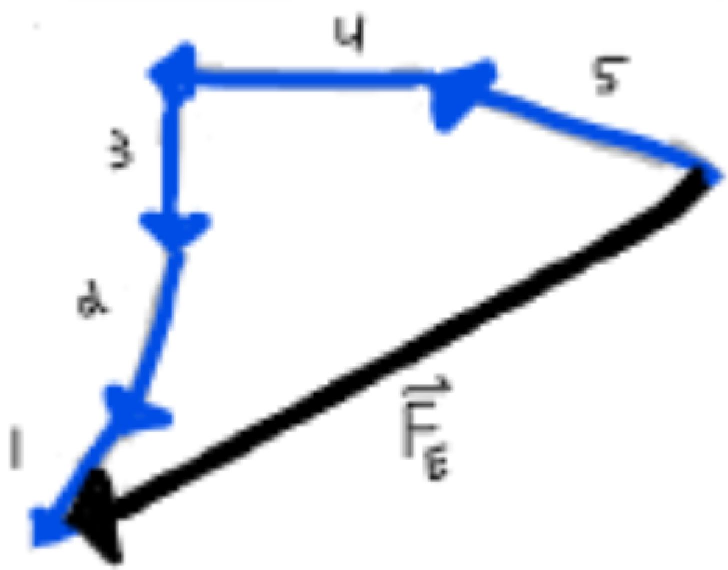
e. The separation vectors between q and the charges q_i are indicated by the thin dotted black lines in panels 1 through 5.

f) If you were to remove the charge q and instead place a charge $-q$ (meaning a charge of equal amount but negative rather than positive) at the location of the open circle, what would the total electric force on it look like (meaning draw a vector on the figure, indicating very roughly the length and direction of that force).

g) What are the (approximate) ratios q_2/q_1, q_3/q_1, q_4/q_1, and q_5/q_1? Note you should be able to get these to within 5 percent relative error fairly straightforwardly.

Note: In each case state clearly your reasoning.

f. If q were exchanged for $-q$, the direction of the force exerted by the charges q_i on $-q$ would be reversed, and thus the total force would be a vector with equal magnitude in the opposite direction:



g. The magnitude of the force due to q_1 on q is given by

$$|\vec{F}_1| = \frac{qq_1}{|r_1 - r_i|}$$

where $|r_1 - r_i|$ is the distance between q_1 and q.

For the force due to q_2 on q:

$$|\vec{F}_2| = \frac{qq_2}{|r_2 - r_i|}$$

We can use these expressions to examine the charge ratios - for q_1 and q_2, we have

$$\frac{\Delta\vec{F}_1}{\Delta\vec{F}_2} = \frac{qq_1(r_2 - r_i)}{qq_2(r_1 - r_i)} \rightarrow \frac{q_2}{q_1} = \frac{\vec{F}_2(r_2 - r_i)}{\vec{F}_1(r_1 - r_i)}.$$

A measurement of the force and separation vectors yields the following ratios:

$$\frac{q_2}{q_1} = \frac{\vec{F}_2(r_2 - r_i)}{\vec{F}_1(r_1 - r_i)} = \frac{(1\text{cm})(2.4\text{cm})}{(0.8\text{cm})(2\text{cm})} = 1.5$$

$$\frac{q_3}{q_1} = \frac{\vec{F}_3(r_3 - r_i)}{\vec{F}_1(r_1 - r_i)} = \frac{(0.7\text{cm})(2.2\text{cm})}{(1.6\text{cm}^2)} = 0.9625$$

$$\frac{q_4}{q_1} = \frac{\vec{F}_4(r_2 - r_i)}{\vec{F}_1(r_1 - r_i)} = \frac{(1.2\text{cm})(2.3\text{cm})}{(1.6\text{cm}^2)} = 1.725$$

$$\frac{q_5}{q_1} = \frac{\vec{F}_5(r_5 - r_i)}{\vec{F}_1(r_1 - r_i)} = \frac{(1.2\text{cm})(3.2\text{cm})}{(1.6\text{cm}^2)} = 2.6$$

EFA-2: There is a thin disk of charge, of radius a, in the xy-plane, centered on the origin. The surface charge density on the disk is σ, a constant, such that the total charge is $Q = \pi a^2 \sigma$. Show that the electric field on the $z-$axis is given by

$$\vec{E}(z) = E_z(z)\hat{z} = \frac{\sigma \hat{z}}{2\epsilon_\circ} \left[\frac{\sqrt{a^2+z^2} - z}{\sqrt{a^2+z^2}} \right] \frac{z}{|z|}$$

Show as well that this expression correctly gives the sign of E_z for both positive and negative z, that for $|z| \gg a$ the limiting behaviour is that of a point charge $Q = \pi r^2 \sigma$ at the origin, and that for $|z| \ll a$, the limiting behaviour is that of an infinite thin sheet of uniform charge density σ on the z=0 plane. *Hint*: sum, via integration over r, concentric rings of dq, using the expression from the previous question.

Since the disk is centered on the origin, we can see via symmetry that the x and y components of the electric field of the disk will cancel along the z-axis, thus

$$\vec{E}(z) = E_z(z)\,\hat{\mathbf{z}}$$

and the electric field is entirely oriented in the $\hat{\mathbf{z}}$ direction.

We can find the form of the field by considering concentric *rings* of charge, with radius r, that contribute some charge Δq to the total charge Q. We can then integrate the expression for a ring from $r = 0$ to $r = a$ to obtain the electric field of the disk.

We start with the electric field $\Delta\vec{E}$ of a thin ring of charge at a point on the z-axis, which has the same symmetry conditions:

$$\Delta\vec{E}_z(z) = \frac{1}{4\pi\epsilon_0} \frac{\Delta q\, z}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}$$

where r is the inner radius of the ring and $r + \Delta r$ is the outer radius. We do not know Δq, but we do know the total charge $Q = \pi a^2 \sigma$, so the charge in the ring must be

$$\begin{aligned} \Delta q &= Q\left(\frac{\text{area of ring}}{\text{area of disk}}\right) \\ &= Q\left(\frac{2\pi r \Delta r}{\pi a^2}\right) \\ &= \cancel{\pi a^2}\,\sigma\left(\frac{2\pi r \Delta r}{\cancel{\pi a^2}}\right) \\ &= 2\pi\sigma r\Delta r \end{aligned}$$

Returning to the $\vec{E}$ equation:

$$\Delta \vec{E}_z(z) = \frac{1}{4\cancel{\pi}\epsilon_0} \frac{2\cancel{\pi}\sigma r \Delta r\, z}{(r^2+z^2)^{3/2}} \hat{\mathbf{z}}$$

Putting this into differential form, we can then integrate to find the total contribution of all the disks from radius 0 to radius a:

$$dE_z(z) = \frac{\sigma}{2\epsilon_0} \frac{z}{(r^2+z^2)^{3/2}} r dr\, \hat{\mathbf{z}}$$

$$E_z(z) = \frac{\sigma z}{2\epsilon_0} \int_0^a \frac{r}{(r^2+z^2)^{\frac{3}{2}}}\, dr\, \hat{\mathbf{z}}$$

where z has been pulled out of the integral at the end because it is the z coordinate of the point we are examining the field at, and is constant for this integral.

We evaluate the integral using a u-substitution. The ugliest looking part is the denominator of the integrand, so we'll set

$$u = r^2 + z^2$$

and

$$\frac{du}{dr} = 2r \to du = 2rdr \to dr = \frac{1}{2r} du$$

leaving us with

$$E_z(z) = \frac{\sigma z}{2\epsilon_0} \int_0^a \frac{1}{2\cancel{r}} \left(\frac{\cancel{r}}{u^{\frac{3}{2}}} \right) \hat{\mathbf{z}}$$

$$= \frac{\sigma z}{2\epsilon_0} \left(\frac{-2}{\sqrt{u}} \right) \Bigg|_0^a$$

$$= \frac{\sigma z}{4\epsilon_0} \left(\frac{-2}{\sqrt{r^2+z^2}} \right) \Bigg|_0^a$$

$$= \frac{\sigma z}{2\epsilon_0} \left(\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}} \right)$$

$$= \frac{\sigma \cancel{z}}{2\epsilon_0} \left(\frac{\sqrt{a^2+z^2} - z}{\cancel{z}\sqrt{a^2+z^2}} \right)$$

$$= \frac{\sigma}{2\epsilon_0} \left(\frac{\sqrt{a^2+z^2} - z}{\sqrt{a^2+z^2}} \right) \frac{z}{|z|}$$

where the last term was added to give the direction (negative z will result in the force being directed downwards, towards the disk, whereas positive z will result in upwards force, away from the disk, as expected). Thus:

$$E_z(z)\, \hat{\mathbf{z}} = \frac{\sigma \hat{\mathbf{z}}}{2\epsilon_0} \left(\frac{\sqrt{a^2+z^2} - z}{\sqrt{a^2+z^2}} \right) \frac{z}{|z|}.$$

ase of $|z| \gg a$, a becomes negligible compared to z, so in r the ring, $(r^2+z^2)^{3/2} \approx z^3$, and if this is integrated to find the disk as before, we end up with an electric field proportional to $\frac{1}{z^2}$, which is the form of the field of a radial point charge. The width of the ring becomes negligible in the limit of large z.

$$E_z(z)\, \hat{\mathbf{z}} = \frac{\sigma \hat{\mathbf{z}}}{2\epsilon_0} \left(\frac{\sqrt{z^2} - z}{\sqrt{a^2+z^2}} \right) \frac{z}{|z|}$$

For very very small $|z| \ll a$, we have

$$E_z(z) \approx \frac{\sigma \hat{\mathbf{z}}}{2\epsilon_0} \cancel{\frac{\sqrt{a^2}}{\sqrt{a^2}}} \frac{z}{|z|} = \frac{\sigma \hat{\mathbf{z}}}{2\epsilon_0} \frac{z}{|z|}$$

..and the dependence on z has vanished entirely, which is the form of an infinite plane of charge.

EFA-3: Show that the for time stationary situations, the curl of the electric field is zero. Starting directly from Coulomb's Law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ} \int \rho(\vec{r}') \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\tau'$$

and taking the curl of the integral, show that the curl of the static electric field is zero.

Here we are integrating over the primed variables, and the unprimed $\vec{r}$ is a constant for the integral. We will need to put both $\vec{r}$ and $\vec{r'}$ in terms of Cartesian coordinates:

$$\vec{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$
$$\vec{r'} = x'\,\hat{\mathbf{x}} + y'\,\hat{\mathbf{y}} + z'\,\hat{\mathbf{z}}$$

We can then begin to transform our integrand into Cartesian coordinates:

$$(\vec{r} - \vec{r'}) = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$
$$|\vec{r} - \vec{r'}|^3 = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{\frac{3}{2}}$$

So the curl we need to find is

$$\vec{\nabla} \times \left(\rho(x', y', z') \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{\frac{3}{2}}} \right)$$

...where the integral has been dropped as we can interchange taking the curl of the integral and taking the integral of the curl - so we are just interested in the curl of the integrand.

However, we can make a simplification here - if there is zero charge in the region (i.e. the integral of $\rho(x', y', z')$ over the region of interest is equal to zero), then of course the curl is trivially zero. This is unhelpful, so assume that there is some charge in the region of interest. Then, the only way the curl can be zero is is if the curl of the other term is equal to zero:

$$\vec{\nabla} \times \left(\frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{\frac{3}{2}}} \right)$$

Let's call the term in brackets $\vec{A}$. The curl of a vector function such as $\vec{A}$ can be written

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)\hat{\mathbf{z}}$$

We will work though one component to demonstrate that it is equal to zero. Identical logic follows for the other components.

We need to find the derivative of the z-component of the integrand with respect to y', and vice versa, to determine if the x-component of the curl is zero.

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$$(\vec{\nabla} \times A)_x = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)$$

The first term here is

$$\frac{\partial A_z}{\partial y} = \frac{\partial}{\partial y}\left[\frac{(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\right]$$

If we rewrite the denominator as $R(y') = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$:

$$\Delta \vec{E}_z(z) = \frac{1}{4\cancel{\pi}\epsilon_0}\frac{2\cancel{\pi}\sigma r \Delta r\, z}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}$$

We use the quotient rule and the chain rule (recalling that R is a function of y')

$$\begin{aligned}\frac{\partial A_z}{\partial y'} &= \frac{(0 \cdot R^3) - (z-z') \cdot 3R^2 \frac{-(y-y')}{R}}{R^6} \\ &= \frac{3(y-y')(z-z')}{R^5} \\ &= \frac{3(y-y')(z-z')}{}\end{aligned}$$

Similar logic follows for $\frac{\partial A_y}{\partial z'}$:

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$$\begin{aligned}\frac{\partial A_y}{\partial z'} &= \frac{(0 \cdot R^3) - (y-y') \cdot 3R^2 \frac{-(z-z')}{R}}{R^6} \\ &= \frac{3(y-y')(z-z')}{R^5} \\ &= \frac{3(y-y')(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{5}{2}}}\end{aligned}$$

Thus

$$\begin{aligned}(\vec{\nabla} \times A)_x &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \\ &= \left(\frac{3(y-y')(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{5}{2}}} - \frac{3(y-y')(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{5}{2}}}\right) \\ &= 0\end{aligned}$$

and the same holds for the other two components of the curl of the integrand.

EFA-4: The following are true:

$$\vec{E}(r < a) = -\frac{Qr\hat{r}}{4\pi\epsilon_\circ a^3}$$
$$\rho(r > a) = 0 \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

What is the total charge?

Answer:

Start with Gauss' Law in differential form:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

The divergence of the electric field, in spherical coordinates, was given in the course notes as

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r)$$

Since our given electric field is exclusively in the radial direction, we can replace E_r with the entire magnitude of the electric field:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{Qr}{4\pi\epsilon_0 a^3}\right)\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{Qr^3}{4\pi\epsilon_0 a^3}\right) \\ &= \frac{1}{r^2}\left(-\frac{3Qr^2}{4\pi\epsilon_0 a^3}\right) \\ &= -\frac{3Q}{4\pi\epsilon_0 a^3}\end{aligned}$$

Setting this equal to the RHS of Gauss's Law:

$$\frac{\rho}{\epsilon_0} = -\frac{3Q}{4\pi\epsilon_0 a^3}$$

$$Q = -\frac{4}{3}\pi a^3 \rho$$

I.e. the volume of the sphere of radius a multiplied by the charge density, which seems like a sensible result.