PHYS435 Assignment 2

Max Stronge (30064749)

1.

An excited atom can lose its excitation energy in the form of spontaneous emission of a photon. The excited electron in the atom is treated as a damped harmonic oscillator with mass m, spring constant k, natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$, and damping constant γ. The time-dependent amplitude of the oscillation can be obtained from the damped harmonic oscillator given by

$$\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0.$$

a) by using the Fourier transform of the above ODE, show that

$$\omega_\pm = \frac{i\gamma}{2} \pm \omega$$

NB: $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ is the frequency of the oscillation.

Solution:

Taking the Fourier transform of each term, we find

$$FT\left(\frac{d^2x(t)}{dt^2}\right) = (i\omega)^2\tilde{X}(\omega) = -\omega^2\tilde{X}(\omega)$$

...where $\tilde{X}(\omega)$ is the Fourier transform of $x(t)$.

Proceeding to the next term:

$$FT\left(\gamma\frac{dx(t)}{dt}\right) = \gamma FT\left(\frac{dx(t)}{dt}\right) = \gamma i\omega\tilde{X}(\omega)$$

and the final one:

$$FT(\omega_0^2 x(t)) = \omega_0^2\tilde{X}(\omega)$$

So we have

$$-\omega^2\tilde{X}(\omega) + \gamma i\omega\tilde{X}(\omega) + \omega_0^2\tilde{X}(\omega) = 0$$

and we can divide out the unknown function

$$-\omega^2 + \gamma i\omega + \omega_0^2 = 0$$

Applying the quadratic formula here leaves us with

$$\omega_\pm = \frac{\gamma i \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2}$$

Recall that $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$. Multiplying this equation by 2 we get

$$2\omega = 2\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \sqrt{4\omega_0^2 - \gamma^2}$$

which we recognize as part of the numerator above - making this substitution, we have

$$\omega_{\pm} = \frac{\gamma i}{2} \pm \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2}$$

$$= \frac{\gamma i}{2} \pm \frac{2\omega}{2} = \frac{\gamma i}{2} \pm \omega$$

...which was to be shown.

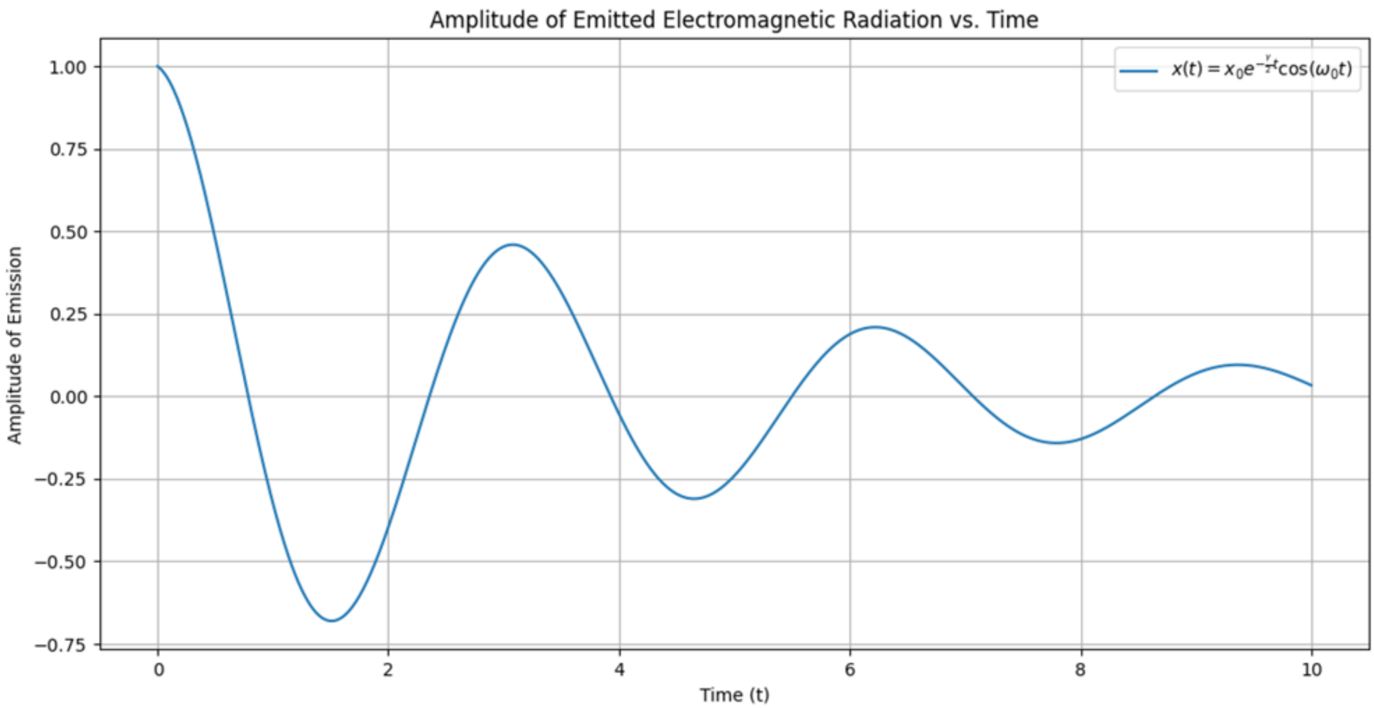
---

b) The solution to the differential equation above for a slightly damped oscillator is

$$x(t) = x_0 e^{-(\gamma/2)t} \cos(\omega_0 t)$$

Plot $x(t)$ for a slightly damped oscillator.

Solution:



---

c) Calculate $A(\omega) = FT[x(t)]$ assuming the spontaneous emission (i.e. the oscillation) occurs at time $t = 0$.

Solution:

We start by substituting the given function $x(t)$ into the definition of the Fourier transform:

$$A(\omega) = \int_{-\infty}^{\infty} x_0 e^{-(\gamma/2)t} \cos(\omega_0 t) e^{-i\omega t} dt$$

Rewriting the $\cos$ term using Euler's formula we have

$$A(\omega) = \frac{x_0}{2} \int_{-\infty}^{\infty} e^{-(\gamma/2)t} (e^{i\omega_0 t} + e^{-i\omega_0 t}) e^{-i\omega t}\, dt$$

We can then separate this into two integrals:

$$A(\omega) = \frac{x_0}{2} \left[\int_{-\infty}^{\infty} e^{-(\gamma/2)t} e^{i\omega_0 t} e^{-i\omega t}\, dt + \int_{-\infty}^{\infty} e^{-(\gamma/2)t} e^{-i\omega_0 t} e^{-i\omega t}\, dt \right]$$

We can then combine the last two exponential terms in each integrand:

$$A(\omega) = \frac{x_0}{2} \left[\int_{-\infty}^{\infty} e^{-(\gamma/2)t} e^{i(\omega_0 - \omega)t}\, dt + \int_{-\infty}^{\infty} e^{-(\gamma/2)t} e^{i(-\omega_0 - \omega)t}\, dt \right]$$

Each of these integrals has the form of the Fourier transform of a decaying exponential function, $e^{-\alpha t}$, which is $\frac{1}{\alpha + i\omega}$.

In this case, for the first integral, $\alpha = \frac{\gamma}{2} - i\omega_0$ and for the second integral, $\alpha = \frac{\gamma}{2} + i\omega_0$.

Upon integration, we should find:

$$A(\omega) = \frac{x_0}{2}\left(\frac{1}{(\gamma/2) - i(\omega_0 - \omega)} + \frac{1}{(\gamma/2) - i(-\omega_0 - \omega)}\right)$$

---

d) Calculate the spectral radiation power density defined as $I(\omega) \propto |A(\omega)|^2$, and normalize the resulting expression using

$$\int_0^\infty I(\omega)\, d\omega = I_0$$

to obtain the normalized line profile as

$$I(\omega) = I_0 \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}^2\right)}$$

This is called the Lorentzian profile.

Solution:

Recall the approximation $\omega - \omega_0 << \omega_0$.

The unnormalized $|A(\omega)|^2$ using the approximation (i.e. replacing $-\omega_0 - \omega$ with $-2\omega_0$) is

$$I_{\text{unnormalized}}(\omega) = \frac{x_0^2}{4}\left[\frac{1}{(\gamma/2)^2 + (\omega_0 - \omega)^2} + \frac{1}{(\gamma/2)^2 + 4\omega_0^2} + \frac{2}{(\gamma/2)^2 + (\omega_0 - \omega)^2} \times \frac{1}{(\gamma/2)^2 + 4\omega_0^2}\right]$$

Combine like terms in $I_{\text{unnormalized}}(\omega)$:

$$I_{\text{unnormalized}}(\omega) = \frac{x_0^2}{4}\left[\frac{1 + 2/\left((\gamma/2)^2 + 4\omega_0^2\right)}{(\gamma/2)^2 + (\omega_0 - \omega)^2} + \frac{1}{(\gamma/2)^2 + 4\omega_0^2}\right]$$

We now need to normalize as directed:

$$\int_0^\infty I_{\text{unnormalized}}(\omega)\, d\omega = \frac{x_0^2\pi}{2\gamma}\left(1 + \frac{2}{(\gamma/2)^2 + 4\omega_0^2}\right) + \frac{x_0^2\pi}{2\gamma}$$

Use this integral to normalize $I_{\text{unnormalized}}(\omega)$.

$$I(\omega) = \frac{I_0}{\frac{x_0^2\pi}{\gamma}\left(1 + \frac{2}{(\gamma/2)^2 + 4\omega_0^2}\right)} \times I_{\text{unnormalized}}(\omega)$$

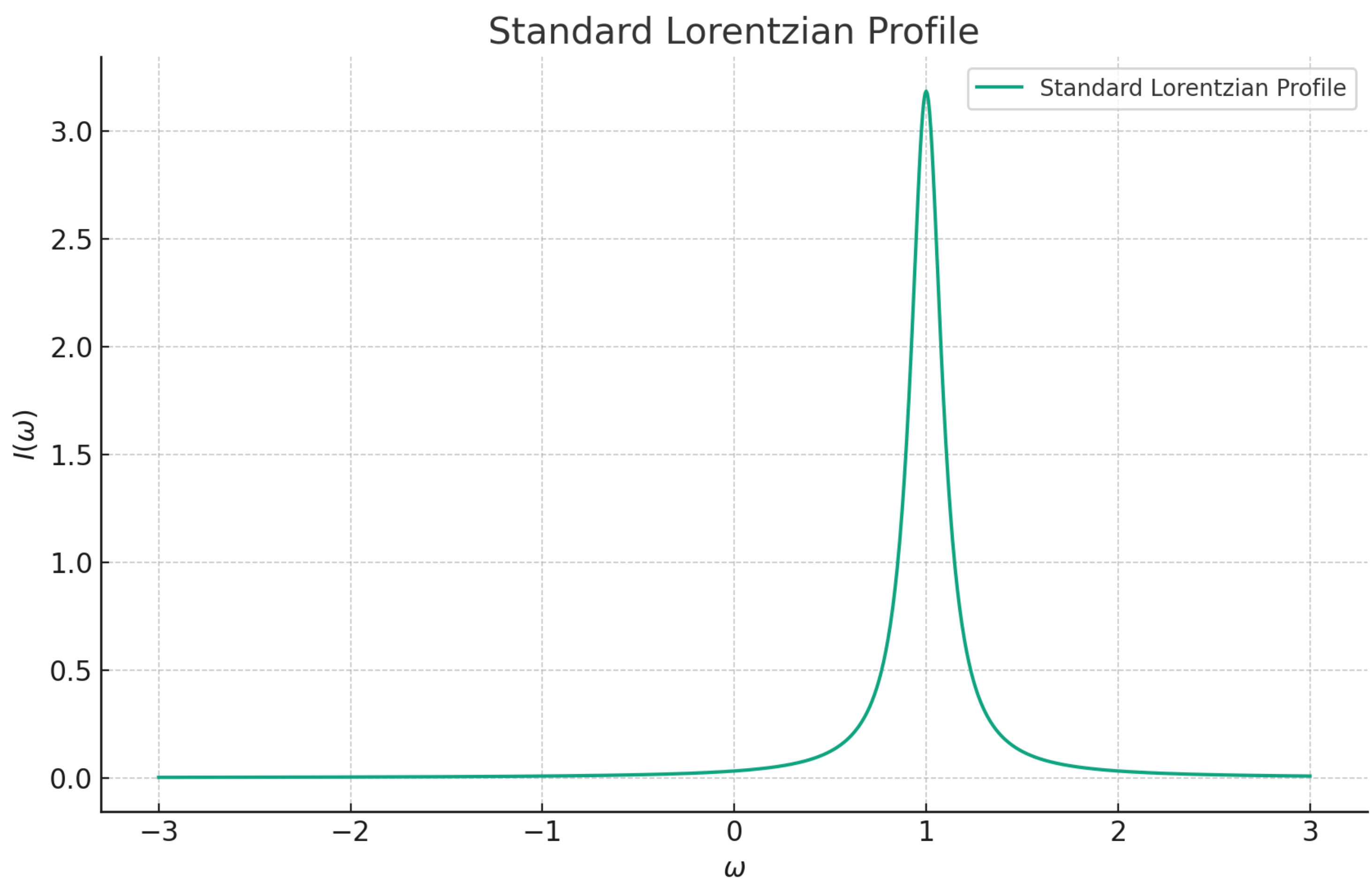
$$= \frac{I_0}{\frac{x_0^2\pi}{\gamma}\cancel{\left(1 + \frac{2}{(\gamma/2)^2 + 4\omega_0^2}\right)}} \quad \frac{x_0^2\pi}{2\gamma}\cancel{\left(1 + \frac{2}{(\gamma/2)^2 + 4\omega_0^2}\right)} + \frac{x_0^2\pi}{2\gamma}$$

$$I(\omega) = I_0 \frac{\gamma/2\pi}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

---

e) Plot the Lorentzian profile and calculate the full width at half maximum (FWHM) of this profile. This is the natural linewidth.

Solution:



For Lorentzian distributions, the FWHM is equal to γ, which in the case of the graph above is 0.2.

2:

a) Show that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, dk$$

where x and k are conjugate variables.

Solution:

We can replace $|f(x)|^2$ with $f(x)f^*(x)$ where $f^*(x)$ is the complex conjugate of $f(x)$:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} f(x)f^*(x) \, dx$$

From here, we can write both $f(x)$ and $f^*(x)$ in terms of the inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx} \, dk$$
$$f^*(x) = \int_{-\infty}^{\infty} \tilde{f}^*(k')e^{-ik'x} \, dk'$$

(where all we have done in the latter expression is replace i with $-i$.) The prime on k is to differentiate it from the other k when we multiply.

We can then substitute these expressions into the above:

$$\int_{-\infty}^{\infty} f(x)f^*(x) \, dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx} \, dk \right) \left(\int_{-\infty}^{\infty} \tilde{f}^*(k')e^{-ik'x} \, dk' \right) dx$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k)\tilde{f}^*(k')e^{i(k-k')x} \, dk \, dk' \, dx$$

We can rearrange the integrals to utilize a property of the delta function:

$$\int_{-\infty}^{\infty} f(x)f^*(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{f}^*(k')\left(\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx\right)dk\,dk'$$

But $\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx = \delta(k-k')$, so we can write

$$\int_{-\infty}^{\infty} f(x)f^*(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{f}^*(k')\,\delta(k-k')\,dk\,dk'$$

We can note that delta function in the integrand will cause the integral to evaluate to zero everywhere where $k \neq k'$, so we can take $k = k'$ and reduce this to a single integral with respect to k:

$$= \int_{-\infty}^{\infty} \tilde{f}(k)\tilde{f}^*(k)\,dk = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,dk$$

which was to be shown.

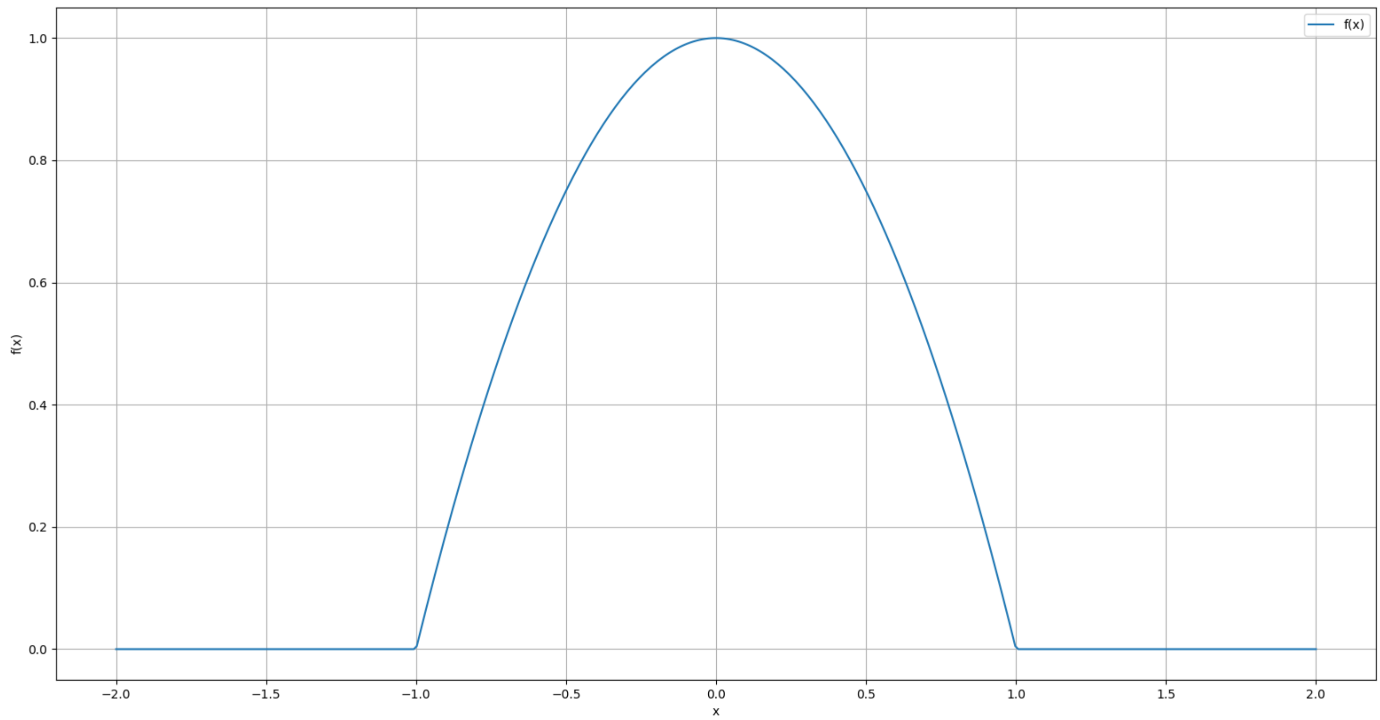
---

b) Calculate the Fourier Transform of the function

$$f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Solution:

The function is plotted below:



Using the definition:

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

but we know that $f(x) = 0$ everywhere except the interval $(-1, 1)$, so we can rewrite the above as

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{-i\omega x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-1}^{1} e^{-i\omega x}\,dx - \int_{-1}^{1} x^2 e^{-i\omega x}\,dx\right)$$

The first integral in the parentheses is easy to evaluate, and reduces to

$$\frac{e^{-i\omega}-e^{i\omega}}{-i\omega}=\frac{2\sin\omega}{\omega}$$

For the second integral, we will need to use integration by parts:

$$u=x^2 \implies du=2xdx, dv=e^{-i\omega x}dx \implies v=\frac{i}{\omega}e^{-i\omega x}$$

Setting up the formula for integration by parts, we find

$$uv|_{-1}^{1}=\frac{2\sin\omega}{\omega}$$

and then we are left with the integral of v du...

$$\frac{2i}{\omega}\int_{-1}^{1}xe^{-i\omega x}\,d$$

which requires another round of integration by parts - the final expression for the above reduces to

$$\frac{-4}{\omega^3}(\omega\cos\omega-\sin\omega)$$

and so we have

$$\tilde{f}(\omega)=\frac{1}{\sqrt{2\pi}}\left(\cancel{\frac{2\sin\omega}{\omega}-\frac{2\sin\omega}{\omega}}+\frac{4}{\omega^3}(\omega\cos\omega-\sin\omega)\right)$$

$$\tilde{f}(\omega)=\frac{4\omega\cos\omega-4\sin\omega}{\sqrt{2\pi}\omega^3}$$

---

c) Using the result from **b**, show that

$$\int_0^\infty\frac{(x\cos x-\sin x)^2}{x^6}\,dx=\frac{\pi}{15}$$

First, note that if we square our function $\tilde{f}\omega$, we find

$$|\tilde{f}(\omega)|^2=\frac{16}{2\pi}\left(\frac{(\omega\cos\omega-\sin\omega)^2}{\omega^6}\right)$$

and if we square the value of our function $f(x)$ for $-1<x<1$ (assuming x is real), we find that the integral from $0\to\infty$ is the same as the integral from $0\to 1$, so:

$$\int_0^\infty|f(x)|^2\,dx=\int_0^1(x^2-1)^2dx$$

$$=\int_0^1(x^4-2x^2+1)\,dx$$

which, broken up into the three constituent terms, evaluates to $\frac{8}{15}$.

We can apply Parseval's Theorem once again here:

$$\int_0^\infty|f(x)|^2\,dx\;=\;\int_{-0}^{\infty}|\tilde{f}(\omega)|^2\,d\omega$$

$$\frac{8}{15}=\frac{16}{2\pi}\int_0^\infty\left(\frac{(\omega\cos\omega-\sin\omega)^2}{\omega^6}\right)d\omega$$

$$\int_0^\infty\left(\frac{(\omega\cos\omega-\sin\omega)^2}{\omega^6}\right)d\omega=\left(\frac{2\pi}{16}\right)\frac{8}{15}$$

$$=\frac{\pi}{15}$$

...which was to be shown.

3.

Using the formalism for the complex Fourier series expansion, calculate the Fourier series for $\delta(x-x_0)$ in the interval $(-\pi,\pi)$ where $-\pi<x_0<\pi$.

Solution:

The complex Fourier expansion is written

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{2\pi i n x}{L}}$$

where L as before is the length of the interval, which here is $L=2\pi$.

The complex coefficients are given by

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx$$

Applying this to the delta function:

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x-x_0)e^{-inx}\,dx$$

From the properties of the δ-function (and given that we were told that the interval contains x_0), this will pick out the value of the integrand where $x=x_0$:

$$c_n=\frac{1}{2\pi}e^{-inx_0}$$

and thus the complex Fourier expansion of the delta function is

$$\begin{aligned}\delta(x-x_0)&=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}e^{-inx_0}e^{inx}\right)\\&=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in(x-x_0)}.\end{aligned}$$

4.

The Fourier transform of the exponential decay function

$$f(t)=\begin{cases}0 & , \quad t<0\\ e^{-\lambda t} & , \quad t\geq 0\text{ and }\lambda>0\end{cases}$$

is given by

$$\tilde{f}(\omega)=\frac{1}{\sqrt{2\pi}}\left[\frac{\lambda-i\omega}{\lambda^2+\omega^2}\right]$$

Show that

$$\int_0^{\infty}\frac{(\lambda\cos\omega t+\omega\sin\omega t)}{(\lambda^2+\omega^2)}\,d\omega=\pi e^{-\lambda t}$$

Solution:

We have the inverse Fourier transform as:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\lambda - i\omega}{\lambda^2 + \omega^2} e^{i\omega t} \, d\omega$$

Expanding $e^{i\omega t}$ into $\cos(\omega t) + i \sin(\omega t)$, we get:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\lambda - i\omega}{\lambda^2 + \omega^2} (\cos(\omega t) + i \sin(\omega t)) \, d\omega$$

Further expanding, we get two integrals:

$$f(t) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \frac{\lambda \cos(\omega t) + \omega \sin(\omega t)}{\lambda^2 + \omega^2} \, d\omega + i \int_{-\infty}^{\infty} \frac{-\omega \cos(\omega t) + \lambda \sin(\omega t)}{\lambda^2 + \omega^2} \, d\omega \right]$$

Since $f(t) = e^{-\lambda t}$ for $t \geq 0$ and $\lambda > 0$, and $e^{-\lambda t}$ is a real function, we can focus on the real part of the above expression to prove the given equation.

The real part is:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\lambda \cos(\omega t) + \omega \sin(\omega t)}{\lambda^2 + \omega^2} \, d\omega$$

Since the function is even in ω, we can rewrite the integral as:

$$\frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{\lambda \cos(\omega t) + \omega \sin(\omega t)}{\lambda^2 + \omega^2} \, d\omega$$

Finally, we get:

$$\sqrt{2\pi} \int_{0}^{\infty} \frac{\lambda \cos(\omega t) + \omega \sin(\omega t)}{\lambda^2 + \omega^2} \, d\omega = e^{-\lambda t}$$

Since $f(t) = e^{-\lambda t}$, we have:

$$\int_{0}^{\infty} \frac{\lambda \cos(\omega t) + \omega \sin(\omega t)}{\lambda^2 + \omega^2} \, d\omega = \frac{e^{-\lambda t}}{\sqrt{2\pi}} \times \sqrt{2\pi} = \pi e^{-\lambda t}$$

which was to be shown.
