Question Bank

Math for Electromagnetism

Some Basics

M-1: What is a scalar?

M-2: What is a vector?

M-3: What is the magnitude of a vector?

M-4: Show that $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$, where $A = |\vec{A}|$ (e.g., the magnitude of $\vec{A}$).

M-5: What is a component of a vector?

M-6: What is a unit vector?

M-7: What are $\hat{x}$, $\hat{y}$, and $\hat{z}$

M-8: What is a field?

M-9: What is a function?

M-10: What does it mean to say an operation is associative?

M-11: What does it mean to say an operation is distributive?

M-12: What does it mean to say an operation is commutative?

M-13: What does it mean to say an operation is linear?

M-14: Scalar multiplication is defined such that if α is a scalar, then $\alpha\vec{A}$ is a vector parallel (if $\alpha > 0$) or anti-parallel (if $\alpha < 0$) to $\vec{A}$, with magnitude αA.

M-15: If $\vec{C} = \vec{A} + \vec{B}$, then $\vec{C}$ is determined by adding $\vec{A}$ and $\vec{B}$ *tip to toe.*

M-16: Show that, if $\vec{C} = \vec{A} + \vec{B}$, then C_x=A_x+B_x.

M-17: Vector subtraction, e.g. $\vec{C} = \vec{A} - \vec{B}$, can be understood as vector addition and scalar multiplication:

$$\vec{C} = \vec{A} - \vec{B} = \vec{\mathrm{A}} + \left(\alpha\vec{B}\right) \qquad \text{where } \alpha = -1$$

M-18: Show that $A_x = \vec{A} \cdot \hat{x}$.

M-19: Show that $A_y = \vec{A} \cdot \hat{y}$.

M-20: Show that $A_z = \vec{A} \cdot \hat{z}$.

M-21: Show that $\hat{z} \cdot \hat{z} = 1$.

M-22: Show that $\hat{x} \cdot \hat{z} = 0$.

M-23: Show that $\hat{z} \times \hat{x} = \hat{y}$.

M-24: Show that $\hat{x} \times \hat{y} = \hat{z}$.

M-25: Show that $\hat{y} \times \hat{z} = \hat{x}$.

M-26: Show that $\hat{x} \times \hat{z} = -\hat{y}$.

M-27: Show that $\hat{y} \times \hat{x} = -\hat{z}$.

M-28: Show that $\hat{z} \times \hat{y} = -\hat{x}$.

Unit Vectors and Coordinate Systems

M-29: Show that

$$\hat{\phi} = \frac{-y\hat{x} + x\hat{y}}{\sqrt{x^2 + y^2}}$$

Answer: Both $\hat{\phi}$ and $\hat{s}$ have zero $z-$components. They are also mutually orthogonal, so their dot product is zero. We know (something we should memorize) that

$$\hat{s} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}}$$

$\hat{\phi}$ is a unit vector, with zero $z-$component, that is perpendicular to $\hat{s}$, so the only two possibilities are

$$\hat{\phi} = \pm\frac{y\hat{x} - x\hat{y}}{\sqrt{x^2 + y^2}}$$

But on the positive $x - axis$, $y = 0$ and $\hat{\phi} = \hat{y}$. Thus,

$$\hat{\phi} = \frac{-y\hat{x} + x\hat{y}}{\sqrt{x^2 + y^2}}$$

M-30: Show that, in cylindrical coordinates, $\hat{z} = \hat{s} \times \hat{\phi}$

Answer: Express $\hat{s}$ and $\hat{\phi}$ in terms of their Cartesian coordinates:

$$\hat{s} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \hat{\phi} = \frac{-y\hat{x} + x\hat{y}}{\sqrt{x^2 + y^2}}$$

Therefore

$$\hat{s} \times \hat{\phi} = \frac{1}{x^2 + y^2}\left[- xy\hat{x} \times \hat{x} + x^2\hat{x} \times \hat{y} - y^2\hat{y} \times \hat{x} + xy\hat{y} \times \hat{y}\right]$$

But, $\hat{x} \times \hat{x} = 0$, $\hat{y} \times \hat{y} = 0$, and $\hat{y} \times \hat{x} = \hat{x} \times \hat{y} = \hat{z}$, so therefore

$$\hat{s} \times \hat{\phi} = \frac{x^2 + y^2}{x^2 + y^2}\hat{z} = \hat{z}$$

M-31: Using a Cartesian representation of $\hat{s}$ and $\hat{z}$, show that $\hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y}$.

Answer: $\hat{s}$ is perpendicular to $\hat{z}$, and points radially away from the $z-$axis. Therefore, at any point (x, y, z), $\hat{s}$ is parallel to the vector $(x, y, 0) = s(\cos(\phi), \sin(\phi), 0)$, where $s = \sqrt{x^2 + y^2}$. Since it is a unit vector we have to normalize it, so

$$\begin{aligned}
\hat{s} &= \frac{(x, y, 0)}{|(x, y, 0)|} \\
&= \frac{(x, y, 0)}{\sqrt{x^2 + y^2}} \\
&= \frac{(x, y, 0)}{s} && \text{where I have used } \; s = \sqrt{x^2 + y^2} \\
&= \frac{(s\cos(\phi), s\sin(\phi), 0)}{s} && \text{where I have used } \; x = s\cos(\phi) \quad \text{and} \quad y = s\sin(\phi), \text{ so} \\
&= (\cos(\phi), \sin(\phi), 0)
\end{aligned}$$

But we know that $\hat{\phi}$ is the cross product of $\hat{z}$ and $\hat{s}$:

$$\begin{aligned}
\hat{\phi} &= \hat{z} \times \hat{s} \\
&= (0, 0, 1) \times (\cos(\phi), \sin(\phi), 0) \\
&= (-\sin(\phi), \cos(\phi), 0)
\end{aligned}$$

or

$$\hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y}$$

Note: I have heard many people, including researchers and students, saying "because $\hat{s}$ is perpendicular to $\hat{z}$, we know it is in the xy-plane," or "we know $\hat{s}$ and $\hat{\phi}$ are perpendicular to $hatz$ or similar statements to justify writing down something like

$$\hat{s} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}}$$

This formula is correct ($\hat{s}$ points in the same direction as the vector from the point $(0, 0, z)$ to the point (x, y, z), meaning $\hat{s}$ is parallel to the vector $(x, y, 0)$ but is normalized), but is is not *in the xy-plane.* A vector is not anywhere, it is simply a quantity that has a direction and a length.

M-32: Using a Cartesian representation of $\hat{s}$ and $\hat{z}$, show that $\hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y}$.

Less long winded answer: We know that since $\hat{s}$ is perpendicular to $\hat{z}$, and that at a point (x, y, z) it points radially away from the $z-$axis. Plus we know it is a normal vector. Therefore

$$\hat{s} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}}$$

We know that $\hat{\phi}$ is the cross product of $\hat{z}$ and $\hat{s}$, so

$$\begin{aligned}
\hat{\phi} &= \hat{z} \times \left[\frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}}\right] \\
&= \frac{x}{\sqrt{x^2 + y^2}}\hat{z} \times \hat{x} + \frac{y}{\sqrt{x^2 + y^2}}\hat{z} \times \hat{y} \\
&= \frac{x}{s}\hat{y} - \frac{y}{s}\hat{x} \\
&= -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}
\end{aligned}$$

since

$$x = s\cos(\phi),\ y = s\sin(\phi) \quad , \quad \text{and} \quad s = \sqrt{x^2 + y^2}$$

M-33: Using a Cartesian representation of $\hat{s}$ and $\hat{z}$, show that $\hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y}$.

Short winded answer: We know that

$$\hat{s} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}}, \quad \text{or that} \quad \hat{s} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}, \quad \text{and that} \quad \hat{\phi} = \hat{z} \times \hat{s}$$

therefore

$$\hat{\phi} = \cos(\phi)(\hat{z} \times \hat{x}) + \sin(\phi)(\hat{z} \times \hat{y}) \quad \text{so therefore} \quad \hat{\phi} = \cos(\phi)\hat{y} - \sin(\phi)\hat{x}$$

M-34: Using a Cartesian representation of $\hat{s}$ and $\hat{z}$, show that $\hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y}$.

Clever answer: I know that $\hat{s} = (\cos(\phi), \sin(\phi), 0)$, the $z-$component of $\hat{\phi}$ is zero, $\hat{\phi}$ is a unit vector, and $\hat{s} \cdot \hat{\phi} = 0$. I can represent $\hat{\phi}$ as $\hat{\phi} = (\alpha, \beta, 0)$, so that $\hat{s} \cdot \hat{\phi} = 0$ means

$$\alpha\cos(\phi) + \beta\sin(\phi) = 0$$

This, together with the fact $\hat{\phi}$ is a unit vector, means

$$\hat{\phi} = \pm(\sin(\phi), -\cos(\phi), 0)$$

Since $\hat{\phi}$ points in the direction of increasing ϕ (think what that means if, e.g., we are looking at a point on the positive $x-$axis where $\hat{\phi} = (0, 1, 0)$), the answer has to be

$$\hat{\phi} = (-\sin(\phi), \cos(\phi), 0)$$

Cross Product and Dot Product

M-35: Using the geometric definition of the cross product, show that $\vec{A} \times \vec{B}$=-$\vec{B} \times \vec{A}$.

Answer: According to the geometric definition, this result is immediately obvious, from the magnitudes of $\vec{A} \times \vec{B}$ and $\vec{A} \times \vec{B}$, and their directions via the right hand rule being opposite.

M-36: Using the determinant definition of the cross product, show that $\vec{A} \times \vec{B}$=-$\vec{B} \times \vec{A}$.

Answer: The cross product is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

However, if exchanging two adjacent rows in a determinant is equivalent to multiplying that depermi-nant by -1, so

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix} = - \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

But

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix} = \vec{B} \times \vec{A}$$

Therefore

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$$

M-37: Show that the cross product is distributive, by which I mean

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \tag{1}$$

Answer: From the definition of the cross product

$$\begin{aligned} \left[\vec{A} \times (\vec{B} + \vec{C})\right]_x &= A_y(\vec{B} + \vec{C})_z - A_z(\vec{B} + \vec{C})_y \\ &= (A_y B_z - A_z B_y) + (A_y C_z - A_z C_y) \\ &= (\vec{A} \times \vec{B})_x + (\vec{A} \times \vec{C})_x \end{aligned}$$

It is easy to show the same is true for the y and z components, so therefore

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

M-38: Show that the cross product is distributive, by which I mean

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Alternative Answer: If you are comfortable with determinants...

$$\begin{aligned}
\vec{A} \times (\vec{B} + \vec{C}) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x + C_x & B_y + C_y & B_z + C_z \end{vmatrix} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} \\
&= \vec{A} \times \vec{B} + \vec{A} \times \vec{C}
\end{aligned}$$

M-39: In words, how do you know $\vec{A} \cdot \left(\vec{A} \times \vec{B}\right) = 0$?

Answer: The vector $\vec{A} \times \vec{B}$ is perpendicular to $\vec{A}$.

M-40: Show that

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{2}$$

Answer: The LHS of the Equation 2 is

$$\begin{aligned}
\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) &= A_x \left[\vec{B} \times \vec{C}\right]_x + A_y \left[\vec{B} \times \vec{C}\right]_y + A_z \left[\vec{B} \times \vec{C}\right]_z \\
&= A_x \left[B_y C_z - B_z C_y\right] + A_y \left[B_z C_x - B_x C_z\right] + A_z \left[B_x C_y - B_y C_x\right] \\
&= A_x \begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix} + A_y \begin{vmatrix} B_z & B_x \\ C_z & C_x \end{vmatrix} + A_z \begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix}
\end{aligned}$$

while the LHS of Equation 2 is

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = A_x \begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix} + A_y \begin{vmatrix} B_z & B_x \\ C_z & C_x \end{vmatrix} + A_z \begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix}$$

M-41: Use the properties of the determinant to show that

$$\vec{A} \cdot \left(\vec{A} \times \vec{B} \right) = 0$$

Answer: Based on the previous question, we have

$$\vec{A} \cdot \left(\vec{B} \times \vec{C} \right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Therefore

$$\vec{A} \cdot \left(\vec{A} \times \vec{B} \right) = \begin{vmatrix} A_x & A_y & A_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

If two rows of a determinant are equal, then the determinant is zero. Therefore,

$$\vec{A} \cdot \left(\vec{A} \times \vec{B} \right) = 0$$

M-42: Show that if we define the dot product such that

$$\vec{A} \cdot \vec{B} = AB\cos(\theta) \tag{3}$$

then it is also the case that

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{4}$$

Consider first the two dimensional case, where $\vec{A}$ and $\vec{B}$ are both in the xy plane, and make angles θ_1 and θ_2 with the x-axis, respectively. In this case

$$\begin{aligned} A_x B_x + A_y B_y &= AB(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)) \\ &= AB\cos(\theta_1 - \theta_2) \\ &= AB\cos(\theta) \\ &= \vec{A} \cdot \vec{B} \end{aligned}$$

where $\theta = \theta_1 - \theta_2$ is the angle between the two vectors. Thus in this case the two quantities are equal.

How about the 3D case? Well this is more difficult, and I will leave it for another venue.

M-43: Show that if we define the cross product such that

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z} \tag{5}$$

then it is also true that

$$\vec{A} \times \vec{B} = AB\sin(\theta)\hat{n} \tag{6}$$

Answer: Consider the quantity

$$\begin{aligned}
A^2B^2 - (\vec{A}\cdot\vec{B})^2 - |\vec{A}\times\vec{B}|^2 &= A^2B^2 - (\vec{A}\cdot\vec{B})^2 - (A_yB_z - A_zB_y)^2 \\
&\quad -(A_zB_x - A_xB_z)^2 - (A_xB_y - A_yB_x)^2 \\
&= (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_xB_x + A_yB_y + A_zB_z)^2 \\
&\quad -(A_yB_z - A_zB_y)^2 - (A_zB_x - A_xB_z)^2 - (A_xB_y - A_yB_x)^2 \\
&= A_x^2B_x^2 + A_x^2B_y^2 + A_x^2B_z^2 + A_y^2B_x^2 \\
&\quad +A_y^2B_y^2 + A_y^2B_z^2 + A_y^2B_x^2 + A_y^2B_y^2 + A_y^2B_z^2 + \\
&\quad -A_x^2B_x^2 - A_xB_xA_yB_y - A_xB_xA_zB_z \\
&\quad -A_yB_yA_xB_x - A_y^2B_y^2 - A_yB_yA_zB_z \\
&\quad -A_zB_zA_xB_x - A_zB_zA_yB_y - A_z^2B_z^2 \\
&\quad -A_y^2B_z^2 + 2A_yB_zA_zB_y - A_z^2B_y^2 \\
&\quad -A_z^2B_x^2 + 2A_zB_xA_xB_z - A_x^2B_z^2 \\
&\quad -A_x^2B_y^2 + 2A_xB_yA_yB_x - A_y^2B_x^2
\end{aligned} \tag{7}$$

Some rearranging gives

$$\begin{aligned}
A^2B^2 - (\vec{A}\cdot\vec{B})^2 - |\vec{A}\times\vec{B}|^2 &= A_x^2B_x^2 + A_x^2B_y^2 + A_x^2B_z^2 + A_y^2B_x^2 \\
&\quad +A_y^2B_y^2 + A_y^2B_z^2 + A_y^2B_x^2 + A_y^2B_y^2 + A_y^2B_z^2 + \\
&\quad -A_x^2B_x^2 - A_y^2B_y^2 - A_z^2B_z^2 - A_y^2B_z^2 - A_z^2B_y^2 \\
&\quad -A_z^2B_x^2 - A_x^2B_z^2 - A_x^2B_y^2 - A_y^2B_x^2 \\
&\quad -2(A_xB_xA_yB_y + A_xB_xA_zB_z + A_yB_yA_zB_z) \\
&\quad +2(A_yB_zA_zB_y + A_zB_xA_xB_z + A_xB_yA_yB_x) \\
&= 0
\end{aligned} \tag{8}$$

Therefore, from Equations 3 and 8, we have

$$|\vec{A}\times\vec{B}|^2 = A^2B^2(1-\cos^2(\theta))^2 \tag{9}$$

and finally, using $1 - \cos^2(\theta) = \sin^2(\theta)$, this gives

$$|\vec{A}\times\vec{B}| = AB\sin(\theta) \quad \text{where we define } 0 \le \theta \le \pi \tag{10}$$

This demonstrates the magnitude aspect of Equation 6, but not the direction. We can always define the xy plane to be the plane defined by $\vec{A}$ and $\vec{B}$. From above, in this frame, $\vec{A}\times\vec{B} = (A_xB_y - A_yB_x)\hat{z}$. Since we can always do this, we know that that $\vec{A}\times\vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$. What about the right hand rule aspect of Equation 6? Well, we are also free to define our x axis such that $\vec{A} = A_x\hat{x}$ and $A_x > 0$. In this case $A_y = 0$ and $\vec{A}\times\vec{B} = A_xB_y\hat{z}$, but with A_x being positive, then $\vec{A}\times\vec{B}$ points in the direction of $\hat{z}$ or $-\hat{z}$ if $B_y > 0$ or $B_y < 0$, respectively. If $B_y > 0$ or $B_y < 0$, then the right hand rule would tell us the result is in the direction of $\hat{z}$ or $-\hat{z}$, respectively. Thus, Equation 10 tells us the magnitude of the cross product as defined by Equation 5 is $AB\sin(\theta)$, and this geometric consideration tells us the direction of the cross product as defined by Equation 5 is given by the right hand rule. Thus, the equivalence represented by Equation 6 holds true.

M-44: Show that given two non-collinear vectors $\vec{A}$ and $\vec{B}$, we can always set up a cartesian coordinate system where $\hat{x}$ and $\hat{y}$ are in the plane defined by $\vec{A}$ and $\vec{B}$.

Answer: The plane of $\vec{A}$ and $\vec{B}$ is defined by its normal vector, which is either

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} \quad \text{or} \quad \hat{n} = -\frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

If the $xy-$plane is the plane defined by $\vec{A}$ and $\vec{B}$, then we can pick

$$\hat{z} = \hat{n}$$

and we can set $\hat{x}$ as

$$\hat{x} = \frac{\vec{A}}{A}$$

because that is in the plane of $\vec{A}$ and $\vec{B}$. Then, with $\hat{x}$ and $\hat{z}$ defined, $\hat{y}$ is determined:

$$\hat{y} = \hat{z} \times \hat{x}$$

or (not that one needs to spell this out)

$$\hat{y} = \left[\frac{\vec{A}}{A}\right] \times \left[\frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}\right]$$

Of course $\hat{y}$ is in the plane of $\vec{A}$ and $\vec{B}$, since $\hat{z}$ is perpendicular to that plane. So we have a triad of mutually orthogonal vectors, $\hat{x}$, $\hat{y}$, and $\hat{z}$, where $\hat{x}$ and $\hat{y}$ are in the plane of $\vec{A}$ and $\vec{B}$.

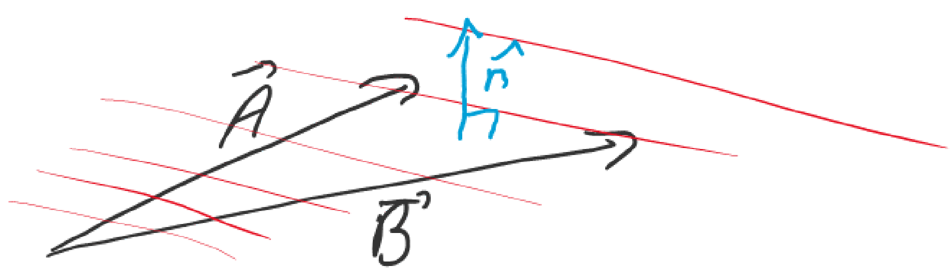


Figure 1: Two non-collinear vectors make a plane. In this case, I have set $\hat{n} = -\vec{A} \times \vec{B}$

M-45: Show that any vector $\vec{C}$ that is co-planar with two other vectors $\vec{A}$ and $\vec{B}$, can be represented as a linear combination of those two vectors

$$\vec{C} = \alpha\vec{A} + \beta\vec{B} \tag{11}$$

unless $\vec{A}$ and $\vec{B}$ are collinear (in which case they do not define a plane).

Answer: Two non-collinear vectors ($\vec{A}$ and $\vec{B}$) define a plane, and we know we can define a Cartesian coordinate system $\hat{x}$ and $\hat{y}$ *that spans the 2D space that is the plane defined by the two vectors.* That means that we can find $\hat{x}$ and $\hat{y}$, such that any vector in that plane can be expressed as a linear combination of those two vectors. Therefore, I can find α and β, such that for any vector $\vec{C}$ in that plane,

$$\vec{C} = \alpha\hat{x} + \beta\hat{y}$$

With the above we have fully answered the question.

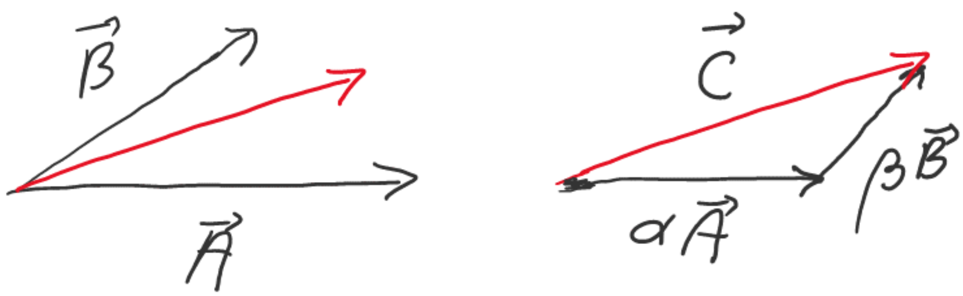


Figure 2: Any vector $\vec{C}$ in the plane defined by $\vec{A}$ and $\vec{B}$ is a linear combination of $\vec{A}$ and $\vec{B}$.

I am going to carry this ball a bit further, to elucidate a point that I think matters. Working in a 2D space, I'm going to show that the vectors $\hat{x}$ and $\hat{y}$ *span that space.* Let's assert that Equation 11 holds. Then it would be true that

$$\begin{bmatrix} C_x \\ C_y \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ A_y & B_y \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

We know that there is a solution for α and β provided the determinant

$$\begin{vmatrix} A_x & B_x \\ A_y & B_y \end{vmatrix} \neq 0$$

This is the condition for collinearity of vectors, namely their cross product is zero. Collinearity is just a fancy word for the vectors being parallel or anti-parallel to one another (the angle between them being 0° or 180°.

M-46: Show that $\vec{A} \times (\vec{B} \times \vec{C})$ is a linear combination of $\vec{A}$ and $\vec{B}$, or, in other words

$$\vec{A} \times (\vec{B} \times \vec{C}) = \alpha\vec{B} + \beta\vec{C} \tag{12}$$

Answer: $\vec{B} \times \vec{C}$ is perpendicular to the plane defined by $\vec{B}$ and $\vec{C}$. Therefore $\vec{A} \times (\vec{B} \times \vec{C})$ is a vector in the plane defined by $\vec{B}$ and $\vec{C}$. Therefore, as is the case for any vector in that plane, it is a linear combination of $\vec{B}$ and $\vec{C}$, or, in other words, Equation 12 is true.

Note, it is often said that the above is true unless $\vec{B}$ and $\vec{C}$ are collinear, however it is true even in that case. In that case $\vec{A} \times (\vec{B} \times \vec{C}) = 0$, and the zero vector is a linear combination of two collinear vectors.

M-47: Show that the $x-$component of the LHS and RHS in Equation 13:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \tag{13}$$

Answer: Let's start with the $x-$component of the LHS

$$\begin{aligned}
\left[\vec{A} \times (\vec{B} \times \vec{C})\right]_x &= A_y(\vec{B} \times \vec{C})_z - A_z(\vec{B} \times \vec{C})_y \\
&= A_y(B_xC_y - C_xB_y) - A_z(B_zC_x - B_xC_z) \\
&= B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z) \\
&= B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z) + B_xA_xC_x - B_xA_xC_x \\
&= B_x(A_xB_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z) \\
&= B_x(\vec{A} \cdot \vec{C}) - C_x(\vec{A} \cdot \vec{B})
\end{aligned}$$

which is the $x-$ component of the RHS. There is no real reason to work through the equivalent exercise for the $y-$ and $z-$components, there is nothing different, except the order of all the subscripts, and you end up with

$$\left[\vec{A} \times (\vec{B} \times \vec{C})\right]_y = \left[\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})\right]_y$$

and

$$\left[\vec{A} \times (\vec{B} \times \vec{C})\right]_z = \left[\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})\right]_z$$

so the $x-$, $y-$, and $z-$ components of the LHS equal those of the RHS, so this triple product rule, as Equation 13 is referred to, is true.

M-48: Prove the Law of Cosines

Answer: Consider the three vectors shown in the figure. The lengths of the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ are A, B, and C, respectively. The internal angles opposite to $\vec{A}$, $\vec{B}$, and $\vec{C}$ are a, b, and c, respectively. We have

$$\vec{A} + \vec{C} = \vec{B}$$

and

$$(\vec{A} + \vec{C}) \cdot (\vec{A} + \vec{C}) = \vec{B} \cdot \vec{B}$$

or

$$\vec{A} \cdot \vec{A} + \vec{C} \cdot \vec{C} + 2\vec{A} \cdot \vec{C} = \vec{B} \cdot \vec{B}$$

which gives

$$A^2 + C^2 + 2\vec{A} \cdot \vec{C} = B^2$$

The dot product

$$\vec{A} \cdot \vec{C} = AC\cos(\theta)$$

where θ is the angle between $\vec{A}$ and $\vec{C}$, which is $\pi - b$. Thus, since $\cos(\pi - b) = -\cos(b)$

we have

$$B^2 = A^2 + C^2 - 2AC\cos(b)$$

which is the Law of Cosines.

The Gradient

M-49: Show that, in cylindrical coordinates,

$$\left(\vec{\nabla} f\right)_\phi = \frac{1}{s}\frac{\partial}{\partial\phi} f$$

Answer: The component of the gradient in any direction is the directional derivative in that direction. In this case, it is the derivative of f with respect to distance along a circle of radius s centered on the $z-$ axis. If we call that distance h, then h is just the distance along the circle of radius s centered on the z-axis that passes through the point where we want to know the ϕ component of the gradient of f. If we measure h from $\phi = 0$, then

$$h = s\phi$$

We can specified the point by setting $s = s_0$ and $z = z_0$, and then imagine the following single variable function

$$g(h) = f(s_0, u, z_0) \quad \text{where} \quad u = \frac{h}{s_0}$$

From the chain rule, we have

$$\frac{dg}{dh} = \frac{df(s_0, u, z_0)}{du}\frac{du}{dh} = \frac{1}{s_0}\frac{df(s_0, u, z_0)}{du} \tag{14}$$

However

$$\frac{df(s_0, u, z_0)}{du} = \frac{df(s_0, \phi, z_0)}{d\phi} \tag{15}$$

In words, the quantity on the right hand side is the rate of change of f with respect to ϕ while keeping s and z constant, which is, in other words, the partial derivative of f with respect to ϕ. Therefore, Equations 14 and 15 give us

$$\left(\vec{\nabla} f\right)_\phi = \frac{1}{s}\frac{\partial f}{\partial\phi}$$

M-50: Working in Cartesian coordinates, show that $s\nabla\phi = \hat{\phi}$.

Answer: Given

$$s^2 = x^2 + y^2, \;\; \hat{\phi} = \frac{-y}{s}\hat{x} + \frac{x}{s}\hat{x}, \;\; \phi = \mathrm{atan}(y/x), \;\text{ and }\; \frac{d}{du}\mathrm{atan}(u) = \frac{1}{1+u^2}$$

we have

$$\begin{aligned}
s\vec{\nabla}\phi &= s\left[\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}\right] \;\text{ but } \phi \text{ does not depend on } z \text{ so} \\
s\vec{\nabla}\phi &= s\left[\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y}\right] \;\text{ so} \\
s\vec{\nabla}\phi &= s\left[-\frac{y}{x^2}\frac{1}{1+\frac{x^2}{y^2}}\hat{x} + \frac{1}{x}\frac{1}{1+\frac{x^2}{y^2}}\hat{y}\right] \\
&= s\left[\frac{-y\hat{x}}{x^2+y^2} + \frac{x\hat{y}}{x^2+y^2}\right] \\
&= s\left[\frac{-y\hat{x}}{s^2} + \frac{x\hat{y}}{s^2}\right] \\
&= \frac{-y\hat{x}}{s} + \frac{x\hat{y}}{s} \\
&= \hat{\phi}
\end{aligned}$$

M-51: Working in Cartesian coordinates, show that $s\nabla\phi = \hat{\phi}$.

Answer (based on intuition): Consider that the *directional derivative*:

$$\hat{n} \cdot \vec{\nabla} F = \text{the rate of change of F in the direction of } \hat{n}$$

As well, we know that the gradient of a function points in the direction of the maximum rate of change (with distance) of that function. Imagine we are looking at a point (s, ϕ, z). Since ϕ does not depend on z, its gradient is perpendicular to z. Intuitively, if we move a small distance away from that point, I expect the change in ϕ to be greatest if I am moving in the direction of $\pm\hat{\phi}$. More concretely, $\Delta\phi$ is zero if we are moving in the direction of $\pm\hat{s}$, so $\vec{\nabla}\phi$ is perpendicular to $\hat{s}$. Further, $\Delta\phi$ will be positive if we move in the direction of $\hat{\phi}$. From all of this I *expect*

$$\vec{\nabla}\phi = F(s, \phi)\hat{\phi} \tag{16}$$

At any point, the directional derivative of ϕ in the direction of $\hat{\phi}$ is the rate of change, with distance along the arc of the circle (centered on the $z - axis$) that point is on. If we call h the distance along the arc from $\phi = 0$, then (think arc length) we have

$$\phi = \frac{h}{s}, \quad \frac{\partial\phi}{\partial h} = \frac{1}{s}, \quad \hat{\phi} \cdot \vec{\nabla}\phi = \frac{\partial\phi}{\partial h}, \quad \text{and consequently} \quad F(s, \phi)\hat{\phi} \cdot \hat{\phi} = \frac{1}{s}$$

Where I have used Equation 16 in that last step. Since $\hat{\phi} \cdot \hat{\phi} = 1$, we have

$$F(s, \phi) = \frac{1}{s}, \quad \text{and therefore} \quad \vec{\nabla}\phi = \frac{1}{s}\hat{\phi}, \quad \text{and finally} \quad s\vec{\nabla}\phi = \hat{\phi}$$

M-52: Working in Cartesian coordinates, show that $s\nabla\phi = \hat{\phi}$.

Answer (super intuitive): At any point, ϕ does not vary in the direction of $\hat{s}$, therefore its gradient points in the direction of $\hat{\phi}$. The rate of change of ϕ in that direction is

$$\frac{d\phi}{dh}$$

where h is distance along the circle centered on the z-axis that passes through the point. This is the ϕ (and only non-zero) component of the gradient of ϕ. If I measure f from where $\phi = 0$, then

$$h = s\phi$$

so that

$$\frac{d\phi}{dh} = \frac{1}{s}$$

Again, this is the ϕ component of, and the only non-zero component of, $\vec{\nabla}\phi$, so we have

$$\vec{\nabla}\phi = \frac{1}{s}\hat{\phi}, \quad \text{and therefore} \quad s\vec{\nabla}\phi = \hat{\phi}$$

The Curl in Cylindrically Symmetric Situations

M-53: Imagine a cylindrically symmetric situation (meaning cylindrically symmetric sources and fields). We can understand the curl in this situation by considering Stokes' Theorem. Consider a vector field, $\vec{A}$, that is cylindrically symmetric. As well, consider a boundary of a surface, between s and $s+h$, shaded in blue in Figure 3. If the surface is in the $z=0$ plane, or any $z = constant$ frame, then the following is true by Stokes' Theorem:

$$2\pi sh\left[\vec{\nabla}\times\vec{A}\right]_z = 2\pi(s+h)B_\phi(s+h) - 2\pi s B_\phi(s)$$

Rearranging, we have

$$\left[\vec{\nabla}\times\vec{A}\right]_z = \frac{B_\phi(s+h)-B_\phi(s)}{h} + \frac{B_\phi(s+h)}{s}$$

In the limit of small h, this becomes

$$\left[\vec{\nabla}\times\vec{A}\right]_z = \frac{\partial B_\phi}{\partial s} + \frac{B_\phi(s)}{s}$$

Therefore,

$$\left[\vec{\nabla}\times\vec{A}\right]_z = \frac{1}{s}\frac{\partial}{\partial s}\left[sB_\phi\right]$$

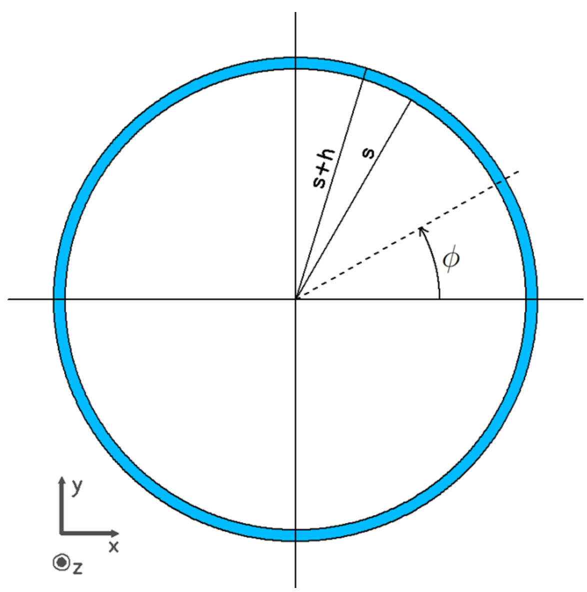


Figure 3: Geometry for the curl in a situation of cylindrical symmetry.

The Divergence in Spherically Symmetric Situations

M-54: Imagine a spherically symmetric situation (spherically symmetric sources and thus fields). We can understand the divergence in this situation by considering the Divergence Theorem. Consider a vector field, $\vec{A} = A_r(r)\hat{r}$, that is spherically symmetric. Show that, in this spherically symmetric situation,

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 A_r$$

Answer: Start with the divergence theorem

$$\int_S \vec{A} \cdot \vec{da} = \int_V \vec{\nabla} \cdot \vec{A} d\tau \tag{17}$$

Consider the volume between r and $r + h$, shaded in blue in Figure 4.

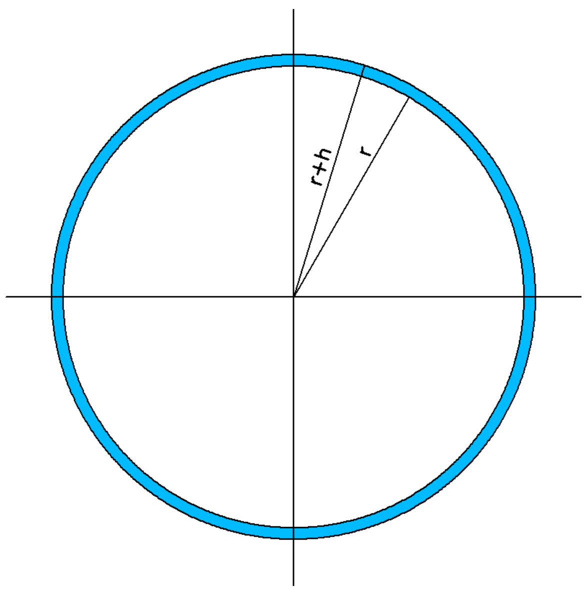


Figure 4: Geometry for the divergence in a situation of spherical symmetry.

The volume integral in Equation 17 is

$$\int_V \vec{\nabla} \cdot \vec{A} d\tau = 4\pi r^2 h \vec{\nabla} \cdot \vec{A}$$

while the flux integral in the same equation is, roughly,

$$f(r + h) - f(r) \quad \text{where} \quad f(r) = 4\pi r^2 A_r(r)$$

Thus

$$f(r + h) - f(r) = 4\pi r^2 h \vec{\nabla} \cdot \vec{A} \quad \text{so} \quad \vec{\nabla} \cdot \vec{A} = \frac{1}{4\pi r^2}\frac{\partial f}{\partial r} \quad \text{or} \quad \vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 A_r$$

Integrals as Averages

M-55: Show that if

$$\int_a^b f(x)dx = 0 \quad \text{for all intervals } [a,b], \text{ then} \quad f(x) = 0 \quad \text{for all } x$$

Answer: Consider the meaning of the integral, namely that

$$\int_a^b f(x)dx = f^*(b-a)$$

where by f^* I mean the *average* of f over the interval (this is the definition of what the average value of a function is). Since the relation

$$\int_a^b f(x)dx = 0 \quad \text{holds for all intervals } [a,b], \text{ we therefore have} \quad (b-a)f^* = 0$$

for all $[a,b]$, where f^* is the average of f on (a,b), so therefor

$$f^* = 0 \quad \text{for all intervals} \quad [a,b].$$

For any given value of x, we can therefore pick increasingly small intervals around that point. As the intervals get smaller and smaller the value of f at every point on the interval, including *at* x, will approach the average, which is 0. So assuming f is *well behaved*, $f(x)$=0.

M-56: Show that

$$\left| \int_V \vec{\nabla} \cdot \vec{C}\, d\tau \right| \leq AC^*$$

where the volume integral is over a volume bounded by a surface of area A, C is the magnitude of $\vec{C}$, and C^* is the average value of C over the surface bounding the volume.

Answer: The volume integral of the divergence can be converted to a surface integral with Gauss's law:

$$\int_V \vec{\nabla} \cdot \vec{C}\, d\tau = \int_S \vec{C} \cdot \hat{n}\, da$$

Re-express the surface integral as the product of the area A of the surface (that bounds the volume) and the average of the integrand:

$$\int_S \vec{C} \cdot \hat{n}\, da = A(\vec{C} \cdot \hat{n})^*$$

where A is the area of the surface and $(\vec{C} \cdot \hat{n})^*$ is the average of $(\vec{C} \cdot \hat{n})$ on the surface. Now, anywhere on the surface, $(\vec{C} \cdot \hat{n})$ has a value $-C < (\vec{C} \cdot \hat{n}) < C$, so its average cannot be less than C^* nor can it be greater than C^*. Therefore

$$\left| \int_V \vec{\nabla} \cdot \vec{C}\, d\tau \right| \leq AC^*$$

Path (Contour, Line) Integrals

M-57: Consider the curve shown in Figure 5. Wht is the length of the curve between x_0 and x_1?

Answer: We can break it into segments each of actual length ΔL_i, and approximate these actual lengths with the straight line distance between the points at the beginning and end of each segment:

$$\Delta L_i \approx \sqrt{\Delta x_i^2 + \Delta y_i^2} = \Delta x_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}$$

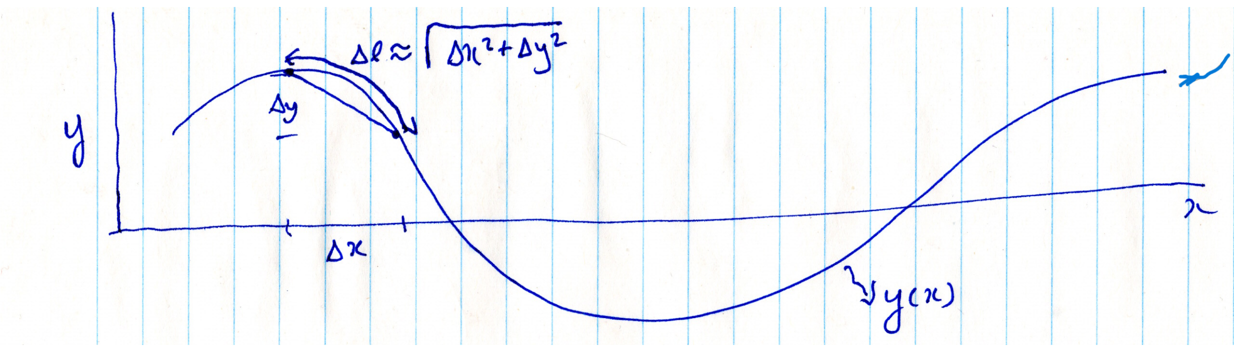


Figure 5: Geometric basis for the First Fundamental Theorem.

We can then approximate the length of the curve by the sum of the lengths of the however many approximating straight line segments we have broken the curve into:

$$L \approx \sum_i \Delta x_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}$$

The length of the curve is then the limit of very large numbers of segments, each of which is diminishingly small, or

$$L = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

How does this work in practice? Consider the contour $y(x) = \sqrt{1-x^2}$ between $x = -1$ and $x = 1$. This is one half of the unit circle, and so its length is π. Carrying out the above integral to determine that length, first note

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$$

so

$$L = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x)\Big|_{-1}^{1} = \pi$$

M-58: Show that the length of the curve $y = (1 - x^2)^{\frac{1}{2}}$, between $x_\circ = 0$ and $x_1 = 1/\sqrt{2}$, is $\pi/4$.

Answer: The length of the curve is

$$L = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^{\frac{1}{\sqrt{2}}} \sqrt{1 + \frac{x}{1 - x^2}} dx = \int_0^{\frac{1}{\sqrt{2}}} \sqrt{1 + \frac{x}{1 - x^2}} dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1 - x^2}} dx$$

But, (x, y) is a point on the unit circle, so $x = \sin(\theta)$, $dx = \cos(\theta)d\theta$, and $\sqrt{1 - x^2} = \cos(\theta)$, where θ is the angle between the y-axis and the line between the origin and a point on the curve (positive clockwise). Thus,

$$L = \int_{\theta=0}^{\theta=\pi/4} \frac{\cos(\theta)}{\cos(\theta)} d\theta = \int_0^{\pi/4} d\theta = \frac{\pi}{4}$$

Answer: So in a way we have just gone through a bit of work to prove the arc length of $1/8^{th}$ of a unit circle is $\pi/4$, or the circumference of a unit circle is 2π, things we have known since at least as far back as junior high school. But, what if we did not know the numerical value of π, and we needed to (in some magical universe where we need to know π but we can't just Google it)? The way we answered this question points us to a way of estimating π (see next question). Plus, a key point of this exercise is to show ourselves that this path integral approach really works (meaning I know $\pi/4$ is the arc length of $1/8^{th}$ of the unit circle, and that's what the integral yielded).

M-59: Knowing that the length of the curve $y = (1 - x^2)^{\frac{1}{2}}$, between $x_\circ = 0$ and $x_1 = 1/\sqrt{2}$, is $\pi/4$, use a numerical estimate of that arc length to estimate the value of π.

Answer: If we numerically estimate that distance, then that estimated distance multiplied by 4 is an estimate of π. I created an Excel spreadsheet where I evaluated ΔL on just five equal-length segments (Δx_i, $i = 0, ..., 4$) between $x_\circ = 0$ and $x_1 = 1/\sqrt{2}$. In that sheet you'd find a very simple calculation, where I picked six evenly space points between and including $x = 0$ and $x = 1/\sqrt{2}$, and the five (Pythagorean) distances (between point 0 and 1, then between point 1 and 2, ... and between point 4 and 5) using $\Delta L = \sqrt{\Delta y^2 + \Delta y^2}$ (can you see that the distance we're going to get by summing these is not just an estimate, but an underestimate?). The value this sum gives, to three significant figures, is 3.14, and it underestimates π by about a tenth of a percent. To me this is a bit amazing, because it seems a remarkably good estimate for such an easy approach.

I'm really trying to drive home the idea of integration as addition. Today in class I spoke about integrating distance along a curve. Doing this analytically is straightforward to set up, although it does require you to really understand how to use the function e.g. $y = f(x)$ as a parameterized curve. The relationship (in this case in 2D) between the variables that f represents is what keeps you on the curve you want.

Although setting up the contour integral of dl (note this is a scalar) to calculate the distance along a segment of the curve is straightforward, except in mostly highly contrived problems the integral is difficult or impossible to do analytically. In most *applications* it does not matter because doing the integral numerically is trivially easy and even very basic approaches yield remarkably good answers.

M-60: However cool it is to use Excel in this way, it is not a practical programming tool for numerical work in physics. There are many reasons for this, but the big one is that over the arc of a career our work has to fit with that of others... so we constantly use functions and subroutines written by others in our programs, so it's best if I we programming in a language or with tools that others also use.

Actually, Excel is a perfectly good tool in, e.g., biological sciences, because this is what many in that field use. In my field (space physics), a language called IDL has been pretty much the standard for decades, though younger space physicists are using Python more and more, and within a decade that might well be the standard.

All of you will ultimately be working in Python, but I don't think this is the course to really dive into that if you have no experience with it. However, I strung five commands together (IDL is an interpreter so one can just type into a command line or create programs that are compiled and run) a simple to do exactly what the Excel spreadsheet does...

```
IDL> x=[0,1,2,3,4,5]/5.0/sqrt(2)
IDL> y=sqrt(1-x^2)
IDL> length=0
IDL> for i=0,4 do length=length+sqrt((x(i+1)-x(i))^2+(y(i+1)-y(i))^2)
IDL> print,'estimated length of curve: '+string(4*length),
estimated length of curve: 3.13827
```

and I hope you will try to implement a program that does this in whatever language or tool you use for computational work. If you have no experience with Python, maybe you could look over the shoulder of one of your colleagues. By the way, using 20, and 50 points yields errors of roughly 0.007 percent and 0.001 percent, respectively. *The reason why I am focusing on this is that these easy computational tasks often make our work tremendously simpler, and give us very easy ways of checking the validity of expressions we might derive or visualizing how certain functions behave.*

M-61: Find the distance along the curve $y = \ln(\cos(x))$ between $x = x_\circ$ and $x = x_1$, where $0 < x_\circ < x_1 < \pi/2$.

Answer: This length is

$$L = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{\sin(x)}{\cos(x)}\right)^2} dx = \int_{x_0}^{x_1} \sqrt{1 + \tan^2(x)} dx$$

However, $1 + \tan^2(x) = \sec^2(x)$, and $\int \sec(x)dx = \ln(\sec(x) + \tan(x))$

Therefore,

$$L = \ln[\sec(x_1) + \tan(x_1)] - \ln[\sec(x_\circ) + \tan(x_\circ)]$$

This is an example where being able to numerically estimate the integral of dl is a great check on the above expression. For example, taking $x_\circ = 0.05\pi$ and $x_1 = 0.495\pi$, the above expression gives a length $L = 4.68896...$, and doing the integral numerically, using 100 evenly space points and the 99 distance elements between adjacent points yields the same answer to five significant figures.

M-62: We can easily extend this *length along a curve* to 3D. A curve can be specified using the formalism

$$\vec{r} = f(t)\hat{x} + g(t)\hat{y} + h(t)\hat{z}$$

where t is a variable (that may or may not be time). For one example

$$\hat{r} = t\hat{x} + t\hat{y} + t\hat{z}$$

is a straight line that passes through the origin ($t = 0$). At $t = 1$, the point on the line is $(1, 1, 1)$. At $t = 2$, it is t$(2, 2, 2)$, etc. Just by geometry and Pythagoras, the distance along the curve from $t = 0$ to $t = a$ is $\sqrt{3}a$. Set this up as a contour integral, and show the distance along the curve between $\vec{r}(t = 0)$ and $\vec{r}(t = a)$ is $\sqrt{3}a$.

Answer: The length along the curve from $\vec{r}(t = 0)$ to $\vec{r}(t = a)$

$$L = \int_{\vec{r}(t=0)}^{\vec{r}(t=a)} \sqrt{dx^2 + dy^2 + dz^2} = \int_0^a dt\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \int_0^a dt\sqrt{\dot{f}^2 + \dot{g}^2 + \dot{h}^2}$$

where the 'dot' indicates the first derivative with respect to t (which I remind us may or may not be time). Here, the three derivatives $\dot{f}$, $\dot{g}$, and $\dot{h}$ are each 1, so

$$L = \int_0^a \sqrt{3}dt = (a - 0)\sqrt{3} = \sqrt{3}a$$

M-63: Consider the curve

$$(x, y, z) = (\cos(\omega t), sin(\omega t), \beta t)$$

where t is the independent variable and ω and β are constants, where if we interpret t as time, the former is an angular frequency, and the latter a (constant) speed in the z direction. The above equation is then the trajectory of e.g. a particle. Show that the distance travelled by the particle between $t = t_\circ$ and $t = t_1$ is given by

$$L = (t_1 - t_\circ)\sqrt{\omega^2 + \beta^2}$$

I find this an interesting result. It turns out if this is the equation of the trajectory of a particle, and t is time, this particle's speed is constant. As an aside, can you visualize this trajectory? It's actually a helix... circular motion in the xy-plane and z increasing linearly with time (speed β).

Vector Calculus

M-64: What is the curl of a vector $\vec{A}$ in a Cartesian frame?

Answer: In Cartesian coordinates, the curl of a vector $\vec{A}$ is

$$\vec{\nabla} \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{x} + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{y} + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{z}$$

Some things are worth the trouble of memorizing. One of these, in my opinion, is the $x-$component of the curl in Cartesian:

$$\left[\vec{\nabla} \times \vec{A}\right]_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Trusting *cyclic permutation*, one can then generate the $y-$ and $z-$components from this.

M-65: Show that the following product rule holds:

$$\vec{\nabla}(fg) = g\vec{\nabla}f + f\vec{\nabla}g$$

Answer: We can expand out the LHS, and apply the standard product rule:

$$\begin{aligned}
\vec{\nabla}(fg) &= \frac{\partial fg}{\partial x}\hat{x} + \frac{\partial fg}{\partial y}\hat{y} + \frac{\partial fg}{\partial z}\hat{z} \\
&= g\frac{\partial f}{\partial x}\hat{x} + f\frac{\partial g}{\partial x}\hat{x} + g\frac{\partial f}{\partial y}\hat{y} + f\frac{\partial g}{\partial y}\hat{y} + g\frac{\partial f}{\partial z}\hat{z} + f\frac{\partial g}{\partial z}\hat{z} \\
&= g\left[\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right] + f\left[\frac{\partial f}{\partial g}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z}\right] \\
&= g\vec{\nabla}f + f\vec{\nabla}g
\end{aligned}$$

M-66: Show that the following product rule holds:

$$\vec{\nabla} \cdot (f\vec{A}) = f\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla}f$$

Answer: We can start from the RHS and write out the divergence:

$$\vec{\nabla} \cdot (f\vec{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

We can apply the standard product rule to each of the three terms on the right

$$\vec{\nabla} \cdot (f\vec{A}) = f\frac{\partial A_x}{\partial x} + A_x\frac{\partial f}{\partial x} + f\frac{\partial A_y}{\partial y} + A_y\frac{\partial f}{\partial y} + f\frac{\partial A_z}{\partial z} + A_z\frac{\partial f}{\partial x}$$

and then group terms

$$\vec{\nabla} \cdot (f\vec{A}) = f\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] + \left[A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right] \quad \text{and so} \quad \vec{\nabla} \cdot (f\vec{A}) = f\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla}f$$

M-67: Show that the following product rule holds:

$$\vec{\nabla} \times \left(f\vec{A} \right) = f\vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla} f$$

Answer: First, show that the $x - \textit{component}$ of the LHS equals the x-component of the RHS. The $x-$component of the LHS is

$$\begin{aligned}
\left[\vec{\nabla} \times \left(f\vec{A} \right) \right]_x &= \frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y) \\
&= \left(f\frac{\partial}{\partial y}A_z - f\frac{\partial}{\partial z}A_y \right) + A_z\frac{\partial f}{\partial y} - A_y\frac{\partial f}{\partial z} \\
&= f\left(\vec{\nabla} \times \vec{A} \right)_x + A_z\left(\vec{\nabla} f \right)_y - A_y\left(\vec{\nabla} f \right)_z \\
&= f\left(\vec{\nabla} \times \vec{A} \right)_x - \left[A_y\left(\vec{\nabla} f \right)_z - A_z\left(\vec{\nabla} f \right)_y \right] \\
&= f\left(\vec{\nabla} \times \vec{A} \right)_x - \left(\vec{A} \times \vec{\nabla} f \right)_x
\end{aligned}$$

You can go through the same exercise for the $y-$ and $z-$components of the LHS, and you will find that they equal the $y-$ and $z-$ components of the RHS gives the following product rule. Summing up, we can say

$$\begin{aligned}
\text{LHS}_x &= \text{RHS}_x \\
\text{LHS}_y &= \text{RHS}_y \\
\text{LHS}_z &= \text{RHS}_z
\end{aligned}$$

and so therefore, we have the following

$$\vec{\nabla} \times \left(f\vec{A} \right) = f\vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla} f$$

M-68: Show that

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$$

Answer: Using the determinant representation of the curl, we have

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla} \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

where I have used

$$\partial_x \equiv \frac{\partial}{\partial x}, \quad \partial_y \equiv \frac{\partial}{\partial y} \quad \text{and} \quad \partial_z \equiv \frac{\partial}{\partial z}$$

for compactness. The 'dell dot' operator acting on the determinant replaces the top row of the determinant by the components of the vector dell operator, so we have

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

We also know that a determinant with two rows being identical is identically zero. To see this, you know you can switch two adjacent rows and the result is to multiply the determinant by minus one. In this case, if we swap the top and middle rows, we end up with the negative of what we started with. But since the top two rows are identical this means that

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = -\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right)$$

so

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$$

If you work it through by brute force it becomes obvious that this result comes from the same property of partial derivatives that allowed us to say the curl of a gradient is zero, namely, that we can switch the order of first partial derivatives. If you go on to learn tensors, you will soon see this as a consequence of the inner product between a symmetric and an anti-symmetric tensor.

M-69: Show that

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0$$

Answer: Expanding out the expression for the curl, we have

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z},\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x},\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

Expanding, we have

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=\frac{\partial}{\partial x}\frac{\partial}{\partial y}A_z-\frac{\partial}{\partial x}\frac{\partial}{\partial z}A_y+\frac{\partial}{\partial y}\frac{\partial}{\partial z}A_x-\frac{\partial}{\partial y}\frac{\partial}{\partial x}A_z+\frac{\partial}{\partial z}\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial z}\frac{\partial}{\partial y}A_x$$

Grouping terms, we have

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}-\frac{\partial}{\partial y}\frac{\partial}{\partial x}\right)A_z-\left(\frac{\partial}{\partial x}\frac{\partial}{\partial z}-\frac{\partial}{\partial z}\frac{\partial}{\partial x}\right)A_y+\left(\frac{\partial}{\partial y}\frac{\partial}{\partial z}-\frac{\partial}{\partial z}\frac{\partial}{\partial y}\right)A_x$$

Since we can reverse the order of partial derivatives, so that, for example

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}=\frac{\partial}{\partial y}\frac{\partial}{\partial x}$$

we have

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0$$

M-70: Using determinants, Show that

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0$$

Answer: Again, this is only for those who are really confident with determinants.

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

But we can switch the second and first rows of the determinant, and thus have

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=-\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)$$

Therfore,

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0$$

M-71: Show that

$$\left[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})\right]_x = \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{A})\right]_x - \left[\nabla^2 \vec{A}\right]_x$$

Answer: Let's start with the LHS:

$$\begin{aligned}\left[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})\right]_x &= \frac{\partial}{\partial y}(\vec{\nabla} \times \vec{A})_z - \frac{\partial}{\partial z}(\vec{\nabla} \times \vec{A})_y \\ &= \frac{\partial}{\partial y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - \frac{\partial}{\partial z}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] \\ &= \frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x}\end{aligned} \tag{18}$$

and consider the RHS:

$$\begin{aligned}\left[\vec{\nabla}(\vec{\nabla} \cdot \vec{A})\right]_x - \left[\nabla^2 \vec{A}\right]_x &= \frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] A_x \\ &= \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} \\ &= \frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \partial z} \\ &= \frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x}\end{aligned} \tag{19}$$

where in that last step I used

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial z}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial z}$$

From Equations 18 and 19, we see that

$$\left[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})\right]_x = \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{A})\right]_x - \left[\nabla^2 \vec{A}\right]_x$$

Doing the same exercise for the $y-$ and $-z$ components of

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

tells us that vector calculus identity is true. We use his in the future, when we start go develop wave equations for the electric and magnetic fields (EM waves).

M-72: Same as above, but more complete... Show that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \tag{20}$$

Answer: Let's start with the LHS:

$$\begin{aligned}
\left[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})\right]_x &= \frac{\partial}{\partial y}(\vec{\nabla} \times \vec{A})_z - \frac{\partial}{\partial z}(\vec{\nabla} \times \vec{A})_y \\
&= \frac{\partial}{\partial y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - \frac{\partial}{\partial z}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] \\
&= \frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x} \\
&= \frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x} + \left[\frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial x^2}\right] \\
&= \frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] A_x \\
&= \frac{\partial}{\partial x}\left[\vec{\nabla} \cdot \vec{A}\right] - \nabla^2 A_x \\
&= \left[\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right)\right]_x - \left[\nabla^2 \vec{A}\right]_x
\end{aligned} \tag{21}$$

Cyclic permutation dictates $x \Rightarrow y$, and $y \Rightarrow z$, so therefore Equation 21 becomes

$$\left[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})\right]_y = \left[\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right)\right]_y - \left[\nabla^2 \vec{A}\right]_y \tag{22}$$

Cyclic permutation dictates $x \Rightarrow y$, and $y \Rightarrow z$, so therefore Equation 21 becomes

$$\left[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})\right]_z = \left[\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right)\right]_z - \left[\nabla^2 \vec{A}\right]_z \tag{23}$$

Thus, the x, y, and z components of the LHS of Equation 20 equal the x, y, and z components of the RHS of Equation 20, respectively, and so,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

The Divergence and Gauss's Theorem

M-73:

M-74: Say that

$$\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$$

What is the divergence of $\vec{A}$?

Answer: The divergence of $\vec{A}$ is

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

Therefore, in this case,

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

and so,

$$\vec{\nabla} \cdot \vec{A} = 3$$

M-75: Say that

$$\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$$

What is the outward flux of $\vec{A}$ from a volume V bounded by a surface S?

Answer: The outward flux of $\vec{A}$ from the volume V (bounded by the surface S) is

$$\Phi_{outward} = \int_S \vec{A} \cdot \vec{da}$$

According to Gauss's Theorem

$$\int_S \vec{A} \cdot \vec{da} = \int_V \vec{\nabla} \cdot \vec{A} d\tau \tag{24}$$

In this case,

$$\vec{\nabla} \cdot \vec{A} = 3 \tag{25}$$

From Equations 24 and 25, we have

$$\Phi_{outward} = 3V$$

M-76: Given

$$\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$$

Show that, applied to a spherical volume of radius R centered on the origin, the Divergence Theorem holds.

Answer: $\vec{A} = r\hat{r}$, and everywhere on the surface of the volume $\vec{A} = R\hat{r}$, and the outward unit normal is $\hat{r}$, the outward flux from the volume is

$$\begin{aligned}
\Phi_{outward} &= \int_S \vec{A} \cdot \vec{da} \\
&= \int_S R\hat{r} \cdot \hat{r} da \\
&= R \int_S ds \\
&= R(4\pi R^2) \\
&= 4\pi R^3
\end{aligned}$$

We have already shown

$$\vec{\nabla} \cdot \vec{A} = 3$$

so

$$\begin{aligned}
\int_V \vec{\nabla} \cdot \vec{A} d\tau &= 3 \int_V d\tau \\
&= 3V \\
&= 3\frac{4\pi R^3}{3} \\
&= 4\pi R^3
\end{aligned}$$

Therefore, in this case,

$$\int_V \vec{\nabla} \cdot \vec{A} d\tau = \int_S \vec{A} \cdot \vec{da}$$

M-77: Derive Gauss's Law from Coulomb's Law.

Answer: According to Coulomb's Law,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\int \rho(\vec{r}')\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}d\tau'$$

Therefore

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \vec{\nabla}\cdot\int \frac{\rho(\vec{r}')}{4\pi\epsilon_\circ}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}d\tau'$$

The $\vec{\nabla}$ involves derivatives with respect to the unprimed variables, so

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \int \frac{\rho(\vec{r}')}{4\pi\epsilon_\circ}\left[\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)\right]d\tau'$$

It is straightforward to show that

$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = 4\pi\delta^3(\vec{r}-\vec{r}')$$

Since

$$\delta^3(\vec{r}-\vec{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$$

and $\delta(u)$ is an even function of its argument

$$\delta^3(\vec{r}-\vec{r}') = \delta(x'-x)\delta(y'-y)\delta(z'-z) \quad \text{or} \quad \delta^3(\vec{r}-\vec{r}') = \delta^3(\vec{r}'-\vec{r})$$

Therefore,

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \int \frac{\rho(\vec{r}')}{4\pi\epsilon_\circ}\Big[4\pi\delta(\vec{r}'-\vec{r})\Big]d\tau'$$

Based on the properties of the Dirac Delta Function, we have, then,

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_\circ} \quad \text{or, more simply,} \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_\circ}$$

This is Gauss's Law.

M-78: Say that, somehow, we know that

$$\int_V \frac{\rho}{\epsilon_\circ} d\tau = \int_S \vec{E} \cdot \vec{da}$$

where V is any volume, and S is the surface bounding that volume. Show that

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_\circ}$$

Answer: Using Gauss's Theorem (from above), I can convert the surface integral into a volume integral over V (again, *any* volume V):

$$\int_V \frac{\rho}{\epsilon_\circ} d\tau = \int_V \vec{\nabla} \cdot \vec{E} d\tau$$

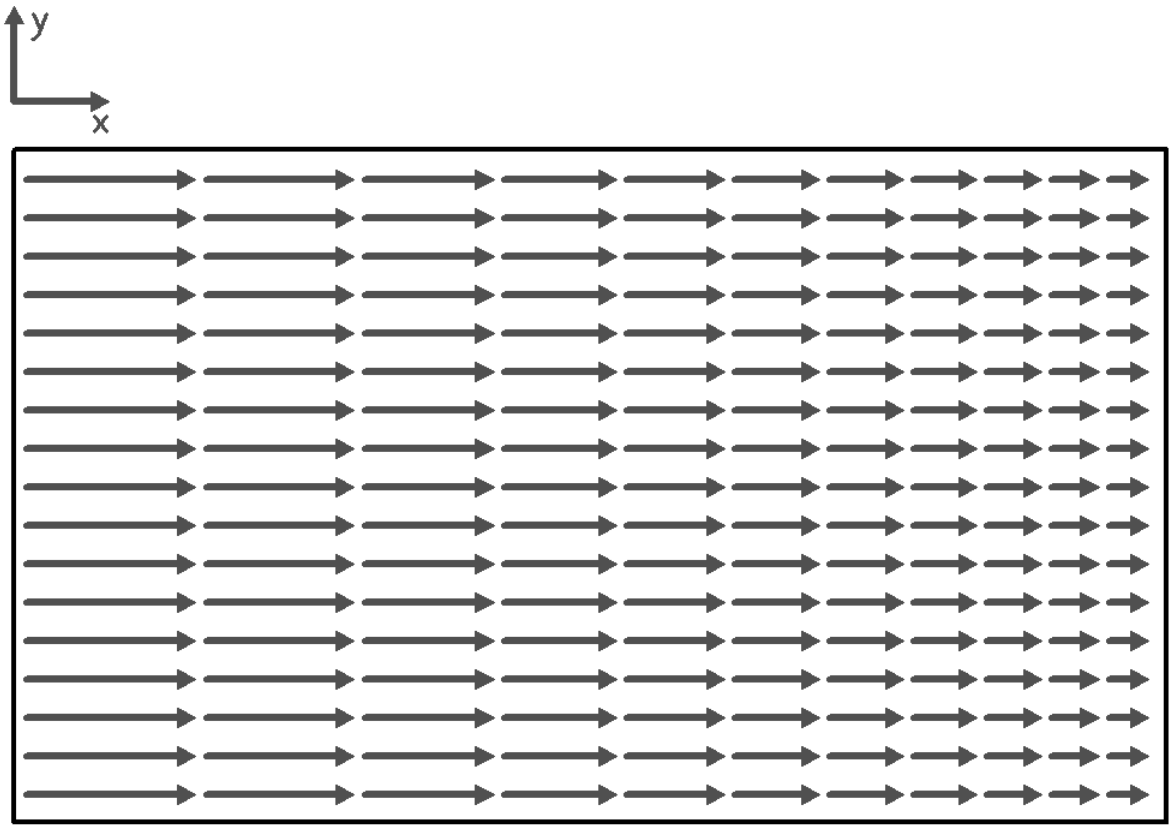
or

$$\int_V \left[\frac{\rho}{\epsilon_\circ} - \vec{\nabla} \cdot \vec{E} \right] d\tau = 0$$

Since this is true for any volume V (in particular think of arbitrarily small volumes), we can say that

$$\frac{\rho}{\epsilon_\circ} - \vec{\nabla} \cdot \vec{E} = 0 \quad \text{and therefore} \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_\circ}$$

M-79: The figure below depicts a field, $\vec{A}$. For this field, $\partial A_x/\partial z = 0$, $\partial A_y/\partial z = 0$, and $\vec{\nabla} \cdot \vec{A} = 0$. Is $\partial A_z/\partial z = 0$ positive, zero, or negative?



Answer: We are given the divergence is zero:

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

and we can see that

$$\frac{\partial A_x}{\partial x} < 0 \quad \text{and} \quad \frac{\partial A_y}{\partial y} = 0$$

These, together, mean that

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 \quad \text{and} \quad \frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x} = 0$$

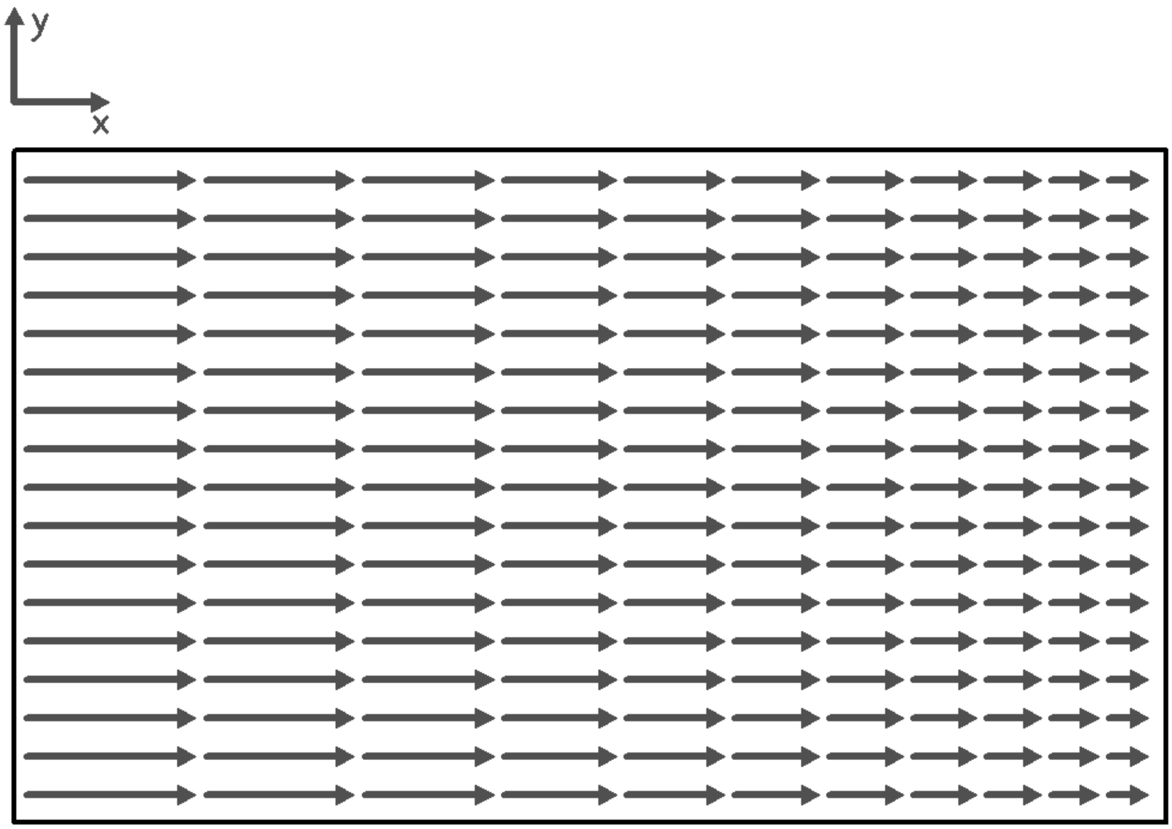
or

$$\frac{\partial A_x}{\partial x} < 0$$

therefore

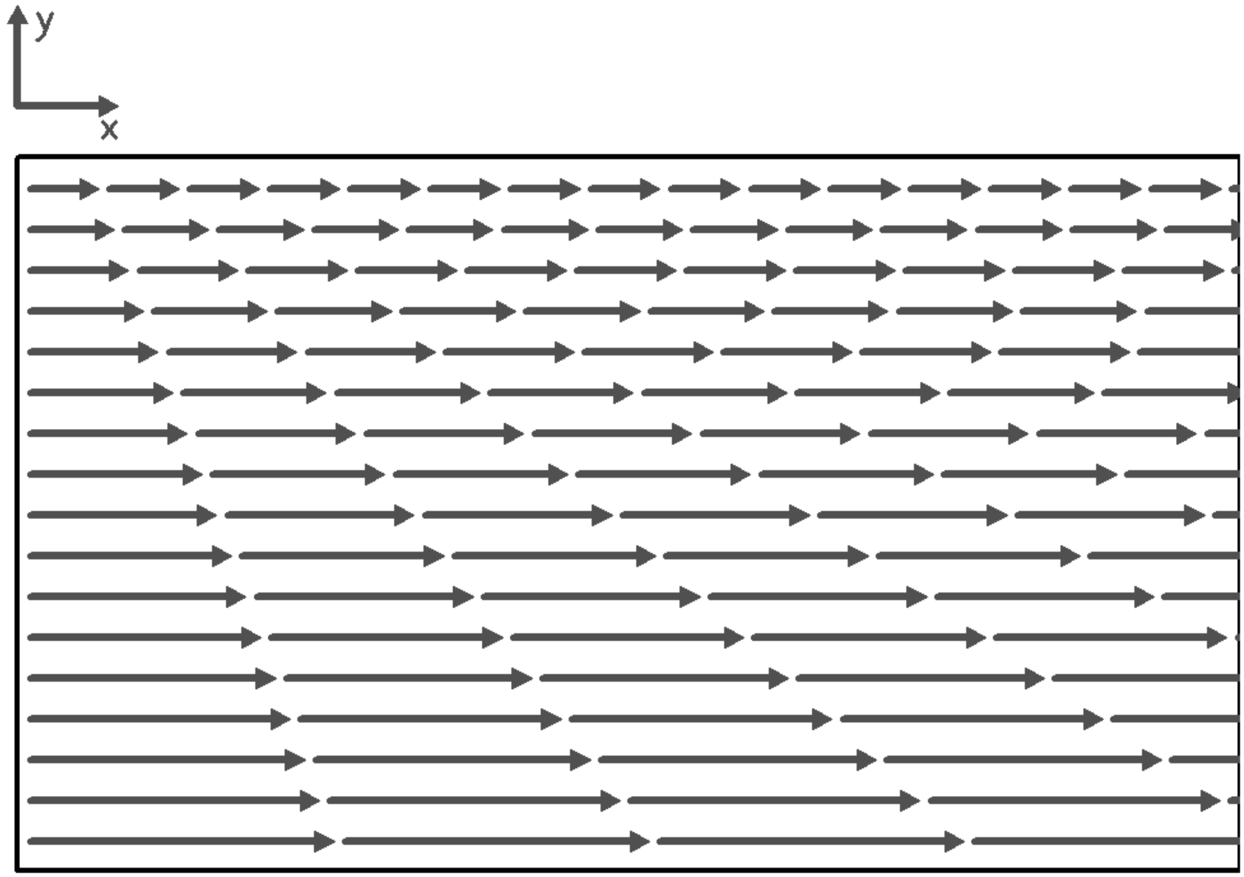
$$\frac{\partial A_z}{\partial z} > 0$$

M-80: The figure below depicts the flow field of a gas, $\vec{v}$. For this field, , $\partial A_y/\partial z = 0$, and $\vec{\nabla} \cdot \vec{A} = 0$. Is the density of the gas increasing with time, decreasing with time, or constant in time?



Answer:

M-81: The figure below depicts a field, $\vec{A}$. In this plane, is $(\vec{\nabla} \times \vec{A})_z$ positive, zero, or negative?



Answer: We have

$$(\vec{\nabla} \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

From the figure, we can see that

$$\frac{\partial A_y}{\partial x} = 0$$

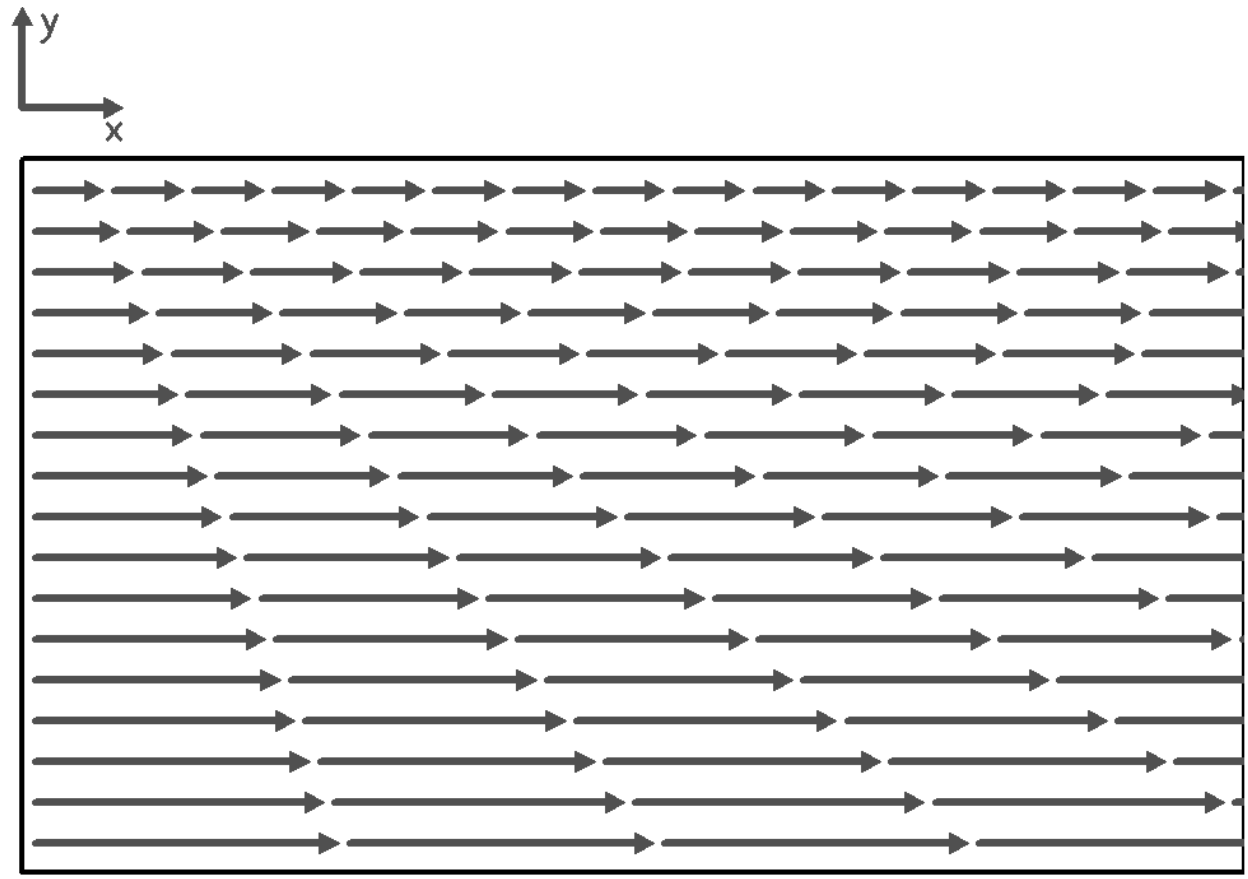
Also from the figure, we can see that

$$\frac{\partial A_x}{\partial y} < 0$$

so therefore

$$(\vec{\nabla} \times \vec{A})_z > 0$$

M-82: The figure below depicts a field, $\vec{A}$. There is no variation of any quantity with respect to z. Is $\vec{\nabla} \cdot \vec{A}$ positive, zero, or negative?



Answer:

The Curl and Stokes' Theorem

M-83: Given a vector $\vec{A} = x\hat{x} + xy\hat{y}$, show by directly evaluating the contour integral on the contour from $(0,0)$ to $(1,0)$ along the x-axis, then from $(1,0)$ to $(1,1)$ along the line $x = 1$, and then back to $(0,0)$ along the curve $y = x^n$, is

$$\oint \vec{A} \cdot \vec{dl} = \frac{1}{2(2n+1)}$$

Answer:

M-84: Given the vector $\vec{A}$ from question 1

a) show that its curl is $y\hat{z}$

b) show, on the surface in the xy-plane bounded by the contour from a, that

$$\oint \oint (\vec{\nabla} \times \vec{a}) \cdot \vec{da} = \frac{1}{2(2n+1)}$$

Answer:

M-85: Using Stokes' Theorem, show that if

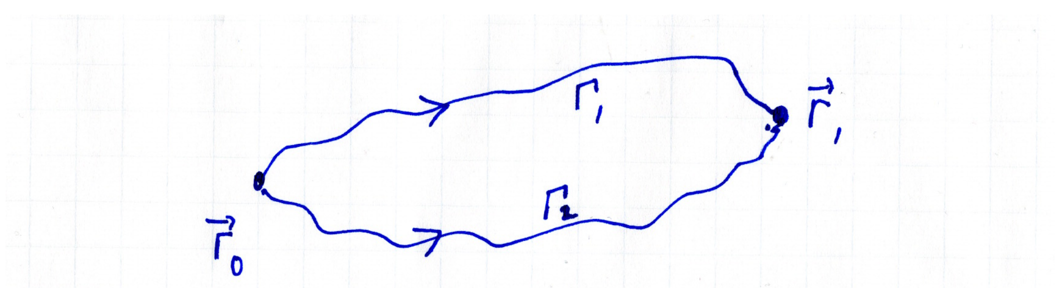
$$\vec{\nabla} \times \vec{A} = 0$$

contour (path) integrals of $\vec{A}$

$$\int \vec{A} \cdot \vec{dl}$$

are path independent.

Answer: Imagine two contours between points $\vec{r}_\circ$ and $\vec{r}_1$, as shown below. The integral along Γ_1 and that along Γ_2 start at a common point $(\vec{r}_\circ)$ and end at a different (but also a common) point $(\vec{r}_1)$.



Consider the contour integrals

$$I_1 = \int_{\Gamma_1} \vec{A} \cdot \vec{dl} \qquad \text{and} \qquad I_2 = \int_{\Gamma_2} \vec{A} \cdot \vec{dl}$$

where I_1 and I_2 are contour integrals from $\vec{r}_\circ$ to $\vec{r}_\circ$ along contours Γ_1 and Γ_2, respectively. Going clockwise around the closed contour made up by the paths of Γ_1 and Γ_2, the closed contour integral is

$$\oint \vec{E}(\vec{r}) \cdot \vec{dl} = \int_{\Gamma_1} \vec{A} \cdot \vec{dl} - \int_{\Gamma_2} = \vec{A} \cdot \vec{dl} = I_1 - I_2$$

With Stokes' Theorem, we can replace the closed contour integral with the flux of $\vec{\nabla} \times \vec{A}$ through *any* surface S bound by that contour:

$$I_1 - I_2 = \int_S \vec{\nabla} \times \vec{A} \cdot da$$

But since we are given $\vec{\nabla} \times \vec{A} = 0$,

$$I_1 - I_2 = 0$$

and since we did not specify anything about the two contours except they both start at $\vec{r}_\circ$ and both end at $\vec{r}_1$, that means the contour integral

$$\int_{\vec{r}_\circ}^{\vec{r}_1} \vec{A} \cdot \vec{dl}$$

is path independent (which I often refer to as "PI" for brevity).

Within the context of electromagnetism, this result is of profound, and I would say even iconic, importance. Since the static electric field is curl free, the path integrals of the static electric field are path independent. On one hand, this means that the amount of work done on a charge moving from one point to another depends only upon those endpoints. On the other hand, it also allows us to construct the scalar potential, from which the static electric field can be derived.

M-86: Given

$$\vec{A} = f(x)\hat{x} + g(y)\hat{y} + h(z)\hat{z}$$

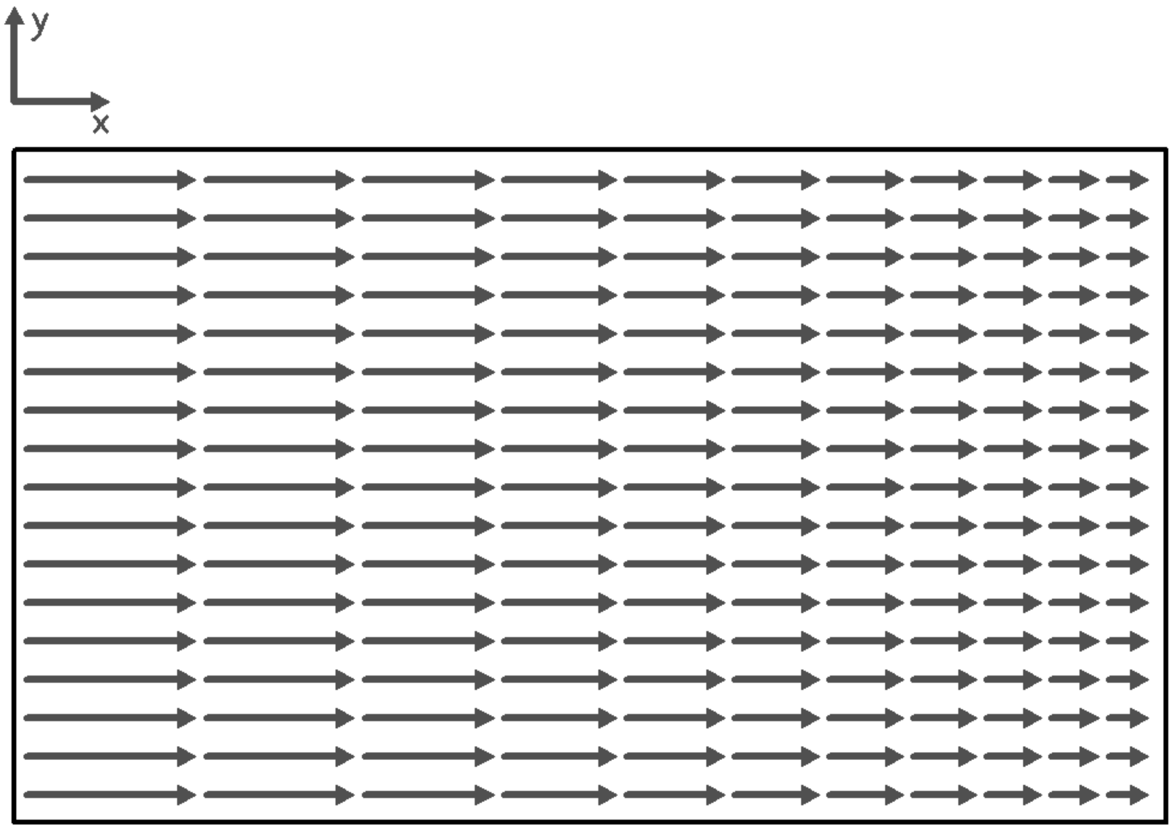
show that path integrals of $\vec{A} \cdot \vec{dl}$ are path independent.

Answer: In this case, trivially, $\vec{\nabla} \times \vec{A} = 0$. By Stokes' Theorem, therefore, we know that path (line) integrals of $\vec{A}$ are path independent.

M-87: Let $\vec{F}_1 = x^2\hat{z}$ and $\vec{F}_2 = x\hat{x} + y\hat{y} + z\hat{z}$. Which of these can be expressed as the gradient of a scalar? What is that scalar?

Answer:

M-88: The figure below depicts a field, $\vec{A}$. For this field, $\partial A_x/\partial z = 0$, $\partial A_y/\partial z = 0$, and $\vec{\nabla} \cdot \vec{A} = 0$. Is $\partial A_z/\partial z = 0$ positive, zero, or negative?



Answer: We are given the divergence is zero:

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

and we can see that

$$\frac{\partial A_x}{\partial x} < 0 \quad \text{and} \quad \frac{\partial A_y}{\partial y} = 0$$

These, together, mean that

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 \quad \text{and} \quad \frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x} = 0$$

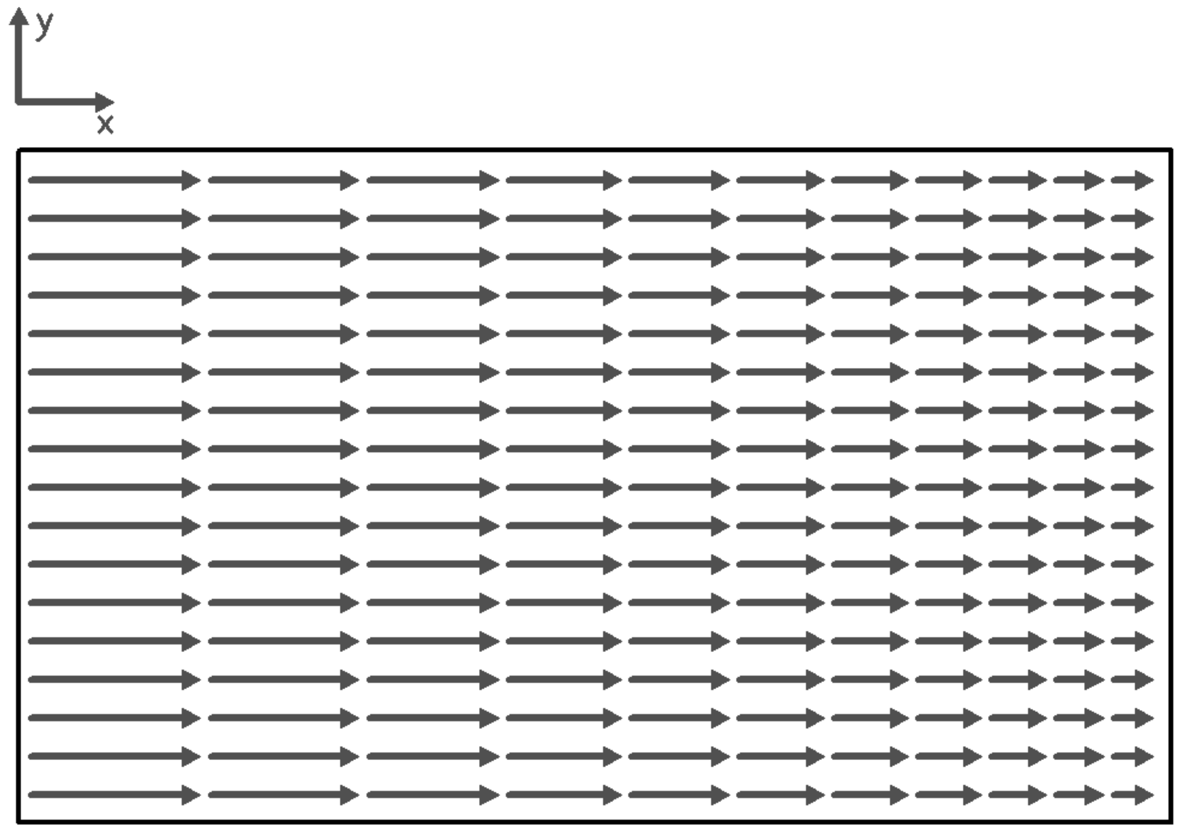
or

$$\frac{\partial A_x}{\partial x} < 0$$

therefore

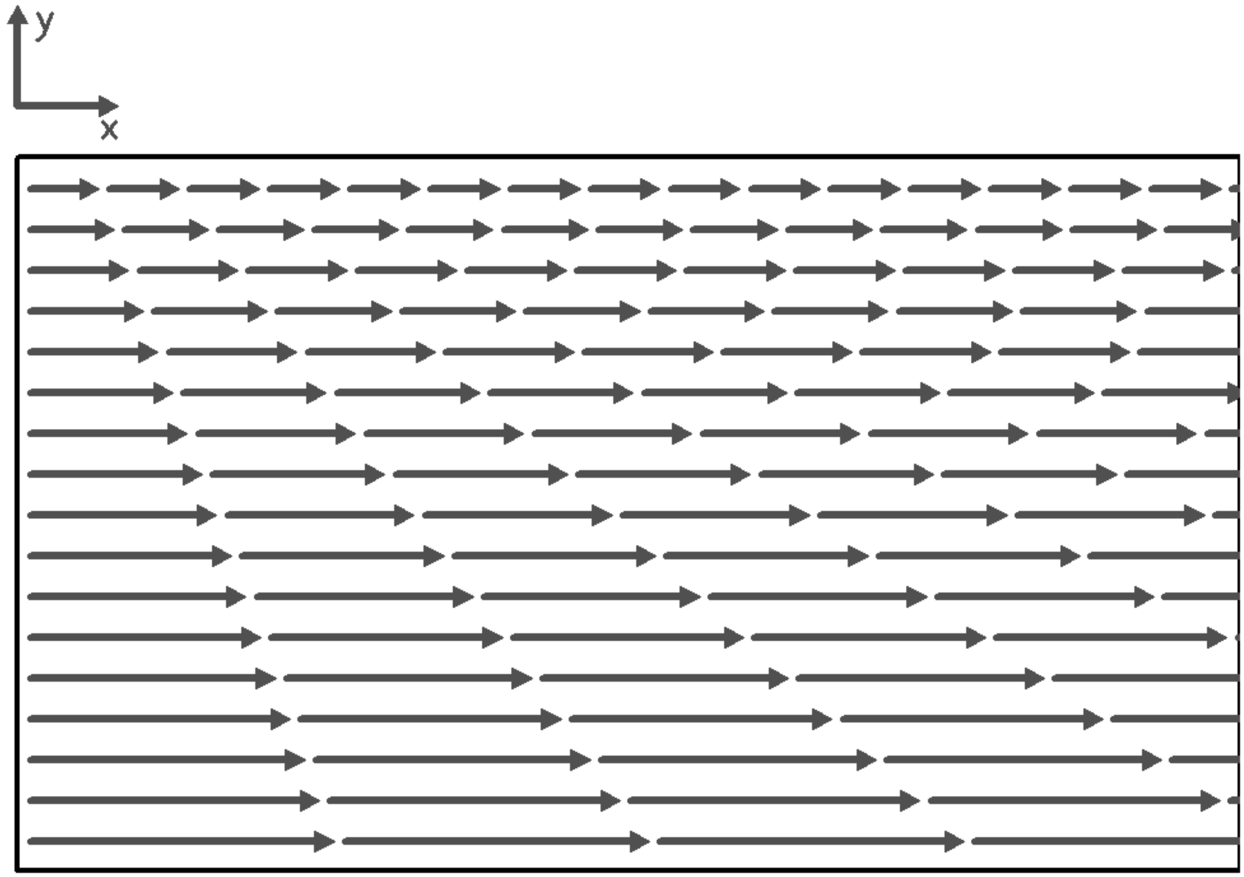
$$\frac{\partial A_z}{\partial z} > 0$$

M-89: The figure below depicts the flow field of a gas, $\vec{v}$. For this field, , $\partial A_y/\partial z = 0$, and $\vec{\nabla} \cdot \vec{A} = 0$. Is the density of the gas increasing with time, decreasing with time, or constant in time?



Answer:

M-90: The figure below depicts a field, $\vec{A}$. In this plane, is $(\vec{\nabla} \times \vec{A})_z$ positive, zero, or negative?



Answer: We have

$$(\vec{\nabla} \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

From the figure, we can see that

$$\frac{\partial A_y}{\partial x} = 0$$

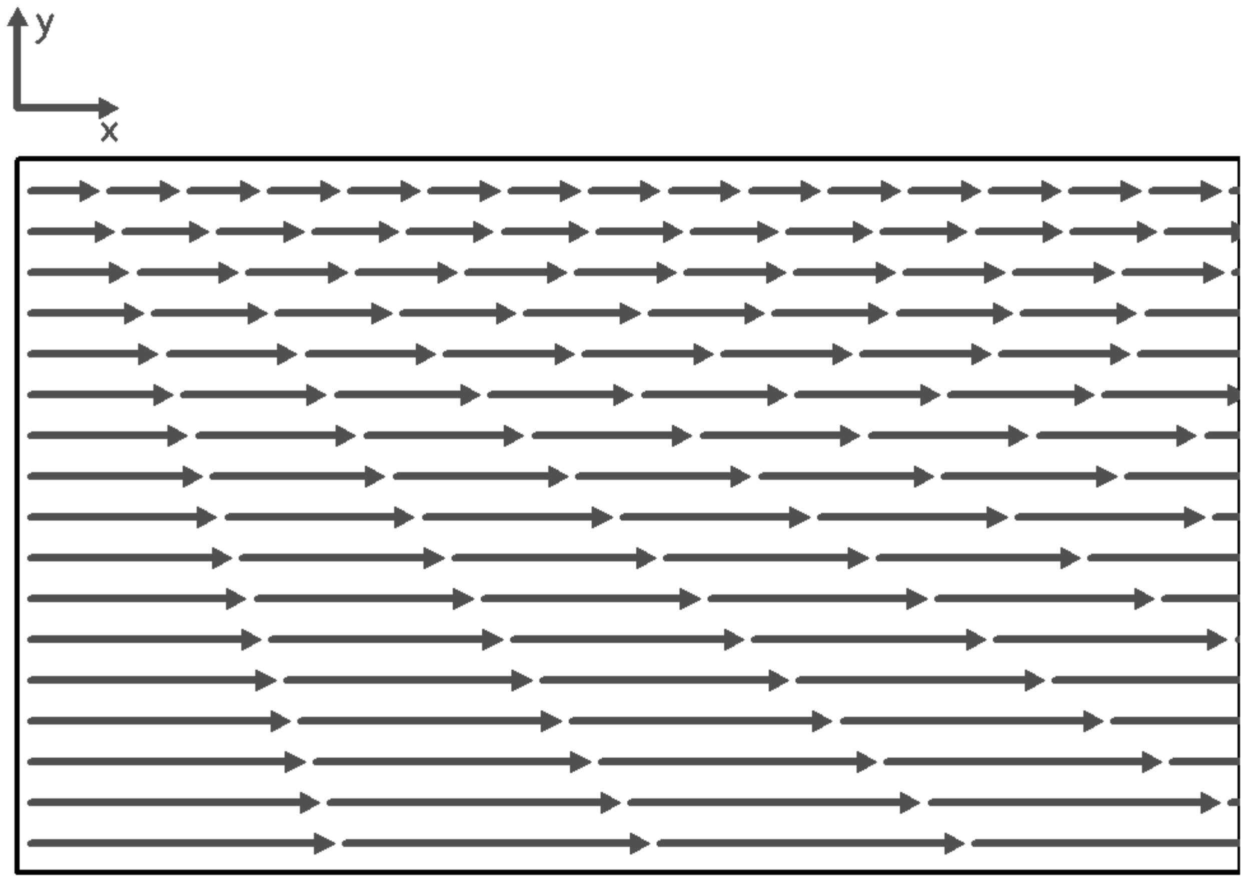
Also from the figure, we can see that

$$\frac{\partial A_x}{\partial y} < 0$$

so therefore

$$(\vec{\nabla} \times \vec{A})_z > 0$$

M-91: The figure below depicts a field, $\vec{A}$. There is no variation of any quantity with respect to z. Is $\vec{\nabla} \cdot \vec{A}$ positive, zero, or negative?



Answer:

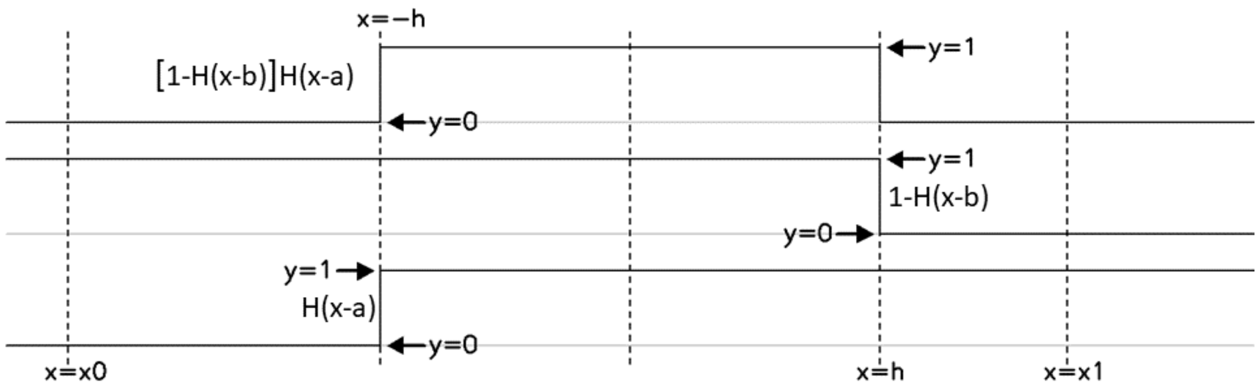
Heaviside Function

M-92: Given $x_\circ < x-h < x+h < x_1$, Show that

$$\int_{x_\circ}^{x_1} \frac{H(x+h)-H(x-h)}{2h} f(x)dx = f^*$$

where f^* is the average of f on the interval $[-h,h]$.

Answer: Consider the properties of $H(x)$ as illustrated in the figure.



It is clear that

$$\int_{x_\circ}^{x_1} \frac{H(x+h)-H(x-h)}{2h} f(x)dx = \frac{1}{2h}\int_{x_\circ}^{x_1} \Big[H(x+h)-H(x-h)\Big] f(x)dx = \frac{1}{2h}\int_{-h}^{h} f(x)dx$$

However,

$$\frac{1}{2h}\int_{-h}^{h} f(x)dx = f^* \quad \text{and therefore} \quad \int_{x_\circ}^{x_1} \frac{H(x+h)-H(x-h)}{2h} f(x)dx = f^*$$

M-93: Show that

$$\lim_{h\to 0} \frac{H(x+h)-H(x-h)}{2h} = \delta(x)$$

Considering the previous question, and retaining the order $x_\circ < x-h < x+h < x_1$, we have

$$\int_{x_\circ}^{x_1} \left[\lim_{h\to 0} \frac{H(x+h)-H(x-h)}{2h}\right] f(x)dx = \lim_{h\to 0}\left(\frac{1}{2h}\int_{x_\circ}^{x_1} \Big[H(x+h)-H(x-h)\Big] f(x)dx\right) = \lim_{h\to 0} f^*$$

where, as before, f^* is the average of f on the interval $[-h,h]$. In the limit of $h\to 0$, f^*=f(0). Therefore

$$\int_{x_\circ}^{x_1} \left[\lim_{h\to 0} \frac{H(x+h)-H(x-h)}{2h}\right] f(x)dx = f(0) \quad \text{and thus} \quad \lim_{h\to 0} \frac{H(x+h)-H(x-h)}{2h} = \delta(x)$$

where we are using the definition, in the sense of generalized functions, of the Dirac delta function.

Dirac Delta Function

M-94: Describe the defining properties of the Dirac Delta Function.

Answer:

M-95: Show that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$$

Answer:

M-96: Show that

$$\delta(x) = \frac{dH(x)}{dx}$$

where $H(x)$ is the Heaviside Function.

Answer: Using integration by parts, and the definition of the Heaviside Function, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left[f(x) \frac{d}{dx} H(x) \right] dx &= \int_{-\infty}^{\infty} \frac{d}{dx} \left[H(x) f(x) \right] dx - \int_{-\infty}^{\infty} H(x) \frac{df(x)}{dx} dx \\
&= H(x) f(x) \Big|_{-\infty}^{\infty} - \int_{o}^{\infty} \frac{df(x)}{dx} dx \\
&= H(x) f(x) \Big|_{-\infty}^{\infty} - f(x) \Big|_{0}^{\infty} \\
&= f(x=\infty) - f(x=\infty) + f(0) \\
&= f(0)
\end{aligned}$$

Therefore

$$\frac{dH(x)}{dx} = \delta(x)$$

One could add *in the sense of generalized functions*, and one could question the $f(x = \infty)$ in the above. For the former, this is not a math class. For the latter, the concept of generalized functions like the Dirac Delta Function requires that the *regular* functions we deal with, and all their derivatives, go to zero as $x \to \pm\infty$. This is a reasonable expectation for any functions we deal with in classical electromagnetism.

You will see that we already answered this question (previous section), since

$$\frac{dH(x)}{dx} = \lim_{h \to 0} \frac{H(x+h) - H(x-h)}{2h}$$

and we showed that, with $x_\circ < x - h < x + h < x_1$,

$$\int_{x_\circ}^{x_1} \left[\lim_{h \to 0} \frac{H(x+h) - H(x-h)}{2h} \right] f(x) dx = f(0)$$

M-97: Show that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)dx$$

Answer: Using integration by parts, and the definition of the Heaviside Function, we have

M-98: Show that

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 0$$

for $\vec{r} \neq 0$

Answer: Working in Cartesian coordinates, where $r = \sqrt{x^2+y^2+z^2}$, we have

$$\hat{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$

Therefor

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \vec{\nabla}\cdot\frac{x\hat{x}+y\hat{y}+z\hat{z}}{r^3}$$

or

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \frac{3}{r^3} - (x\hat{x}+y\hat{y}+z\hat{z})\cdot\left[\hat{x}\frac{\partial}{\partial x}\frac{1}{r^3} + \hat{y}\frac{\partial}{\partial y}\frac{1}{r^3} + \hat{z}\frac{\partial}{\partial z}\frac{1}{r^3}\right]$$

where

$$r = \sqrt{x^2+y^2+z^2}$$

Given that

$$\frac{\partial}{\partial x}\frac{1}{r^3} = -\frac{3x}{r^5}, \quad \frac{\partial}{\partial y}\frac{1}{r^3} = -\frac{3y}{r^5}, \quad \text{and} \quad \frac{\partial}{\partial z}\frac{1}{r^3} = -\frac{3z}{r^5}$$

we have

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \frac{3}{r^3} - 3\left[\frac{x^2}{r^5}+\frac{y^2}{r^5}+\frac{z^2}{r^5}\right] = \frac{3}{r^3} - 3\left[\frac{x^2+y^2+z^2}{r^5}\right] = \frac{3}{r^3} - 3\left[\frac{r^2}{r^5}\right] = \frac{3}{r^3} - \frac{3}{r^3} = 0$$

M-99: Show that

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$$

Answer: It is fairly straightforward to show that

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 0$$

for $\vec{r} \neq 0$

and we can use the Divergence Theorem to show that for a sphere of any radius centered on the origin

$$\int_V \vec{\nabla} \cdot \frac{\hat{r}}{r^2} d\tau = 4\pi$$

For a function $\psi(\vec{r})$ which is zero everywhere except the origin, we have

$$\int_V f(\vec{r})\psi(\vec{r})d\tau = f(0)\int_V \psi(\vec{r})d\tau$$

Putting all of this together we have

$$\int_V f(\vec{r})\vec{\nabla} \cdot \frac{\hat{r}}{r^2} d\tau = 4\pi f(0)$$

Therefore,

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$$

M-100: Show that

$$\int_{all\ space} f(\vec{r}) \left[\vec{\nabla} \cdot \frac{\hat{r}}{r^2}\right] d\tau = f(0)\int_{all\ space} \left[\vec{\nabla} \cdot \frac{\hat{r}}{r^2}\right] d\tau$$

Answer:

Develop a simple argument to the effect that

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$$

Answer: First, note that this is of the form of the electric field due to a point charge and so its divergence is zero except at the origin, or note that in this case

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = (\partial/\partial r[r^2/r^2])/r^2 = 0$$

except at the origin where I don't know what it is. This is a colloquial way of describing the Dirac delta function: zero everywhere, except where it's infinite, and we need to think very carefully about how to quantify that infinity so that it does something we understand when it is integrated over.

Consider the quantity

$$I = \int_{volume} f(\vec{r'}) \vec{\nabla} \cdot \frac{\hat{r'}}{r'^2} d\tau'$$

where the volume is a sphere of radius r centered on the origin. Note that the integrand is zero everywhere except at the origin, so without any loss of generality we can say

$$I = f(0) \int_{volume} \vec{\nabla} \cdot \frac{\hat{r'}}{r'^2} d\tau'$$

Using Gauss's Theorem

$$\int_{volume} \vec{\nabla} \cdot \frac{\hat{r'}}{r'^2} d\tau' = \frac{1}{r^2} \int_{surface} da' = \frac{4\pi r^2}{r^2} = 4\pi$$

where the surface integral on the right is over the spherical shell or radius r bounding the volume (which is just and I have used $\vec{da'} = \hat{n'} da'$. Note too my use of primed and unprimed, it's subtle but worth thinking about a bit. This shows

$$\begin{aligned} \int_{volume} \vec{\nabla} \cdot \frac{\hat{r'}}{r'^2} d\tau' &= \frac{1}{r^2} \int_{surface} da' \\ &= \frac{4\pi r^2}{r^2} \\ &= 4\pi \end{aligned} \tag{26}$$

and ultimately

$$I = 4\pi f(0)$$

so

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3(\vec{r})$$

The following two problems could equally be in other sections. In particular, these could be questions under the headings of fundamental quantities (e.g., charge distributions) or under electric fields. However, I'm including them here as examples of how the Dirac delta function gives us a mathematical formalism for dealing with quantities such as charge distributions. This is a great example of how our learning is becoming multi-threaded. As you move forward in your education the material becomes increasingly *non-local* (a term you will encounter in years to come).

M-101: Write down the *volume* charge density $\rho(x, y, z)$ for an infinitesimally thin current sheet in the $z = 0$ plane with surface charge density $\sigma(x, y)$.

Answer: $\rho = \delta(z)\sigma(x, y)$

M-102: Show that the electric field due to an infinitesimally thin current sheet in the $z = 0$ plane with surface charge density $\sigma(x, y)$ is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\int_{-\infty}^{\infty}\sigma(x', y')\frac{\vec{r} - x'\hat{x} - y'\hat{y}}{|\vec{r} - x'\hat{x} - y'\hat{y}|^3}dx'dy'$$

Answer: The electric field is (Coulomb's Law)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\int\rho(\vec{r}')\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}d\tau'$$

Using the charge density from the previous question, we have

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\int\int\sigma(x', y')\left[\int\delta(z')\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}dz'\right]dx'dy'$$

However,

$$\int\delta(z')\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}dz' = \frac{\vec{r} - x'\hat{x} - y'\hat{y}}{|\vec{r} - x'\hat{x} - y'\hat{y}|^3}$$

so therefore,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\int_{-\infty}^{\infty}\sigma(x', y')\frac{\vec{r} - x'\hat{x} - y'\hat{y}}{|\vec{r} - x'\hat{x} - y'\hat{y}|^3}dx'dy'$$

The following problem could equally be in other sections. In particular, this could be questions under the heading of electric fields. However, I include it here as an example of how the Heaviside and Dirac delta functions give us a mathematical formalism for dealing with stepwise functions.

M-103: What is the charge density, given that the potential is

$$\phi(r < a) = \frac{Qr^2}{8\pi\epsilon_\circ a^3} + \frac{Q}{8\pi\epsilon_\circ a} \quad \text{and} \quad \phi(r > a) = \frac{Q}{4\pi\epsilon_\circ r} \tag{27}$$

where r is the radius in spherical coordinates: $r = \sqrt{x^2 + y^2 + z^2}$.

Answer: Here we approach this problem in a systematic way, one that I do not recommend you adopt during for example a test, but one which I believe supports a better understanding of the mathematical framework that is the relationship between the potential and the charge distribution.

We start with the electric field, determined from the potential using the gradient which in this case (spherical symmetry) is $\vec{\nabla} = \hat{r}\partial/\partial r$. Here, however, we will express the electric field as a function that is valid everywhere:

$$\vec{E}(r) = \frac{Q}{4\pi\epsilon_\circ}\left[\frac{H(r-a)}{r^2} - \frac{rH(a-r)}{a^3}\right]\hat{r}$$

Where $H(r-a)$ is the Heaviside Function, where the step from zero to one is at $r = a$. We can then get the charge distribution from Gauss's Law, noting that

$$\begin{aligned}
\rho(r) &= \epsilon_\circ \vec{\nabla}\cdot\vec{E}(r) \\
&= \frac{Q}{4\pi r^2}\frac{\partial}{\partial r}\left[H(r-a) - \frac{r^3 H(a-r)}{a^3}\right] \\
&= \frac{Q}{4\pi r^2}\left[\frac{\partial}{\partial r}H(r-a) - \frac{3r^2 H(a-r)}{a^3} - \frac{r^3}{a^3}\frac{\partial}{\partial r}H(a-r)\right] \\
&= \frac{Q}{4\pi r^2}\left[\delta(r-a) + \delta(a-r)\frac{r^3}{a^3} - \frac{r^3 H(a-r)}{a^3}\right] \\
&= \frac{Q}{2\pi r^2}\delta(r-a) - \frac{3Q}{4\pi a^3}H(a-r)
\end{aligned}$$

Here, I have used, again because of the spherical symmetry,

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 E_r\right]$$

As well, $\delta(r-a)$ is the Dirac Delta Function, and I have used the following of its properties and those of the Heaviside Function:

$$\frac{\partial}{\partial r}H(r-a) = \delta(r-a), \quad \frac{\partial}{\partial r}H(a-r) = -\delta(a-r), \quad \delta(r-a) = \delta(a-r) \quad \text{and} \quad \delta(r-a)f(r) = \delta(r-a)f(a)$$

Dummy Variables

A dummy variable in an integral (or dummy index in a sum) does not *survive* the integration (or summation), and thus does not appear in the final answer. We will be working with functions of, e.g., position, that are themselves integrals or summations over dummy variables and dummy indices. Coulomb's Law and that of Biot and Savart are two examples.

M-104: Show that

$$\frac{d}{dx}\int_{x_0}^{x} x du = 2x - x_0$$

Answer:

M-105: Show that

$$\frac{d}{dx}\int_{g(x)}^{x_\circ} x^2 du = 2x\Big[x_\circ - g(x)\Big] - x^2\frac{d}{dx}g(x)$$

Answer:

M-106: Let V be a sphere of radius R, centered on the origin. Show that

$$\int_V \Big[x^2 + x'^2 + y'^2 + z'^2\Big] d\tau' = 4\pi R^3 \left[\frac{5x^2 - 3R^2}{15}\right]$$

Answer:

M-107: My sense of this is that most of us struggle with definite integrals of the following form:

$$g(x) = \int_a^b f(x, x')dx'$$

As an example, evaluate the integral

$$g(x) = \int_a^b (x - x')dx'$$

Where is this function zero?

Answer:

M-108: Show that

$$\vec{\nabla}\int_V \sqrt{x^2+x'y'}\,d\tau' = x\hat{x}\int_V \frac{d\tau'}{\sqrt{x^2+x'y'}}$$

Answer: The gradient operator here is with respect to the unprimed variables. The intgeral is a function of x, but not of y or z.

$$\int_V \sqrt{x^2+x'y'}\,d\tau' = f(x)$$

For the gradient of $f(x)$ we have

$$\vec{\nabla}f = \hat{x}\frac{\partial f}{\partial x} \quad \text{so that} \quad \vec{\nabla}\int_V \sqrt{x^2+x'y'}\,d\tau' = \hat{x}\frac{\partial}{\partial x}\int_V \sqrt{x^2+x'y'}\,d\tau'$$

However, we can take the partial derivative into the integrand, so we have

$$\vec{\nabla}\int_V \sqrt{x^2+x'y'}\,d\tau' = \hat{x}\int_V \left[\frac{\partial}{\partial x}\sqrt{x^2+x'y'}\right]d\tau'$$

The partial derivative

$$\frac{\partial}{\partial x}\sqrt{x^2+x'y'} = \frac{x}{\sqrt{x^2+x'y'}} \quad \text{so therefore} \quad \vec{\nabla}\int_V \sqrt{x^2+x'y'}\,d\tau' = x\hat{x}\int_V \frac{d\tau'}{\sqrt{x^2+x'y'}}$$

M-109: Show that, if V' is the cube with vertices at $(0,0,0)$ and $(1,1,1)$, so that

$$\int_{V'} f(\vec{r}')d\tau' = \int_0^1\int_0^1\int_0^1 f(\vec{r}')dx'dy'dz'$$

and if $\vec{A}(\vec{r},\vec{r}') = xx'\hat{x} + yy'\hat{y} + zz'\hat{z}$, then

$$\vec{\nabla}\cdot\int_{V'} \vec{A}(\vec{r},\vec{r}')d\tau' = \frac{3}{2}$$

Answer: We can switch the order of differentiation and integration, such that

$$\vec{\nabla}\cdot\int_{V'} \vec{A}(\vec{r},\vec{r}')d\tau' = \int_{V'}\left[\vec{\nabla}\cdot\vec{A}(\vec{r},\vec{r}')\right]d\tau' = \int_0^1\int_0^1\int_0^1\left[x'+y'+z'\right]dx'dy'dz'$$

However,

$$\int_0^1\int_0^1\int_0^1 x'dx'dy'dz' = \left[\int_0^1 x'dx'\right]\left[\int_0^1\int_0^1 dy'dz'\right] = \left[\frac{1}{2}\right]\left[1\right] = \frac{1}{2}$$

Similarly,

$$\int_0^1\int_0^1\int_0^1 y'dy'dx'dz' = \frac{1}{2} \quad \text{and} \quad \int_0^1\int_0^1\int_0^1 z'dz'dy'dz' = \frac{1}{2}$$

so that,

$$\vec{\nabla}\cdot\int_{V'} \vec{A}(\vec{r},\vec{r}')d\tau' = \int_0^1\int_0^1\int_0^1\left[x'+y'+z'\right]dx'dy'dz' = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$$

Mathematics of the Scalar Potential

M-110: Starting directly from Coulomb's Law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\int \rho(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}d\tau'$$

and taking the curl of the integral, show that the curl of the static electric field is zero.

Answer We can take the curl of the electric field, noting that here $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ are derivatives with respect to unprimed independent variables. First we can bring the curl inside the integral, and since in the integrand ρ is a function only of $\vec{r}'$,

$$\vec{\nabla}\times\vec{E} = \frac{1}{4\pi\epsilon_\circ}\int \rho(\vec{r}')\left(\vec{\nabla}\times\left[\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right]\right)d\tau' \tag{28}$$

Given that

$$\vec{r}-\vec{r}' = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

and

$$|\vec{r}-\vec{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \quad \text{and} \quad |\vec{r}-\vec{r}'|^3 = \left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{3}{2}}$$

Let

$$\vec{A} = \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}, \quad \text{and thus} \quad A_x = \frac{x-x'}{|\vec{r}-\vec{r}'|^3}, \quad A_x = \frac{y-y'}{|\vec{r}-\vec{r}'|^3}, \quad \text{and} \quad A_z = \frac{z-z'}{|\vec{r}-\vec{r}'|^3}$$

So,

$$\left[\vec{\nabla}\times\vec{A}\right]_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{3}{|\vec{r}-\vec{r}'|^5}\Big[(y-y')(z-z')-(z-z')(y-y')\Big] = 0$$

$$\left[\vec{\nabla}\times\vec{A}\right]_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{3}{|\vec{r}-\vec{r}'|^5}\Big[(z-z')(x-x')-(x-x')(z-z')\Big] = 0$$

$$\left[\vec{\nabla}\times\vec{A}\right]_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{3}{|\vec{r}-\vec{r}'|^5}\Big[(x-x')(y-y')-(y-y')(x-x')\Big] = 0$$

Since the $x-$, $y-$ and $z-$components of $\vec{\nabla}\times\vec{A}$ are all zero, and

$$\vec{A} = \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

then

$$\vec{\nabla}\times\left[\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right] = 0$$

and so, looking at Equation 28, we see that for time independent situations,

$$\vec{\nabla}\times\vec{E}(\vec{r}) = 0 \tag{29}$$

M-111: Let

$$\vec{A}(\vec{r}-\vec{r}') = (\vec{r}-\vec{r}')f(|\vec{r}-\vec{r}'|)$$

where $\vec{r}'$ is a constant. Show that $\vec{A}$ is curl free.

Answer: The three Cartesian components of $\vec{A}$ are

$$\left[\vec{\nabla}\times\vec{A}\right]_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\dot{f}(|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}\left[(y-y')(z-z') - (z-z')(y-y')\right] = 0$$

where $\dot{f}$ indicates the derivative of f with respect to its argument. Given the two cyclic permutations, we have $\vec{\nabla}\times\vec{A} = 0$.

M-112: Let

$$\vec{A}(\vec{r}) = (\vec{r}-\vec{r}')f(|\vec{r}-\vec{r}'|) \tag{30}$$

where $\vec{r}'$ is a constant. Shift your origin to $\vec{r}'$, and then work in spherical coordinates *in that shifted coordinate system*, and show that $\vec{A}$ is curl free.

Answer: Let $\vec{r}'' = \vec{r}-\vec{r}'$. Then

$$\vec{A}(\vec{r}) = r'' f(r'')\hat{r}'' \quad \text{where} \quad r'' = |\vec{r}-\vec{r}'| \quad \text{and} \quad \hat{r}'' = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

Thus, in this double-primed coordinate system, the vector $\vec{A}$ is both spherically symmetric, and radial, with the latter meaning

$$\vec{A} = A_{r''}\hat{r}'', \quad A_{\theta''} = 0, \quad A_{\phi''} = 0, \quad \frac{\partial}{\partial\theta''} \equiv 0, \quad \text{and} \quad \frac{\partial}{\partial\phi''} \equiv 0 \tag{31}$$

In this double-primed coordinate system, $\vec{A}$ is spherically symmetric. The curl in that coordinate system is thus

$$\vec{\nabla}''\times\vec{A} = \frac{1}{r''\sin(\theta'')}\left[\frac{\partial}{\partial\theta''}\left[\sin(\theta'')A_{\phi''}\right] - \frac{\partial A_{\theta''}}{\partial\phi''}\right]\hat{r}'' + \left[\frac{1}{\sin(\theta'')}\frac{\partial A_{r''}}{\partial\phi''} - \frac{\partial}{\partial r''}\left[r''A_{\phi''}\right]\right]\hat{\theta}'' + \frac{1}{r''}\left[\frac{\partial}{\partial r''}\left[r''A_{\theta''}\right] - \frac{\partial A_{r''}}{\partial\theta''}\right]\hat{\phi}''$$

Given Equation 31,

$$\vec{\nabla}''\times\vec{A} = 0 \quad \text{and Thus,} \quad \vec{\nabla}\times\vec{A} = 0$$

Note: In a way, this question is overkill. In reality, one can look at Equation 30, and simply (and justifiably) say "we can obviously shift the origin, and then we'll have a curl-free function, so therefore $\vec{A}$ is curl-free". That is fine, but I believe it is worth thinking this through in detail, at least once.

M-113: Above, we have shown that if

$$\vec{A}(\vec{r}) = (\vec{r} - \vec{r}')f(|\vec{r} - \vec{r}'|)$$

where $\vec{r}'$ is a constant, then

$$\vec{\nabla} \times \vec{A}(\vec{r}) = 0$$

Using this result, and the *functional form* of the integrand in Coulomb's Law, show that the curl of the static electric field is zero.

Answer: We can express Coulomb's Law as

$$\vec{E}(\vec{r}) = \int \rho(\vec{r}')(\vec{r} - \vec{r}')f(|\vec{r} - \vec{r}'|)d\tau'$$

where

$$f(|\vec{r} - \vec{r}'|) = \frac{1}{|\vec{r} - \vec{r}'|^3}$$

Thus

$$\vec{\nabla} \times \vec{E}(\vec{r}) = \int \rho(\vec{r}')\vec{g}(\vec{r}, \vec{r}')d\tau' \tag{32}$$

where

$$\vec{g}(\vec{r}, \vec{r}') = \vec{\nabla} \times \left[(\vec{r} - \vec{r}')f(|\vec{r} - \vec{r}'|) \right]$$

which we have already shown to be identically zero. Thus, the integrand in Equation 32 is everywhere zero, and so we have, for the static electric field

$$\vec{\nabla} \times \vec{E}(\vec{r}) = 0 \tag{33}$$

Note: We did the same question, meaning we showed the static electric field is curl free, above. There we used what I would call brute force. For me this is a much more elegant and concise way of showing the same result.

M-114: Provided contour (path) integrals of $\vec{E}$ are path independent, the following is a single valued function of $\vec{r}$ (do you see why that is true?).

$$V(\vec{r}) = V(\vec{r}_\circ) - \int_{\vec{r}_\circ}^{\vec{r}} \vec{E}(\vec{r}\,') \cdot \vec{dl}\,' \tag{34}$$

Show that, provided contour integrals of $\vec{E}$ are path independent,

$$\vec{E} = -\vec{\nabla} V$$

where V is the function given above.

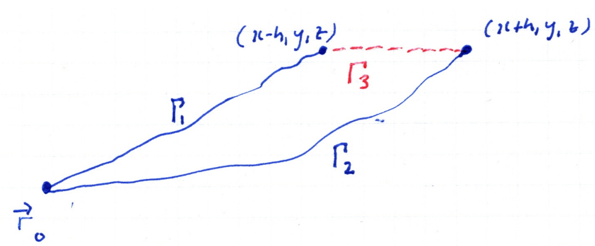
Answer: The above is true if and only if

$$-\frac{\partial V}{\partial x} = E_x, \quad -\frac{\partial V}{\partial y} = E_y, \quad \text{and} \quad -\frac{\partial V}{\partial z} = E_z \tag{35}$$

A good place to start, then, is to determine what the first partial derivative of V is with respect to x. The spatial dependence in $V(\vec{r})$ in Equation 34 is in the upper limit of the integral, which is $\vec{r}$. By definition, then,

$$-\frac{\partial V}{\partial x} = \lim_{h\to 0}\left[\frac{1}{2h}\left(\int_{(x_\circ,y_\circ,z_\circ)}^{(x+h,y,z)} \vec{E}(\vec{r}\,')\cdot\vec{dl}\,' - \int_{(x_\circ,y_\circ,z_\circ)}^{(x-h,y,z)} \vec{E}(\vec{r}\,')\cdot\vec{dl}\,'\right)\right] \tag{36}$$

The quantity in brackets is the difference between two contour integrals, both that start at that start at $\vec{r}_\circ$, one (Γ_1) ending at $(x-h,y,z)$ and the second (Γ_2) at $(x+h,y,z)$ I illustrate these two contours in the first line in the figure, below. The difference between them is the contour integral of $\vec{E}$ from $(x-h,y,z)$ to $(x+h,y,z)$. We are given that contour integrals of $\vec{E}$ are path independent, so we can pick a contour that is particularly helpful here, name Γ_3, the red-dashed contour in the Figure.



Γ_3 is the straight line contour ($\vec{dl} = dx\hat{x}$) between $(x-h,y,z)$ and $(x+h,y,z)$. Equation 36 then becomes

$$-\frac{\partial V}{\partial x} = \lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h} E_x(x',y,z)dx' = \lim_{h\to 0}\frac{2h < E_x >}{2h} = \lim_{h\to 0} < E_x > \tag{37}$$

where $< E_x >$ is the average of the x-component of $\vec{E}$ along the $2h$ long *straight line* contour between $(x-h,y,z)$ and $(x+h,y,z)$. It is clear that for diminishing (very small) h, this average is $E_x(x,y,z)$.

Therefore, from Equation 37 we can see that

$$\frac{\partial V}{\partial x} = -E_x \tag{38}$$

Note that there is nothing in the above argument that differentiates x from y or z, so we have shown that the condition in Equation 35 is true.

Dot Product as Distributive

M-115: It is easy to show the dot product is distributive provided $\vec{A}$, $\vec{B}$, and $\vec{C}$ are coplanar. If the three vectors are in the xy plane, and we let $\vec{D} = \vec{B} + \vec{C}$, we have

$$\begin{aligned} \vec{A} \cdot \vec{D} &= A_x D_x + A_y D_y \\ &= A_x(B_x + C_x) + A_y(B_y + C_y) \\ &= (A_x B_x + A_y B_y) + (A_x C_x + A_y C_y) \\ &= \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \end{aligned} \tag{39}$$

Since we can always call whatever plane the vectors are in the xy plane, then this is a general result for coplanar vectors. What about the case of non-coplanar vectors? In particular, imagine $\vec{A} \cdot \vec{D}$ where $\vec{D} = \vec{B} + \vec{C}$, for the case where $\vec{A}$ has a component that is orthogonal to the plane defined by $\vec{B}$ and $\vec{C}$ (the "BC plane"). $\vec{A}$ can be thought of as the sum of two vectors, namely

$$\vec{A} = \vec{A}_\perp + \vec{A}_\| \tag{40}$$

where $\vec{A}_\perp$ is that part of $\vec{A}$ in the direction perpendicular to the BC plane. Then, $\vec{A}_\| = \vec{A} - \vec{A}_\perp$ is in the BC plane. Equation 3 means the dot product is the projection of one of the vectors in the direction of the second, multiplied by the length of the second. If we consider $\vec{A} \cdot \vec{D}$, $\vec{A} \cdot \vec{B}$, and $\vec{A} \cdot \vec{C}$, the part of the vector $\vec{A}$ that is perpendicular to the BC plane does not contribute or affect the projection of $\vec{A}$ in the direction of a vector in the BC plane $\vec{B}$. Thus, if I consider $\vec{A} \cdot \vec{B}$, the projection of $\vec{A}$ along $\vec{B}$ will be the same as the projection of $\vec{A}_\|$ along $\vec{B}$. For the three quantities that are needed here, we have

$$\begin{aligned} \vec{A} \cdot \vec{D} &= \vec{A}_\| \cdot \vec{D} \\ \vec{A} \cdot \vec{B} &= \vec{A}_\| \cdot \vec{B} \\ \vec{A} \cdot \vec{C} &= \vec{A}_\| \cdot \vec{C} \end{aligned} \tag{41}$$

$\vec{A}_\|$ is in the BC plane, so $\vec{A}_\|$, $\vec{B}$, and $\vec{C}$ (and of course $\vec{D} = \vec{B} + \vec{C}$) are coplanar. Thus, from Equation 39

$$\vec{A}_\| \cdot \vec{D} = \vec{A}_\| \cdot (\vec{B} + \vec{C}) = \vec{A}_\| \cdot \vec{B} + \vec{A}_\| \cdot \vec{C} \tag{42}$$

Using Equation 41 in Equation 42, we have

$$\vec{A} \cdot \vec{D} = \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \tag{43}$$

and therefore we have shown Equation 39.

> This is not an argument I would expect a student at *any* level, or for that matter a university physics instructor, to be able to conjure up on the spot. I include this question because in my opinion working through it advances one's facility with vectors. As well, I think it's good to remember, now and then, that we really need to be able to find out why we say something like "the dot product is distributive."

Fourier Series

M-116: Show that

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}dx = \frac{a}{2} \quad (m \neq n)$$
$$= 0 \quad (m \neq n) \tag{44}$$

Answer: The result for $m = n$ can be shown using the average of $\sin(x)$ on $(0, \pi)$ $(1/2)$, and noting that for $m \neq n$

$$\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a} = \frac{\cos[(m-n)\pi x/a] - \cos[(m+n)\pi x/a]}{2}$$

M-117: Use graphs to argue that

$$\int_0^\pi \sin^2(\theta)d\theta = \int_0^\pi \cos^2(\theta)d\theta$$

Answer:

M-118: Show that, if n is an integer,

$$\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{a}{2}$$

Answer: Let

$$\theta = \frac{n\pi x}{a}, \quad \text{so that} \;\; dx = \frac{a}{n\pi}d\theta$$

Then,

$$\int_{x=0}^{x=a} \sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{a}{n\pi}\int_{\theta=0}^{\theta=n\pi} \sin^2(\theta)d\theta$$

Based on the result of the previous question, this means that

$$\int_{x=0}^{x=a} \sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{a}{n\pi}\frac{n\pi}{2} = \frac{a}{2} \tag{45}$$

M-119: Show that

$$\int_0^{\pi} \sin^2(\theta) d\theta = \frac{\pi}{2}$$

Answer: Based on the result of the previous question, we have

$$\int_0^{\pi} \sin^2(\theta) d\theta = \frac{1}{2}\left[\int_0^{\pi} \sin^2(\theta) d\theta + \int_0^{\pi} \cos^2(\theta) d\theta\right] = \frac{1}{2}\int_0^{\pi} \left[\sin^2(\theta) + \cos^2(\theta)\right] d\theta$$

and we know

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Therefore,

$$\int_0^{\pi} \sin^2(\theta) d\theta = \frac{1}{2}\int_0^{\pi} d\theta = \frac{\pi}{2}$$

M-120: Show that

$$\int_0^{n\pi} \sin^2(\theta) d\theta = \frac{n\pi}{2}$$

Answer:

M-121: Show that, if n is any integer and $n \neq 0$, then

$$\int_0^{\pi} \cos(n\theta) d\theta = 0$$

Answer: Let $u = n\theta$, so that $d\theta = du/n$, and

$$\int_{\theta=0}^{\theta=\pi} \cos(n\theta) d\theta = \frac{1}{n} \int_{u=0}^{u=n\pi} \cos(u) du = \frac{1}{n} \sin(u) \Big|_0^{n\pi} = \frac{\sin(n\pi) - \sin(0)}{n} = 0 \tag{46}$$

M-122: Show that

$$\sin(u)\sin(v) = \frac{1}{2}\Big[\cos(u-v) - \cos(u+v)\Big]$$

Answer: We can apply the double angle formulae to the quantity in parentheses:

$$\begin{aligned}
\cos(u-v) - \cos(u+v) &= \cos(u)\cos(v) + \sin(u)\sin(v) - \Big[\cos(u)\cos(v) - \sin(u)\sin(v)\Big] \\
&= \cos(u)\cos(v) - \cos(u)\cos(v) + \sin(u)\sin(v) + \sin(u)\sin(v) \\
&= 2\sin(u)\sin(v)
\end{aligned}$$

Therefore

$$\sin(u)\sin(v) = \frac{1}{2}\Big[\cos(u-v) - \cos(u+v)\Big]$$

M-123: Show that, if m and n are integers, and $m \neq n$,

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = 0$$

Answer: Based on the results of the previous question, we have

$$\begin{aligned}
\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx &= \frac{1}{2} \int_0^a \left[\cos\left(\frac{(n-m)\pi x}{a}\right) - \cos\left(\frac{(n+m)\pi x}{a}\right)\right] dx \\
&= \frac{a}{2(n-m)\pi} \sin\left(\frac{(n-m)\pi x}{a}\right)\Bigg|_0^a - \frac{a}{2(n+m)\pi} \sin\left(\frac{(n+m)\pi x}{a}\right)\Bigg|_0^a \\
&= \frac{a}{2(n-m)\pi}\left[\sin\Big((n-m)\pi\Big) - \sin(0)\right] - \frac{a}{2(n+m)\pi}\left[\sin\Big((n+m)\pi\Big) - \sin(0)\right] \\
&= \frac{a}{2(n-m)\pi}\Big[0 - 0\Big] - \frac{a}{2(n+m)\pi}\Big[0 - 0\Big] \\
&= 0
\end{aligned} \tag{47}$$

M-124: Show that

$$\int_{x=0}^{x=a} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \frac{a}{2}\delta_{nm}$$

Answer: According to Equations 45 and 47, if m and n are integers

$$\int_{x=0}^{x=a} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = 0 \quad \text{if} \quad m \neq n$$

and

$$\int_{x=0}^{x=a} \sin^2\left(\frac{n\pi x}{a}\right)\sin dx = \frac{a}{2}$$

Therefore,

$$\int_{x=0}^{x=a} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \frac{a}{2}\delta_{nm} \tag{48}$$

M-125: In an 'N-space', the vector $\vec{A}$ is

$$\vec{A} = \sum_{i=1}^{i-N} A_i \hat{x}_i$$

If the set of unit vectors $\hat{x}_i$ spans the space, and is an *orthogonormal* set, what is A_j (meaning the component of $\vec{A}$ in the direction of $\hat{x}_j$)?

Answer: Noting that the orthogonality condition, for an *orthonormal* set, is $\hat{x}_i \hat{x}_j = \delta_{ij}$, we have

$$\hat{x}_j \cdot \vec{A} = \hat{x}_j \cdot \left[\sum_{i=1}^{i-N} A_i \hat{x}_i \right] = \sum_{i=1}^{i-N} A_i (\hat{x}_i \cdot \hat{x}_j) = \sum_{i=1}^{i-N} A_i \delta_{ij} = A_j$$

M-126: The Fourier sine series for $f(x)$ on [0,a] is

$$f(x) = \sum_{m=1}^{m=\infty} A_m \sin\left(\frac{m\pi x}{a} \right)$$

What is A_n?

Answer: This is called "Fourier's Trick'... employing a kind of equivalent to the dot product in the previous question:

$$\begin{aligned}
\int_0^a f(x) \sin\left(\frac{n\pi x}{a} \right) dx &= \int_0^a \left[\sin\left(\frac{n\pi x}{a} \right) \sum_{m=1}^{m=\infty} A_m \sin\left(\frac{n\pi x}{a} \right) \right] dx \\
&= \sum_{m=1}^{m=\infty} A_m \int_0^a \left[\sin\left(\frac{m\pi x}{a} \right) \sin\left(\frac{n\pi x}{a} \right) \right] dx \\
&= \sum_{m=1}^{m=\infty} \frac{a}{2} \delta_{mn} A_m \\
&= \frac{a}{2} A_n
\end{aligned}$$

Therefore,

$$A_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a} \right) dx$$

Do you see the connection with the previous question? Fourier's trick is essentially determining the 'projection' of $f(x)$ on the 'basis' function $\sin(n\pi x/a)$.

M-127: What is the Fourier Sine series of $f(x) = 1$ on the interval $[0, a]$?

Answer: The Fourier Sine series, up to the Mth term, is

$$f(x) \approx \sum_{m=1}^{m=M} A_m \sin\left(\frac{m\pi x}{a}\right)$$

For the interval $[0, a]$, we have

$$\begin{aligned} A_m &= \frac{2}{a}\int_0^a f(x)\sin\left(\frac{m\pi x}{a}\right)dx \\ &= \frac{2}{a}\int_0^a (1)\sin\left(\frac{m\pi x}{a}\right)dx \\ &= -\frac{2}{a}\frac{a}{m\pi}\cos\left(\frac{m\pi x}{a}\right)\Big|_0^a \\ &= -\frac{2}{a}\frac{a}{m\pi}\Big[\cos(m\pi) - \cos(0)\Big] \\ &= \frac{4}{m\pi} \text{ for odd } m, \text{ for even } m \end{aligned}$$

Thus, the Fourier Sine series for 1 is (up to the the Mth term, with M being odd)

$$f(x) \approx \frac{4}{\pi}\sin\left(\frac{\pi x}{a}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi x}{a}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi x}{a}\right) + \cdots + \frac{4}{M\pi}\sin\left(\frac{M\pi x}{a}\right)$$

A compact way of writing this without the need to restrict the equation to odd terms is

$$f(x) \approx \sum_{m=1}^{m=M} \frac{2[1-(-1)^m]}{m\pi}\sin\left(\frac{m\pi x}{a}\right)$$

Commentary

We develop a Fourier (in our case) Sine series, which in practice is always truncated at $n = N$, to represent (or approximate) a function $f(x)$ on an interval (which in our case is) $[0, a]$. The Fourier Sine series is

$$f(x) \approx \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{a}\right) \quad \text{where} \quad A_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx \tag{49}$$

Although the coefficients in Equation 49 are for approximating the function on $[0, a]$, we can evaluate the truncated series anywhere, including for $x < 0$ and $x > a$. We can, for example, plot the series on an interval that extends beyond $[0, a]$, as I do for some of the questions below (in those cases on $[-2a, 2a]$).

Since the sine is an odd function of its argument, we know that the Fourier Sine series will be an odd function of x. As well, since

$$\sin\left(\frac{n\pi x}{a}\right) = 0 \text{ at } x = 0 \text{ and at } x = a$$

we know that regardless of what the function $f(x)$ does at it approaches 0^+ or a^-, the series will be exactly zero at $x = 0$ and $x = a$.

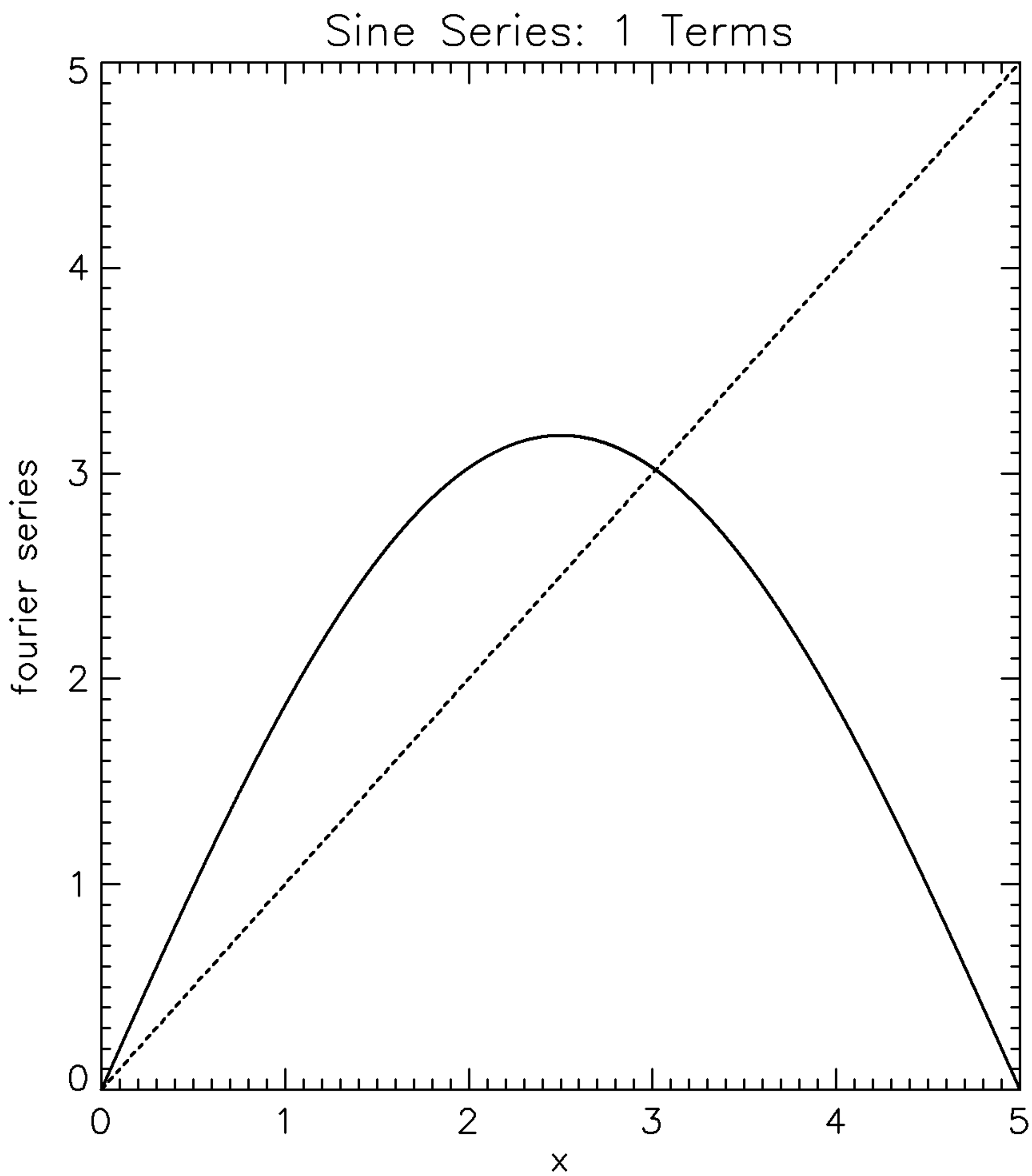
It is true that the Fourier Sine series converges *uniformly* to the function $f(x)$ on $[0, a]$. This means that at any value of x on $(0, a)$ (note the *round* brackets), in the limit of $N \to \infty$ the limit of the series at that value of x is $f(x)$. However, a consequence of the aforementioned constraint of the series being zero at $x = 0$ and $x = a$ is the so-called Gibbs phenomenon, where the series overshoots and undershoots the function as x approaches discontinuities in the function and its derivatives and as x approaches $0+$ and a^- depending on the behavior of $f(x)$ around those endpoints. The Gibbs phenomenon always occurs at such boundaries and is why we have only *uniform*, rather than *absolute* convergence.

For the next questions, I again am not suggesting you create these graphs yourselves. Rather, again, I want you to look at these graphs to further develop your understanding of how a Fourier series actually behaves. Identify for yourself the Gibbs phenomenon and try to develop a feel for when it does and does not occur. Other things to look for are the obviousness of the Fourier Sine series being odd, and the differences in the qualitative look of series truncated at low (e.g., 3, 5, 7), mid (e.g., 25 or 30), and high (e.g., 100 or 500) numbers of terms. A benefit of the following questions is the graphs lend credence to the idea the Fourier Sine series is an arbitrarily accurate representation of a function. I other words, Fourier Series work.

Finally, generally speaking, the analytical evaluation of the integrals to obtain the coefficients for Fourier series such as in the preceding question is not something one does in this day and age, although one it is important to be able to set up the integrals and otherwise frame the series and how they are used in solving larger questions. We have such ready access to computers that it is very easy to carry out those integrals numerically in just three or four lines of code. Being able to do such integrals numerically is a tremendously valuable skill, something that most working physicists do countless times as a matter of course in their jobs.

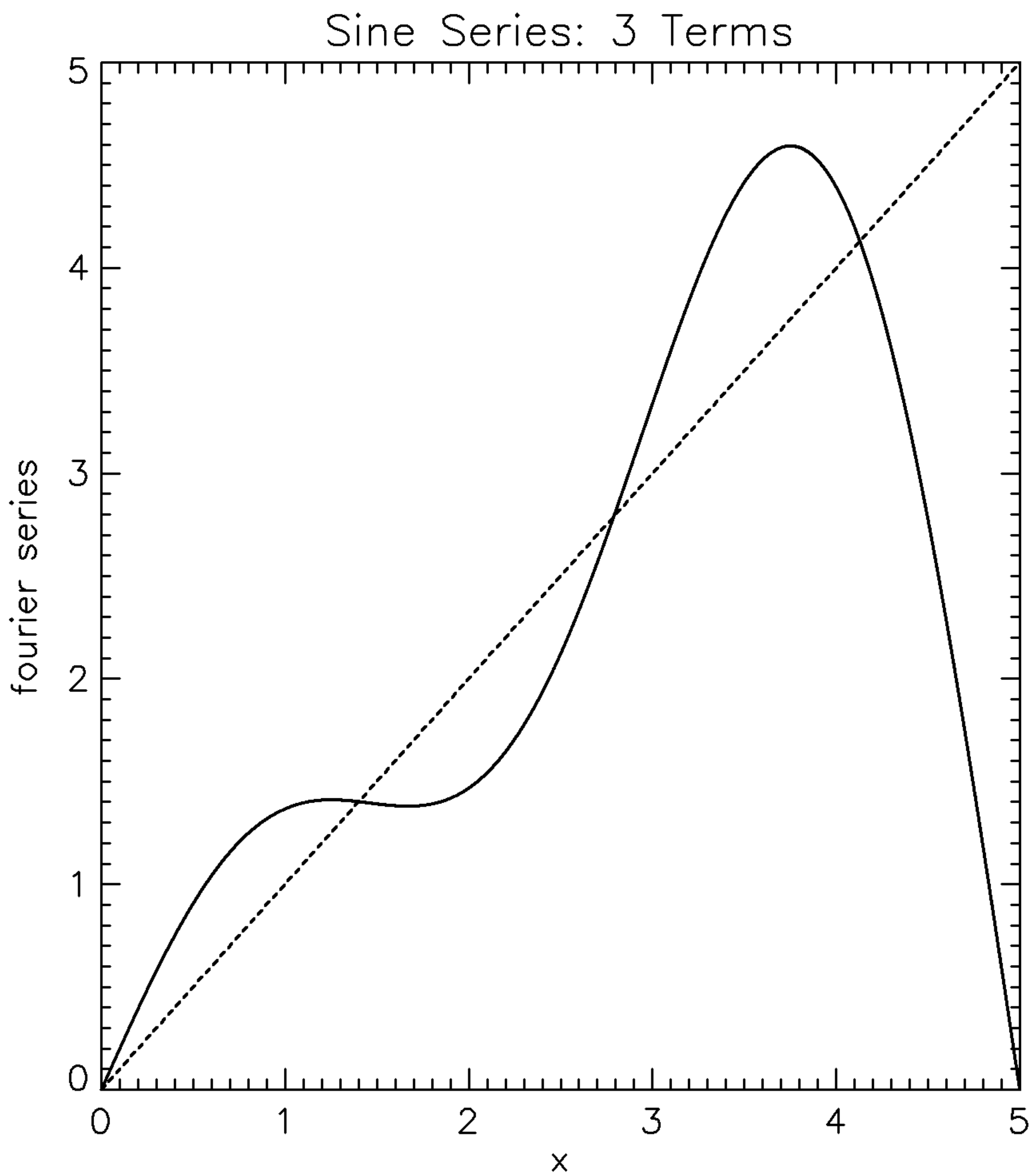
M-128: Create a graph of the truncated Fourier Sine series of $f(x) = x$ on the interval $[0, 5]$, where the series is truncated after the first term.

Answer: Here is the graph:



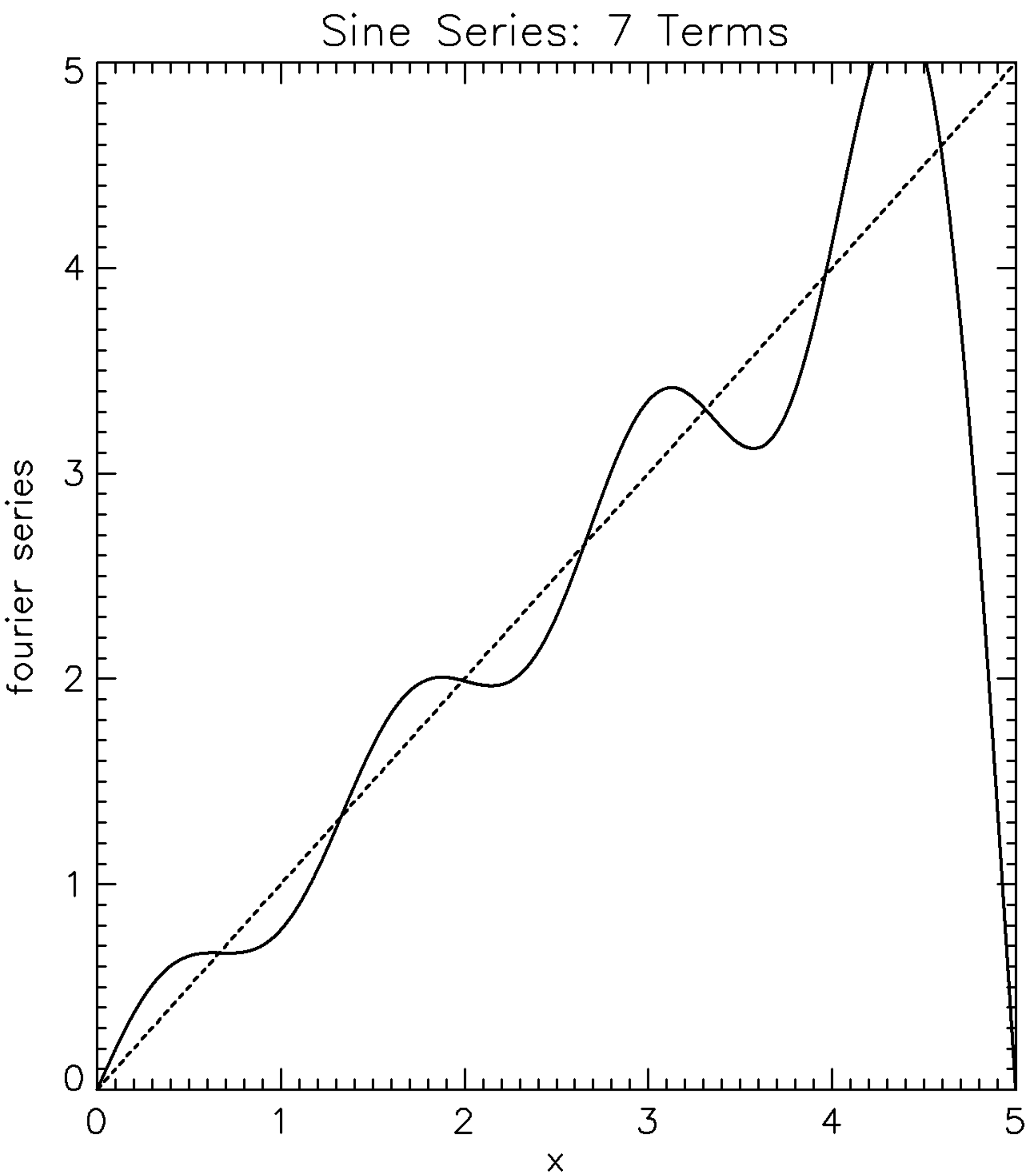
M-129: Create a graph of the truncated Fourier Sine series of $f(x) = x$ on the interval $[0, 5]$, where the series is truncated after the third term.

Answer: Here is the graph:



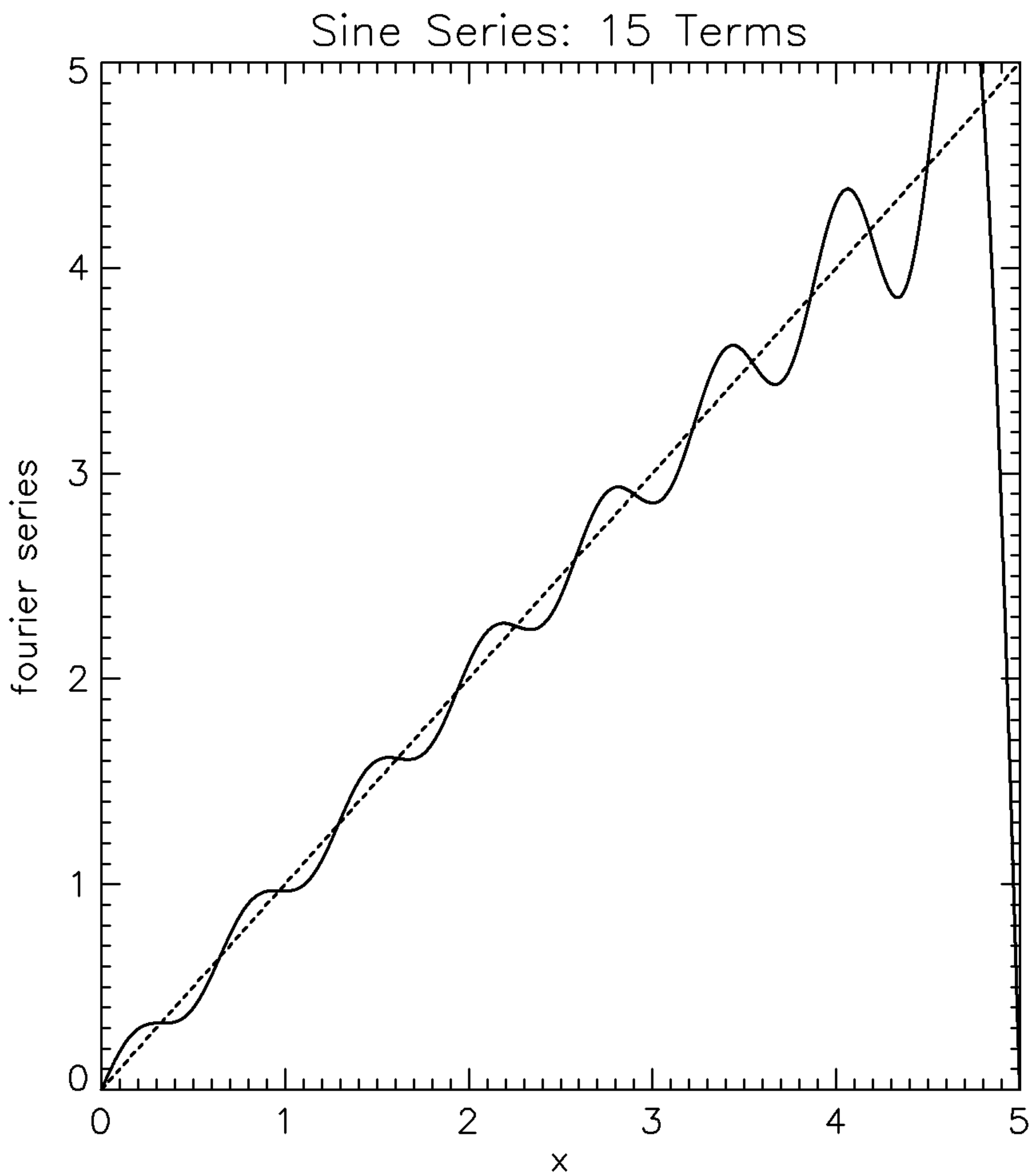
M-130: Create a graph of the truncated Fourier Sine series of $f(x) = x$ on the interval $[0, 5]$, where the series is truncated after the 7th term.

Answer: Here is the graph:



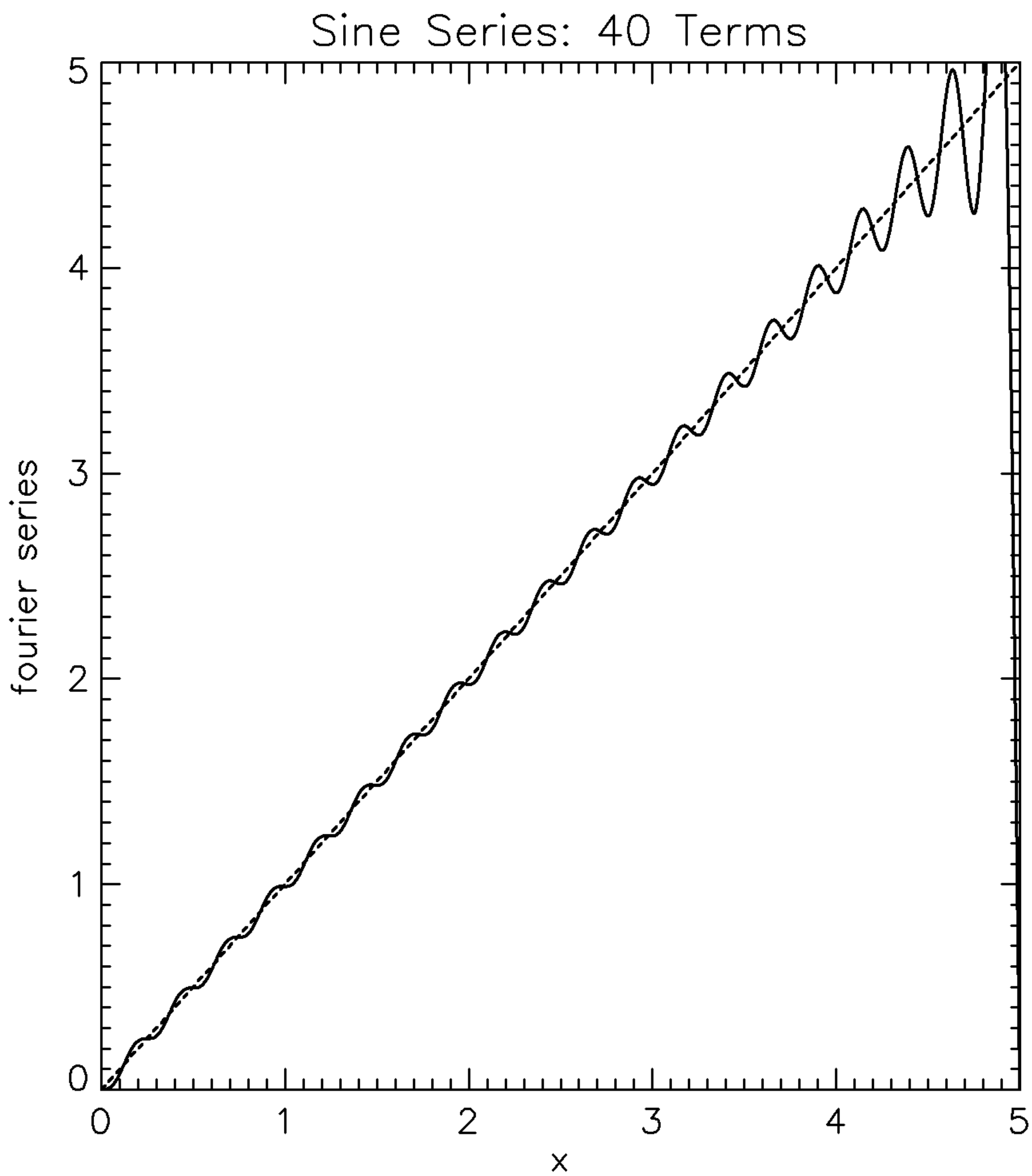
M-131: Create a graph of the truncated Fourier Sine series of $f(x) = x$ on the interval $[0, 5]$, where the series is truncated after the 15th term.

Answer: Here is the graph:



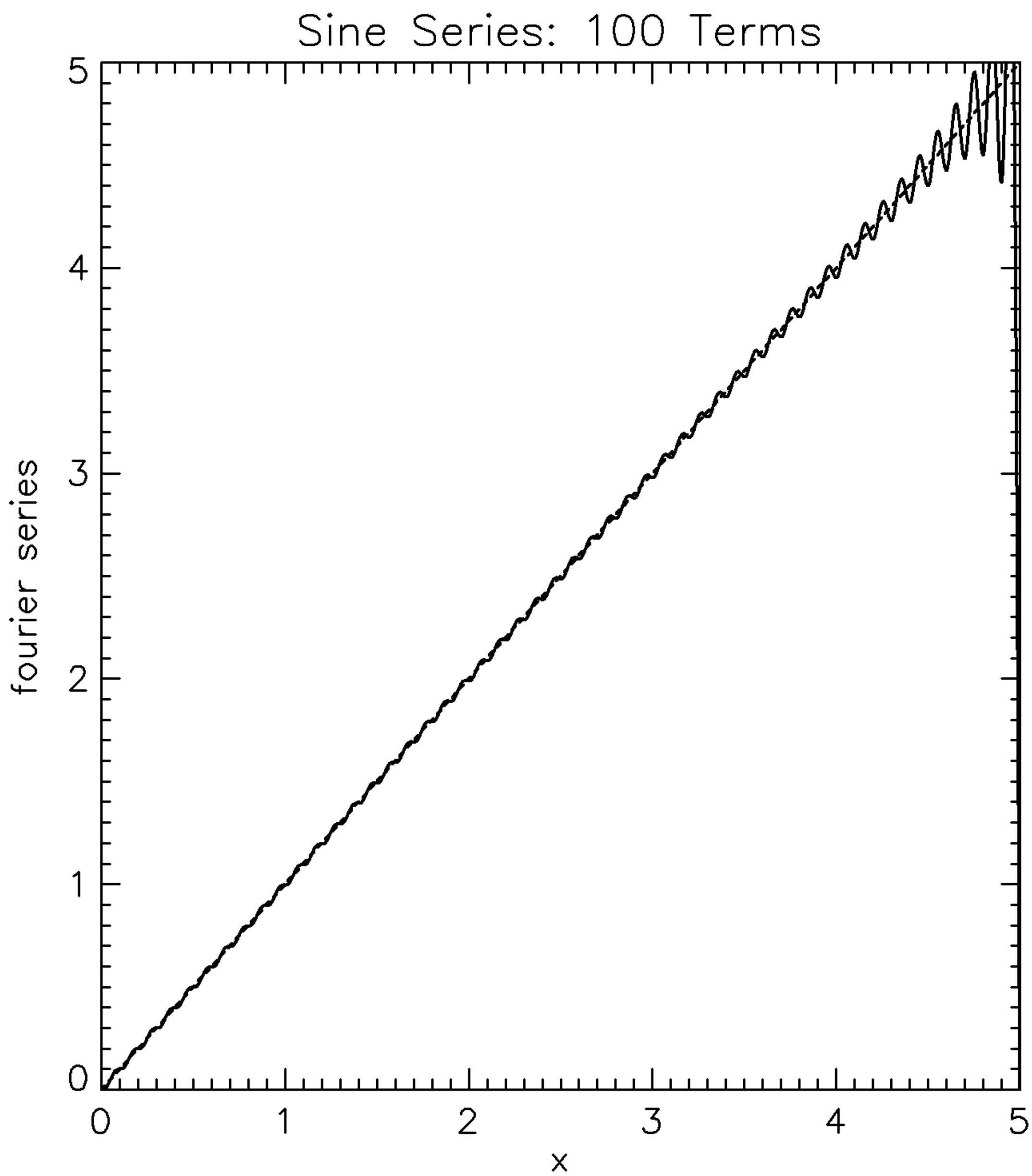
M-132: Create a graph of the truncated Fourier Sine series of $f(x) = x$ on the interval $[0, 5]$, where the series is truncated after the 40th term.

Answer: Here is the graph:



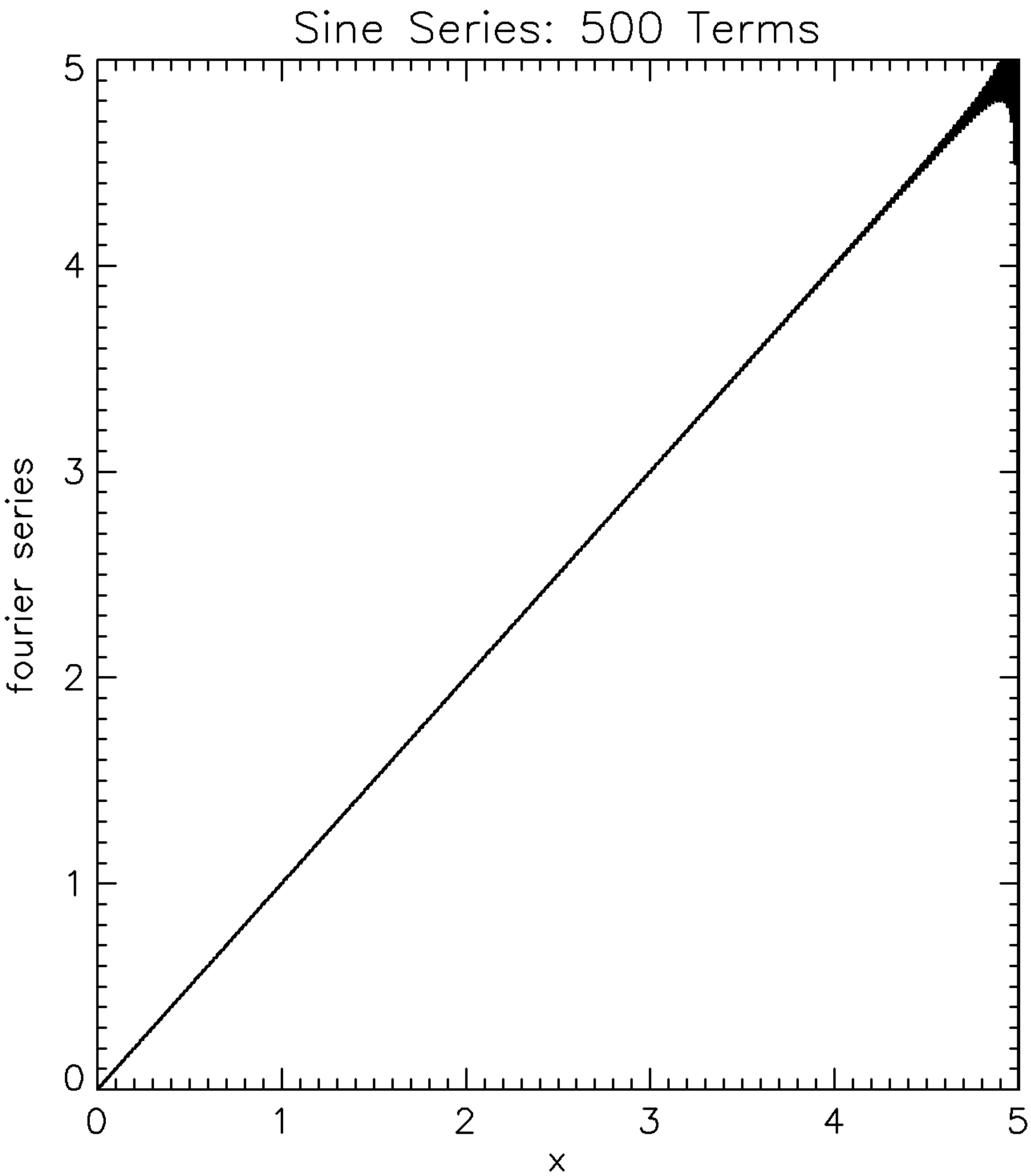
M-133: Create a graph of the truncated Fourier Sine series of $f(x) = x$ on the interval $[0, 5]$, where the series is truncated after the 100th term.

Answer: Here is the graph:



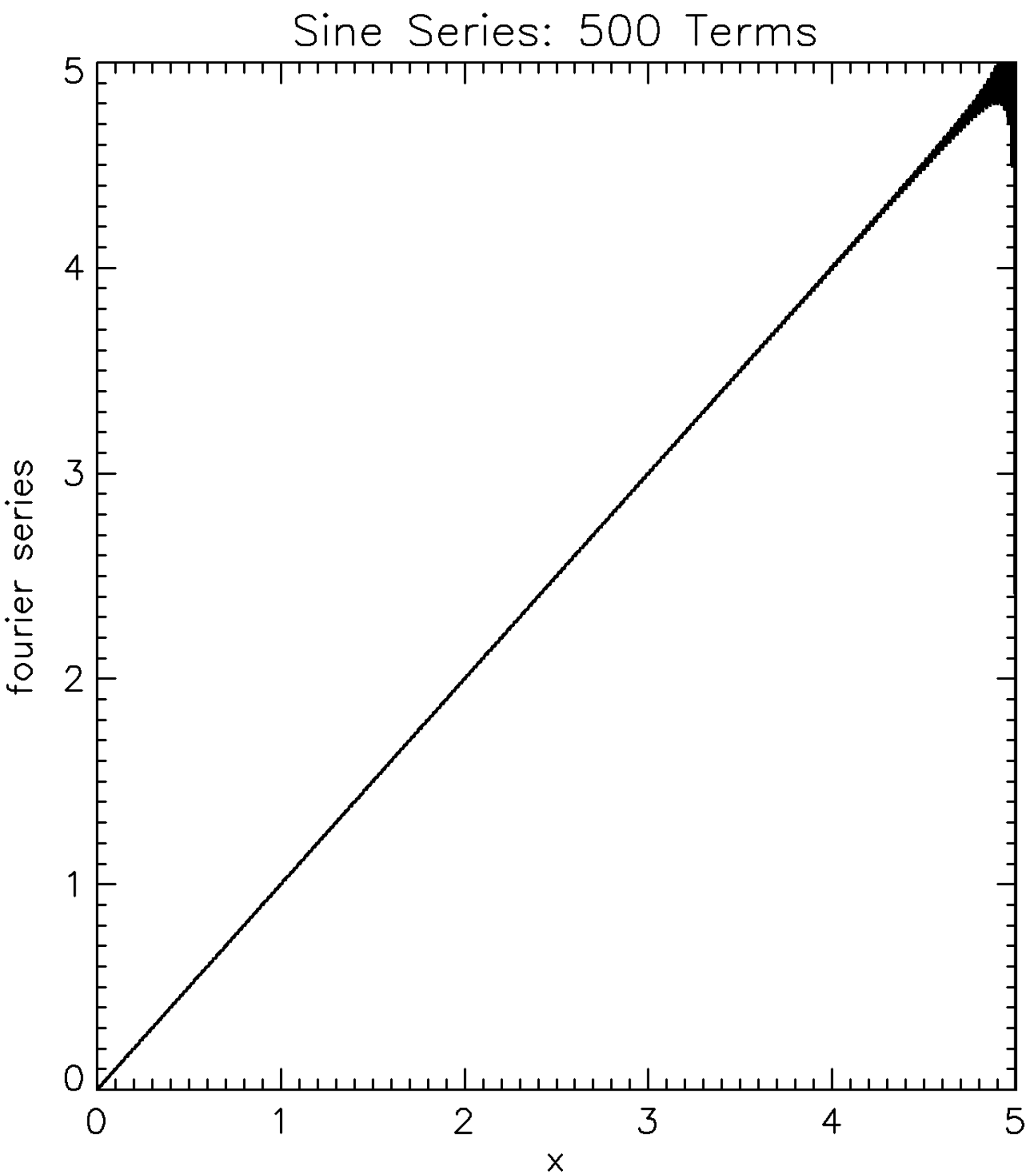
M-134: Create a graph of the truncated Fourier Sine series of $f(x) = x$ on the interval $[0, 5]$, where the series is truncated after the 500th term.

Answer: Here is the graph:



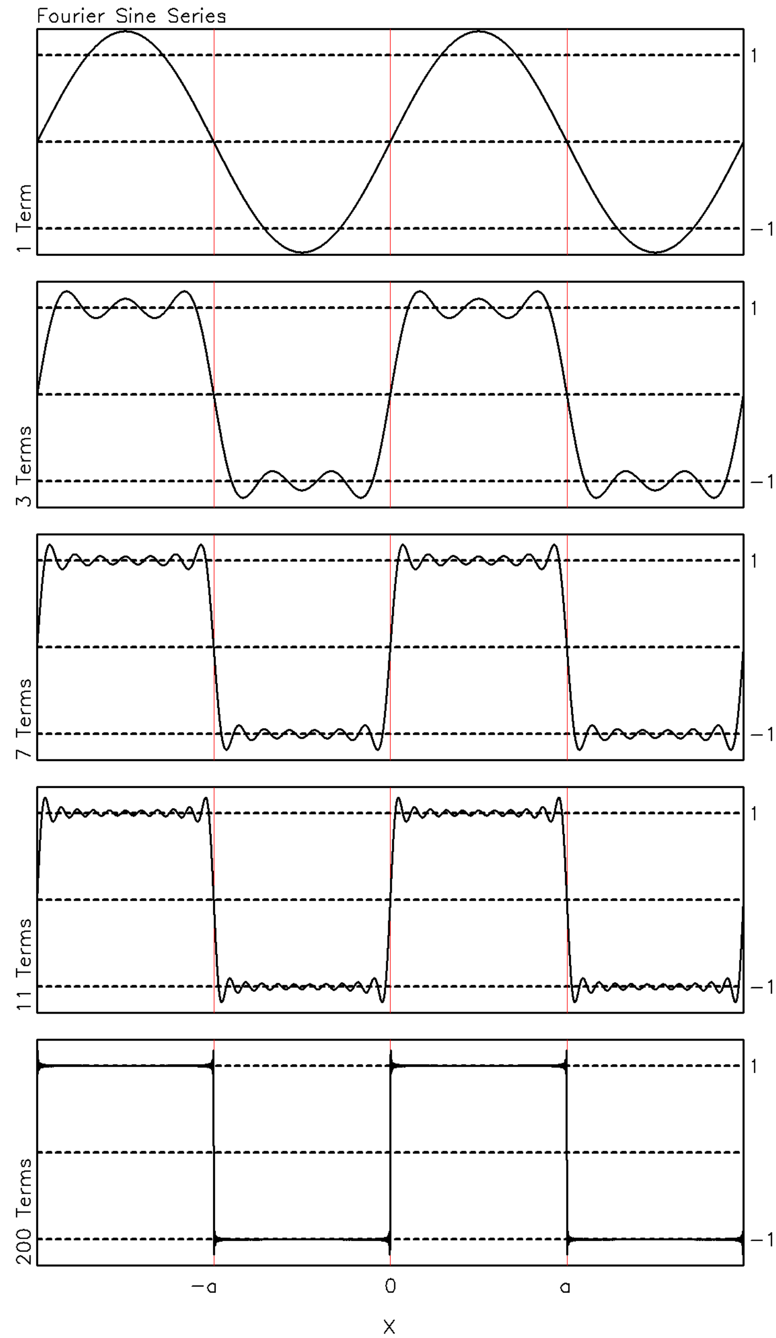
M-135: Create a graph of the truncated Fourier Sine series of $f(x) = x(1-x)$ on the interval $[0, a]$, where the series is truncated after the 500th term.

Answer: Here is the graph:



M-136: Determine the coefficients of the Fourier Sine series of $f(x) = 1$ on the interval $[0, a]$. Plot truncated series where there are 1, 3, 7, 11, and 200 terms on the interval [-2a,2a].

Answer: Here are those graphs:

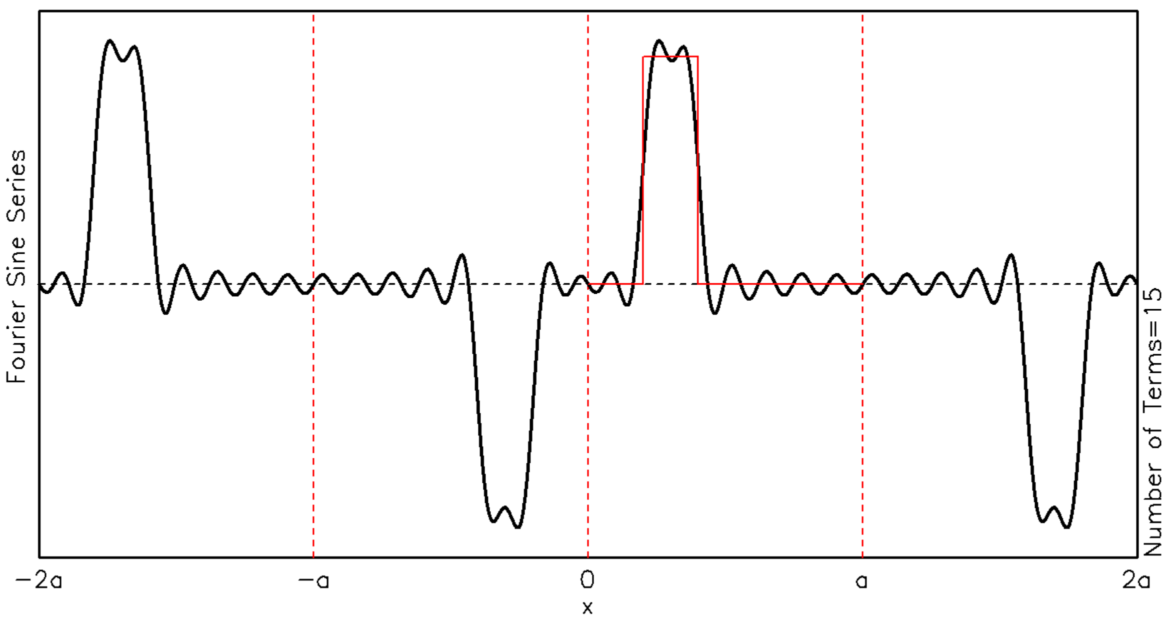


M-137: Determine the coefficients of the Fourier Sine series of

$$f(x) = \begin{pmatrix} 0 & 0 < x < 0.2a \\ 1 & 0.2a < x < 0.4a \\ 0 & 0.4a < x < a \end{pmatrix}$$

on the interval $[0, a]$. Plot (in black) the Fourier Sine series on the interval [-2a,2a], where the series is truncated at 15 terms. Overplot (in red) the function $f(x)$ on $[0, a]$.

Answer: Here is the graph:

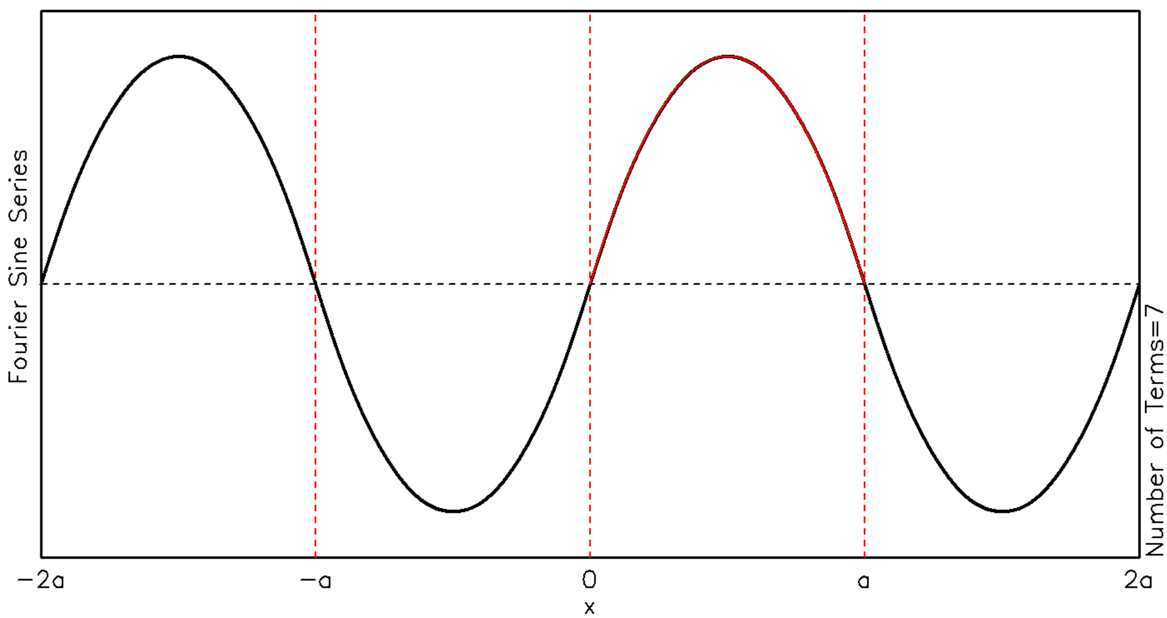


M-138: Determine the coefficients of the Fourier Sine series of

$$f(x) = \frac{x}{a}(1 - \frac{x}{a})$$

on the interval $[0, a]$. Plot (in black) the Fourier Sine series on the interval [-2a,2a], where the series is truncated at 7 terms. Overplot (in red) the function $f(x)$ on $[0, a]$.

Answer: Here is the graph:

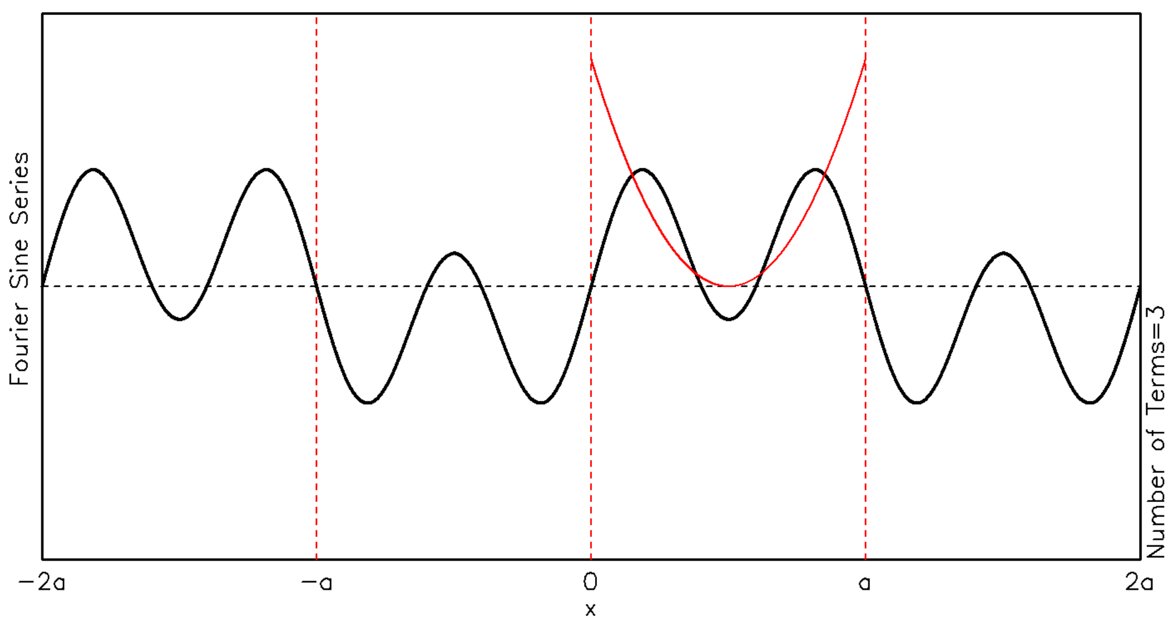


M-139: Determine the coefficients of the Fourier Sine series of

$$f(x) = 4\left(\frac{x}{a} - \frac{1}{2}\right)^2$$

on the interval $[0, a]$. Plot (in black) the Fourier Sine series on the interval [-2a,2a], where the series is truncated at 3 terms. Overplot (in red) the function $f(x)$ on $[0, a]$.

Answer: Here is the graph:

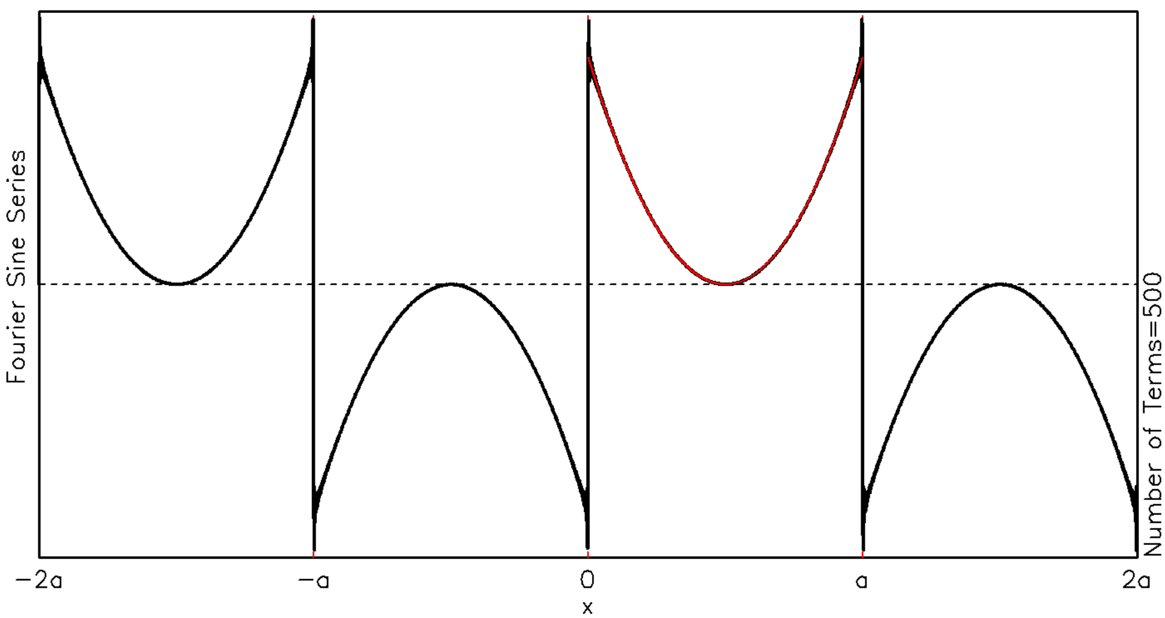


M-140: Determine the coefficients of the Fourier Sine series of

$$f(x) = 4\left(\frac{x}{a} - \frac{1}{2}\right)^2$$

on the interval $[0, a]$. Plot (in black) the Fourier Sine series on the interval [-2a,2a], where the series is truncated at 500 terms. Overplot (in red) the function $f(x)$ on $[0, a]$.

Answer: Here is the graph:

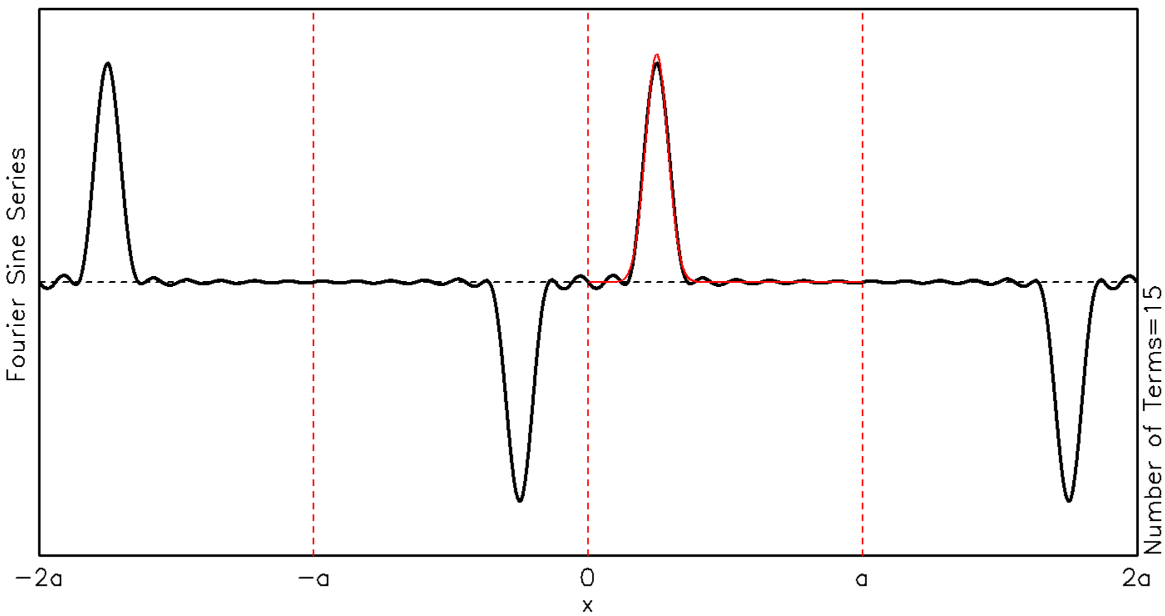


M-141: Determine the coefficients of the Fourier Sine series of

$$f(x) = \mathrm{e}^{-300(x/a-0.25)^2}$$

on the interval $[0, a]$. Plot (in black) the Fourier Sine series on the interval [-2a,2a], where the series is truncated at 15 terms. Overplot (in red) the function $f(x)$ on $[0, a]$.

Answer: Here is the graph:



Laplace's Equation

Solutions of Laplace's Equation have some very interesting and useful properties. One of these is that the value of a solution at a point is an average of the values of the solution at points around that point. This simple property is fairly easy to understand and to prove. In the next few questions we will first explore why this makes sense, and then we will prove it to be true in a general sense, where we need to be careful with how we define *average*.

M-142: We have a function f that depends only on x, that is solution to Laplace's Equation in a volume V, and that satisfies a given set of boundary conditions on the surface bounding V. Show that at any point within V (meaning *not on* S), f is the average of its value at two *closely spaced* points around the point. In particular, show that, for small Δ,

$$f(x,y) = \frac{f(x+\Delta) + f(x-\Delta)}{2}$$

Answer: For small h,

$$\frac{df}{dx} = \frac{f(x+h) - f(x-h)}{2h} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{f(x,y+h) - f(x,y-h)}{2h}$$

Then, for the second derivative, we have, for small h,

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left[\frac{df}{dx}\right] = \frac{1}{2h}\frac{df}{dx}\bigg|_{(x-h)}^{(x+h)} = \frac{f(x+2h) - f(x) - f(x) + f(x-2h)}{4h^2}$$

Since f is a solution of Laplace's Equation

$$\nabla^2 f = 0$$

Therefore, with the proviso that h is small, we have

$$f(x) = \frac{f(x+2h) + f(x-2h)}{2}$$

There is nothing special about h, as long as it is *small*, so defining $\Delta = 2h$ we have

$$f(x) = \frac{f(x+\Delta) + f(x-\Delta)}{2}$$

M-143: Given a *solution of Laplace's Equation* that does not depend on z, show that, provided h is small (meaning small as in the case of determining derivatives),

$$f(x,y) = \frac{f(x+h,y) + f(x,y+h) + f(x-h,y) + f(x,y-h)}{4}$$

Answer: The partial derivative of f with respect to x is

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)} = \lim_{h\to 0} \frac{f(x+h,y) - f(x-h,y)}{2h}$$

Here we can think like physicists, and drop the limit on the understanding that h is very small (that's what a derivative is anyway). The second partial derivative with respect to x is then

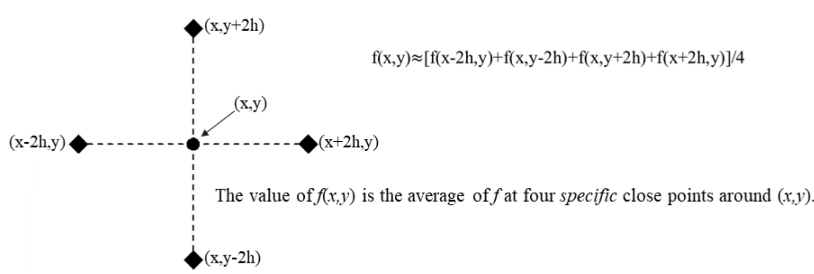
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)} = \frac{1}{2h}\left[\left.\frac{\partial f}{\partial x}\right|_{(x+h,y)} - \left.\frac{\partial f}{\partial x}\right|_{(x-h,y)}\right] = \frac{f(x+2h,y) - f(x,y) - f(x,y) + f(x-2h,y)}{4h^2}$$

We can make the same argument for the second partial derivative with respect to y, so we have

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)} + \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x,y)} = \frac{f(x+2h,y) + f(x-2h,y) + f(x,y+2h) + f(x,y-2h) - 4f(x,y)}{4h^2}$$

The left hand side is the Laplacian (remember, this function does not depend on z), which is zero since this function is a solution of Laplace's equation. Thus, for small h (why can I replace $2h$ with h?),

$$f(x,y) = \frac{f(x+h,y) + f(x,y+h) + f(x-h,y) + f(x,y-h)}{4}$$



M-144: Given a *solution of Laplace's Equation*, show that, provided h is small,

$$f(x,y,z) = \frac{f(x+h,y,z) + f(x,y+h,z) + f(x-h,z,y,z) + f(x,y-h,z) + f(x,y,z+h) - f(x,y,z-h)}{6}$$

Answer: Use the same argument as above (not really worth working through).

M-145: A function f is a solution of Laplace's equation. Show that, the average of f on any sphere centered on (x, y, z) is $f(x, y, z)$.

Answer: Let $f^*(R)$ be the average of f on a sphere of radius R centered on the point (x, y, z).

$$f^*(R) = \frac{1}{4\pi R^2} \int_S f(\vec{r}')da' = 4\pi R^2 f^*(R)$$

Consider, for small ΔR, the difference

$$f^*(R + \Delta R) - f^*(R) = \frac{1}{4\pi R^2 \Delta R} \int_V \nabla^2 f d\tau = 0 \quad \text{since } \nabla^2 f = 0$$

Thus, for solutions f of Laplace's equation, $f^*(R)$ is independent of R, where $f^*(R)$ is the average of f on a sphere of radius R centered on (x, y, z). Since (make sure you know why this is actually obvious)

$$\lim_{R \to 0} f^*(R) = f(x, y, z)$$

the average of a solution of Laplace's equation on a sphere is equal to the value of that solution at the center of the sphere.

M-146: Show that if f is a solution of Laplace's Equation, then the average of f over the surface of a sphere (of any radius) is equal to its value at the center of the sphere.

Answer: Let V be the volume of the sphere and $S = 4\pi r^2$ be its surface area. If f is a solution to Laplace's Equation, then ${\nabla'}^2 f = 0$. Alternately, since ${\nabla'}^2 = \vec{\nabla} \cdot \vec{\nabla}$, we have

$$\int_V \vec{\nabla}' \cdot \left(\vec{\nabla}' f(\vec{r}') \right) d\tau' = 0 \tag{50}$$

where V is the volume of the sphere. Using the Divergence Theorem, we have

$$\int_S \left[\vec{\nabla}' f(\vec{r}') \right] \cdot \hat{r}' da' = 0 \tag{51}$$

where S is the spherical surface bounding V. Now, on S,

$$\left[\vec{\nabla}' f(\vec{r}') \right] \cdot \hat{r}' = \left. \frac{\partial f(r', \theta', \phi')}{\partial r'} \right|_{r'=R} \tag{52}$$

With Equations 51 and 52, and $da = R^2 \sin(\theta) d\theta d\phi$, we then have

$$R^2 \int_S \left. \frac{\partial f(r', \theta', \phi')}{\partial r'} \right|_{r'=R} \sin(\theta) d\theta' d\phi' = 0 \tag{53}$$

We can divide by R^2, because the expression is equal to zero, and I can take the partial derivative out of the integral, so

$$\frac{\partial}{\partial r} \left. \left(\int_S f(r, \theta', \phi') \sin(\theta) d\theta' d\phi' \right) \right|_{r=R} = 0 \tag{54}$$

The quantity in brackets is $4\pi f^*$, where f^* is the average value of f on S. Thus,

$$\frac{df^*}{dR} = 0$$

M-147: A function f satisfies Laplace's equation in a region. Show that, within that region, f can have no extrema (e.g., no maxima or minima).

Answer: The average of f on any sphere centered on any point in the region is the value of f at that point. Suppose $f(x, y, z)$ is larger than it is at any point on the sphere of radius R centered on (x, y, z), or that it is smaller than it is at any point on the sphere of radius R centered on (x, y, z). In those cases, it cannot be that the average of f on any sphere centered on any point in the region is the value of f at that point. Thus, there cannot be a local maxima, nor a local minima, of f, anywhere in the region wherein f is a solution of Laplace's equation.

M-148: There is a volume V bounded by surface S. Within V, the function f is a solution of Laplace's Equation. It's value everywhere on S is known (the *boundary condition*). Show that the solution to f is unique, meaning there is one and only one solution of Laplace's Equation in V that satisfies the boundary condition on S. In other words, prove *uniqueness*.

Answer: Assume there are two solutions, f and g, that satisfy Laplace's Equation in V and the boundary condition on S. Then the function

$$h = f - g$$

satisfies Laplace's Equation in V (because Laplace's Equation is linear), and the boundary condition $h = 0$ on S.

Since h is a solution to Laplace's Equation, its extrema are on the boundary S. But $h = 0$ everywhere on S, so at every point in V, h is neither greater than, nor less than, 0. Thus,

$$h = 0$$

and therefore

$$g = f$$

In other words if f and g both satisfy the boundary condition on S, and are both solutions of Laplace's Equation in V, *they are the same function.*

These results are profound and have far reaching consequences. On one hand, solutions of Laplace's equation have maxima and minima *only* on their boundaries. There are no extrema in any region with which the solutions hold.

Further, the fact that solutions of Laplace's equation have no extrema within regions wherein they are valid means that solutions are unique. On one hand, I want to say that this *uniqueness* is to be expected: we are talking about solutions to real world physical problems so *of course* they are unique, and so a matrhematical proof to that effect is at the very least reassuring. On the other hand, it means that *guessing* is a perfectly valid approach to seeking solution to problems.

M-149: Guess the solutions to Laplace's Equation in the region $0 < x < a$ and $0 < y < a$ that satisfy the following boundary conditions $f = (0, y) = 0$, $f(x, 0) = 0$, $f(a, y) = ay$ and $f(x, a) = ax$.

Answer:

M-150: Guess the solutions to Laplace's Equation in the region $0 < x < a$ and $0 < y < a$ that satisfy the following boundary conditions $f(0, y) = \cos(y)$, $f(x, 0) = e^x$, $f(a, y) = e^a \cos(x)$ and $f(x, a) = e^x \cos(a)$.

Answer:

M-151: The function $f(x)$ (e.g., depends only on x) is a solution of Laplace's Equation in the region between $x = x_0$ and $x = x_1$, and satisfies the boundary condition $f(x_0) = c_0$ and $f(x_1) = c_1$. Show that on $[x_0, x_1]$ (or (x_0, x_1), it does not matter).

$$f(x) = \left[\frac{c_1 - c_0}{x_1 - x_0}\right] x + c_1 - \left[\frac{c_1 - c_0}{x_1 - x_0}\right] x_1$$

Answer: Since f is a solution of Laplace's Equation, and only depends on x, we have

$$\begin{aligned}
\nabla^2 f &= 0 \quad \text{therefore...} \\
\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] f &= 0 \quad \text{therefore...} \\
\frac{d^2}{dx^2} f &= 0 \quad \text{therefore...} \\
\frac{d}{dx} f &= m \quad \text{where m is some constant so,} \\
f(x) &= mx + b \quad \text{where m and b are constants TBD by BCs}
\end{aligned}$$

We get the constants m and b from the fact that the solution to *this* problem must satisfy the boundary conditions, for which we often use shorthand 'BC'. In this case, we have

$$\begin{aligned}
c_0 &= mx_0 + b \quad \text{and} \\
c_1 &= mx_1 + b
\end{aligned}$$

so,

$$m = \frac{c_1 - c_0}{x_1 - x_0}, \quad b = c_1 - \left[\frac{c_1 - c_0}{x_1 - x_0}\right] x_1, \quad \text{and finally...} \quad f(x) = \left[\frac{c_1 - c_0}{x_1 - x_0}\right] x + c_1 - \left[\frac{c_1 - c_0}{x_1 - x_0}\right] x_1$$

My answer is kind of ridiculous. I hope you understand that this is a long-winded and overly pedantic answer to an exceptionally simple question. In terms of the level of detail, and its formality, it would be ridiculous to require such answer by a student to a text question. I'll get back to the point I'm trying to make but I want to clarify something. Your answer could be as simple as the following...

Expected Student Answer: The function depends only on x, and so $\nabla^2 \to d^2/dx^2$ and so

$$\frac{d^2 f}{dx^2} = 0 \quad \text{so} \quad \frac{df}{dx} = C \quad (\text{note } C \text{ is a constant})$$

Therefore $f = mx + b$ is a line, and from the boundary conditions we get $m = (c_1 - c_0)/(x_1 - x_0)$ and $b = c_1 - mx_1$. QED.

Point I am trying to make: We are going to be applying a rather formulaic, but very powerful, approach to finding solutions to a class of electrostatic problems. We will first determine the form of the solution (in this case a line... actually it's a plane in 3D whose normal vector is perpendicular to $\hat{z}$, something I hope you can see), then get our particular solution by using the boundary conditions.

M-152: Given the function (from the preceding question)

$$f(x) = mx + b$$

show that

a: contours of constant f are a family of planes perpendicular to $\hat{z}$;

b: if we consider a volume bound by the surface S, that the extreme values of f in V are on S (where are they)?

Answer:

M-153: Given

$$f(\vec{r}) = ax + by + cz \quad \text{and} \quad \sqrt{a^2 + b^2 + c^2} = 1$$

a: show that f is a solution of Laplace's Equation;

b: find the locations of the maximum and minimum values of f in the volume V, where V a sphere of radius R centered on the origin.

Answer:

M-154: Given

$$f(\vec{r}) = xyz$$

a: show that f is a solution of Laplace's Equation;

b: find the two locations of the extreme values of f in the volume V, where V is a cube, centered on the origin, with side length 2, and sides perpendicular to $\hat{x}$, $\hat{y}$, and $\hat{z}$.

Answer: b) If we include the boundary, there are eight extrema in V, and they are all on its bounding surface. They are the eight vertices, at four of which $f = 8$, and at the other four $f = -8$.

M-155: In order for

$$f(\vec{r}) = A\sin(kx)e^{\kappa y}$$

to be a solution of Laplace's Equation, what must be the relationship between κ and k?

Answer:

M-156: In order for

$$f(\vec{r}) = A\sin(k_n x)\sin(k_m z)\mathrm{e}^{\kappa y}$$

to be a solution of Laplace's Equation, what must be κ in terms of k_n and k_m?

Answer:

M-157: Show that

$$f(\vec{r}) = A + \frac{B}{r}$$

where A and B are constants, is a solution of Laplace's Equation.

Answer 1: In spherical coordinates, for a spherically symmetric function (this f is), we have

$$\begin{aligned}
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left[A + \frac{B}{r}\right]\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \left[-\frac{B}{r^2}\right]\right) \\
&= -\frac{1}{r^2}\frac{\partial}{\partial r}B \\
&= 0
\end{aligned}$$

Answer 2: In Cartesian coordinates,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right) \\
&= -\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
&= -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}
\end{aligned}$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \quad \text{and} \quad \frac{\partial^2 f}{\partial z^2} = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Therefore,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -\frac{3}{(x^2+y^2+z^2)^{\frac{3}{2}}} + 3\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 3\left(\frac{r^2}{r^5} - \frac{1}{r^3}\right) = 0$$

M-158: Show that if

$$f(x) + g(y) = 0 \tag{55}$$

then f and g are both constant.

Answer:

M-159: Show that if $f(x,y)$ is a solution to Laplace's Equation in two dimensions, and if it is *separable*, by which I mean

$$f(x,y) = X(x)Y(y) \tag{56}$$

then

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0 \tag{57}$$

then f and g are each constant.

Answer: The fact that $f(x,y)$ is a solution to Laplace's equation, and that I am assuming it is *separable* means

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] X(x)Y(y) = 0$$

Thus,

$$\frac{1}{X(x)Y(y)}\left[Y(y)\frac{d^2}{\partial x^2}X(x) + X(x)\frac{d^2}{\partial y^2}Y(y)\right] = 0$$

and so,

$$\frac{1}{X(x)}\frac{d^2}{\partial x^2}X(x) + \frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y) = 0$$

M-160: Show that if $f(x,y)$ is a solution to Laplace's Equation in two dimensions, and if it is *separable* in the sense of Equation 56, then the set of solution(s) $f(x,y)$ *include* the set of functions of the form

$$f(x,y) = \Big[A\sin(kx) + B\cos(kx)\Big]\Big[C\mathrm{e}^{ky} + D\mathrm{e}^{-ky}\Big] \text{ or } f(x,y) = \Big[C\mathrm{e}^{kx} + D\mathrm{e}^{-kx}\Big]\Big[A\sin(ky) + B\cos(ky)\Big]$$

Answer: We can show this by applying the Laplacian operator

$$\begin{aligned}
\nabla^2 f(x,y) &= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\Big[A\sin(kx) + B\cos(kx)\Big]\Big[C\mathrm{e}^{ky} + D\mathrm{e}^{-ky}\Big] \\
&= \Big[C\mathrm{e}^{ky} + D\mathrm{e}^{-ky}\Big]\frac{\partial^2}{\partial x^2}\Big[A\sin(kx) + B\cos(kx)\Big] + \Big[A\sin(kx) + B\cos(kx)\Big]\frac{\partial^2}{\partial y^2}\Big[C\mathrm{e}^{ky} + D\mathrm{e}^{-ky}\Big] \\
&= (-k^2 + k^2)\Big[A\sin(kx) + B\cos(kx)\Big]\Big[C\mathrm{e}^{ky} + D\mathrm{e}^{-ky}\Big] \\
&= 0
\end{aligned}$$

The same is obviously true for

$$f(x,y) = \Big[C\mathrm{e}^{kx} + D\mathrm{e}^{-kx}\Big]\Big[A\sin(ky) + B\cos(ky)\Big]$$

Thus both forms are obviously solutions of Laplace's Equation.

M-161: Show that if $f(x,y)$ is a solution to Laplace's Equation in two dimensions, and if it is *separable* in the sense of Equation 56, then the set of solution(s) $f(x,y)$ *include* the set of functions of the form

$$f(x,y) = \Big[A\sin(kx)+B\cos(kx)\Big]\Big[Ce^{ky}+De^{-ky}\Big] \text{ or } f(x,y) = \Big[Ce^{kx}+De^{-kx}\Big]\Big[A\sin(ky)+B\cos(ky)\Big]$$

Answer: We can show this by using what we've shown above, namely that if the solutions to Laplace's equation are separable, as in e.g.,

$$f(x,y) = X(x)Y(y) \tag{58}$$

then

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0 \tag{59}$$

But this is of the form

$$h(x) + g(y) = 0$$

so that $h(x)$ and $g(y)$ are constants. Anticipating the form of the solutions, we can set

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k^2 \text{ and } \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k^2$$

Where the k is the same in both differential equations. These are straightforward to solve, and we have

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k^2 \quad\rightarrow\quad \frac{d^2X(x)}{dx^2} = -k^2X(x) \quad\rightarrow\quad X(x) = A\sin(kx) + B\cos(kx)$$

and

$$\frac{d^2Y(x)}{dy^2} = k^2 \quad\rightarrow\quad \frac{d^2Y(x)}{dy^2} = k^2Y(y) \quad\rightarrow\quad Y(y) = Ce^{ky} + De^{-ky}$$

Thus

$$f(x,y) = \Big[A\sin(kx) + B\cos(kx)\Big]\Big[Ce^{ky} + De^{-ky}\Big]$$

The other form, namely

$$f(x,y) = \Big[Ce^{kx} + De^{-kx}\Big]\Big[A\sin(ky) + B\cos(ky)\Big]$$

can be obtained by recognizing we can swap x and y in the preceding steps. Thus both forms are obviously solutions of Laplace's Equation.

M-162: There is a rectangular cavity bounded by $(0, 0, z)$, $(0, 1, z)$, $(1, 0, z)$, and $(1, 1, z)$. In this cavity

$$\nabla^2 \vec{A} = 0 \tag{60}$$

and on the boundaries, the following hold:

$$\vec{A}(x, 0, z) = z\hat{x}, \quad \vec{A}(1, y, z) = z\hat{x} + y\hat{z}, \quad \vec{A}(x, 1, z) = z\hat{x} + x\hat{z}, \quad \text{and} \quad \vec{A}(0, y, z) = z\hat{x}$$

What is $\vec{A}$ in the rectangular cavity.

Answer: Equation 60 is actually three equations, namely

$$\nabla^2 A_x = 0, \quad \nabla^2 A_y = 0, \quad \text{and} \quad \nabla^2 A_z = 0$$

such that A_x, A_y, and A_z are each, and independently of each other, solutions of Laplace's Equation. Additionally, they are subject to the boundary conditions

$$A_x = z \quad \text{everywhere boundary the boundary},$$

$$A_y = 0 \quad \text{everywhere boundary the boundary}, \quad \text{and}$$

$$A_z = 0 \text{ on } y = 0, \quad A_z = y \text{ on } x = 1, \quad A_z = x \text{ on } y = 1, \quad \text{and} \quad A_z = 0 \text{ on } x = 0$$

In other words, this is actually three independent solutions of Laplace's Equation subject to boundary conditions. The solutions to the three independent problems are

$$A_x = z$$

$$A_y = 0$$

$$A_z = xy$$

By inspection we can see that each satisfies the required boundary conditions. The solution to the overall problem is

$$\vec{A} = z\hat{x} + xy\hat{z}$$

Lines and Planes

M-163: Given a plane $ax + by + cz = d$, where $a^2 + b^2 + c^2 = 1$, show that the perpendicular distance between the point $\vec{r}_\circ = x_\circ\hat{x} + y_\circ\hat{y} + z_\circ\hat{z}$ and the plane is

$$distance = |d - ax_\circ - by_\circ - cz_\circ|$$

Answer: The perpendicular distance from the plane is the distance between the point on the plane that is closest to $\vec{r}_\circ$ to $\vec{r}_\circ$. Let's call this closest point $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. We know that the vector between $\vec{r}_\circ$ and $\vec{r}$ (or $\vec{r} - \vec{r}_\circ$) is perpendicular to the plane and is therefore a multiple of $\hat{n} = a\hat{x} + b\hat{y} + c\hat{z}$ ($\hat{n}$ is the unit normal to the plane, and a, b, and c are its direction cosines), or, in other words

$$\vec{r} - \vec{r}_\circ = \lambda\hat{n}$$

We can rewrite this as

$$x\hat{x} + y\hat{y} + z\hat{z} = (x_\circ + a\lambda)\hat{x} + (y_\circ + b\lambda)\hat{y} + (z_\circ + c\lambda)\hat{z} \tag{61}$$

But, the point $\vec{r}$ is on the plane, so

$$ax + by + cz = d \tag{62}$$

and, from Equations 61 and 62, we have

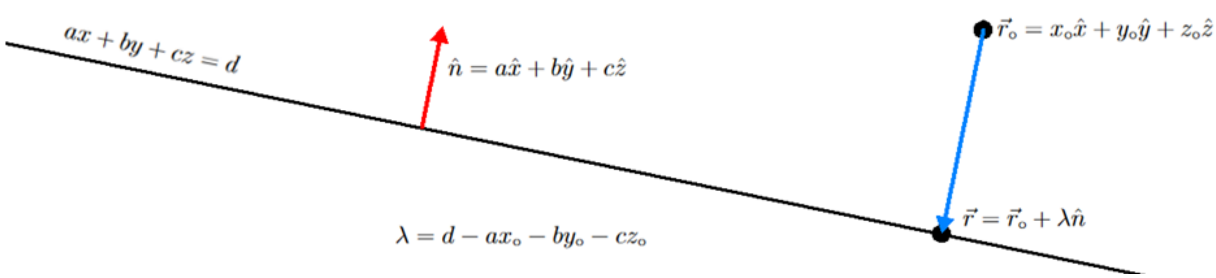
$$a(x_\circ + a\lambda) + b(y_\circ + b\lambda) + c(z_\circ + c\lambda) = d$$

Since $a^2 + b^2 + c^2 = 1$, we can rewrite this as

$$\lambda = d - ax_\circ - by_\circ - cz_\circ$$

If $\vec{r}_\circ$ is on one side of the plane, λ will be positive, while on the other, it is negative. Thus, the perpendicular distance is the absolute value of λ, so

$$distance = |d - ax_\circ - by_\circ - cz_\circ|$$



M-164: Given a line $ax + by = d$, where $a^2 + b^2 = 1$, show that the perpendicular distance between the point $\vec{r}_\circ = x_\circ\hat{x} + y_\circ\hat{y}$ and the line is

$$distance = |d - ax_\circ - by_\circ|$$

Answer: The perpendicular distance from the line is the distance between the point on the line that is closest to $\vec{r}_\circ$ to $\vec{r}_\circ$. Let's call this closest point $\vec{r} = x\hat{x} + y\hat{y}$. We know that the vector between $\vec{r}_\circ$ and $\vec{r}$ (or $\vec{r} - \vec{r}_\circ$) is perpendicular to the line and is therefore a multiple of $\hat{n} = a\hat{x} + b\hat{y}$ ($\hat{n}$ is the unit normal to the line, and a, and b are its direction cosines), or, in other words

$$\vec{r} - \vec{r}_\circ = \lambda\hat{n}$$

We can rewrite this as

$$x\hat{x} + y\hat{y} = (x_\circ + a\lambda)\hat{x} + (y_\circ + b\lambda)\hat{y} \tag{63}$$

But, the point $\vec{r}$ is on the plane, so

$$ax + by = d \tag{64}$$

and, from Equations 63 and 64, we have

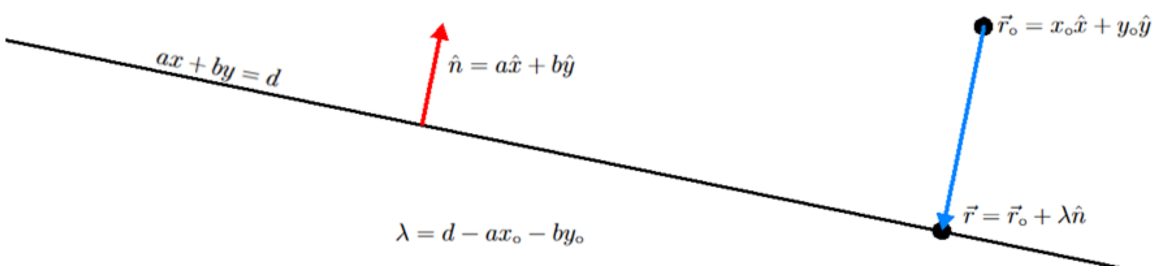
$$a(x_\circ + a\lambda) + b(y_\circ + b\lambda) = d$$

Since $a^2 + b^2 = 1$, we can rewrite this as

$$\lambda = d - ax_\circ - by_\circ$$

If $\vec{r}_\circ$ is on one side of the line, λ will be positive, while on the other, it is negative. Thus, the perpendicular distance is the absolute value of λ, so

$$distance = |d - ax_\circ - by_\circ|$$



M-165: Convince yourself that the perpendicular distance between the line $ax + by = d$ and the point $\vec{r}_\circ = x_\circ\hat{x} + y_\circ\hat{y}$ in the previous question, is also the perpendicular distance between the point $\vec{r}_\circ = x_\circ\hat{x} + y_\circ\hat{y}$ and the plane $ax + by = d$.

Answer:

> One point I am trying to make here (and in the preceding question) is that the equation for a line is the equation for a plane whose normal vector is $(a, b, 0)$. Often we think of ourselves as working in 2D, but physics is no more 2D than is the world.

M-166: What is the perpendicular distance between the point $\vec{r}_\circ = x_\circ\hat{x} + y_\circ\hat{y} + z_\circ\hat{z}$ and the plane $ax + by = d$?

Answer:

M-167: Along what locus of points does the plane

$$ax + by + cz = d$$

cut through the plane $z = 0$?

Answer: Set $z = 0$ in the equation for the plane. The result is the equation of a line *in the* $z = 0$ *plane*:

$$ax + by = d$$

This is the locus of points along which the plane $ax + by + cz = d$ cuts through the plane $z = 0$.

M-168: Two non-coplanar planes both cut through the $z = 0$ plane. Their equations are

$$a_1x + b_1y + c_1z = d_1 \quad \text{and} \quad a_2x + b_2y + c_2z = d_2$$

What is the location of the intersection of the three planes

$$a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2 \quad \text{and} \quad z = 0?$$

Answer: This intersection is the intersection of the two lines where planes 1 and 2 cut through the $z = 0$ plane. That is, the intersection of the three planes is the intersection of these two lines in the $z = 0$ plane:

$$a_1x + b_1y = d_1 \quad \text{and} \quad a_2x + b_2y = d_2$$

This point of intersection is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{a_1b_2 - a_2b_1} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

M-169: The intersection of two planes that are not coplanar is a line. What is the parametric equation of the line of intersection?

Answer: The two planes are specified by

$$a_1x + b_1y + c_1z = d_1 \quad \text{and} \quad a_2x + b_2y + c_2z = d_2$$

The vectors (a_1, b_1, c_1) and (a_1, b_1, c_1) are normal to plane 1 and plane 2, respectively. Without loss of generality, we can scale each equation such the normal vectors are also unit vectors (in which case, for each $\sqrt{a^2 + b^2 + c^2} = 1$).

The parametric equation for a line in 3D is

$$(x, y, z) = (x_\circ, y_\circ, z_\circ) + \lambda\hat{u}$$

It is not necessary that the vector in the parametric equation be a unit vector, but it makes things easier and the problem is not over-specified if we require it to be.

The vector $\hat{n}$ must be in both plane 1 and plane 2. This will be true if

$$\hat{u} = \frac{\hat{n}_1 \times \hat{n}_2}{|\hat{n}_1 \times \hat{n}_2|}$$

Note, this is well defined provided $\hat{n}_1$ and $\hat{n}_2$ are not colinear, which will be true provided plane 1 and plane 2 are not coplanar, which we were told. Now,

$$\hat{n}_1 = a_1\hat{x} + b_1\hat{y} + c_1\hat{z} \quad \text{and} \quad \hat{n}_2 = a_2\hat{x} + b_2\hat{y} + c_2\hat{z}$$

so

$$\hat{u} = \frac{(b_1c_2 - b_2c_1)\hat{x} + (c_1a_2 - c_2a_1)\hat{y} + (a_1b_2 - a_2b_1)\hat{z}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

We thus have the vector $\hat{u}$ for the parametric equation, so we just need one point on the line (e.g., $(x_\circ, y_\circ, z_\circ)$). This last bit can be finicky. For example it will not work if one of the planes is $z = \textit{constant}$ where $\textit{constant} \neq 0$. If that is the case one needs to find another plane on which to carry out this calculation, but that's just a little more work. It does not change the approach. So let's presume this is well posed, and find the point $(x_\circ, y_\circ, z_\circ = 0)$ where the line cuts through the $z = 0$ plane.

This point is the intersection of the two lines

$$a_1x + b_1y = d_1 \quad \text{and} \quad a_2x + b_2y = d_2$$

in the plane $z = 0$. From the previous question, we can say this point is

$$(x_\circ, y_\circ, 0)$$

where

$$\begin{bmatrix} x_\circ \\ y_\circ \end{bmatrix} = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Parameterized Curves

My above comments about the equation for a line being also the equation for a plane, and that we tend to think about problems in 2D wherein the real world and its physics are inherently 3D notwithstanding, there is such a thing as a line. A line through space is an example of a parameterized curve, which is a continuous locos of points, each specified by a parameter t which might be time, but also (and much more often than not) might not be.

M-170: At $t = 0$, an object is at $\vec{r}(0)$ and moving with velocity $\vec{v}_\circ$. There is zero net force acting on the object. What is it's position as a function of time?

Answer: By Newton's First Law, the object moves in a straight line, at constant speed. In this case, the object's trajectory is a parameterized curve that is a line through 3D space:

$$\vec{r}(t) = \vec{r}(0) + \vec{v}_\circ t$$

M-171: A *helix* is a trajectory wherein a particle (mass, or object) is moving with constant component of velocity in one direction, and uniform circular motion in the plane perpendicular to that direction (this describes the motion of a charged particle in a uniform magnetic field, in the absence of an electric field). What is the most general parameterized curve corresponding to a particle moving with constant velocity in the z direction, and circular motion in in the xy plane (maybe better to say 'perpendicular to the z axis')?

Answer: Let the z component of the velocity of the particle be v_z. As well, let the radius of the particle corresponding to its motion perpendicular to z be $R_\perp$, and its period of *gyration* be T. The most general parameterized curve corresponding to the trajectory of the particle is then

$$\vec{r}(t) = R_\perp \Big[\cos(2\pi t/T + \delta)\hat{x} \pm \sin(2\pi t/T + \delta)\hat{y} \Big] + (z_\circ + v_z t)\hat{z}$$

M-172: A particle (mass or object) executes circular motion about a point (center) which translates with uniform velocity (imagine the trajectory of a dot on a car or bike tire with the car or bike moving with constant speed). This trajectory is called a *cycloid*, and is, for example, the trajectory of a charged particle undergoing what we call 'E cross B' motion (or drift). Envisage a particle (mass or object) orbiting a center that is moving in the positive $\hat{x}$ direction at speed v, where that orbit is in the xy plane, clockwise when looking down the z axis (from positive z), and where at $t = 0$ the particle was moving in the positive y direction as it passed through the origin. If the radius of its orbit about the center is R, the orbital period is T, and if it is in the $z = 0$ plane, what is its trajectory?

Answer: Let $\omega = 2\pi/T$. The trajectory of this particle is

$$\vec{r}(t) = \left[\frac{R}{2} + vt \right] \hat{x} + R \Big[-\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y} \Big]$$

M-173: This problem could equally well be in the section on magnetic fields, but I hope you'll agree with me that, provided you know the magnetic force, this is really a question pertaining to vector algebra. The electric field is zero, and the magnetic field is

$$\vec{B} = B_\circ \hat{z}$$

At $t = 0$, a proton is at

$$\vec{r} = x_\circ \hat{x} + y_\circ \hat{y}$$

and is moving with velocity

$$\vec{v} = v_\circ \cos(\theta_\circ)\hat{x} + v_\circ \sin(\theta_\circ)\hat{y}$$

What is the trajectory of the proton, as a function of time?

Answer: The proton is moving in the equatorial plane perpendicular to the magnetic field. Its speed is $v_\circ$ (why?). Since the electric field is zero, the trajectory is a circle of (Larmor or gyro) radius

$$R_G = \frac{m_p v_\circ}{eB}$$

where e and m_p are the proton charge and mass, respectively. The instantaneous force on the proton is the *central* force that keeps the proton moving on its circular trajectory. Thus, the instantaneous force on the proton points towards the center of the circular trajectory. Given that, if we identify the location of that center as $\vec{r}_c$, we have

$$\vec{r}_c = x_\circ \hat{x} + y_\circ \hat{y} + R_G \hat{n}$$

From the equation for the magnetic force, we know

$$\vec{F} = e\vec{v} \times \vec{B} = e v_\circ B_\circ \Big[\sin(\theta_\circ)\hat{x} - \cos(\theta_\circ)\hat{y}\Big] \quad \text{and so} \quad \hat{n} = \sin(\theta_\circ)\hat{x} - \cos(\theta_\circ)\hat{y}$$

Therefore,

$$\vec{r}_c = x_\circ \hat{x} + y_\circ \hat{y} + R_G \hat{n} \quad \text{or} \quad (x_c, y_c) = \left[x_\circ + \frac{m_p v_\circ}{eB}\sin(\theta_\circ), y_\circ - \frac{m_p v_\circ}{eB}\cos(\theta_\circ)\right]$$

The proton orbits this point, on a circle of radius R_G, so its trajectory, parameterized as $\vec{r}(t)$, is

$$(x, y) = (x_c, y_c) + R_G \Big[\cos(\phi(t)), \sin(\phi(t))\Big]$$

where

$$\phi(t = 0) = \theta_\circ - \pi \;\; \text{and} \;\; \phi(t) = \phi(t = 0) - \omega_G t \;\; \text{where} \;\; \omega_G = \frac{eB}{m_p}$$

Thus,

$$(x, y) = \left[x_\circ + \frac{m_p v_\circ}{eB}\sin(\theta_\circ), y_\circ - \frac{m_p v_\circ}{eB}\cos(\theta_\circ)\right] + \frac{m_p v_\circ}{eB}\left[\cos(\theta_\circ - \pi - \frac{eB}{m_p}t), \sin(\theta_\circ - \pi - \frac{eB}{m_p}t)\right]$$

M-174: There is a satellite in a polar, circular orbit around Earth. It's geocentric distance, or alternately its altitude, is such that its orbital period T is $T = 0.25 \times (\text{one sidereal day})$. At midnight UT the satellite is at the equator, moving northward, on the Greenwich Meridian (0° longitude). Develop a parameterized expression for the position of the satellite, in a frame of reference rotating with the Earth (the $z-$axis is the Earth's rotational axis, with north being positive z, the $x-$axis passes through the Earth's center and the equator on the Greenwich Meridian, and x, y, and z make a right handed coordinate system), presuming its orbital plane is inertially fixed.

Answer: The Earth rotates under the satellite orbit. The longitude of the satellite increases linearly in time, from 0° at midnight to 360° (also 0°) in one orbital period, so the spacecraft longitude is

$$\phi(t) = \frac{\pi}{180^\circ}\left[\left[t \times 360^\circ/T\right] \bmod 360^\circ\right]$$

The spacecraft latitude (maybe think about what I mean when I say the latitude is 270°) is

$$\lambda(t) = \frac{\pi}{180^\circ}\left[\left[t \times 360^\circ/T\right] \bmod 360^\circ\right]$$

The parameterized curve describing the satellite position is then

$$\vec{r}(t) = R\cos(\phi(t))|\cos(\lambda(t))|\hat{x} + R\sin(\phi(t))|\cos(\lambda(t))|\hat{y} + R\sin(\lambda(t))\hat{z}$$

where R is the geocentric radius that corresponds to an orbit of period T.

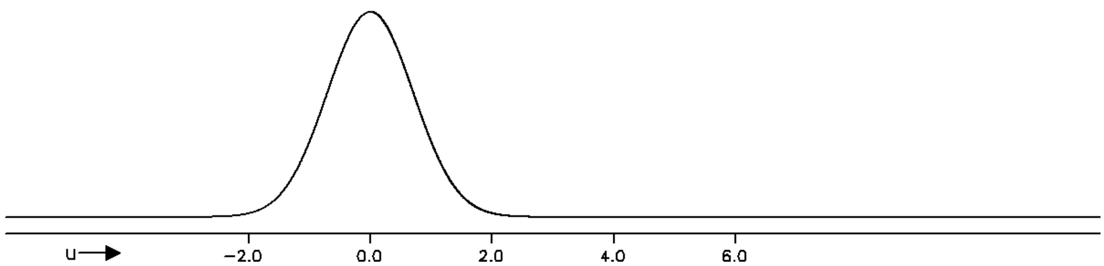
Think about units.

Infinite Plane Waves

M-175: Consider $f(u)$, which is a function of one variable, namely u. Graph $f(u)$, if

$$f(u) = \mathrm{e}^{-u^2}$$

Answer: Here is the graph:



Comment: Have you thought much about what a graph of a function, or even a function, is? Without getting philosophical or mathematical, a graph, and a function, is simply a shape. It could be, for example, the instantaneous shape (along the direction of propagation) of a solitary wave (look up tidal bores). But whatever, it is simply a 'shape.'

M-176: Consider the fame function of one variable from the previous question:

$$f(u) = \mathrm{e}^{-u^2}$$

Let $u = x - vt$, where x is position along a horizontal (or at least linear) axis, and t is time. Show that the location of the maximum of f is propagating in the positive $x-$ direction at speed v.

Answer: Let u_{max} be the location of the maximum value of $f(u)$. It is obvious that $u_{max} = 0$, but we can show that rigorously by noting that

$$\frac{df}{du} = -2u\mathrm{e}^{-u^2} \quad \text{and} \quad \frac{d^2f}{du^2} = -2\left[1 - 2u\right]\mathrm{e}^{-u^2}$$

There is only one value of u for which the first derivative is zero, namely at $u = 0$, and at that location the second derivative is negative. Therefore, the global maximum of $f(u)$ is at $u = 0$. The location of the maximum is at $u_{max} = 0$. Let's represent the location of the maximum on the $x-$axis by x_{max}. It must be that

$$u_{max} = x_{max} - vt \quad \text{so that} \quad x_{max} = u_max + vt \quad \text{or} \quad x_{max} = vt$$

While $f(u)$ is a stationary shape, $f(x,t) = f(x - vt)$ has the same shape. At $t = 0$, the of $f(x - vt)$ is at $x = 0$ (like the peak of $f(u)$ is at $u = 0$. But, that peak is propagating towards positive x with speed v.

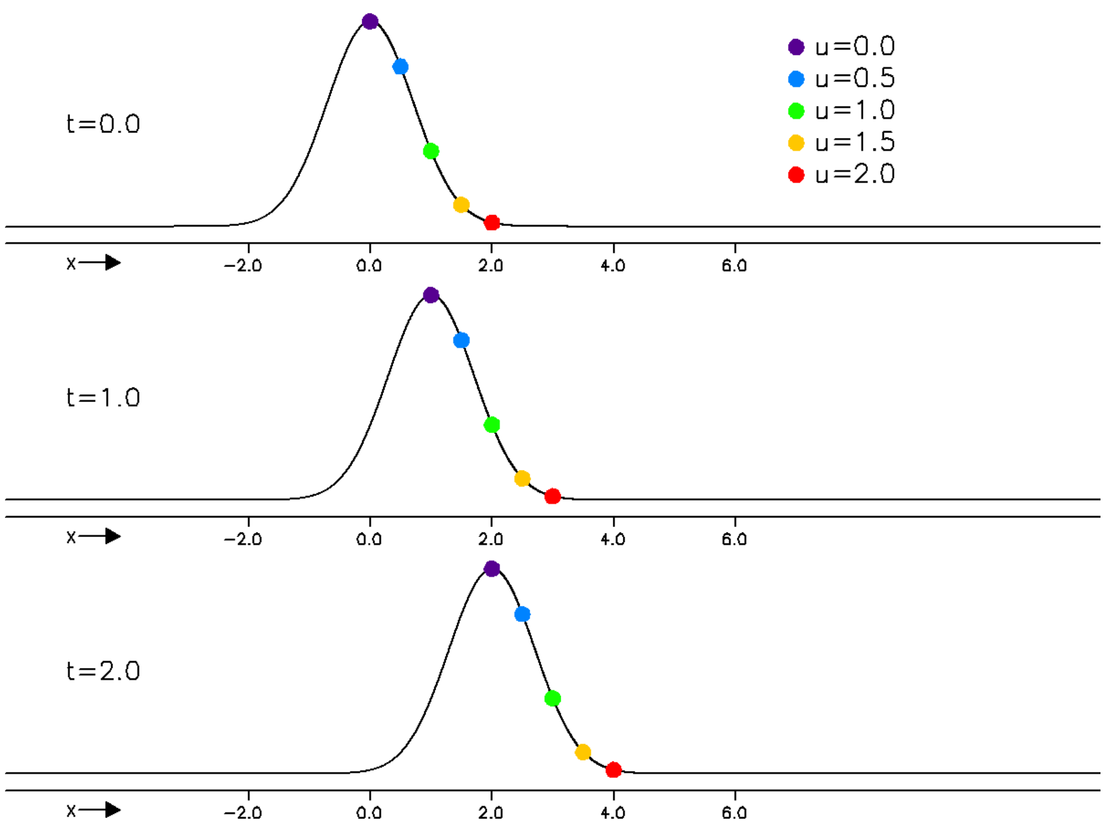
M-177: Consider the same function from the previous questions:

$$f(u) = \mathrm{e}^{-u^2}$$

Again, let $u = x - vt$. Pick five values of u, in particular $u = [0.0, 0.5, 1.0, 1.5, 2.0]$

(a) Make a stack plot of three graphs of $f(x,t)$, one for each of $t = 0.0$, $t = 1.0$ and $t = 2.0$. In other words, plot $f(x, 0.0)$, $f(x, 1.0)$ and $f(x, 2.0)$ as functions of x. On each plot indicate the five 'places' on the curve (shape) that correspond to the five values of u listed above. **(b)** Show that the speed of the location in x where $u = 1.5$ is propagating towards positive x at speed v. **(c)** Shgw that the location of a point corresponding to any specific value of u is propagating towards positive x at speed v.

Answer (a): Here is the graph. On it I have indicated u =0, 0.5, 1.0, 1.5, and 2.0 with purple, dark blue, light blue, mustard, and red filled in circles, respectively.



Answer (b): We can think of x as a function of $u = 1.5$ and t:

$$x(1.5, t) = 1.5 + vt$$

But, this is a function only of time, and its derivative with respect to time is v. The position in x corresponding to $u = 1.5$ is moving towards positive x with speed v.

Answer (c): More generally than for (b), for any *specific* u (e.g., fixed u), $x = u + vt$ is a function only of time, and its derivative with respect to time is v. The position in x corresponding to any specific u is moving towards positive x with speed v.

Additional Thoughts on Parameterized Curves

M-178: To say that a curve is parameterized is simply to say that

$$\vec{r}(\lambda) = x(\lambda)\hat{x} + y(\lambda)\hat{y} + z(\lambda)\hat{z}$$

Show that

$$\left.\frac{d\vec{r}}{d\lambda}\right|_{\lambda_\circ} \text{ is the tangent to the curve at } \vec{r}(\lambda_\circ)$$

Answer: Starting from the definition of the derivative,

$$\left.\frac{d\vec{r}}{d\lambda}\right|_{\lambda_\circ} = \frac{\vec{r}(\lambda_\circ + h) - \vec{r}(\lambda_\circ)}{h}$$

where h is infinitesimally small. Thus

$$\vec{r}(\lambda_\circ + h) = \vec{r}(\lambda_\circ) + h\left.\frac{d\vec{r}}{d\lambda}\right|_{\lambda_\circ}$$

Therefore,

$$\left.\frac{d\vec{r}}{d\lambda}\right|_{\lambda_\circ} \text{ is the tangent to the curve at } \vec{r}(\lambda_\circ)$$

M-179: Given the unit circle centered on the origin, show that

a: the circle can be represented as a curve parameterized by θ, e.g., as $\vec{r}(\theta)$ (work in $2D$), and

b: that the tangent to the curve at $\vec{r}(\theta_\circ)$ is the derivative of $\vec{r}$ with respect to θ, evaluated at $\theta_\circ$.

Answer (a): The circle can be parameterized as

$$\vec{r}(\theta) = \cos(\theta)\hat{x} + \sin(\theta)\hat{y}$$

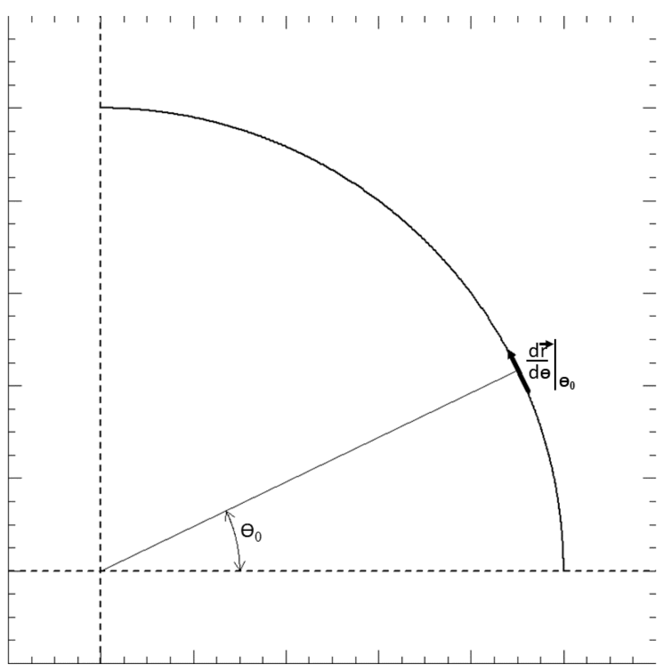
Answer (b): At a point on the circle, the curve is perpendicular to the vector from the origin to the point. In other words, $\vec{r}$ is perpendicular to the tangent to the curve. Any vector that satisfies this equation is a valid tangent ("the" is probably too restrictive). Consider

$$\left.\frac{d\vec{r}}{d\theta}\right|_{\theta_\circ} = -\sin(\theta_\circ)\hat{x} + \cos(\theta_\circ)\hat{x}$$

Thus

$$\hat{r}(\theta_\circ) \cdot \left.\frac{d\vec{r}}{d\theta}\right|_{\theta_\circ} = -\sin(\theta_\circ)\cos(\theta_\circ) + \sin(\theta_\circ)\cos(\theta_\circ) = 0$$

Therefore the derivative of $\vec{r}$ with respect to θ, evaluated at $\theta_\circ$, is the (a?) tangent to the curve at $\vec{r}(\theta_\circ)$.



M-180: We *can* define a line to be a curve where the distance along the curve between two points is the Euclidean distance between those to points. That is, if the curve is parameterized by distance along it, s, then

$$|\vec{r}(s_1) - \vec{r}(s_\circ)| = (s_1 - s_\circ)$$

Show that the parameterized curve

$$x(s_1)\hat{x} + y(s_1)\hat{y} + z(s_1)\hat{z} = x(s_\circ)\hat{x} + y(s_\circ)\hat{y} + z(s_\circ)\hat{z} + (s_1 - s_\circ)\left[a\hat{x} + b\hat{y} + c\hat{z}\right] \qquad (65)$$

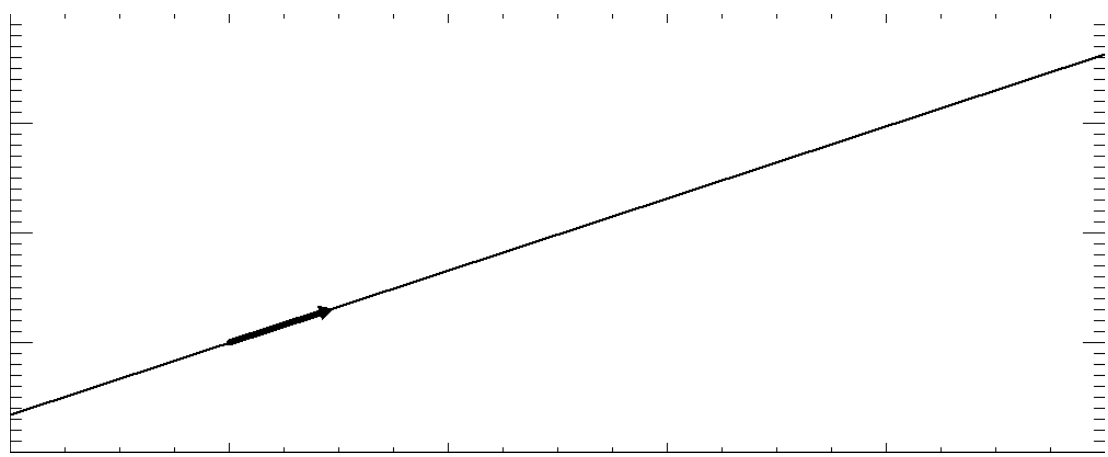
where $\sqrt{a^2 + b^2 + c^2} = 1$ is a line according to the above definition.

Answer: Consider the Euclidean distance between the points $\vec{r}(s_\circ)$ and $\vec{r}(s_\circ + \Delta s)$:

$$\textit{Euclidean distance} = \sqrt{[x(s_1) - x(s_\circ)]^2 + [y(s_1) - y(s_\circ)]^2 + [z(s_1) - z(s_\circ)]^2}$$

According to Equation 65, the RHS of the above is $(s_1 - s_\circ)\sqrt{a^2 + b^2 + c^2} = s_1 - s_\circ$

Thus, the distance along the curve between $\vec{r}(s_\circ)$ and $\vec{r}(s_1)$ is the Euclidean distance between $\vec{r}(s_\circ)$ and $\vec{r}(s_1)$. This is true for all pairs of points on the curve. Therefore, the curve is a line.



Some Thoughts on Tensors

M-181: Show that

$$\vec{A} \cdot \vec{B} = A_i B_i$$

M-182: Show that

$$\delta_{ii} = 3$$

M-183: Show that

$$A^2 = A_i A_i$$

where $A = |\vec{A}|$.

M-184: Show that

$$\delta_{ij} = \delta_{ji}$$

M-185: Show that

$$A_i = \delta_{ij} A_j$$

M-186: Show that

$$\vec{\nabla} \cdot \vec{A} = \partial_i A_i$$

M-187: Show that

$$\epsilon_{i23} \epsilon_{i13} = 0$$

M-188: Show that

$$\epsilon_{jik} = -\epsilon_{ijk}$$

M-189: Show that

$$\epsilon_{iik} = 0$$

M-190: Show that the following very useful identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

is true.

Answer: This is a four dimensional quantity with $3^4 = 81$ values. There are various ways to show this relationship holds, but I find doing this brute force (with the help of a computer because we are not in the dark ages like when I was in 3rd and 4th year!) to be most instructive about what is going on. I just write a program that shows that for each of the 81 "elements" of this quantity, the expression on the left hand side equals that on the right. It is included below, written in IDL, which a few of you might have, but I believe the code is fairly readable and you could look at this and easily construct your own equivalent in whatever programming language (e.g., Python) you prefer. I find that going through this exercise helps me to better visualize what we are dealing with.

```
function lc,i,j,k
    lvs=((j-i)*(k-i)*(k-j))/2
 return,lvs
 end

 function kronicker_delta,i,j
 return,fix((i eq j))
 end

 function lhs,i,j,l,m
    lhs=0
    for k=1,3 do lhs=lhs+levi_cevita(k,i,j)*levi_cevita(k,l,m)
 return,lhs
 end

 function rhs,i,j,l,m
    rhs=kronicker_delta(i,l)*kronicker_delta(j,m)-kronicker_delta(i,m)*kronicker_delta(j,l)
 return,rhs
 end

 pro test_levi_cevita_kronicker_delta_identity
    for i=1,3 do for j=1,3 do for l=1,3 do $
        for m=1,3 do s=s+strcompress(fix(rhs(i,j,l,m) eq lhs(i,j,l,m)),/remove_all)
    print,' string: '+s
 return
 end
```

When I was in first year Calculus, for the first month or so I was getting scared. The professor kept stopping and saying “what do we need to do now?”. The answer, time and again, was “we need to use the trick where...”. After a few weeks it seemed like the number of “tricks” we would need to remember was endless. But of curse, it closes off pretty quickly, and it turns out there are a finite number of such tricks, and everyone past a certain point knows them all. This is the same here. These are not just random mathematical quantities and identities. They emerge as useful because of how they can be used in representations of complicated vector relationships. As is often the case, this is more complicated than necessary for simple operations, but a simpler approach for more complicated operations. For example, we gain nothing. Certainly we gained nothing with a somewhat obtuse representation of the dot product or the trace of the identity matrix. But those are easy.

M-191: Show that the cross product can be represented as follows:

$$\vec{A} \times \vec{B} = \left[\vec{A} \times \vec{B}\right]_i = \epsilon_{ijk} A_j B_k$$

Answer: To see that this is true, we just need to think a bit about the quantity on the right hand side. First, we can see that it is a one dimensional quantity, since while there are five indices, there are two sets of two that are repeated. Double indices indicate summation (also called *contraction*), so this quantity has only three elements (we can think of them as components of the vector). If we set i=1 (which would be the x-component of the vector) and perform the summation, we have

$$\begin{aligned}
\left[\vec{A} \times \vec{B}\right]_1 &= \epsilon_{1jk} A_j B_k \\
&= \epsilon_{111} A_1 B_1 + \epsilon_{112} A_1 B_2 + \epsilon_{113} A_1 B_3 + \epsilon_{121} A_2 B_1 + \epsilon_{122} A_2 B_2 \\
&\qquad + \epsilon_{123} A_2 B_3 + \epsilon_{131} A_3 B_1 + \epsilon_{132} A_3 B_2 + \epsilon_{133} A_3 B_3 \\
&= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 \\
&= A_2 B_3 - A_3 B_2
\end{aligned}$$

To get this result, I have used the fact that the Levi-Cevita symbol is zero if any one of its three indices equals any other of its three indices ($\epsilon_{111} = 0$, $\epsilon_{112} = 0$, $\epsilon_{113} = 0$, $\epsilon_{121} = 0$, $\epsilon_{122} = 0$, $\epsilon_{131} = 0$, and $\epsilon_{133} = 0$), $\epsilon_{123} = 1$, and $\epsilon_{132} = -1$. Noting that 1, 2, and 3 correspond to the x, y, and z components, respectively, this means that the first (x) component of the right hand side is $A_y B_z - A_z B_y$ which is the x-component of $\vec{A} \times \vec{B}$. I recommend working through this for the 2nd and 3rd components, which you will find are equal to the y and z components of $\vec{A} \times \vec{B}$. Thus, the quantity on the right hand side is a set of three numbers that are the x, y, and z components of $\vec{A} \times \vec{B}$, and so by definition is $\vec{A} \times \vec{B}$.

As you work through the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ components of the above, you should start to see the pattern I refer to as *circular permutation*. The second component is just the first component with every index permuted: $1 \to 2$, $2 \to 3$, and $3 \to 1$. For each you end up with the two terms non-zero on the right hand side with the indices permuted. This way of inferring the 2nd component from the 1st, and the 3rd from the 2nd, and the 1st from the 3rd holds in general.

M-192: Prove the double cross product identity, $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$, is true.

$$
\begin{aligned}
\textbf{Answer}: \quad \vec{A}\times(\vec{B}\times\vec{C}) &= \left[\vec{A}\times(\vec{B}\times\vec{C})\right]_i \\
&= \epsilon_{ijk}A_j(\epsilon_{klm}B_lC_m) \\
&= \epsilon_{ijk}\epsilon_{klm}A_jB_lC_m \\
&= \delta_{il}\delta_{jm}A_jB_lC_m - \delta_{im}\delta_{jl}A_jB_lC_m \text{ using the identity, above} \\
&= B_iA_jC_m - A_jB_jC_i \text{ (using } \delta_{ij}G_j = G_i) \\
&= B_i\vec{A}\cdot\vec{C} - C_i\vec{A}\cdot\vec{B} \text{ (using } G_jH_j = G_1H_1 + G_2H_2 + G_3H_3 = \vec{G}\cdot\vec{H}) \\
&= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})
\end{aligned}
$$

I hope you can start to see a hint of the power of this approach. If you just know the rules that the Levi-Cevita and Kronecker Delta symbols follow, understand how repeated indices mean summation over that index, and know the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, then it is tedious but very straightforward to derive this vector identity. The key point is that this does not get more complicated, so for more complicated vector identities, this becomes relatively easier.

Now we can look at some other identities. For example

$$
\begin{aligned}
\vec{A}\cdot(\vec{B}\times\vec{C}) &= \vec{A}_i(\vec{B}\times\vec{C})_i \\
&= A_i(\epsilon_{ijk}B_jC_k) \\
&= B_j(\epsilon_{jki}C_kA_i) \quad \text{e.g.,} = \vec{B}\cdot(\vec{C}\times\vec{A}) \\
&= C_k(\epsilon_{kij}A_iB_j) \quad \text{e.g.,} = \vec{C}\cdot(\vec{A}\times\vec{B})
\end{aligned}
$$

The fact that the curl of a gradient and divergence of a curl are zero is widely used in E&M. These are very useful identities that are trivially proven using this approach. The key is to really understand what you can do when the quantity is symmetric or antisymmetric with respect to two indices

$$
\begin{aligned}
\nabla\cdot(\nabla\times\vec{A}) &= \partial_i\epsilon_{ijk}\partial_jA_k \\
&= \epsilon_{ijk}\partial_i\partial_jA_k \\
&= \epsilon_{ijk}\partial\partial_iA_k \quad \text{switch order of differentiation} \\
&= -\epsilon_{jik}\partial_j\partial_iA_k \quad \epsilon_{ijk}\text{is antisymmmetric in } ij \\
&= -\epsilon_{ijk}\partial_i\partial_jA_k \quad i \text{ and } j \text{ are dummy variables} \\
&= -\nabla\cdot(\nabla\times\vec{A}) \\
&= 0 \qquad\qquad \text{e.g.,} \ \nabla\cdot(\nabla\times\vec{A}) = -\nabla\cdot(\nabla\times\vec{A})
\end{aligned}
$$

In this case we have used the antisymmetry of ϵ_{ijk} with respect to i and j (so $\epsilon_{ijk} = -\epsilon_{jik}$) and the symmetry of $\partial_i\partial_j$ with respect to i and j (which follows from the fact that we can reverse the order of integration). As well, we have used a key property of "dummy indices". These are indices that are repeated and thus imply summation (also called contraction). So switching i and j in just the Levi-Cevita symbol introduces a minus sign, because it is antisymmetric, but i and j throughout $\epsilon_{ijk}\partial_i\partial_j$ does not change anything. In other words

$$\epsilon_{ijk} = -\epsilon_{jik}$$

however

$$\epsilon_{ijk}\partial_i\partial_j = \epsilon_{jik}\partial_j\partial_i$$

This is a subtle but very important point. It is perhaps easier to see if one considers that one can use another letter to replace a dummy index (use p for example instead of i).

$$\epsilon_{ijk}\partial_i\partial_j = \epsilon_{pjk}\partial_p\partial_j$$

Similarly we can replace j with q

$$\epsilon_{ijk}\partial_i\partial_j = \epsilon_{pqk}\partial_p\partial_q$$

and then in succession switch j for p and then i for q so that

$$\epsilon_{ijk}\partial_i\partial_j = \epsilon_{jik}\partial_j\partial_i$$

Showing that the curl of a gradient is zero is equally straightforward with this approach:

$$\begin{aligned}
\nabla \times \nabla G &= \epsilon_{ijk}\partial_i\partial_j A_k \\
&= \epsilon_{ijk}\partial_j\partial_i A_k \\
&= -\epsilon_{jik}\partial_j\partial_i A_k \\
&= -\epsilon_{ijk}\partial_i\partial_j A_k \\
&= -\nabla \times \nabla G \\
&= 0
\end{aligned}$$

Again, this uses the fact we can switch the order of differentiation, the antisymmetric nature of the Levi-Cevita symbol, and the meaning of dummy variables to show that $\nabla \times \nabla G = -\nabla \times \nabla G$ so that $\nabla \times \nabla G = 0$.

A fairly complicated identity is

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + \vec{A} \cdot \nabla\vec{B} + \vec{B} \cdot \nabla\vec{A}$$

I find this easier to approach as follows

$$\begin{aligned}
\vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) &= \epsilon_{kij}\epsilon_{klm}(A_j\partial_l B_m + B_j\partial_l A_m) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(A_j\partial_l B_m + B_j\partial_l A_m) \\
&= A_m\partial_i B_m + B_m\partial_i A_m - A_j\partial_j B_i - B_j\partial_j A_i \\
&= \partial_i(A_m B_m) - A_j\partial_j B_i - B_j\partial_j A_i \\
&= \nabla(\vec{A} \cdot \vec{B}) - \vec{A} \cdot \nabla\vec{B} - \vec{B} \cdot \nabla\vec{A} \qquad \text{so therefore} \\
\nabla(\vec{A} \cdot \vec{B}) &= \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + \vec{A} \cdot \nabla\vec{B} + \vec{B} \cdot \nabla\vec{A}
\end{aligned}$$

I suggest looking in the front cover of Griffiths or searching the web for other examples of such identities and working those out for yourself.

M-193: Show that

$$\vec{\nabla}\cdot(\vec{B}\times\vec{E}) = \vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})$$

Answer (if you are comfortable with tensors): Using the Einstein Summation Convention, and the Levi-Cevita symbol, we have

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{B}\times\vec{E}) &= \partial_i \epsilon_{ijk} B_j E_k \\
&= \epsilon_{ijk}\partial_i B_j E_k \\
&= \epsilon_{ijk} B_j \partial_i E_k + \epsilon_{ijk} E_k \partial_i B_j \\
&= -B_j \epsilon_{jik}\partial_i E_k + E_k \epsilon_{kij}\partial_i B_j \\
&= \vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})
\end{aligned}$$

M-194: Show that

$$\vec{\nabla}\cdot(\vec{B}\times\vec{E}) = \vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})$$

Answer (if you are not comfortable with tensors): This is the brute force, but foolproof, solution...

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{B}\times\vec{E}) &= \frac{\partial}{\partial x}\Big[B_y E_z - B_z E_y\Big] + \frac{\partial}{\partial y}\Big[B_z E_x - B_x E_z\Big] + \frac{\partial}{\partial z}\Big[B_x E_y - B_y E_x\Big] \\
&= B_y \frac{\partial E_z}{\partial x} + E_z \frac{\partial B_y}{\partial x} + B_z \frac{\partial E_x}{\partial y} + E_x \frac{\partial B_z}{\partial y} + B_x \frac{\partial E_y}{\partial z} + E_y \frac{\partial B_x}{\partial z} \\
&\quad - \left[B_z \frac{\partial E_y}{\partial x} + E_y \frac{\partial B_z}{\partial x} + B_x \frac{\partial E_z}{\partial y} + E_z \frac{\partial B_x}{\partial y} + B_y \frac{\partial E_x}{\partial z} + E_x \frac{\partial B_y}{\partial z}\right] \\
&= E_x\left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right] + E_y\left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right] + E_z\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right] \\
&\quad - \left[B_x\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] + B_y\left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right] + B_z\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]\right] \\
&= \vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})
\end{aligned}$$

Estimating

M-195: Estimate 1/49

Answer: Use $(1+x)^\alpha \simeq 1+\alpha x$, for small *alpha*:

$$\frac{1}{49} = \frac{1}{50(1-\frac{1}{50})} \simeq \frac{1}{50}\left[1+\frac{1}{50}\right] = 0.02(1+0.02) = 0.02+0.0004$$

Thus, my estimate is

$$\frac{1}{49} \simeq 0.0204$$

M-196: Estimate the error in the estimate in the previous question.

Answer: As stated, we used $(1+x)^\alpha \simeq 1+\alpha x$, for small x. In particular, we used $(1-x)^{-1} \simeq 1+x$, with $x = 0.02$. This is the McLauren series

$$f(x) = f(0) + x\left.\frac{df}{dx}\right|_0 + \frac{x^2}{2!}\left.\frac{d^2f}{dx^2}\right|_0 + \frac{x^3}{3!}\left.\frac{d^3f}{dx^3}\right|_0 + \frac{x^4}{4!}\left.\frac{d^4f}{dx^4}\right|_0 + \cdots$$

for the specific case of $f = 1/(1-x)$, truncated after the first two terms. We can get *something* of an estimate of how accurate this might be by looking at the first term we dropped. The MacLauren series (estimate) for $1/(1-x)$, truncated at the third term, is

$$\frac{1}{1-x} \simeq 1+x+x^2$$

For $x = 0.02$, that third term, the first dropped term in our estimate, is 0.0004. So my *guestimate* of the error in our estimate is 0.0002. What this means is that we could report our estimate to be 0.0204 ± 0.0004. You ought to be able to convince yourself that this is an *upper bound* on the error, but even without thinking that through, I think it's reasonable to say this is a defensible estimate of the error.

How did we do? The actual answer, to five significant figures, is $1/49 = 0.02041$, so I would say both our estimate and our understanding of its accuracy were both good.

The MacLauren series above is widely known as an example of a geometric series, namely

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+x^5+\cdots$$

This is a useful series to remember. You can derive it via the MacLauren series expansion, or note the following. If we call the series *infinite* series $S = 1+x+x^2+x^3+\cdots$, then

$$S - xS = 1+x+x^2+x^3+x^4+x^5+\cdots - x\left[1+x+x^2+x^3+x^4+x^5+\cdots\right]$$

or

$$(1-x)S = 1 \quad \text{so that} \quad S = \frac{1}{1-x}$$

Truncated at some finite term, xS is not exactly $S-1$, but that difference becomes diminishingly small as we keep more and more terms. In the limit, the difference is zero.

M-197: The charge and mass of a proton are $e \simeq 1.60 \times 10^{-19}C$ and $m_p \simeq 1.67 \times 10^{-27}kg$, respectively. In a region of space, the magnitude of the magnetic field is $B = 10^{-8}T$. Estimate the gyroperiod (T) and gyrofrequency (ν) of protons in this region.

Find the gyroradius (R) by equating the magnetic force with mass times the centripetal acceleration, and then the gyroperiod from distance travelled ($2\pi R$) in one gyration:

$$evB = \frac{m_p v^2}{R} \Rightarrow R = \frac{m_p v}{eB} \quad \cdots \quad 2\pi R = vT \Rightarrow \frac{2\pi m_p v}{eB} = vT \Rightarrow T = \frac{2\pi m_p}{eB}$$

Thus,

$$T \simeq \frac{2(3.14)1.67 \times 10^{-27}kg}{1.6 \times 10^{-19-8}C \cdot T} = 2(3.14)\frac{1.7}{1.6}s \simeq 2(3.14)(1.05)s = 2.1(3.14)s \simeq 6.59s$$

I should explain some of my rationale. From memory, I know that $\pi \simeq 3.14159$ and I imagine that keeping three significant figures is more than enough for an *estimate*, so I used $\pi \simeq 3.14$. What about 1.67/1.6? Well... that is roughly 17/16, and 11/10 and 21/20 are 1.1 and 1.05, respectively, so I know 17/16 is between 1.05 and 1.1. I also know 1.67/1.6 is a *bit* smaller than 17/16, so I took 1.05 as my estimate of 1.67/1.6. Finally, 2.1×3.14=6.28+0.314 or 2.1×3.14≃6.59, where I have kept three significant figures (and I note the third figure may not be exactly correct but I expect it to be close).

You will also note that I replaced kg/CT with s. I can justify this in two ways. First, if I am putting every quantity into my equation in proper SI units, then the result will be in the proper SI unit. The quantity we are after is a time, so in this case the unit will be seconds. Alternately, I could check, noting that the force equation I started with (to get the gyroradius) is $e/B = m_p v/R$, and thus the units involved are related according to

$$\frac{C}{T} = \frac{kg \cdot m}{s \cdot m} \quad \text{or} \quad \frac{kg}{C \cdot T} = s$$

The gyrofrequency is the inverse of the gyroperiod:

$$\nu = \frac{1}{T} \simeq \frac{1}{6.59s} \simeq \frac{1}{6.60}Hz = \frac{1}{6(1+0.1)}Hz \simeq \frac{1-0.1}{6}Hz = 0.9(0.167)Hz \simeq 0.150Hz$$

I will not justify the particular choices I made, but one thing I will suggest is that you estimate the difference between 1/6.59 and 1/6.60, and to consider whether or not the other choices I made make sense to you.

So how did I do? Well, to three significant figures, $2\pi(1.67)/1.6$=6.56, and also to three significant figures, 1/6.59=0.152.

This will strike some of you, and perhaps even all of you, as a bit of an odd exercise in this day and age. For me, this is an essential part of thinking like a physicist. As physicists, we should be able to arrive at a result that is close enough to be useful without having to look very much up, and without computational aid. On one hand it demonstrates that you actually know what you are doing, and that you have enough tools at your fingertips to meaningfully engage in impromptu discussions about things you are working on... I've never read "How to Win Friends and Influence People," but I see this as a kind of physics version of that concept.

On the other (actually far more important) hand, a lifetime of working things through in this way helps one develop a sense of what good answers to questions look like. As a student, I was often left with a combination of incredulity and annoyance when I would bring my carefully thought out solution to a professor who would glance at it and say "that doesn't look right." More often than not, they were right. After a while, I came to the conclusion this intuition was hard won on their part... the result of decades of working things through.

M-198: What is the gyroradius of a 1 keV proton whose velocity is perpendicular to a magnetic field of strength $10^{-8}T$?

The kinetic energy of a 1 keV proton is the electron charge, e, times 1000 $Joules/Coulomb$:

$$KE = 1000eJ/C \quad \text{Therefore} \quad \frac{1}{2}m_p v^2 = 1000eJ/C \quad \text{or} \quad v = \sqrt{\frac{2000eJ/C}{m_p}}$$

The gyroradius is, then,

$$R = \frac{m_p v}{eB} \quad \text{or} \quad R = \frac{m_p}{eB}\sqrt{\frac{2000eJ/C}{m_p}} \quad \text{so} \quad R = \frac{1}{B}\sqrt{\frac{2000m_pJ/C}{e}}$$

Putting some numbers in,

$$R = \frac{10^8}{T}\sqrt{\frac{(2000)1.67\times10^{-27}kg\cdot J}{1.6\times10^{-19}C^2}} \quad \text{or} \quad R = \sqrt{\frac{3.34}{1.6}\times10^{-27+19+3+4}}\frac{1}{T\cdot C}\sqrt{kg\cdot J} = \sqrt{\frac{3.34}{16}}\frac{1}{T\cdot C}\sqrt{kg\cdot J}$$

Here I introduce an approximation, namely

$$\sqrt{\frac{3.34}{16}} = 0.25\sqrt{3.34} \simeq 0.25\times1.8 = 0.45 \quad \text{so} \quad R \simeq 0.45m$$

Double checking the units, we have

$$\frac{1}{T\cdot C}\sqrt{kg\cdot J} = \frac{1}{T\cdot C}\sqrt{\frac{kg^2m^2}{s^2}} = \frac{1}{T\cdot C}\frac{kg\cdot m}{s} = \frac{s}{m\cdot T\cdot C}\frac{kg\cdot m^2}{s^2} = \frac{s^2}{kg\cdot m}\frac{kg\cdot m^2}{s^2} = m$$

M-199: Given $G = 6.67\times10^{-11}m^3/kg/s^2$, $m_p = 1.67\times10^{-27}kg$, $e = 1.6\times10^{-19}C$, and $\epsilon_\circ = 8.85\times10^{-12}C^2/N/m^2$, estimate the ratio of the electrostatic force between two protons to the gravitational force between those two protons.

Answer: Because both forces are *inverse square*, the ratio does not depend on the distance between the protons:

$$Ratio = \frac{F_{electrostatic}}{F_{gravitational}} = \frac{e^2}{4\pi\epsilon_\circ r^2}\frac{r^2}{Gm_p^2} = \frac{1.6^2 10^{-38+12+11+54}}{4(3.14)(8.85)(6.67)1.67^2}\frac{C^2N\cdot m^2kg\cdot s^2}{C^2kg^2m^3}$$

This is a ratio of one force to another, so should be dimensionless. Let's start by looking at the units:

$$\frac{C^2N\cdot m^2kg\cdot s^2}{C^2kg^2m^3} = \frac{N\cdot s^2}{kg\cdot m} = N\frac{s^2}{kg\cdot m} = N\frac{1}{N} = 1$$

Therefore, our ratio is dimensionless, and is

$$Ratio = \frac{1.6^2 10^{39}}{4(3.14)(8.85)(6.67)1.67^2}$$

Now,

$$1.6^2 = 2.56, \quad 4(3.14)\simeq 12.6, \quad 8.85\times6.67\simeq 59.0, \quad \text{and} \quad 1.67^2\simeq 2.79$$

Thus,

$$Ratio \simeq \frac{2.56}{(1.26)(5.90)(2.79)}10^{37} \simeq \frac{1}{7.43}\frac{2.56}{2.79}10^{37} \simeq \frac{0.95}{3}\frac{1}{25}10^{38} \simeq 0.33(0.95)(0.04)10^{38} \simeq 1.3\times10^{36}$$

Note, the actual answer is closer to 1.2×10^{36}, so I would judge this an excellent estimate.

Dummy Variables

M-200: In the Question Bank I asked the following: Show that

$$\frac{d}{dx}\int_{x_0}^{x} x du = 2x - x_0$$

Answer: This is at once *extremely* simple, and at the same time a bit of a brain twister. Give it a try.

M-201: Show that

$$\vec{\nabla}\int_V \sqrt{x^2 + x'y'}d\tau' = x\hat{x}\int_V \frac{d\tau'}{\sqrt{x^2 + x'y'}}$$

Answer: The gradient operator here is with respect to the unprimed variables. The intgeral is a function of x, but not of y or z.

$$\int_V \sqrt{x^2 + x'y'}d\tau' = f(x)$$

For the gradient of $f(x)$ we have

$$\vec{\nabla}f = \hat{x}\frac{\partial f}{\partial x}$$

Therefore

$$\vec{\nabla}\int_V \sqrt{x^2 + x'y'}d\tau' = \hat{x}\frac{\partial}{\partial x}\int_V \sqrt{x^2 + x'y'}d\tau'$$

However, we can take the partial derivative into the integrand, so we have

$$\vec{\nabla}\int_V \sqrt{x^2 + x'y'}d\tau' = \hat{x}\int_V \left[\frac{\partial}{\partial x}\sqrt{x^2 + x'y'}\right]d\tau'$$

The partial derivative

$$\frac{\partial}{\partial x}\sqrt{x^2 + x'y'} = \frac{x}{\sqrt{x^2 + x'y'}}$$

Therefore

$$\vec{\nabla}\int_V \sqrt{x^2 + x'y'}d\tau' = x\hat{x}\int_V \frac{d\tau'}{\sqrt{x^2 + x'y'}}$$

M-202: Show that

$$\vec{\nabla} \times \vec{\nabla} f = 0$$

Answer: Let's look at the x-component of this equation:

$$\begin{aligned}
\left(\vec{\nabla} \times \vec{\nabla} f\right)_x &= \frac{\partial}{\partial y}\frac{\partial}{\partial z} f - \frac{\partial}{\partial z}\frac{\partial}{\partial y} f \\
&= \frac{\partial}{\partial y}\frac{\partial}{\partial z} f - \frac{\partial}{\partial y}\frac{\partial}{\partial z} f \quad \text{since I can reverse the order of first partial differentiation} \\
&= \left[\frac{\partial}{\partial y}\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\frac{\partial}{\partial z}\right] f \\
&= 0.
\end{aligned}$$

The same is true for the y- and z-components of this equation. Thus

$$\begin{aligned}
\left(\vec{\nabla} \times \vec{\nabla} f\right)_x &= 0, \\
\left(\vec{\nabla} \times \vec{\nabla} f\right)_y &= 0 \text{ and} \\
\left(\vec{\nabla} \times \vec{\nabla} f\right)_z &= 0
\end{aligned}$$

and therefore

$$\vec{\nabla} \times \vec{\nabla} f = 0$$

M-203: Show that

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A} \right) = 0$$

Answer: As for the above, this relies on the fact we can reverse the order of first partial differentiation.

M-204: What is one way you can show two vectors are equal?

Answer (1): You can show that they have the same direction and magnitude.

Answer (2): The two vectors $\vec{A}$ and $\vec{B}$ are equal if and only if

$$A_x = B_x, \quad A_y = B_y \quad \text{and} \quad A_z = B_z$$

Note there are non-Cartesian versions of this second answer.

M-205: Show that

$$\left[\vec{A}\times\left(\vec{B}\times\vec{C}\right)\right]_x = \left[\vec{B}\left(\vec{A}\cdot\vec{C}\right)\right]_x - \left[\vec{C}\left(\vec{A}\cdot\vec{B}\right)\right]_x$$

Answer: Take the LHS and *work towards* the RHS:

$$\begin{aligned}
\left[\vec{A}\times\left(\vec{B}\times\vec{C}\right)\right]_x &= A_y\left(\vec{B}\times\vec{C}\right)_z - A_z\left(\vec{B}\times\vec{C}\right)_y \\
&= A_y\left(B_xC_y - B_yC_x\right) - A_z\left(B_zC_x - B_xC_z\right) \\
&= B_x\left(A_yC_y + A_zC_z\right) - C_x\left(A_yB_y + A_zB_z\right) \\
&= B_x\left(\vec{A}\cdot\vec{C} - A_xC_x\right) - C_x\left(\vec{A}\cdot\vec{B} - A_xB_x\right) \\
&= B_x\left(\vec{A}\cdot\vec{C}\right) - C_x\left(\vec{A}\cdot\vec{B}\right) + A_xB_xC_x - A_xB_xC_x \\
&= B_x\left(\vec{A}\cdot\vec{C}\right) - C_x\left(\vec{A}\cdot\vec{B}\right) \\
&= \text{RHS}
\end{aligned}$$

Therefore the RHS equals the LHS, so

$$\left[\vec{A}\times\left(\vec{B}\times\vec{C}\right)\right]_x = \left[\vec{B}\left(\vec{A}\cdot\vec{C}\right)\right]_x - \left[\vec{C}\left(\vec{A}\cdot\vec{B}\right)\right]_x$$

It is not part of the question, however this works identically for the y- and z- components of

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right) \tag{66}$$

so that Equation 66 is one of our vector identities.

M-206: Say that

$$\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$$

What is the divergence of $\vec{A}$?

Answer: The divergence of $\vec{A}$ is

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

Therefore, in this case,

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

and so,

$$\vec{\nabla} \cdot \vec{A} = 3$$

M-207: Say that

$$\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$$

What is the outward flux of $\vec{A}$ from a volume V bounded by a surface S?

Answer: The outward flux of $\vec{A}$ from the volume V (bounded by the surface S) is

$$\Phi_{outward} = \int_S \vec{A} \cdot \vec{da}$$

According to Gauss's Theorem

$$\int_S \vec{A} \cdot \vec{da} = \int_V \vec{\nabla} \cdot \vec{A} d\tau$$

In this case,

$$\vec{\nabla} \cdot \vec{A} = 3$$

From Equations 24 and 25, we have

$$\Phi_{outward} = 3V$$

M-208: Given a vector $\vec{A} = x\hat{x} + xy\hat{y}$, show by directly evaluating the contour integral on the contour from $(0,0)$ to $(1,0)$ along the x-axis, then from $(1,0)$ to $(1,1)$ along the line $x = 1$, and then back to $(0,0)$ along the curve $y = x^n$, is

$$\oint \vec{A} \cdot \vec{dl} = \frac{1}{2(2n+1)}$$

2) Given the vector $\vec{A}$ from question 1

a) show that its curl is $y\hat{z}$

a) show, on the surface in the xy-plane bounded by the contour from a, that

$$\oint \oint (\vec{\nabla} \times \vec{a}) \cdot \vec{da} = \frac{1}{2(2n+1)}$$

M-209: The relationship between $\vec{B}$ and the vector potential $\vec{A}$ is

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

If the vector potential is given by

$$\vec{A} = \vec{r} \times \vec{C}$$

where $\vec{C}$ is a constant, then what is $\vec{B}$?

Answer: We can use the vector identity

$$\vec{\nabla} \times \left[\vec{M} \times \vec{N}\right] = (\vec{N} \cdot \vec{\nabla})\vec{M} - (\vec{M} \cdot \vec{\nabla})\vec{N} + \vec{M}(\vec{\nabla} \cdot \vec{N}) - \vec{N}(\vec{\nabla} \cdot \vec{M})$$

We have

$$\vec{C}(\vec{\nabla} \cdot \vec{r}) = 3\vec{C},$$

$$\vec{r}(\vec{\nabla} \cdot \vec{C}) = 0,$$

$$(\vec{r} \cdot \vec{\nabla})\vec{C} = 0,$$

and

$$(\vec{C} \cdot \vec{\nabla})\vec{r} = C_x \frac{\partial}{\partial x} x\hat{x} + C_y \frac{\partial}{\partial y} y\hat{y} + C_z \frac{\partial}{\partial z} z\hat{z} = C_x\hat{x} + C_y\hat{y} + C_z\hat{z} = \vec{C}$$

Therefore,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left[\vec{r} \times \vec{C}\right] = -2\vec{C}$$

Question: Given

$$f(x) + g(y) = 0$$

show $f(x) = C$ and $g(y) = D$, where C and D are constants.

Answer: Pick a value of x, such as $x = 7$. Then $g(y) = -f(7)$ (e.g., if $f(x) = x^2$ then $g(y) = -49$ for all y). Therefore $g(y)$ is a constant. The same argument holds for $f(x)$.