Max Stronge
30064749

1. Evaluate

$$I = \lim_{x\to\infty} \frac{1-\cos^2(x)}{x^2}$$

- a) using L'Hôpital's rule
- b) by Taylor expansion of $\cos(x)$

Solution:

To use L'Hôpital's rule, the limit I must be in indeterminate form. Check the behaviour of the numerator and denominator as $x \to 0$:

$$\begin{aligned} 1-\cos^2 x = 1-1 = 0 \\ x^2 = 0 \end{aligned}$$

It is in indeterminate form, so we differentiate both numerator and denominator with respect to x:

$$\frac{d}{dx}\left[1-\cos^2 x\right] = -\frac{d}{dx}\left[\cos^2 x\right] = 2\cos x \sin x$$
$$\frac{d}{dx}\left[x^2\right] = 2x$$

Now we have

$$\frac{\cancel{2}\cos x \sin x}{\cancel{2}x}$$

The function is still indeterminate, so we repeat the process:

$$\frac{d}{dx}[\cos x \sin x] = \cos^2 x - \sin^2 x$$
$$\frac{d}{dx}[x] = 1$$

so

$$\lim_{x\to 0} \frac{\cos^2 x - \sin^2 x}{1} = 1.$$

Using the Taylor series of $\cos x$:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}\dots$$

We can square this series to find $\cos^2 x$:

$$\begin{aligned} \cos^2 x &= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}\dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}\dots\right) \\ &= 1 - \frac{2x^2}{2} + \frac{2x^4}{4!} - \frac{2x^6}{6!} \\ &= 1 - x^2 + \frac{x^4}{12} + \frac{x^6}{180} + \dots \end{aligned}$$

The form of the numerator of our function is $1-\cos^2 x$, so let's subtract the RHS of the above from 1:

$$1-\cos^2 x = 1 - \left(1 - x^2 + \frac{x^4}{12} + \frac{x^6}{180} + \dots\right)$$

$$1-\cos^2 x = 0 + x^2 - \frac{x^4}{12} + \dots$$

In the limit as $x \to 0$, the higher powers of x will become negligible, leaving us with :

$$\lim_{x\to 0}\left(1-\cos^2 x\right) = x^2$$

and thus the limit of the original function

$$I = \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

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2. Consider the function given by

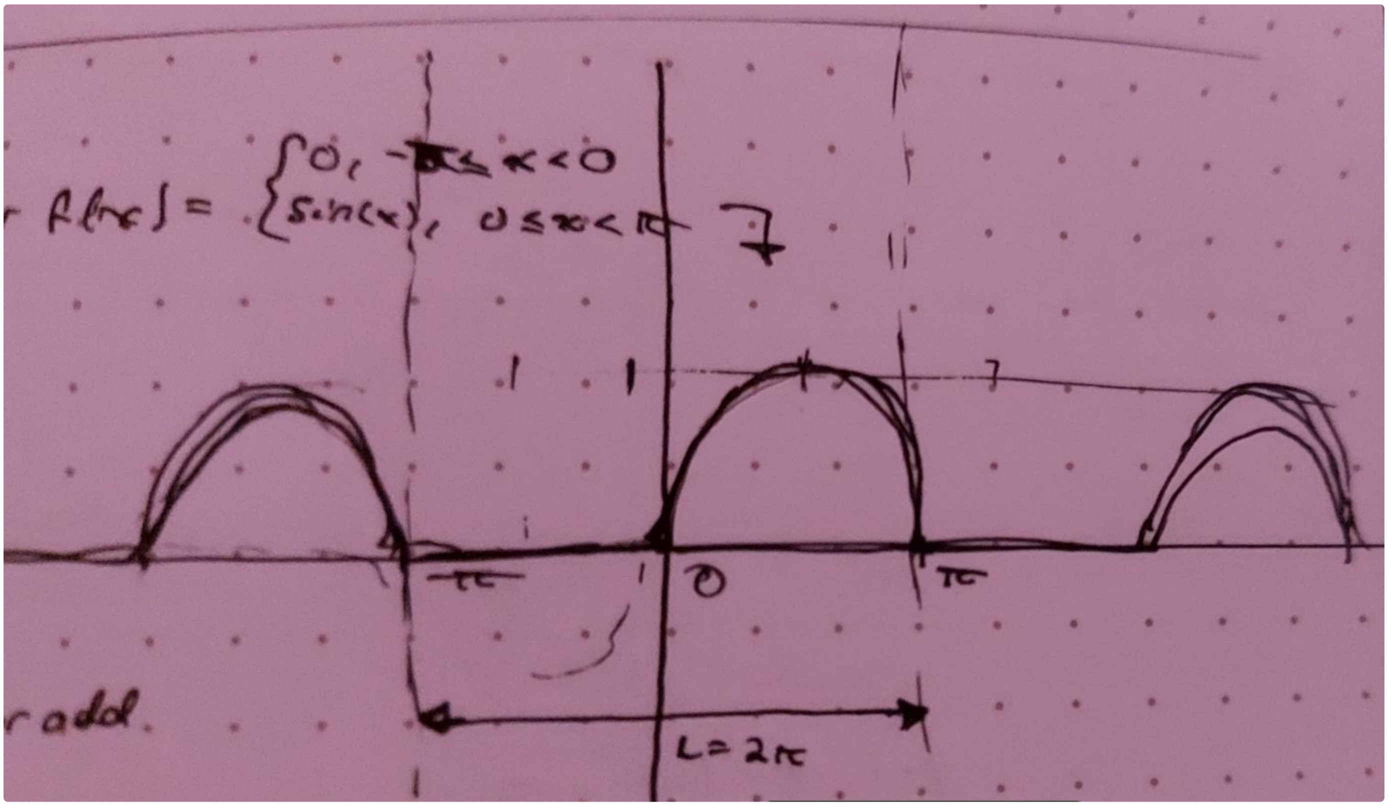
$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi \leq x \leq 0 \\ \sin(x) & \text{if} \quad 0 \leq x \leq \pi \end{cases}$$

- a) Plot $f(x)$ and comment on its parity,
- b) Calculate the Fourier representation of $f(x)$
- c) Evaluate $f(x)$ and its series representation at $x = 0$ to show that

$$\sum_{n=1,2,3,\ldots}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

- d) Show directly (via partial fractions) that the infinite sum in part c) sums to $\frac{1}{2}$.

Solution:



The function is neither even nor odd, as it does not satisfy $f(x) = f(-x)$ or $f(x) = -f(-x)$.

We thus need to compute all the coefficients a_n and b_n.

Our period is periodic with $L = 2\pi$. Thus:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos{(nx)}\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \sin x \cos nx\, dx$$

where we have noted that the function is 0 over $-\pi \leq x < 0$ and neglected that part of the integral.

The definite integral resolves to

$$a_n = \frac{1}{\pi} \frac{\cos(n\pi) + 1}{1 - n^2}$$

We can note that, for even n, this evaluates to

$$\frac{2}{\pi} \frac{1}{1 - n^2}$$

and for odd n it is simply zero,, so we can replace n with $(2n)$ in the Fourier series.

For b_n, we set up the expression:

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx$$

We can note from the orthogonality conditions on the sin terms that, for $n \neq 1$, this will be zero. For $n = 1$, we have the integral of $\sin^2 x$ which is familiar - the above integral evaluates to $\frac{\pi}{2}$, leaving us with

$$b_n = \begin{cases} 0, & n \neq 1 \\ \frac{1}{2}, & n = 1 \end{cases}$$

So the Fourier series is:

$$f(x) = \frac{1}{\pi} + \left[\sum_{n=1}^{\infty} \frac{2}{\pi}\frac{1}{1-(2n)^2}\cos((2n)x)\right] + \frac{1}{2}\sin(1x)$$

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos((2n)x)}{1-4n^2}\right) + \frac{1}{2}\sin(x)$$

At $x = 0$, the $\sin$ terms go to zero and the $\cos$ term becomes 1, leaving

$$0 = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{1-4n^2}\right)$$

and rearranging we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-4n^2}\right) = -\frac{\pi}{2\pi}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-4n^2}\right) = \frac{1}{2}$$

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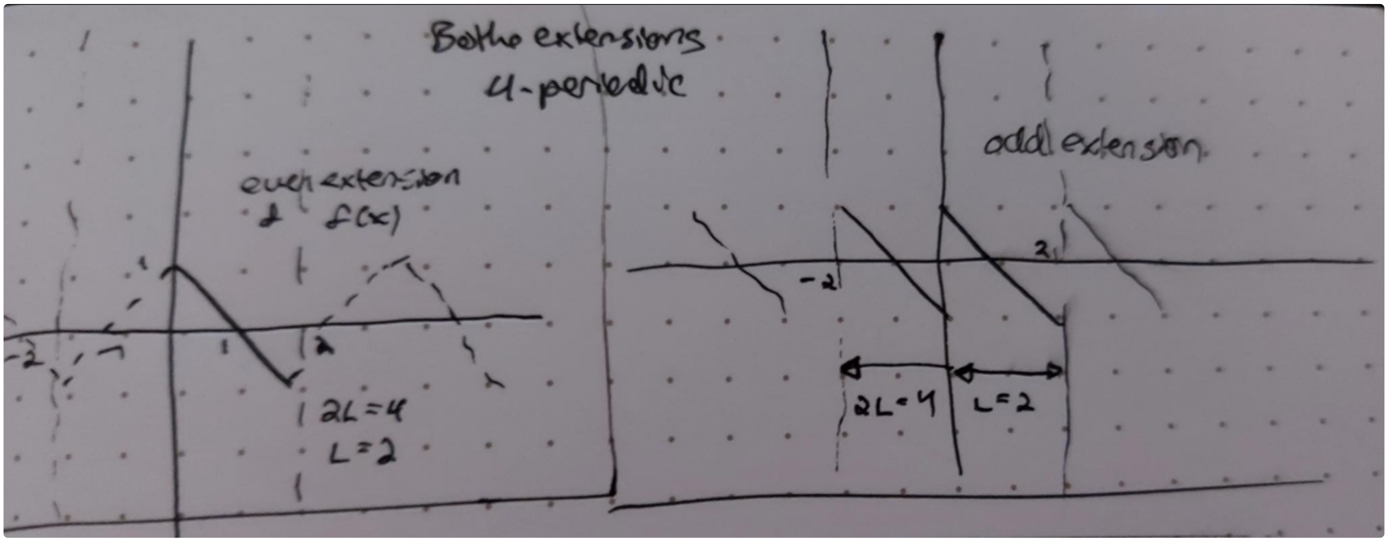
3. Consider the non-periodic function $f(x) = 1 - x$ over the interval $0 \leq x \leq 2$.

- a) Construct an odd periodic extension of $f(x)$ for the interval $-2 \leq x \leq 2$ and plot the function.
- b) Evaluate the Fourier series for the odd periodic extension in part a).
- c) Now construct an *even* periodic extension of $f(x)$ and plot the function.
- d) Evaluate the Fourier series for the even periodic extension in part c).
- e) Examine the two series for the even and odd extensions and comment on their degree of convergence and the dependence on the shape of the original functions, i.e., which series converges more slowly and why?

Solution:



The odd periodic extension of $f(x)$ is

$$f_{odd}(x) = \begin{cases} 1 - x, & 0 \leq x < 2 \\ -x - 1, & -2 \leq x < 0 \end{cases}$$

For the Fourier series of the odd extension, we need only the b_n terms:

$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{2\pi n x}{L}\right)dx$$

$$b_n = \int_{-2}^{0} (x-1)\sin(\pi n x)\,dx + \int_{0}^{2} (1-x)\sin(\pi n x)\,dx$$

This ends up being

$$b_n = \begin{cases} \frac{2(1-\cos(\pi n))}{\pi n}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

So the Fourier series for the odd extension is

$$f(x) = \sum_{n=1}^{\infty} \frac{(1-\cos(\pi n))}{\pi n} \sin(n\pi x)$$

For the even extension we construct a Fourier sine series, using only a_n:

$$a_n = 1nt_0^2(1-x)\cos(n\pi x)\,dx$$

(where we have used the property of even functions that we can multiply by two and then only integrate over half the domain). We can use integration by parts, setting $u = (1-x)$, $dv = \cos n\pi x dx$, we find that the above resolves to

$$a_n = \frac{2\cos(n\pi + \pi)}{\pi n} = \frac{2(-1)^n}{\pi n}$$

and so the Fourier series is

$$f_{odd}(x) = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n \sin(\pi n x)}{\pi n}$$

From the graphs above, it can be noted that the odd extension contains more discontinuities than the even extension. We have seen that the Gibbs overshoot phenomenon is evident near discontinuities and can never be fully eliminated, even in the limit of infinite terms - therefore, discontinuities in a function ensure the function will never fully converge, and will always converge at a slower rate than functions that are fully continuous, like the even extension.

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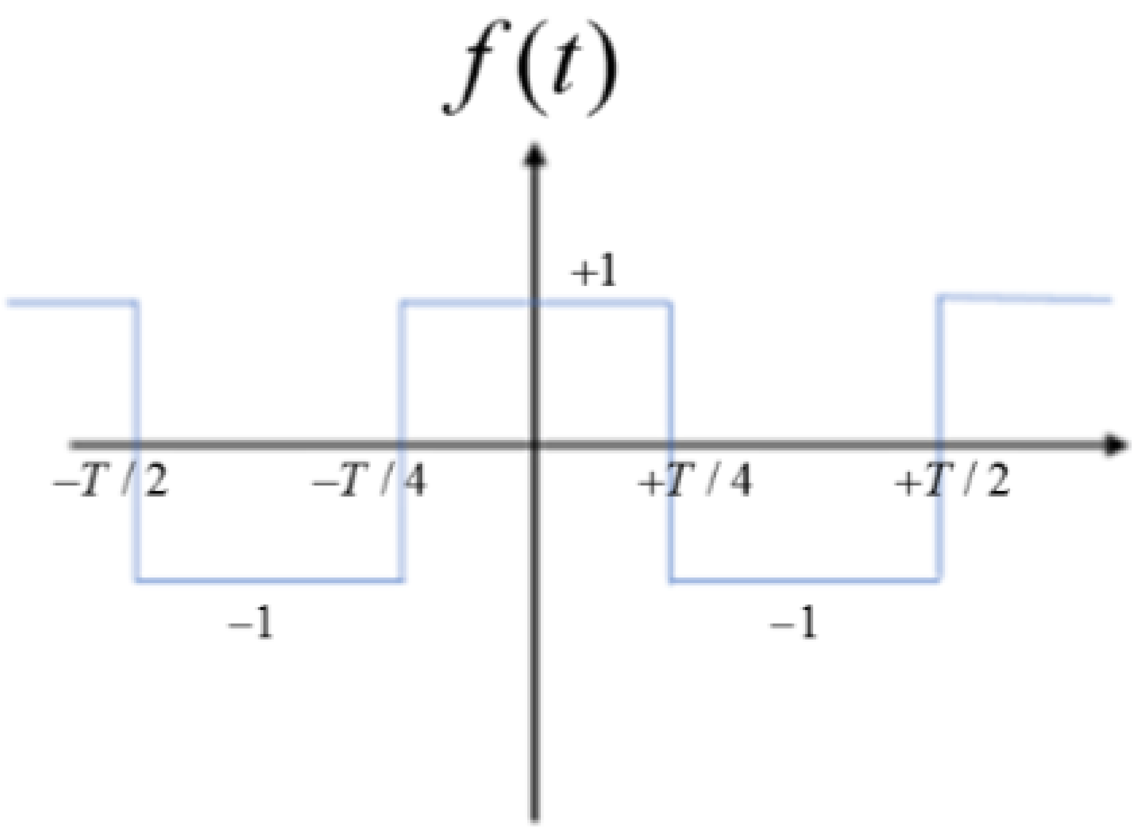
4. Consider the square-wave function $f(t)$ illustrated in the figure below:



- a) Express this square-wave function as a Fourier series.
- b) Compare your results to the shifted square wave studied in lecture 3 and show that the two are equivalent.

Solution:

The function can be described as a piecewise function

$$f(t) = \begin{cases} -1 & \text{for} \quad -\frac{T}{2} \le t < -\frac{T}{4} \\ +1 & \text{for} \quad -\frac{T}{4} \le t < \frac{T}{4} \\ -1 & \text{for} \quad \frac{T}{4} \le t < \frac{T}{2} \end{cases}$$

It can be clearly seen that this function is symmetric about the origin, and so it is an even function. The period is T. We calculate a_n (using immediately the same property of even functions as earlier):

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos\left(\frac{2\pi n t}{T}\right) dt$$

We still need to break up the integral however:

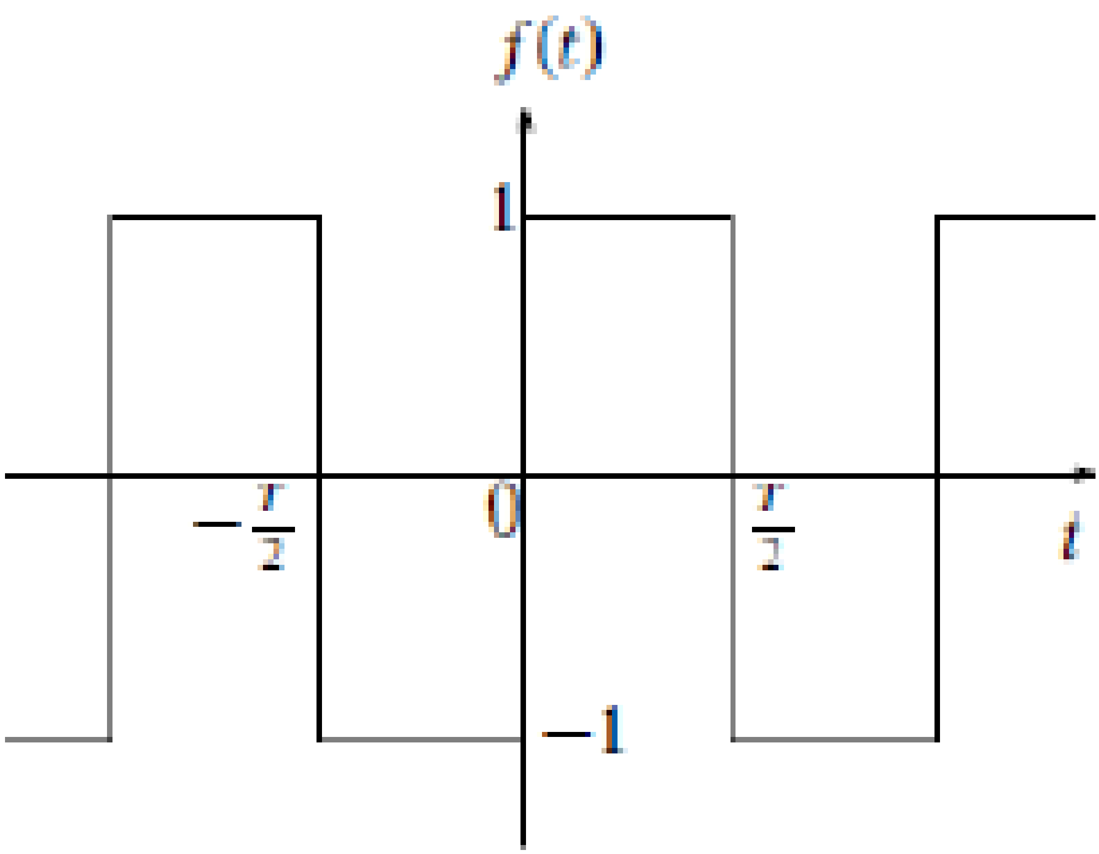
$$a_n = \frac{4}{T} \left(\int_0^{\frac{T}{4}} \cos\left(\frac{2\pi n t}{T}\right) dt + \int_{\frac{T}{4}}^{T/2} (-1) \cos\left(\frac{2\pi n t}{T}\right) dt \right)$$

This evaluates to

$$a_n = \frac{4}{\pi n} \sin\left(\frac{\pi n}{2}\right)$$

We can see that a_0 = 0 - this makes sense visually as the DC level is evidently zero from the given graph.

This function is equivalent to the square-wave studied in the lectures shifted by -$\frac{T}{4}$ - this is self-evidently true if one compares the graphs:



If one shifts the origin of the coordinate system with the transformation $(x, y) \to (x - \frac{T}{4}, y)$, the function is invariant, and thus the functions (and their corresponding Fourier series) are equivalent.