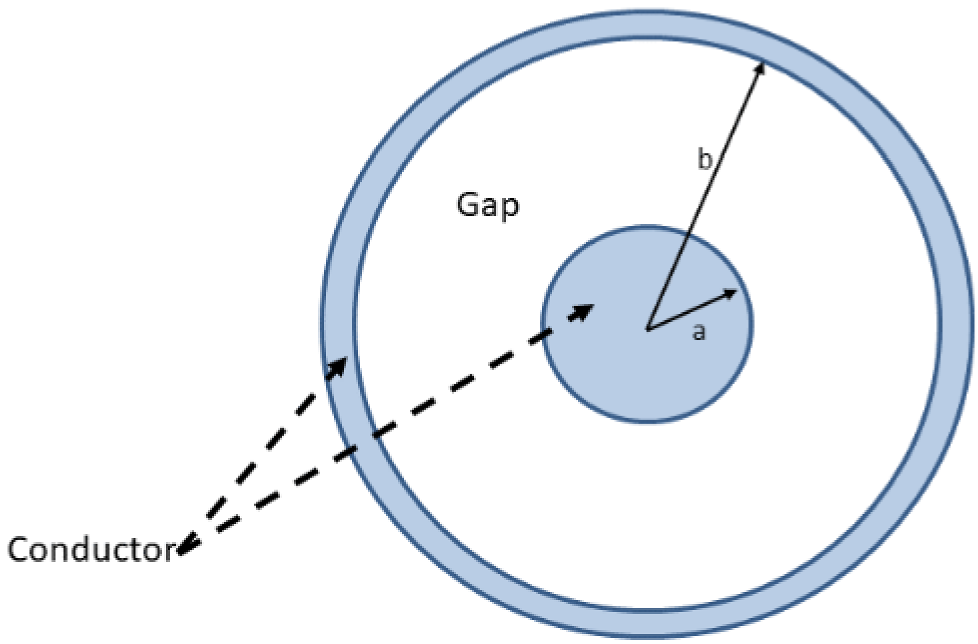
PHYS455 Assignment 3 - Electric Fields II

Max Stronge (30064749)

Q1. There is a spherical capacitor as shown in the figure. The gap is a vaccuum. What is its capacitance?



Solution:

The definition of capacitance is the ratio of the charge on each capacitor to the potential difference between them:

$$C = \frac{Q}{\Delta\phi}$$

We can use Gauss's Law to find the electric field between the conductors by using a Gaussian surface of radius $a < r < b$, which will be that of a point charge of magnitude Q:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

To find the potential difference between $r = a$ and $r = b$, we take the negative of the integral of the electric field from a to b:

$$\Delta\phi = \phi(b) - \phi(a) = -\int_a^b \frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

$$\begin{aligned}\Delta\phi &= -\frac{Q}{4\pi\epsilon_0}\int_a^b r^{-2}\,dr \\ &= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r^2}\right]_a^b \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)\end{aligned}$$

If we substitute this into the original expression for capacitance we find that the charge cancels out and we are left with

$$\begin{aligned}C &= \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)} \\ &= \frac{1}{\frac{1}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right)} \\ &= 4\pi\epsilon_0\left(\frac{ab}{b-a}\right).\end{aligned}$$

Q2. Show that the total bound charge on an isolated dielectric is zero.

Solution:

For an *isolated* dielectric, there is no externally applied electric field, so there is no induced polarization from any external source. There may, however, be some net polarization due to the intrinsic geometry of the molecular constituents of the dielectric material, so we cannot claim that $\vec{P} = 0$ simply because the dielectric is isolated. We do not know that it is homogenous

The total bound charge density in the dielectric is the sum of the bound volume charge density ρ_b and any bound surface charge density σ_b.

$$\text{Total bound charge density } = \rho_b + \sigma_b$$

We can give the bound volume charge density as the negative divergence of the polarization vector $\vec{P}$:

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

Again, $\vec{P}$ may be nonzero due to the molecular properties of the material. We also have an expression for the bound surface charge density:

$$\sigma_b = \vec{P} \cdot \hat{n}$$

where $\hat{n}$ is the unit normal vector pointing out of the surface of the dielectric.

To find the total bound charge, then, we integrate these expressions over the volume and surface of the dielectric, respectively:

$$\sum Q_b = -\int_V \left(\vec{\nabla} \cdot \vec{P}\right) dV + \int_S \left(\vec{P} \cdot \hat{n}\right) dA$$

where the second integral is carried over the closed surface S that represents the outer surface of the dielectric. Because $\hat{n}$ is everywhere perpendicular to this surface we can rewrite the above expression as

$$\sum Q_b = -\int_V \left(\vec{\nabla} \cdot \vec{P}\right) dV + \int_S \vec{P} \cdot\, dA$$

This looks very similar to the divergence theorem, and indeed using the divergence theorem we can say that the integral of the divergence of the polarization over the volume of the dielectric

$$\int_V \left(\vec{\nabla} \cdot \vec{P}\right) dV = \int_S \vec{P} \cdot\, dA$$

and therefore the magnitude of bound volume charge and bound surface charge are equal:

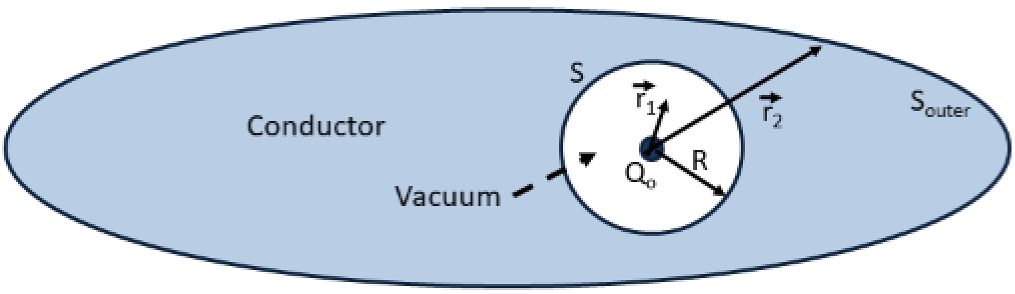
$$\int_S \sigma_b \, dA = \int_V \rho_b \, dV$$

and therefore

$$\sum Q_b = -\int_V \rho_b \, dV + \int_S \sigma_b \, dA = 0$$

and there is zero bound charge on the isolated dielectric.

Q3: There is a spherical vacuum cavity of radius R inside a conductor. There is a charge Q_0 at the center of the cavity, which is located at the origin. The point $\vec{r}_1$ is inside the cavity, and $\vec{r}_2$ is a point on the outer surface of the conductor.



What is the potential difference $\Delta\phi = \phi(\vec{r}_2) - \phi(\vec{r}_1)$?

Solution:

We can use Gauss's Law to investigate the electric field both inside the vacuum cavity and inside the conductor. We can quickly find the electric field (and thus the potential) at $\vec{r}_1$, because the cavity is a vacuum, a Gaussian sphere with radius $|\vec{r}_1|$ gives us the electric field magnitude (noting the spherical symmetry at this point inside the cavity)

$$E(\vec{r}_1) = \frac{1}{4\pi\epsilon_0} \frac{Q_0}{|\vec{r}_1|^2}$$

and thus the potential at $\vec{r}_1$ is

$$\phi(\vec{r}_1) = \frac{1}{4\pi\epsilon_0} \frac{Q_0}{|\vec{r}_1|}$$

Because conductors are equipotentials (i.e. there is no electric field with in the conductor to cause variation in the potential), the potential on the outer surface will be identical to the potential on the inner surface, which is much easier to calculate due to the simpler geometry and spherical symmetry of the inner surface.

The radius of the hollow cavity is R, so we have

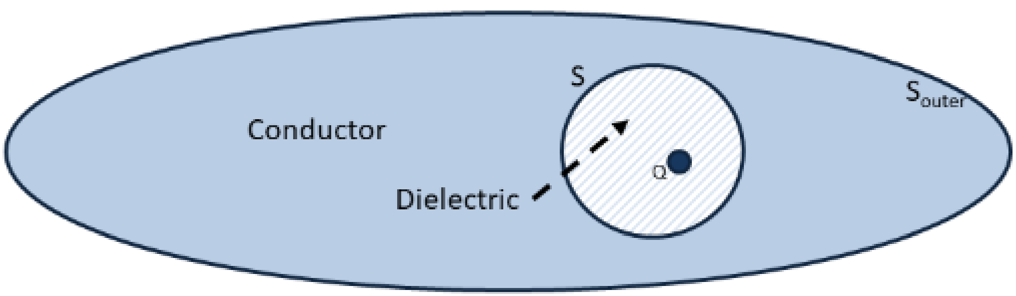
$$\phi(\vec{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{Q_0}{R}$$

and so the potential difference is

$$\Delta\phi = \phi(\vec{r}_2) - \phi(\vec{r}_1) = \frac{Q_0}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{|\vec{r}_1|}\right)$$

for any point $\vec{r}_2$ on the outer surface of the conductor.

Q4: There is a cavity inside of a conductor. The cavity is filled with a linear, isotropic, homogenous dielectric with electric susceptibility χ_e. There is a free charge Q fixed in place inside the dielectric, as shown.



Show that except for at the location of Q, and at the outer boundary of the dielectric, the volume bound charge density is zero.

Solution:

Because of the presence of the dielectric, we do not know the entirety of the charge distribution - the charge Q will produce an electric field which induces polarization in the dielectric, and this will result in some bound charge somewhere in the system. With the assumption that the dielectric is linear, isotropic, and homogenous (and we are told it is) the polarization is given by

$$\vec{P} = \chi_e\epsilon_0\vec{E}$$

where χ_e is some constant not dependent on position.

We can investigate the free charge using Gauss's Law in the follwoing form:

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

where $\vec{D}$ is the electric displacement, given in this case by

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Therefore, rewriting $\vec{P}$ in terms of the electric susceptibility and $\vec{E}$ as above, we can show

$$\begin{aligned}\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E}\right) &= \rho_f \\ \vec{\nabla} \cdot \epsilon_0 \left(1 + \chi_e\right) \vec{E} &= \rho_f \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho_f}{\epsilon_0 (1 + \chi_e)}\end{aligned}$$

We also have the relation between the bound charge and the polarization

$$\vec{\nabla} \cdot \vec{P} = -\rho_b \rightarrow \vec{\nabla} \cdot \left(\chi_e \epsilon_0 \vec{E}\right) = -\rho_b$$

which we can rearrange into

$$-\chi_e \vec{\nabla} \cdot \vec{E} = \frac{\rho_b}{\epsilon_0}$$

which we can combine with our result above to find

$$-\chi_e \frac{\rho_f}{\cancel{\epsilon_0} (1 + \chi_e)} = \frac{\rho_b}{\cancel{\epsilon_0}}$$

or

$$\rho_b = -\frac{\chi_e}{1 + \chi_e} \rho_f$$

and so anywhere in the dielectric where the free charge density ρ_f is zero, the bound volume charge density will also be zero at that point. The only location that the free charge density is nonzero is at the exact location of the free charge Q.

We can use a similar argument to show that there will be a very thin layer of nonzero bound charge at the boundary of the dielectric - a regular Gaussian analysis with a sphere embedded in the conductor reveals that the electric field is zero in a conductor, as usual, and thus there must be no net charge enclosed, therefore there must be an induced negative charge on the inner surface of the conductor - the free charge of the conductor rearranged itself to ensure that the field inside the conductor is zero.

Thus we will have a surface bound charge density on the outer surface of the dielectric:

$$\sigma_b = -\frac{\chi_e}{1+\chi_e}\sigma_f$$

where σ_f is the free charge redistributed on the surface of the conductor.

To satisfy Gauss's law completely, the free charge on the inner surface of the conductor must equal the free charge inside the cavity, and the bound charge at the surface of the dielectric (positive) and at the point of Q (negative) must cancel entirely. Everywhere else there is zero bound charge.

Q5:

Answered in the above analysis as I thought it was part of that question.

Q6:

Via Gauss's Law, there must be an induced charge of $-Q$ on S (to satisfy Gauss's Law and have zero $\vec{E}$ in the conductor). Thus to remain electrically neutral there must be an induced charge of Q on S_{outer}, which also satisfies Gauss's Law (a Gaussian surface enclosing the entire conductor would then have total enclosed charge $Q - Q + Q = Q$.)

Q7:

Again as above:

$$\sigma_b = -\frac{\chi_e}{1+\chi_e}\sigma_f$$

and so the free charge on S is given by

$$\sigma_f = -\frac{1+_{\chi_e}}{\chi_e}\sigma_b$$