Introductory Real Analyis Exercises

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Chaper 1.3 Exercises

Problem 1

Exhibit both a partial ordering and a simple ordering of the set of all complex numbers.

There is the trivial partial ordering

$$a \leq b \Leftrightarrow a = b,$$

a kind of partial ordering by rays from the origin where nonzero points on different rays are incomparable, and zero is the minimal element

$$a \leq b \Leftrightarrow a = sb, s \in [0, 1],$$

another ordering that compares numbers equidistant to the origin according to their argument

$$a \leq b \Leftrightarrow |a| = |b| \wedge \mathrm{arg(a)} \leq \mathrm{arg(b)},$$

and one last example

$$a \leq b \Leftrightarrow \Re(a) = \Re(b) \wedge \Im(a) \leq \Im(b).$$

Finally, as an example of a simple ordering, consider

$$a \leq b \Leftrightarrow \Im(a) < \Im(b) \vee \left(\Im(a) = \Im(b) \wedge \Re(a) \leq \Re(b)\right).$$

Problem 2

What is the minimal element of the set of all subset of a given set X, partially ordered by set inclusion? What is the maximal element?

In $(2^M, \subset)$ a minimal element is $\emptyset$ and a maximal element is M itself. To see that the maximal element is unique suppose that there is another maximimal element M' and construct $S = M \cup M'$. Then $M \subset S$ and $M' \subset S$ together imply $M = S = M'$. A similar argument shows $\emptyset$'s uniqueness.

Problem 3

A partially ordered set M is said to be a *directed set* if, given any two elements $a, b \in M$, there is an element $c \in M$ such that $a \leq c$, $b \leq c$. Are the partially ordered sets in Examples 1-4, Sec 3.1 all directed sets?

1. The trivial partial ordering on a set is directed. Any element a is comparable only to itself, so this is not directed.

2. The continuous functions on $[\alpha, \beta]$, ordered by $f \leq g \Leftrightarrow f(t) \geq g(t)$ for all $t \in [\alpha, \beta]$. Given two continuous functions f and g we have $f(t) \leq \mathrm{max(f(t), g(t))}$ and similar for g, so this set is directed.

3. The subsets of a set M, ordered by inclusion. For $A, B \in 2^M$, $A \subset M$ and $B \subset M$, so this is a directed set.

4. The integers greater than 1, ordered by divisibility. For numbers $a, b > 1$, we have $a \leq ab$ and $b \leq ab$, so this is a directed set.

Problem 4

Prove that the set of all subsets of a given set X, ordered by set inclusion, is a lattice. What is the set theoretic meaning of the greatest lower bound and least upper bound of two elements of this set?

Let A and B be subsets of X.

a) I claim that the least upper bound of A and B under inclusion is $L = A \cup B$. Suppose there is another least upper bound L'. Then $A \subset L'$ and $B \subset L'$, which means that $A \cup B \subset L'$ and thus L' must be equal to L or else it would be greater.

b) I also claim that $G = A \cap B$ is the greatest lower bound of A and B. To show this suppose there is some other greatest lower bound G'. Then $G' \subset A$ and $G' \subset B$ but then $G' \subset G$ so $G' = G$.

Thus the greatest lower bound and lowest upper bound for two subsets of a set are exactly the operations of set intersection and union, respectively.

Problem 5

Prove that an order preserving mapping of one ordered set onto another is automatically an isomorphism.

Since both sets are ordered then each element a is comparable to any other element b. But then for any $f(a)$ and $f(b)$ where $f(a) \leq f(b)$ we must have that $a \leq b$, since if $b \leq a$ then f would not be order preserving.

Problem 6

Prove that ordered sums and products of ordered sets are associative.

Let A, B and C be disjoint ordered sets. Then the order relation on $(A + B) + C$ is

$$a \leq b \Leftrightarrow \ (a \leq_A b) \vee (a \leq_B b) \vee (a \in A \wedge b \in B) \vee \ (a \leq_C b) \vee (a \in A \cup B \wedge b \in C).$$

On $A + (B + C)$ the relation is

$$a \leq b \Leftrightarrow (a \leq_A b) \vee (a \leq_B b) \vee (a \leq_C b) \vee (a \in B \wedge b \in C) \vee (a \in A \wedge a \in B \cup C).$$

But it's easy to see that both of these can be turned into

$$a \leq b \Leftrightarrow \ (a \leq_A b) \vee (a \leq_B b) \vee \ (a \leq_C b) \vee (a \in A \wedge b \in B) \vee (a \in A \wedge b \in C) \vee (a \in B \wedge b \in C).$$

To show that the ordered product is associative, let A, B, C again be disjoint ordered sets. The order relation on $(A \times B) \times C$ is

$$(a_1, a_2, a_3) \leq (b_1, b_2, b_3) \Leftrightarrow (a_1 \leq b_1) \vee (a_1 = a_2 \wedge b_1 \leq b_2) \vee ((a_1, a_2) = (b_1, b_2) \wedge a_3 \leq b_3).$$

For $A \times (B \times C)$ the relation is

$$(a_1, a_2, a_3) \leq (b_1, b_2, b_3) \Leftrightarrow (a_1 \leq b_1) \vee (a_1 = b_1 \wedge (a_2 \leq b_2 \vee (a_2 = b_2 \wedge a_3 \leq b_3))).$$

But these both expand to

$$(a_1, a_2, a_3) \leq (b_1, b_2, b_3) \Leftrightarrow (a_1 \leq b_1) \vee (a_1 = b_1 \wedge a_2 \leq b_2) \vee (a_1 = b_1 \wedge a_2 = b_2 \wedge a_3 \leq b_3).$$

Problem 7

Construct well ordered sets with ordinals

$$\omega + n, \omega + \omega, \omega + \omega + n, \omega + \omega + \omega, \ldots$$

The set $\{n+1, n+2, \ldots\} + \{1, 2, \ldots, n\}$ is of the first order type.
The set $\{1, 3, 5, \ldots\} + \{2, 4, 6, \ldots\}$ is of the second order type.
In general $\omega + \omega + \ldots + \omega = m \times \omega$. We can construct a set with ordinal $m \times \omega + n$ as

$$\{1, 1+m, 1+2m, \ldots\} + \{2, 2+m, \ldots\} + \ldots + \{m, 2m, 3m, \ldots\} + \{-1, -2, \ldots, -n\}$$

Problem 8

Construct well ordered sets with ordinals

$$\omega \cdot n, \omega^2, \omega^2 \cdot n, \omega^3, \ldots$$

Show that the sets are all countable.

Problem 9

Show that

$$\omega + \omega = \omega \cdot 2, \omega + \omega + \omega = \omega \cdot 3, \ldots$$

Problem 10

Prove that the set $W(\alpha)$ of all ordinals less that a given ordinal α is well-ordered.

Problem 11

Prove that any nonempty set of ordinals is well-ordered.

Problem 12

Prove that the set M of all ordinals corresponding to a countable set is itself uncountable.

Suppose that M is countable. Then the M's ordinal α must be a memeber of M.

Problem 13

Let $\aleph_1$ be the power of the set M in the preceding problem. Prove that there is no power m such that $\aleph_0 < m < \aleph_1$.