Introductory Real Analyis Exercises

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Chaper 1.1 Exercises

Problem 1

Prove that if $A \cup B = A$ and $A \cap B = A$ then $A = B$.

To show this we observe that $A \cup B = A$ implies $B \subset A$, while $A \cap B = A$ implies $A \subset B$. But then we have the definition of equality of two sets and $A = B$.

Problem 2

Show that in general $(A - B) \cup B \neq B$.

Suppose that A is not a subset of B. Then $(A - B)$ contains at least one element contained in A alone. Thus the expression $(A - B) \cup B \neq B$ is valid in the general case.

Problem 3

Let $A = \{2, 4, ..., 2n, ...\}$ and $B = \{3, 6, ..., 3n, ...\}$. Find $A \cap B$ and $A - B$.

$$A \cap B = \{a \in \mathbb{N} \mid (2 \mid a \wedge 3 \mid a)\} = \{6, 12, ..., 6n, ...\}$$

$$A - B = \{a \in \mathbb{N} \mid (2 \mid a \wedge 3 \nmid a)\} = \{2, 4, 8, 10, ..., 6n - 4, 6n - 2\}$$

Problem 4

Prove that

a) a) $(A - B) \cap C = (A \cap C) - (B \cap C))$

b) $A \triangle B = (A \cup B) - (A \cap B)$

a) We use the notation AB to denote intersections and $A + B$ to denote unions, and A^c to denote the complement of A relative to some set Ω which is a superset of the sets in our discourse. Then the difference between two sets may be written as an intersection, $A - B = AB^c$. Then

$$(A - B) \cap C = AB^cC = AB^cC + \emptyset = ACB^c + ACC^c = AC(B^c + C^c) = AC(BC)^c = (A \cap C) - (B \cap C)$$

b) To show this we state the definition $A \triangle B = AB^c + A^cB$. Then

$$(A \cup B) - (A \cap B) = (A + B)(AB)^c = (A + B)(A^c + B^c) = AA^c + AB^c + A^cB + BB^c = AB^c + A^cB = A \triangle B$$

Problem 5

Prove that

$$\bigcup_\alpha A_\alpha - \bigcup_\alpha B_\alpha \subset \bigcup_\alpha (A_\alpha - B_\alpha)$$

Problem 5 continued on next page...

First we rewrite the problem as

$$\left(\bigcup_{\alpha} A_\alpha\right) \cap \left(\bigcup_{\alpha} B_\alpha\right)^c \subset \bigcup_{\alpha} (A_\alpha \cap B_\alpha^c)$$

Through deMorgan's law and the distributive property we see that

$$\left(\bigcup_{\alpha} A_\alpha\right) \cap \left(\bigcup_{\alpha} B_\alpha\right)^c = \bigcup_{\alpha} \left(A_\alpha \ \cap \ \bigcap_{\alpha} B_\alpha^c\right).$$

Here it's clear that $\bigcap_\alpha B_\alpha^c \subset B_\alpha^c$ for each, α. But then $A_\alpha \cap (\bigcap_\alpha B_\alpha^c)$ is a subset of $A_\alpha \cap B_\alpha^c$ for each α, proving the formula.

Problem 6

Let A_n be the set of all positive integers divisible by n. Find the sets

a) $\displaystyle\bigcup_{n=2}^{\infty} A_n$

b) $\displaystyle\bigcap_{n=2}^{\infty} A_n$.

a) Our set here is the set of positive integers which are divisible by any integer $n \geq 2$, which is $\mathbb{N} \setminus \{1\}$.

b) Here the set contains positive integers which must be divisible by every integer $n \geq 2$. Such integers would be divisible by an infinite number of primes, which is absurd, so the answer is $\emptyset$.

Problem 7

Find the sets

a) $\displaystyle\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$

b) $\displaystyle\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$

a) Since $\left[a + \frac{1}{n}, b - \frac{1}{n}\right] \subset \left[a + \frac{1}{m}, b - \frac{1}{m}\right]$ if $n \leq m$, it suffices to consider

$$\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = \lim_{n\to\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = (a, b)$$

.

b) Similarly, here $\left(a - \frac{1}{n}, b + \frac{1}{n}\right) \supset \left(a - \frac{1}{m}, b + \frac{1}{m}\right)$ if $n \leq m$, so

$$\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = \lim_{n\to\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = [a, b]$$

Problem 8

Let A_α be the set of points lying on the curve

$$y = \frac{1}{x^\alpha} \qquad (0 < x < \infty).$$

What is

$$\bigcap_{\alpha \geq 1} A_\alpha \ ?$$

We consider any two curves A_α and A_β, where, $\alpha \neq \beta$, and $\alpha, \beta \geq 1$. A point $(a, a^{-\alpha}) \in A_\alpha$ is a member of A_β only if $a^{-\alpha} = a^{-\beta}$. But then we must have $a = 1$, and so our result is

$$\bigcap_{\alpha \geq 1} A_\alpha = (1, 1).$$

Problem 9

Let $y = f(x) = \langle x \rangle$ for all real x, where $\langle x \rangle$ is the fractional part of x. Prove that every closed interval of length 1 has the same image under f. What is this image? If f one-to-one? What is the pre-image of the interval $\frac{1}{4} \leq y \leq \frac{3}{4}$? Partition the real line into classes of points with the same image.

We choose an interval $I = [a, a+1]$ and partition I into two intervals, $I = L \cup R$, where

$$L = [a, a + 1 - \langle a \rangle)$$

$$R = [a + 1 - \langle a \rangle, a + 1].$$

We observe that $f(L) = [\langle a \rangle, 1)$ and $f(R) = [0, \langle a \rangle]$. In other words the image of the unit interval is

$$f(I) = f(L \cup R) = f(L) \cup f(R) = [0, 1).$$

Since every unit interval maps to $[0, 1)$, $f : \mathbb{R} \to [0, 1)$ is not one-to-one. We have that

$$f^{-1}\left[\frac{1}{4}, \frac{3}{4}\right] = \bigcup_{n \in \mathbb{Z}} \left[\frac{1}{4}, \frac{3}{4}\right].$$

In general we can partition the real line into a family of sets A_α, where $\alpha \in [0, 1)$, such that

$$A_\alpha = \{a \mid a = \alpha + n, \ n \in \mathbb{Z}\}$$

Problem 10

Given a set M, let $\mathcal{R}$ be the set of all ordered pairs of of the form (a, a) where $a \in M$, and let aRb if and only if $(a, b) \in \mathcal{R}$. Interpret the relation R.

An interpretation of R is simple, since it is just the equality relation restricted to the elements of the set M. To see this we note that $(a, b) \in \mathcal{R}$ if and only if $a = b$.

Problem 11

Give and example of a binary relation which is

a) Reflexive and symmetric, but not transitive;

b) Reflexive, but neither symmetric nor transitive;

c) Symmetric, but neither reflexive nor transitive;

d) Transitive, but neither reflexive nor symmetric.

a)
$$aRb \Leftrightarrow \mathrm{gcd(a, b)} \neq 1$$

R is reflexive; $\mathrm{gcd(a, a)} = \mathrm{a} \neq 1$. The rule $\mathrm{gdc(a, b)} = \mathrm{gdc(b, a)}$ implies that we have symmetry. Finally, for the transitive property we see that for example $20R6$ and $6R21$ together do not imply $20R21$.

b)
$$aRb \Leftrightarrow a \leq b \vee a = 0 \vee b = 0$$

R is reflexive and non symmetric by definition. R is not transitive since we have that $aR0$ and $0Rb$ for any $a > b > 0$, while aRb does not hold.

c)
$$aRb \Leftrightarrow \mathrm{gcd(a, b)} = 1$$

This relation is the complement of a) so it is not reflexive, while preserving symmetry. Consider numbers a, b such that $\mathrm{gcd(a, b)} = 1$. Since aRb we have bRa, but aRa is false, so R is not transitive.

d) The relation is $>$ on the real numbers. It's only transitive and neither reflexive nor symmetric.