Pseudocode

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July 2024

Algorithm 0.1: MCMC Algorithm

input: observations $(x_i(s_0), \ldots, x_i(s_M))_{i=1,\ldots,n_{\text{obs}}}$, risk functional ℓ, threshold u, parametric model,
prior π, proposal $q(\cdot,\cdot)$, chain length n_{MCMC}, fine grid points $t_0, \ldots, t_N$
output: Markov chain of parameters of length n_{MCMC}
begin
  initialize parameter θ_0 from prior π
  for $n \in \{1, \ldots, n_{\text{MCMC}}\}$
    sample $\ell(X_i) \mid X_i(s_0) = x_i(s_0), \ldots, X_i(s_M) = x_i(s_M)$ under parameter θ_{n-1}
    using Algorithm 2
    set $I = \{i \mid \ell(X_i) > u\}$
    sample $\theta_{\text{prop}} \sim q(\theta_{n-1}, \cdot)$
    sample $r \sim \text{Unif}[0,1]$
    if $r < \alpha(\theta_{n-1}, \theta_{\text{prop}})$
      set $\theta_n = \theta_{\text{prop}}$
    else
      set $\theta_n = \theta_{n-1}$
    end if
  end for
  return $\theta_0, \ldots, \theta_{n_{\text{MCMC}}}$
end

Algorithm 0.2: Cond-X Algorithm

input: observations $(x_i(s_0), \ldots, x_i(s_M))_{i=1,\ldots,n_{\text{obs}}}$, risk functional ℓ, fine grid points $t_0, \ldots, t_N$,
parametric model with parameter θ, chain length n_{condX}
output: sample $\ell(X_i) \mid X_i(s_0) = x_i(s_0), \ldots, X_i(s_M) = x_i(s_M)$ under parameter θ
begin
  for $i \in \{1, \ldots, n_{\text{obs}}\}$
    initialize $w \sim W \mid W(s_1) = \frac{x_i(s_1)}{x_i(s_0)}, \ldots, W(s_M) = \frac{x_i(s_M)}{x_i(s_0)}$ under parameter θ
    for $n \in \{1, \ldots, n_{\text{condX}}\}$
      sample $w_{\text{prop}} \sim W \mid W(s_1) = \frac{x_i(s_1)}{x_i(s_0)}, \ldots, W(s_M) = \frac{x_i(s_M)}{x_i(s_0)}$ under parameter θ
      sample $r \sim \text{Unif}[0,1]$
      if $r < \alpha(w, w_{\text{prop}}) = \min\left\{\frac{\ell^\alpha(w_{\text{prop}})}{\ell^\alpha(w)}, 1\right\}$
        set $w = w_{\text{prop}}$
      end if
    end for
    set $w_i = w$
  end for
  return $\ell(X_i) = x_i(s_0) \cdot \ell(w_i)$
end

Algorithm 0.3: CondGaussian Simulation Algorithm

input: observations $(x_i(s_0), \ldots, x_i(s_M))_{i=1,\ldots,n_{\text{obs}}}$, fine grid points $t_0, \ldots, t_N$,
parametric model with parameter θ
output: sample $W \mid W(s_1) = \frac{x_i(s_1)}{x_i(s_0)}, \ldots, W(s_M) = \frac{x_i(s_M)}{x_i(s_0)}$ under parameter θ
begin
  sample $w_{\text{uncond}}(t_1), \ldots, w_{\text{uncond}}(t_N)$ unconditional on fine grid with parameter θ via algorithm 4
  set $\Sigma_{ts} = (\gamma_\theta(t_i - s_0) + \gamma_\theta(s_j - s_0) - \gamma_\theta(t_i - s_j))_{i \in \{1,\ldots N\}, j \in \{1,\ldots M\}}$

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    set  Σ_ss = (γ_θ(s_i − s_0) + γ_θ(s_j − s_0) − γ_θ(s_i − s_j))_{i∈{1,...M},j∈{1,...M}}
    return  exp(
    log w_uncond(t_1, ..., t_N) + Σ_ts Σ_ss^{-1} (log (x_i(s_1)/x_i(s_0), ... x_i(s_M)/x_i(s_0)) − log w_uncond(s_1, ..., s_M))
    )
end
```

$$\Sigma_{ss} = (\gamma_\theta(s_i - s_0) + \gamma_\theta(s_j - s_0) - \gamma_\theta(s_i - s_j))_{i\in\{1,\dots M\},j\in\{1,\dots M\}}$$

$$\log w_{\text{uncond}}(t_1,\dots,t_N) + \Sigma_{ts}\Sigma_{ss}^{-1}(\log\left(\frac{x_i(s_1)}{x_i(s_0)},\dots\frac{x_i(s_M)}{x_i(s_0)}\right) - \log w_{\text{uncond}}(s_1,\dots,s_M))$$

Algorithm 0.4: log-Gaussian Simulation Algorithm

input: fine grid points $t_0, \dots, t_N$, parametric model with parameter θ
output: sample from W under parameter θ on fine grid
begin
  sample $g(t_1), \dots, g(t_N) \sim$ standard FBM under parameter θ
  return $w_j = \exp(g(t_j) - g(s_0) - \gamma_\theta(t_j, s_0))$
end

Algorithm 0.5: Acceptance Rate MCMC Algorithm

input: exceedance observations $(x_i(s_0), \dots, x_i(s_M))_{i \in I_{\text{exceed}}}$, risk functional ℓ,
prior π, proposal $q(\cdot, \cdot)$, parametric model with parameter θ_{prop} and θ_{old},
sample size $n_{\text{CondGauss}}$, sample size n_{Gauss}, fine grid points $t_0, \dots, t_N$
output: acceptance rate $\alpha(\theta_{\text{old}}, \theta_{\text{prop}})$
begin
  for $\theta \in \{\theta_{\text{prop}}, \theta_{\text{old}}\}$
    for $i \in I_{\text{exceed}}$
      set $a = \text{logGaussianDensity}(\theta, \left(\frac{x_i(s_1)}{x_i(s_0)}, \dots, \frac{x_i(s_M)}{x_i(s_0)}\right))$
      estimate $b = \mathbb{E}(l(W)^\alpha)$ under parameter θ via n_{Gauss} times algorithm 4
      estimate $c = \mathbb{E}(l(W)^\alpha \mid W(s_1) = \frac{x_i(s_1)}{x_i(s_0)}, \dots, W(s_M) = \frac{x_i(s_M)}{x_i(s_0)}$ under parameter θ
      via $n_{\text{CondGauss}}$ times algorithm 3
      set $\log \text{likelihood}(\theta, i) = \log a - \log b + \log c$
    end for
    set $\log \text{likelihood}(\theta) = \sum_{i \in I_{\text{exceed}}} \log \text{likelihood}(\theta, i)$
  end for
  return $\min\left\{1, \frac{\pi(\theta_{\text{prop}}) q(\theta_{\text{prop}}, \theta_{\text{old}}) \text{ likelihood}(\theta_{\text{prop}})}{\pi(\theta_{\text{old}}) q(\theta_{\text{old}}, \theta_{\text{prop}}) \text{ likelihood}(\theta_{\text{old}})}\right\}$
end