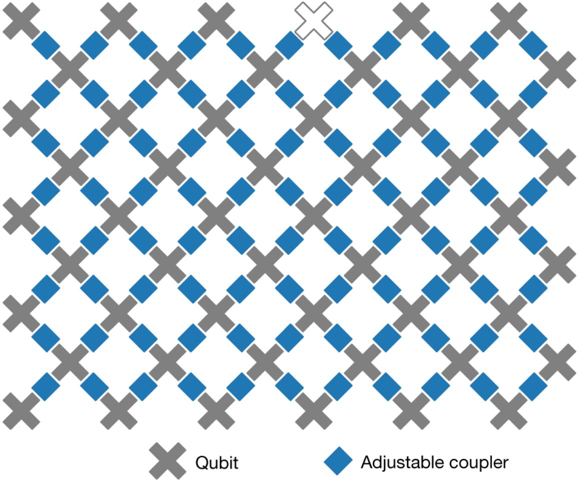
Can we implement good quantum LDPC codes on near-term devices?

Maxime Tremblay[1], Michael Beverland[2], Nicolas Delfosse[2]

[1]Université de Sherbrooke, [2]Microsoft Research

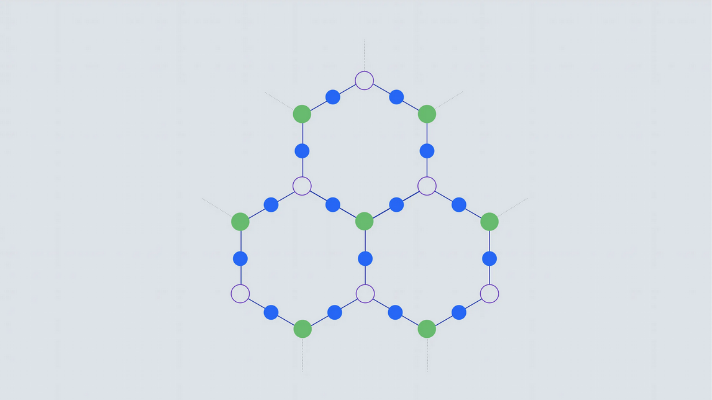
February 4, 2022



Arute et al. Nature 574, 505–510 (2019)

The IBM Quantum heavy hex lattice

As of August 8, 2021, the topology of all active IBM Quantum devices will use the heavy-hex lattice, including the IBM Quantum System One's Falcon processors installed in Germany and Japan.



Near-term quantum computers will be locally connected.

Can we achieve large scale fault-tolerant quantum computing on locally connected devices?

Tradeoffs for reliable quantum information storage in 2D systems

Sergey Bravyi,[1] David Poulin,[2] and Barbara Terhal[1]

[1]IBM Watson Research Center, Yorktown Heights NY 10598, USA
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(Dated: September 11, 2018)

Long range interactions from local operations

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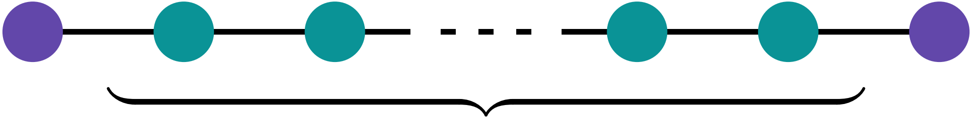
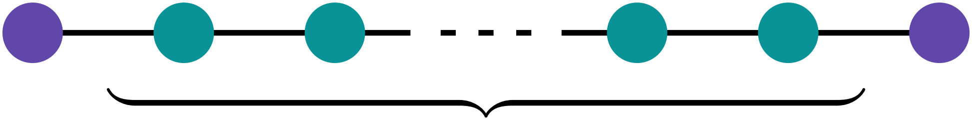


$$|\bar{0}\rangle \equiv |+ + \cdots + +\rangle + |- - \cdots - -\rangle$$

Long range interactions from local operations



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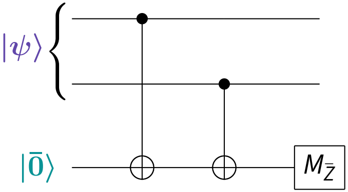

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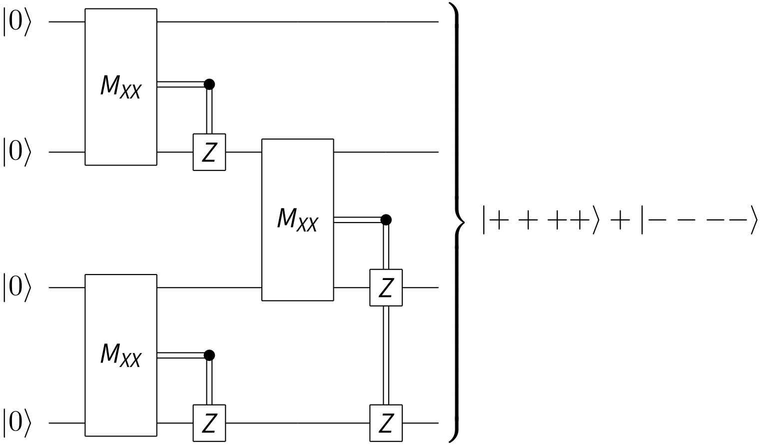

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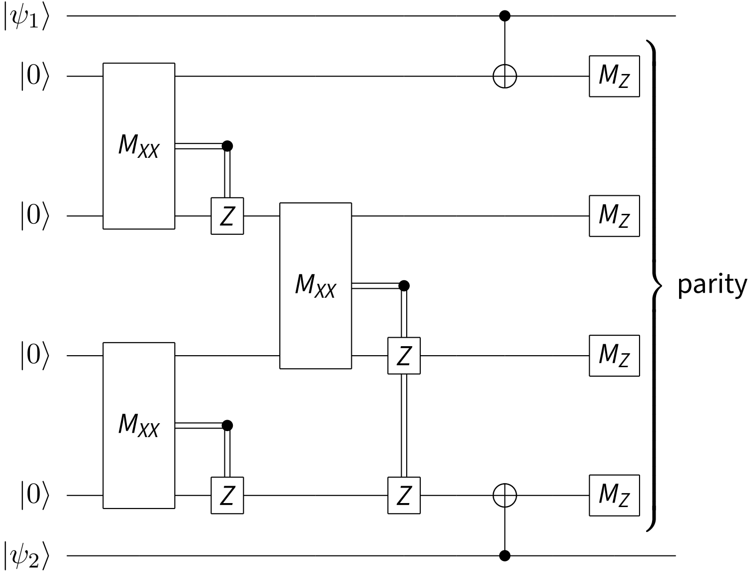

Main results

Theorem

Let C be a Clifford circuit measuring computing Pauli operators $S_1, \ldots, S_r$. Then, for any subset of qubits L, we have

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For families of local-expander quantum LDPC codes of length n, a syndrome extraction circuit C implemented as a local Clifford circuit on a $\sqrt{N} \times \sqrt{N}$ grid of qubits satisfies

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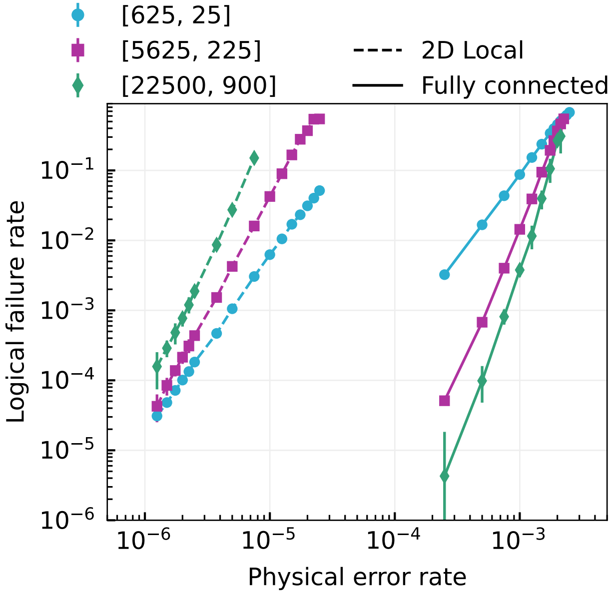

[625, 25]
[5625, 225]
[22500, 900]
2D Local
Fully connected
Logical failure rate
Physical error rate
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-6}
10^{-5}
10^{-4}
10^{-3}

References

- Bounds on stabilizer measurement circuits and obstructions to local implementations of quantum LDPC codes
  arXiv 2109.14599
- Constant-overhead quantum error correction with thin planar connectivity
  arXiv 2109.14609

Outline

1. Background and definitions
2. Proof of the main theorem
3. Circuit implementations
4. Numerical experiments

Background and definitions

Clifford circuit

- Preparations of $|0\rangle$ and $|+\rangle$ and classical bits.

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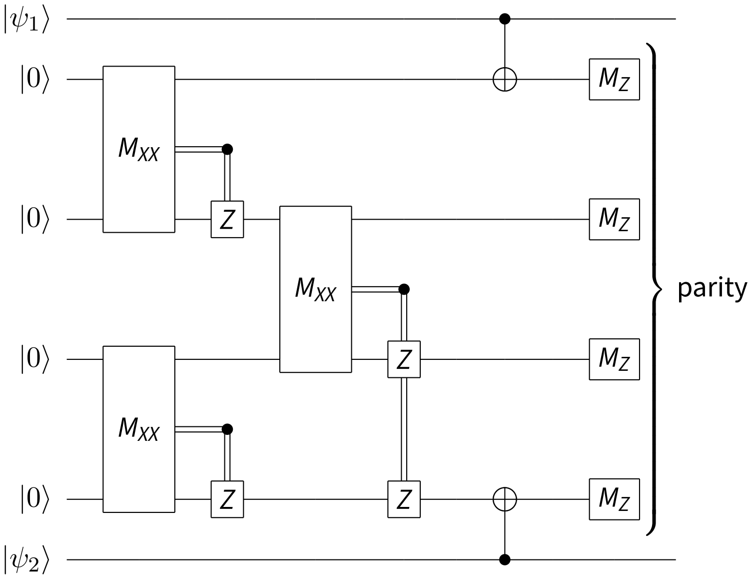
- Ouput the parity of some subsets of measurement outcomes.

Clifford circuit



Local circuit

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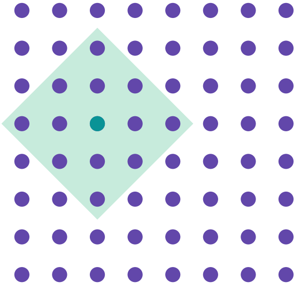
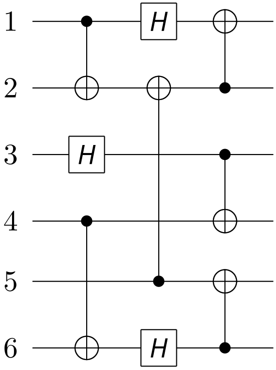


Connectivity graph



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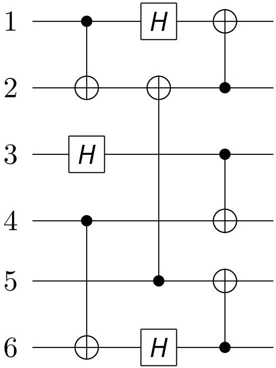

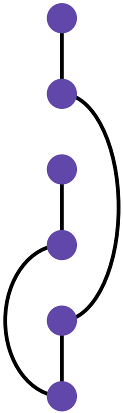

Stabilizer code

Stabilizer group

$$\mathcal{S} = \langle S_1, S_2, \ldots, S_r \rangle \subset \mathcal{P}_n \setminus \{-I\} \text{ such that } S_i S_j = S_j S_i$$

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Example

The five qubits code

$$\mathcal{S} = \langle XXZIZ, ZXXZI, IZXXZ, ZIZXX \rangle$$

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- The circuit use $N = n + a$ qubits where a is the number of ancilla qubits.

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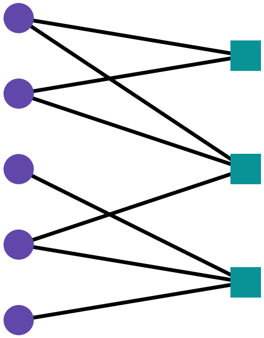

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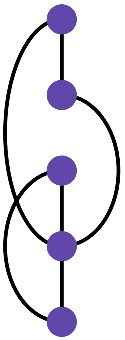

Quantum LDPC codes

A family of quantum LDPC codes $(Q_i)_i$ is a family of stabilizer codes such that the degree of the Tanner graph T_i are bounded by some constant independent of i.

Local-expansion

The Cheeger constant of a graph $G = (V, E)$ is defined as

$$h_\varepsilon(G) = \min_{\substack{L \subseteq V \\ |L| \leq \varepsilon |V|/2}} \frac{|\partial L|}{|L|}.$$

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A family of quantum local-expander codes is a family of stabilizer codes with (α, ε)-expander contracted Tanner graphs.

Lemma

Let T be the Tanner graph of a stabilizer code with length n and with r stabilizer generators. Let $\bar{T}$ be its contracted Tanner graph. Then, for all $\varepsilon \in [0, 1]$, we have

$$h_{\varepsilon'}(\bar{T}) \geq \frac{h_{\varepsilon}(T)}{\deg(T)}$$

where

$$\varepsilon' = \left(1 + \frac{r}{n}\right) \frac{\varepsilon}{1 + \deg(T)}.$$

- Show that

$$|\partial_{\bar{T}} L| \geq \frac{|\partial_T (L \cup N_T(L))|}{\deg(T)}.$$

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$$\left|\partial_T (L \cup N_T(L))\right| \geq h_\varepsilon(T) \left|L \cup N_T(L)\right| \geq h_\varepsilon(T) \left|L\right|.$$

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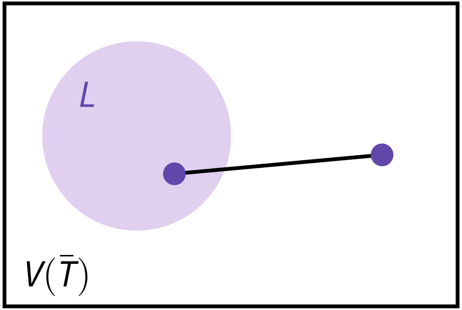
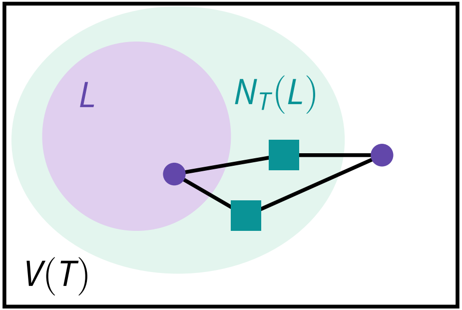
- Combine the two

$$\frac{\left|\partial_{\bar{T}} L\right|}{|L|} \geq \frac{h_\varepsilon(T)}{\deg(T)}.$$

$$|\partial_{\overline{T}} L| \, \mathrm{deg}(T) \geq |\partial_T (L \cup N_T(L))|$$





Review

- (Local) Clifford circuits
- Connectivity graphs
- Stabilizer codes and LDPC codes
- Pauli measurement circuits
- (Contracted) Tanner graphs
- Expander codes

Proof of the main theorem

Theorem

Let C be a Clifford circuit measuring computing Pauli operators $S_1, \dots, S_r$. Then, for any subset of qubits L, we have

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Corollary

For families of local-expander quantum LDPC codes of length n, a syndrome extraction circuit C implemented as a local Clifford circuit on a $\sqrt{N} \times \sqrt{N}$ grid of qubits satisfies

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Strategy

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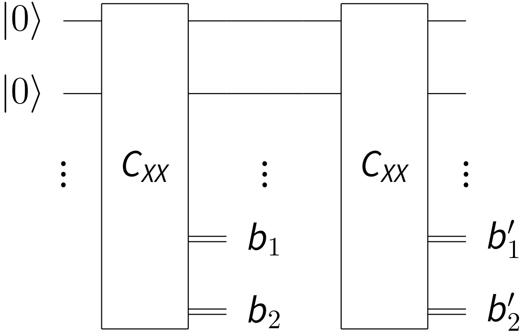
- Combine both arguments to derive a lower bound for the depth of the circuit.

Measuring correlations



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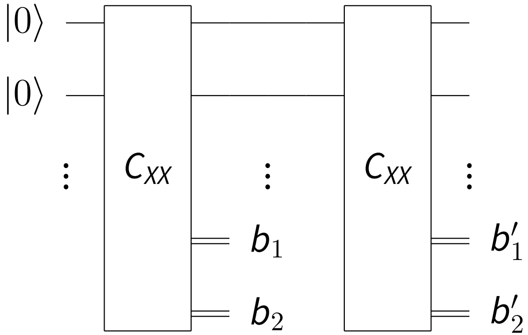

Mutual information

$$I(b_1; b_2) = 0$$

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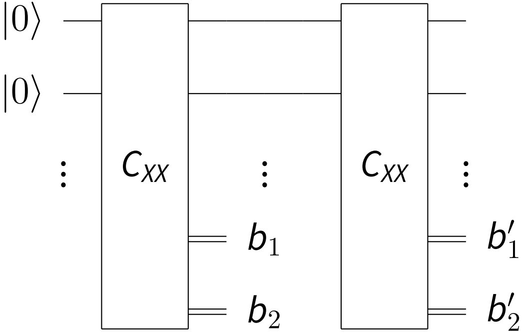
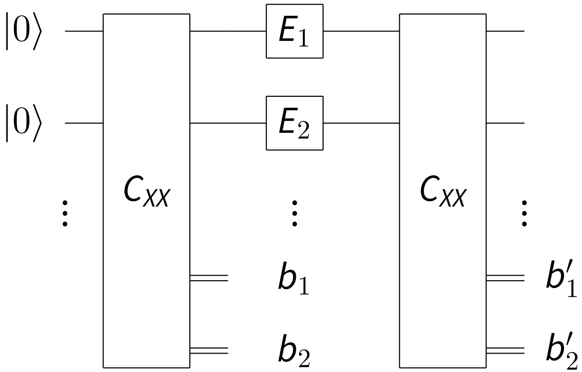


Mutual information

$$I(b_1; b_2) = 0$$

$$I(b'_1; b'_2) = 1$$

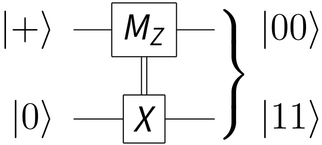
Measuring correlations



Mutual information

$$I(b_1', b_2', E_1; E_2) = 1$$

Measuring correlations



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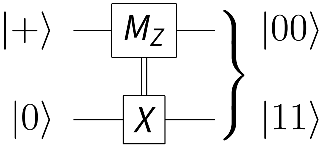

Classical operations can artificially boost mutual information.

Measuring correlations

- Build a circuit C' with the same action and similar depth as C by pushing all measurements and classical operations at the end.

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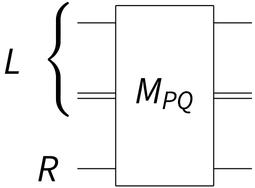
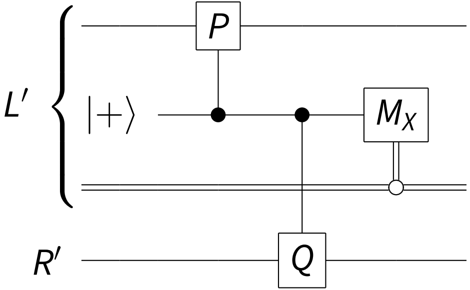
- Consider the circuit $C' \circ E \circ C'$.
- Compute the mutual information

$$I(O_L^{(2)}, E_L; O_R^{(2)}, E_R | O^{(1)}).$$

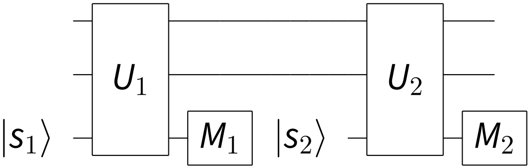
Circuit transformations

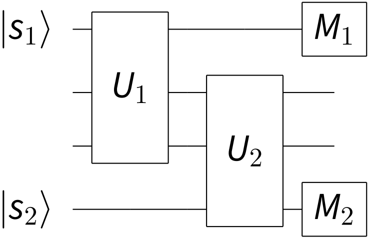




$$\mathrm{depth}(C') \leq 4 \cdot \mathrm{depth}(C) + 2$$

Circuit transformations





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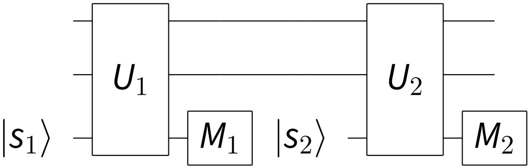
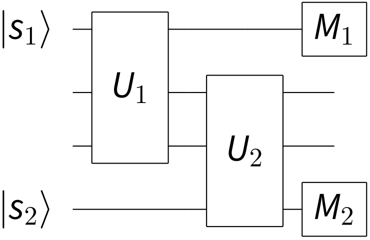
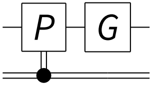
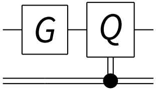




Both ancillas are the same node in the connectivity graph and in the same partition.

Circuit transformations





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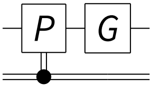
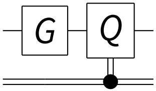
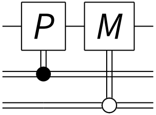
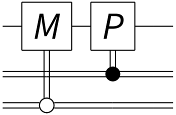
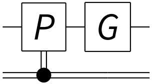


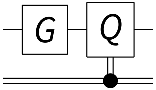


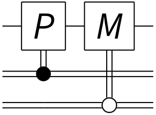


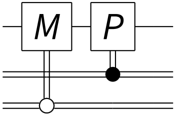


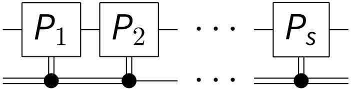
Circuit transformations

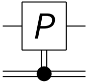




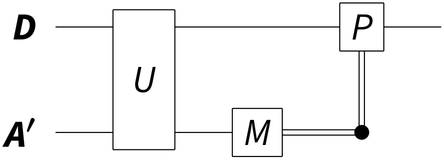




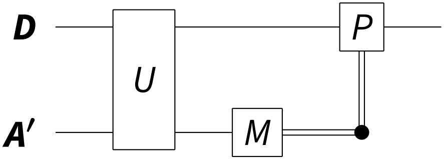




Circuit transformations

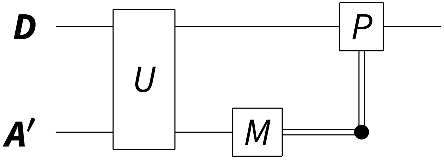


Circuit transformations



$$\mathsf{depth}(U) \leq 4 \cdot \mathsf{depth}(C)$$

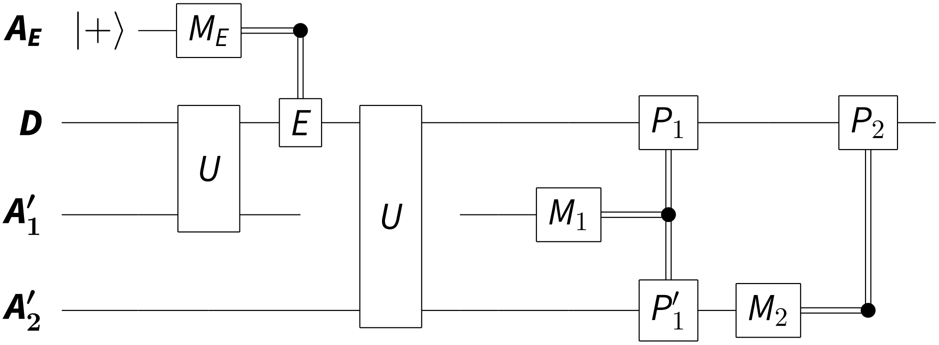
Circuit transformations



$$\text{depth}(U) \leq 4 \cdot \text{depth}(C)$$

$$|\partial L| = |\partial L'|$$

The double measurement circuit



$$I(O^{(2)}_{\bar{L}}, E_{\bar{L}}; O^{(2)}_{\bar{R}}, E_{\bar{R}} | O^{(1)}) \text{ with } \bar{L} = L' \cup A_E$$

Bounds on the mutual information

Lower bound

For the double measurement circuit, we have

$$I(O_{\overline{L}}^{(2)}, E_{\overline{L}}; O_{\overline{R}}^{(2)}, E_{\overline{R}} | O^{(1)}) \geq \frac{n_{\mathrm{cut}}}{2}.$$

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Upper bound

For the double measurement circuit, we have

$$I(O_{\bar{L}}^{(2)}, E_{\bar{L}}; O_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) \leq 32|\partial L| \mathrm{depth}(C).$$

Proof of the lower bound

Note $m_i^{(t)} = \bigoplus_{o \in O_i^{(t)}} o$, the outcome for the measurement of S_i in circuit t.

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From the data processing inequality

$$I(O_{\bar{L}}^{(2)}, E_{\bar{L}}; O_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) \geq I(M_{\bar{L}}^{(2)}, E_{\bar{L}}; M_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)})$$

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From the data processing inequality

$$
\begin{aligned}
I(O_{\bar{L}}^{(2)}, E_{\bar{L}}; O_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) &\geq I(M_{\bar{L}}^{(2)}, E_{\bar{L}}; M_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) \\
&= H(M_{\bar{L}}^{(2)}, E_{\bar{L}} | O^{(1)}) - H(M_{\bar{L}}^{(2)}, E_{\bar{L}} | M_{\bar{R}}^{(2)}, E_{\bar{R}}, O^{(1)})
\end{aligned}
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$$\begin{aligned}
I(O_{\bar{L}}^{(2)}, E_{\bar{L}}; O_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) &\geq I(M_{\bar{L}}^{(2)}, E_{\bar{L}}; M_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) \\
&= H(M_{\bar{L}}^{(2)}, E_{\bar{L}} | O^{(1)}) - H(M_{\bar{L}}^{(2)}, E_{\bar{L}} | M_{\bar{R}}^{(2)}, E_{\bar{R}}, O^{(1)}) \\
&= H(M_{\bar{L}}^{(2)}, E_{\bar{L}}, O^{(1)}) - H(O^{(1)}) - H(E_{\bar{L}})
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I(O_{\bar{L}}^{(2)}, E_{\bar{L}}; O_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) &\geq I(M_{\bar{L}}^{(2)}, E_{\bar{L}}; M_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) \\
&= H(M_{\bar{L}}^{(2)}, E_{\bar{L}} | O^{(1)}) - H(M_{\bar{L}}^{(2)}, E_{\bar{L}} | M_{\bar{R}}^{(2)}, E_{\bar{R}}, O^{(1)}) \\
&= H(M_{\bar{L}}^{(2)}, E_{\bar{L}}, O^{(1)}) - H(O^{(1)}) - H(E_{\bar{L}}) \\
&= H(M_{\bar{L}}^{(2)} | E_{\bar{L}}, O^{(1)}).
\end{aligned}
$$

Proof of the lower bound

Note S_{cut} the operators with support on both L and R.

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From the data processing inequality

$$H(M^{(2)}_{\bar{L}}|E_{\bar{L}}, O^{(1)}) \geq H(M^{(2)}_{\bar{L},\mathrm{cut}}|E_{\bar{L}}, O^{(1)})$$

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From the data processing inequality

$$\begin{aligned} H(M^{(2)}_{\bar{L}}|E_{\bar{L}}, O^{(1)}) &\geq H(M^{(2)}_{\bar{L},\text{cut}}|E_{\bar{L}}, O^{(1)}) \\ &\geq H(M^{(2)}_{\bar{L},\text{cut}}|E_{\bar{L}}, O^{(1)}, M^{(2)}_{\bar{R}}) \end{aligned}$$

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From the data processing inequality

$$\begin{aligned} H(M^{(2)}_{\bar{L}}|E_{\bar{L}}, O^{(1)}) &\geq H(M^{(2)}_{\bar{L},\text{cut}}|E_{\bar{L}}, O^{(1)}) \\ &\geq H(M^{(2)}_{\bar{L},\text{cut}}|E_{\bar{L}}, O^{(1)}, M^{(2)}_{\bar{R}}) \\ &= H(m_i(E_{\bar{R}}) \ : \ S_i \in S_{\text{cut},\bar{L}}|E_{\bar{L}}, O^{(1)}, M^{(2)}_{\bar{R}}) \end{aligned}$$

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Note S_{cut} the operators with support on both L and R.

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$$
\begin{aligned}
H(M^{(2)}_{\bar{L}} | E_{\bar{L}}, O^{(1)}) &\geq H(M^{(2)}_{\bar{L},\text{cut}} | E_{\bar{L}}, O^{(1)}) \\
&\geq H(M^{(2)}_{\bar{L},\text{cut}} | E_{\bar{L}}, O^{(1)}, M^{(2)}_{\bar{R}}) \\
&= H(m_i(E_{\bar{R}}) \ : \ S_i \in S_{\text{cut},\bar{L}} | E_{\bar{L}}, O^{(1)}, M^{(2)}_{\bar{R}}) \\
&= H(m_i(E_{\bar{R}}) \ : \ S_i \in S_{\text{cut},\bar{L}})
\end{aligned}
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&= H(m_i(E_{\bar{R}}) \ : \ S_i \in S_{\text{cut},\bar{L}}|E_{\bar{L}}, O^{(1)}, M^{(2)}_{\bar{R}}) \\
&= H(m_i(E_{\bar{R}}) \ : \ S_i \in S_{\text{cut},\bar{L}}) \\
&= |S_{\text{cut},\bar{L}}|.
\end{aligned}$$

Proof of the lower bound

By symmetry

$$I(O_{\bar{L}}^{(2)}, E_{\bar{L}}; O_{\bar{R}}^{(2)}, E_{\bar{R}} | O^{(1)}) \geq \max\{|S_{\mathrm{cut},\bar{L}}|, |S_{\mathrm{cut},\bar{R}}|\} \geq \frac{n_{\mathrm{cut}}}{2}.$$