SURFACE DEFECTS IN THE $O(N)$ MODEL

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INTRODUCTION

Why surfaces?

One of the simplest example of an interacting CFT is the d dimensional $O(N)$ model

$$S = \int \mathrm{d}^d x \left[\frac{1}{2} \left(\partial_\mu \varphi^i \right)^2 + \frac{\lambda}{4!} \left(\varphi^i \varphi^i \right)^2 \right] . \tag{1}$$

This theory is conformal when setting λ to a critical value λ_*, and, for $d = 3$, is believed to describe many critical phenomena.

In addition to local operators, the theory contains extended operators describing defects. Here I am interested in **surface defects**, which are operators defined over a two-dimensional plane. At $d = 3$ these objects correspond to interfaces and have attracted a lot of attention in the past, in particular there are three (conformal) interfaces known as the **special**, **ordinary** and **extraordinary** transitions. By studying surface defects more generally for any d and N, I reproduce these 3 known defects and find new ones in the ϵ expansion around $d = 4, 6$. Perhaps they play a role in $d = 3$ as well in describing critical phenomena.

The strategy we adopt to construct surface defects is simple. We can integrate local operators over a plane with coefficients u^I

$$D = \exp \left[- \int_{\mathbb{R}^2} \mathrm{d}^2 \tau \, u^I \mathcal{O}_I \right] . \tag{2}$$

The coupling constants u^I get renormalised and give rise to a defect RG flow. Their beta function at small u is known to be

$$\beta_{u^I} = (\Delta_{\mathcal{O}_I} - 2) u^I + \pi C^I{}_{JK} u^J u^K + \dots \tag{3}$$

where $\Delta_{\mathcal{O}}$ are the conformal dimensions and C the structure constants. We then obtain the conformal defects we want by tuning u^I to the zeros of these beta function.

The anomaly coefficients

One of the simplest characterisation of surface defects is by their conformal anomaly. For a defect defined over a surface Σ, the anomaly appears a UV divergence

$$\log \langle D_\Sigma \rangle \sim \frac{\log \tilde{\epsilon}}{4\pi} \int \mathrm{vol}_\Sigma \left[a \, \mathcal{R}^\Sigma + b_1 \operatorname{tr} \tilde{\Pi}^2 + b_2 \operatorname{tr} W + c (\partial n)^2 \right] , \tag{4}$$

where $\tilde{\epsilon}$ is a UV cutoff, and $\mathcal{R}^\Sigma$, $\operatorname{tr} \tilde{\Pi}^2$, $\operatorname{tr} W$, $(\partial n)^2$ are conformal invariants depending on the geometry of Σ.

The anomaly coefficients a, b_1, b_2, c do not depend on the geometry and are numbers that appear in many observables and can be calculated using e.g. perturbative methods. They are also constrained by unitarity, and in particular, a satisfies an a-theorem

$$a_{UV} > a_{IR} , \tag{5}$$

while $b_1, -b_2, c$ are constrained to be positive.

SYMMETRIC DEFECT

Large N analysis

To study defects across dimensions, a convenient tool is the large N expansion. Using the Hubbard-Stratanovich transformation, we can rewrite (1) in terms of the auxiliary field σ

$$S = \int \mathrm{d}^d x \left[\frac{1}{2} \left(\partial_\mu \varphi^i \right)^2 + \frac{1}{2\sqrt{N}} \sigma \varphi^i \varphi^i - \frac{3}{2N\lambda} \sigma^2 \right] . \tag{6}$$

At large N, σ is promoted to a dynamical field, and to leading order it has dimension 2, so naturally gives rise to a surface operator

$$D_N = \exp \left(- \int \mathrm{d}^2 \tau \, h \, \sigma(\tau) \right) . \tag{7}$$

From the beta function (3) we find a fixed point of the dRG flow

$$h_* = \frac{2 - \Delta_\sigma}{\pi C^\sigma{}_{\sigma\sigma}} + \dots \tag{8}$$

where $\Delta_\sigma, C^\sigma{}_{\sigma\sigma}$ are known quantities (see the result in figure 1) on the left.

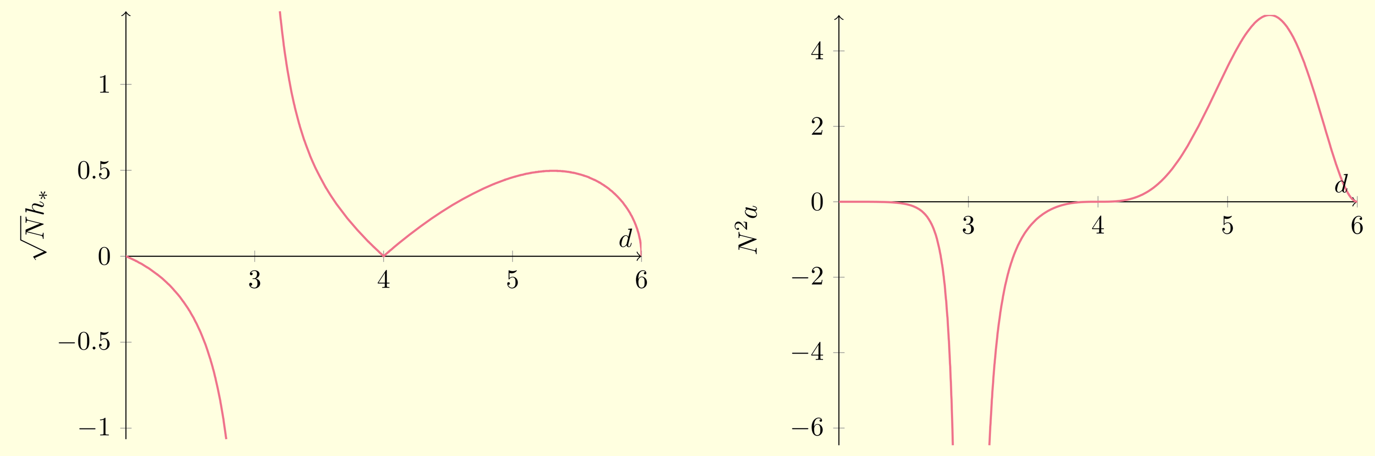

Figure 1: On the left, a plot of $\sqrt{C_\sigma N} h_*(d)$ to leading order at large N. The factor $\sqrt{C_\sigma}$ is introduced to ensure that the value is independent of the choice of normalisation of σ. On the right, the anomaly coefficient $N^2 a$ to leading order at large N.

- h_* is akin to a mass term for φ^i, and should be positive. When $h_* < 0$ we expect spontaneous breaking where φ^i acquires a VEV.
- The theory can be studied perturbatively using $\epsilon = 4 - d$ ($\epsilon = 6 - d$) as a small parameter. The defect operator (7) can be studied there as well, and agrees with the large N result.
- An interesting feature of figure 1 is the divergence at $d = 3$. This happens because $C^\sigma{}_{\sigma\sigma}$ vanishes to leading order at large N. The divergence signals a change in scaling with respect to N, and we expect that including subleading corrections to the beta function would yield a finite value $h_* \sim \sqrt{N}$, in agreement with the **ordinary** transition.
- The anomaly coefficient a can be obtained by calculating the expectation value of the spherical defect. To leading order in N, the result is

$$a = \frac{C_\sigma^3 (\Delta_\sigma - 2)^3}{6 C_{\sigma\sigma\sigma}^2} . \tag{9}$$

A plot of $a(d)$ is presented in figure 1 on the right. This is in agreement with the a-theorem. Between $2 < d < 4$, the coupling h flows from $h = 0$ (UV fixed point) to $h = h_*$ (IR fixed point). Correspondingly $a_{IR} < 0$. Between $4 < d < 6$ the defect is instead the UV fixed point of a dRG flow, and $a_{UV} > 0$.

SYMMETRY BREAKING DEFECTS

We can find more defects by allowing couplings breaking the $O(N)$ symmetry. Here I discuss two cases.

Breaking to $O(N-1)$

The breaking $O(N) \to O(N-1)$ is naturally realised by coupling the defect to one of the scalar field φ^i.

$$D_{=} \exp \left[- \int \mathrm{d}^2 \tau \left(h \sigma + u^i \varphi^i \right) \right] . \tag{10}$$

Such defects are expected to describe the **extraordinary** transition in $d = 3$. They are easy to analyse near $d = 6$, since there both σ, φ^i have $\Delta \sim 2$, so the zeros of the beta functions (3) are perturbative. We find the perturbative fixed point (with $\epsilon = 6 - d$)

$$h_* = -\frac{1}{2} \sqrt{\frac{\pi \epsilon}{6N}} \left[1 - \frac{24}{N} - \frac{286}{N^2} + \dots \right] , \qquad u_*^2 = -\frac{\pi N \epsilon}{12} \left[3 + \frac{356}{N} + \frac{184652}{N^2} + \dots \right] . \tag{11}$$

Breaking to $O(p) \times O(N-p)$

Below $d = 4$, one can construct a new class of defects. Consider the general coupling

$$D_p = \exp \left[- \int_{\mathbb{R}^2} \mathrm{d}^2 \tau \, h_{ij} \varphi^i \varphi^j \right] , \tag{12}$$

We can reduce h_{ij} to its set of eigenvalues. It turns out that there are fixed points with two distinct eigenvalues: $h_{p,+}$ and $h_{N-p,-}$, respectively of multiplicity p and $N - p$, where (with $\epsilon = 4 - d$)

$$h_{p,\pm} = (2\pi\epsilon) \frac{(N + 3 - p) \pm \Delta}{N + 8} + \dots , \qquad \Delta^2 = 9 - p(N - p) . \tag{13}$$

- For $N < N_c$ (and $N_c = 6$ to first order in perturbation theory), there are real fixed points for any $0 \leqslant p \leqslant N$. These are new unitary defects that are saddle points of the dRG flow (see figure 2).
- Above N_c, there is a window $6 < N < 10$ where the defects with $p = N - 1$ are unitary and stable.
- At $N = 6$, the defects with $p = 3, 4, 5, 6$ all coincide. This is the first example of four fixed points colliding that I know of!

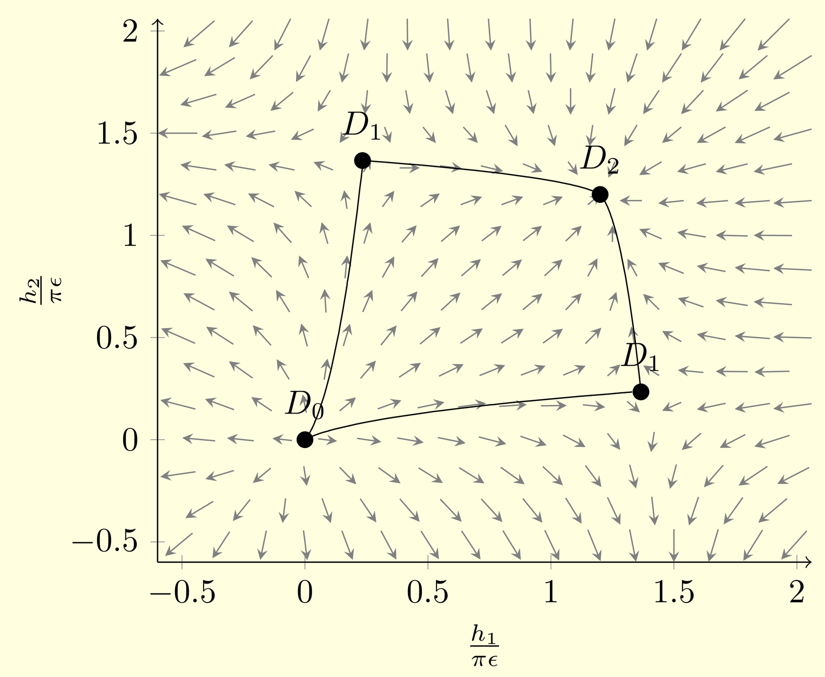

Figure 2: Example of a defect RG flow for surface defects in the $O(2)$ model. The vector field is $-\beta(h_1, h_2)$ given in terms of the eigenvalues h_1, h_2. There is a $\mathbb{Z}_2$ symmetry exchanging $h_1 \leftrightarrow h_2$. The 3 fixed points are D_0, D_1 and D_2, their values of h's are given in (13). The black lines indicate the stable manifold.

OUTLOOK & HOLOGRAPHY..?

Outlook

- The $O(N)$ model is the simplest example of an interacting CFT, yet it still contains a rich array of defects. Can they be classified?
- The value of h_* for the symmetric defect changes drastically at $d = 3$, and it would be interesting to understand if there is a deeper reason behind it.
- In addition to reproducing the known defects in 3d (special = trivial defect, ordinary, extraordinary), we found new symmetry breaking defects in the $\epsilon = 4 - d$ expansion, and some indications of fixed points near $d = 6$. Do they exist also in 3d?
- Symmetry breaking defects seem to be unitary only for small values of N. It would be interesting to clarify if this is a general feature of symmetry breaking defects, and what is the physical interpretation.

What about holography?

Both the free and critical 3d $O(N)$ models are conjectured to be dual to Vasiliev's higher spin theory on $(A)dS_4$. This is another (perhaps less understood) example of a holographic duality.

Unlike the better known cases of holography in string theory, we don't know of any extended objects in higher-spin theory, so finding the holographic dual to any defect is challenging! The symmetric surface defect introduced here may be an excellent starting point for such a programme because they are invariant under the $O(N)$ symmetry.

Reference

[1] M. Trépanier, "Surface defects in the $O(N)$ model," arXiv:2305.10486.

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