

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

MASTER THESIS

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**Smoothed Particle  
Hydrodynamics Simulations for  
Asteroid Deflection**

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*Author:*  
Maximilian Rutz  
Matrikelnr. 4255042

*Supervisor:*  
Dr. Christoph Schäfer  
Prof. Dr. Wilhelm Kley

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## **Declaration**

I hereby declare that I have written the present thesis independently, without assistance from external parties and without use of other resources than those indicated. The ideas taken directly or indirectly from external sources (including electronic sources) are duly acknowledged in the text. The material, either in full or in part, has not been previously submitted for grading at this or any other academic institution.

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Location, date

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Maximilian Rutz

## **ABSTRACT**

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# 1 Introduction

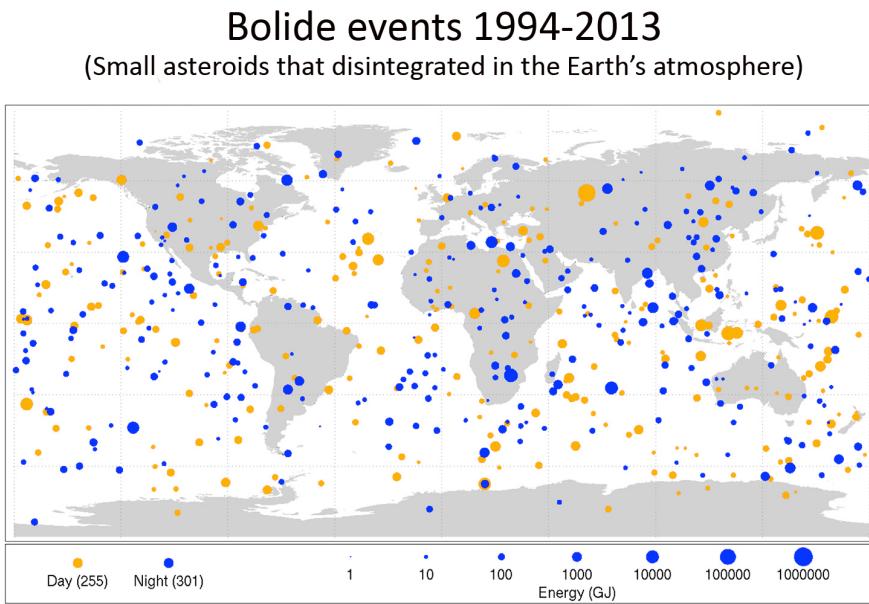


Figure 1: Past impacts [10]

- Dart and Hera Missions

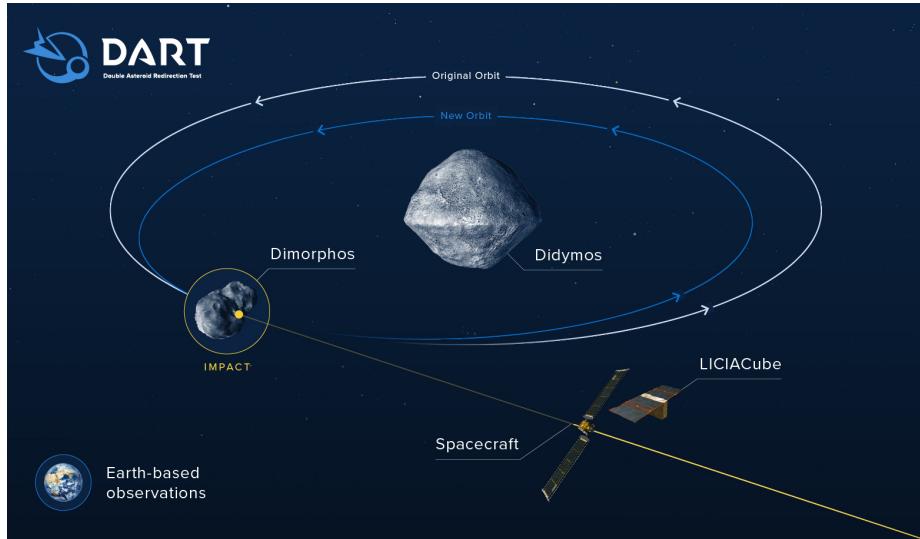


Figure 2: Dart mission [9]

Previous work: - Raducan [8] grid based 2d sims - Stickle [7] sph sims  
 Improvement of previous work: - Effect of impact angle on beta factor  
 Outline of the rest of the paper

## 2 Theory

- A very good explanation of the theory behind the Miluphcode code can be found in [6] and [13]. The explanations in these two papers are concise, well written and difficult to improve upon. The aim of this theory section is therefore only to present the subset of these equations that apply to the particular solid model used in this study. With the given equations, all parameters in Table 1 of Section 3.2.2 should be explained.

### 2.1 Material derivative

- In SPH we want to know the local velocity change at each particle location - Show that:  $\frac{dv_i}{dt} = \frac{\partial v}{\partial t} + v \cdot \nabla v$

-  $v \cdot \nabla v$  holds for any scalar field property  $y$  that a particle traverses with velocity  $u$   $u \cdot \nabla y$  since it is the projection  $u \cdot \nabla y = |u| \cdot |\nabla y| \cdot \cos(\Theta)$  where  $\Theta$  is the angle between  $u$  and  $\nabla y$ . - Thus, the local change in the field property  $\frac{dy_i}{dt}$  at the point i in the Lagrangian view corresponds to the sum of the change of the field property at a point i over time  $\frac{\partial y_i}{\partial t}$  and the change  $u_i \cdot \nabla y_i$  of the field property due to the movement of the particle along the field with velocity  $u$  in the Eulerian view:  $\frac{dy_i}{dt} = \frac{\partial y_i}{\partial t} + u_i \cdot \nabla y_i$

This expression is called the Material Derivative  $\frac{Dy}{dt}$ :  $\frac{Dy}{dt} \equiv \frac{\partial y}{\partial t} + (u \cdot \nabla y)$

$$\frac{D\mathbf{F}}{Dt} = \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{F} \quad (1)$$

It gives us the change of the field property in the reference frame of the particle (denoted by the subscript i):

- In next sections  $\frac{d}{dt}$  denotes material derivative and greek letters are used to denote spacial coordinates which go from 1 to 3. Einstein summation convention is used

### 2.2 Conservation equations

#### 2.2.1 Conservation of mass

- continuity equation

$$\frac{d\rho}{dt} + \rho \frac{\partial v^\alpha}{\partial x^\alpha} = 0 \quad (2)$$

-  $\rho$  velocity,  $v^\alpha$  material velocity

#### 2.2.2 Conservation of momentum

$$\frac{dv^\alpha}{dt} - \frac{1}{\rho} \frac{\partial \sigma^{\alpha\beta}}{\partial x^\beta} = 0 \quad (3)$$

-  $\sigma$  stress tensor

$$\sigma^{\alpha\beta} = -p\delta^{\alpha\beta} + S^{\alpha\beta} \quad (4)$$

- p Druck,  $\delta^{\alpha\beta}$  Kronecker Delta,  $S^{\alpha\beta}$  deviatoric stress tensor

### 2.2.3 Conservation of energy

$$\frac{du}{dt} + \frac{p}{\rho} \frac{\partial v^\alpha}{\partial x^\alpha} - \frac{1}{\rho} S^{\alpha\beta} \dot{\epsilon}^{\alpha\beta} = 0 \quad (5)$$

- u internal energy,  $\dot{\epsilon}$  strain rate tensor

$$\dot{\epsilon}^{\alpha\beta} = \frac{1}{2} \left( \frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha} \right) \quad (6)$$

- Set of Odes can be solved with integration schemes, only remaining unknown quantities at each step are pressure p and deviatoric stress tensor  $S^{\alpha\beta}$
- deviatoric stress tensor  $S^{\alpha\beta}$ : Strength model - pressure: Equation of State + porosity model

## 2.3 Plasticity model

### 2.3.1 Constitutive equations

- the deviatoric stress tensor is the main aspect of modeling a solid, for liquids it is 0.
- Constitutive equations describe time evolution of S - elastic and plastic regime:

  - this form of const equation takes care of elastic regime based on Hookes Law
  - plastic regime modeled by damage and strength model

$$\frac{dS^{\alpha\beta}}{dt} = 2\mu(\dot{\epsilon}^{\alpha\beta} - \frac{1}{3}\delta^{\alpha\beta}\dot{\epsilon}^{\gamma\gamma}) + S^{\alpha\gamma}R^{\gamma\beta} - R^{\alpha\gamma}S^{\gamma\beta} \quad (7)$$

-  $\mu$  shear modulus as in Table 1, rotation rate sensor  $R^{\alpha\beta}$

$$R^{\alpha\beta} = \frac{1}{2} \left( \frac{\partial v^\alpha}{\partial x^\beta} - \frac{\partial v^\beta}{\partial x^\alpha} \right) \quad (8)$$

### 2.3.2 Fracture model

- damage
  - Weibull distribution

### 2.3.3 Pressure dependent yield strength

- ideas in [3] - implementation in [5] - Transition from linearity in elastic case to the plastic case - Von Mises model limits the deviatoric stress at a constant threshold.
- Main invariants of a tensor are invariant under base changes - They can therefore be used for sph particles regardless of their motion and relative orientation -  $J_2$  is the second main invariant of  $S^{\alpha\beta}$  calculated as:

$$J_2 = \frac{1}{2} S^{\alpha\beta} S_{\alpha\beta} \quad (9)$$

It is used to compute the limiting factor  $f_Y$  for each particle:

$$f_Y = \min \left[ \frac{Y}{\sqrt{J}}, 1 \right] \quad (10)$$

$$S^{\alpha\beta} \rightarrow f_Y S^{\alpha\beta} \quad (11)$$

$Y$  is the yield strength and can be determined in different ways. In the simplest case of a von Mises strength model it is constant. In a more sophisticated model developed by [3] and later improved by [5]  $Y$  depends upon the pressure and the damage from the fracture model introduced in Section 2.3.2. In this approach, the yield strength is a linear combination of the yield strength for intact material  $Y_I$  and for completely damaged material  $Y_D$ .

$$Y = (1 - D) Y_I + D Y_D \quad (12)$$

$$Y_I = Y_0 + \frac{\mu_I p}{1 + \frac{\mu_I p}{Y_M - Y_0}} \quad (13)$$

$$Y_D = \mu_D p \quad (14)$$

The coefficients of internal friction  $\mu = \tan(\alpha)$  depend upon the angles  $\alpha_{intact}$  and  $\alpha_{damaged}$ .  $Y_0$  is the cohesion or shear strength at  $p = 0$  and  $Y_M$  the shear strength at  $p = \infty$ .

## 2.4 Equation of State

- calculating pressure  $p$  for high velocity impacts - [1]

1)  $u \leq E_{iv}$

$$p = (a_T + \frac{b_T}{1 + \frac{u}{E_0 \eta^2}}) \rho u + A_T \chi + B_T \chi^2 \quad (15)$$

2)  $u \geq E_{cv}$

$$p = a_T \rho u + \left( \frac{b_T \rho u}{1 + \frac{u}{E_0 \eta^2}} + A_T \chi \cdot \exp \left[ -\beta_T \left( \frac{\rho_0}{\rho} - 1 \right) \right] \right) \cdot \exp \left( -\alpha_T \left[ \frac{\rho_0}{\rho} - 1 \right]^2 \right) \quad (16)$$

with  $eta = \frac{\rho}{\rho_0}$  and  $\chi = eta - 1$ .  $rho_0, A_T, B_T, E_0, E_{iv}, E_{cv}, a_T, b_T, \alpha_T$  and  $\beta_T$  are material dependent parameters (see Table 1 in Section 3.2.2).

3)  $u$  between  $E_{iv}$  and  $E_{cv}$ : linear interpolation

## 2.5 Porosity

There are different ways in which porosity can be modeled depending on the pore size. Macro porosity with pore sizes above the resolution of the simulation can be accounted for in the initial conditions. This however becomes impossible for granular material with sub-resolution sized grains and pores.

Microporosity models porosity as an additional material property and can be applied independently of the resolution.

In these simulations, a microporosity p- $\alpha$  model as outlined in [4] is used. The distention  $\alpha \in [1, \infty)$  relates the current density  $\rho$  to the solid density  $\rho_s$  which is reached if the material is fully compressed. For a non-porous material  $\alpha$  equals one.

$$\alpha \equiv \frac{\rho_s}{\rho} = \frac{1}{1 - \phi} \quad (17)$$

with the porosity  $\phi \in [0, 1)$ .

- Quadratic crush curve

### 3 Numerics

- how numerics are used to solve Equations from Theory section

#### 3.1 Smoothed Particle Hydrodynamics

- before finite difference schemes with spherical coordinates - spherical coordinates bad for collisions

Smoothed Particle Hydrodynamics is a numerical simulation method first introduced by [2] in 1977. It is a Lagrangian particle method and as such often used when the geometry of the underlying problem makes it difficult to apply Eulerian grid-based methods like finite difference schemes. Although SPH is most often used to model liquids, it is possible to add physical models for solids as well.

- explanations apply to Miluphcuda

Miluphcuda is a smoothed particle hydrodynamics code that has been developed over several years at the University of Tuebingen by Christoph Schaefer and others. Its general use is well documented in [6].

##### 3.1.1 Main SPH concepts

A SPH simulation is composed of many individual SPH particles. Each particle moves through space with a velocity  $\vec{v}$  and a mass  $m$ . In contrast to particle methods used for N-body simulations or molecular dynamics, SPH particles carry information about continuous variables such as the density  $\rho$  or energy  $e$ . The particles only act as computational points at which equations from hydrodynamics/continuum mechanics such as the Euler or Navier-Stokes Equations are evaluated. Solving such equations comes down to converting partial differential equations to a system of first order ordinary differential equations in time.

$$\frac{d\vec{y}}{dt} = f(t, \vec{y}(t), A_1, \dots, A_n) \quad (18)$$

In equation 18  $\vec{y}$  is a vector of quantities to be projected forward in time and  $A_1$  through  $A_n$  are quantities that are calculated at every step. Once  $A_1$  through  $A_n$  are known for every particle and the right hand side of quation 18 can be evaluated, standard integrators such as Runge-Kutta or Predictor-Corrector methods are used to update  $\vec{y}$ .

To calculate a quantity  $A$  at a particle location, SPH uses a weighted average of  $A$  over all particles in the neighborhood:

$$A(\vec{r}) \approx \int A(\vec{r}') W(\vec{r} - \vec{r}', h) d\vec{r}' \quad (19)$$

### 3.1.2 Smoothing kernel

The kernel function  $W(|\vec{r} - \vec{r}'|, h)$  at the particle location  $\vec{r}$  depends upon the distance to the other particles and a specific length  $h$  called the smoothing length. Most kernels used today have compact support within a radius of  $h$ . To ensure normalization of the kernel

$$\int W(\vec{r} - \vec{r}', h) d\vec{r}' = 1 \quad (20)$$

- Smoothing kernel image: 3

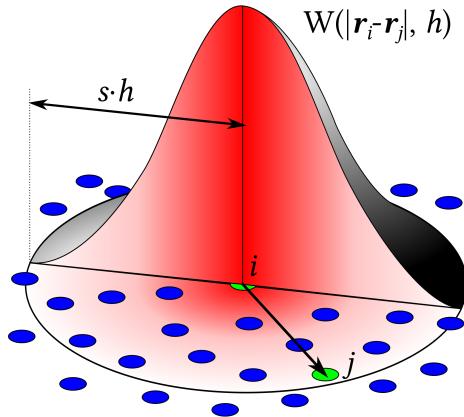


Figure 3: Kernel function [11]

### 3.1.3 Smoothing length

- can be variable - great strength of SPH - material density and  $h$  have to fit together

### 3.1.4 Artificial viscosity

- what it is - why it is needed

### 3.1.5 Time Integration

## 3.2 Initial conditions

### 3.2.1 Particle setup

- Target basalt halfsphere
- Impactor aluminium sphere
- resolution bound to variable smoothing length
- Uniform macro structure but random micro structure to avoid
- seagen [12] used to create initial conditions

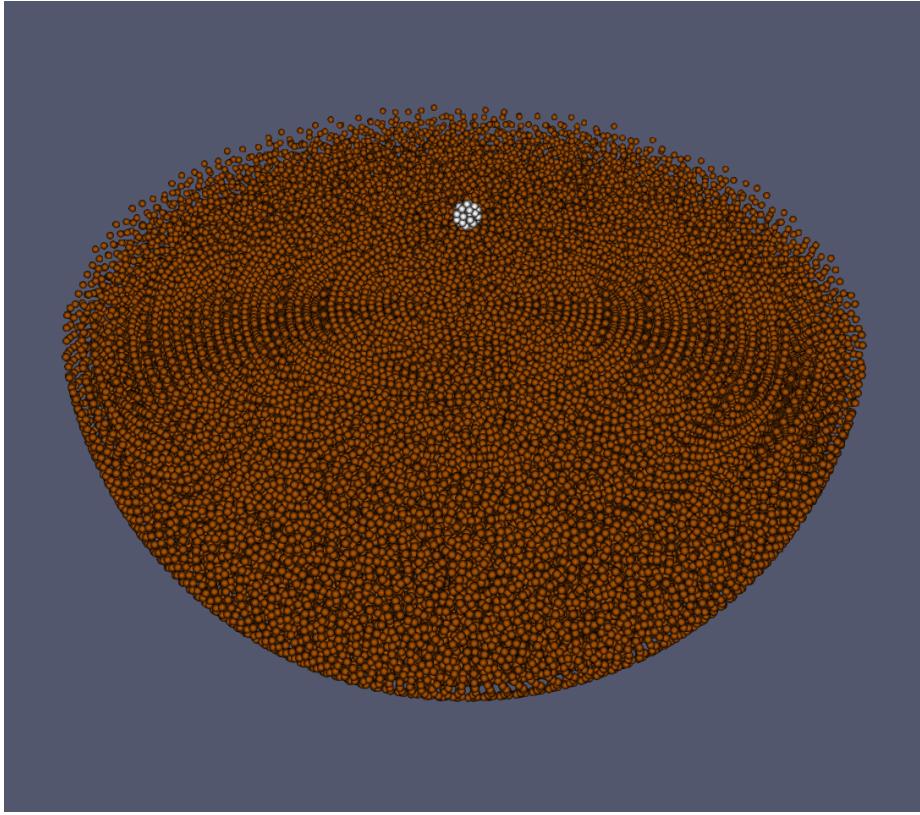


Figure 4: start of example simulation

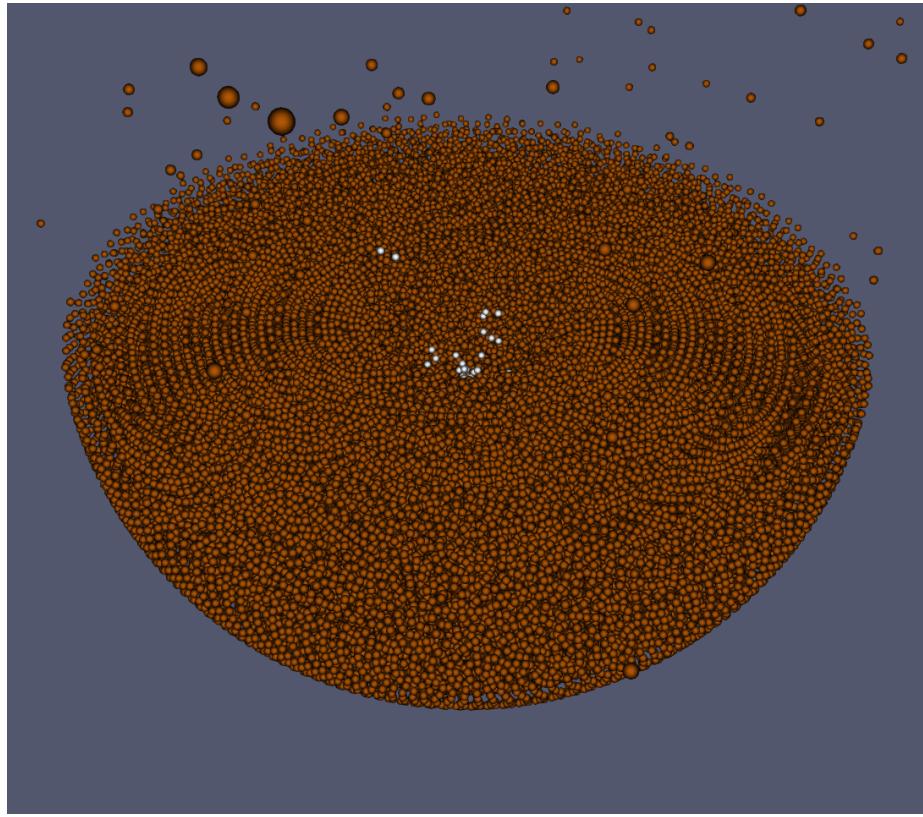


Figure 5: end of example simulation

### 3.2.2 Material parameters

- variable smoothing length min 0.1 bis max 10.0 - rholimit 0.95

		Target	Projectile
Tillotson EOS	$\rho_0$	$2.86 \cdot 10^3 g \cdot cm^{-3}$	$2.70 \cdot 10^3 g \cdot cm^{-3}$
	$A_T$	$2.67 \cdot 10^{10} Pa$	$7.52 \cdot 10^{10} Pa$
	$B_T$	$2.67 \cdot 10^{10} Pa$	$6.50 \cdot 10^{10} Pa$
	$E_0$	$4.87 \cdot 10^8 Jkg^{-1}$	$5.00 \cdot 10^6 Jkg^{-1}$
	$E_{iv}$	$4.72 \cdot 10^6 Jkg^{-1}$	$3.00 \cdot 10^6 Jkg^{-1}$
	$E_{cv}$	$1.82 \cdot 10^7 Jkg^{-1}$	$1.39 \cdot 10^7 Jkg^{-1}$
	$a_T$	0.5	0.5
	$b_T$	1.5	1.63
	$\alpha_T$	5.0	5.0
	$\beta_T$	5.0	5.0
Porosity	$\alpha_0$	<b>varying</b>	not porous
	$p_{elastic}$	$1.0 \cdot 10^6 Pa$	not porous
	$p_{transition}$	$6.8 \cdot 10^7 Pa$	not porous
	$p_{compacted}$	$2.13 \cdot 10^8 Pa$	not porous
	$\alpha_e$	4.64	not porous
	$\alpha_t$	1.90	not porous
	$c_s$	$100.0 m \cdot s^{-1}$	not porous
Strength	cohesive strength $Y_c$	<b>varying</b>	$1.0 \cdot 10^9 Pa$
	$\alpha_{intact}$	0.982793 rad	0 rad
	$\alpha_{damaged}$	0.540419 rad	0 rad
	shear modulus $\mu$	$2.27 \cdot 10^{10} Pa$	$2.69 \cdot 10^{10} Pa$
	bulk modulus $K_0$	$2.67 \cdot 10^{10} Pa$	$5.23 \cdot 10^{10} Pa$
	shear strength $Y_M$	$3.5 \cdot 10^9 Pa$	$2.76 \cdot 10^8 Pa$
Fragmentation	Weibull m	16	no damage
	Weibull k	$1.0 \cdot 10^{61}$	no damage
Artificial viscosity	$\alpha$	1.0	1.0
	$\beta$	2.0	2.0

Table 1: Material parameters for basalt target and aluminium Impactor

## 4 Results

The simulations were run with porosities of 0%, 17%, 33% and 50% and cohesive strengths of 1kPa, 10kPa, 100kPa and 1MPa under impact angles of 0 and 45 degrees. This yields 32 simulations in total.

### 4.1 Cratering

- qualitative analysis
  - head on

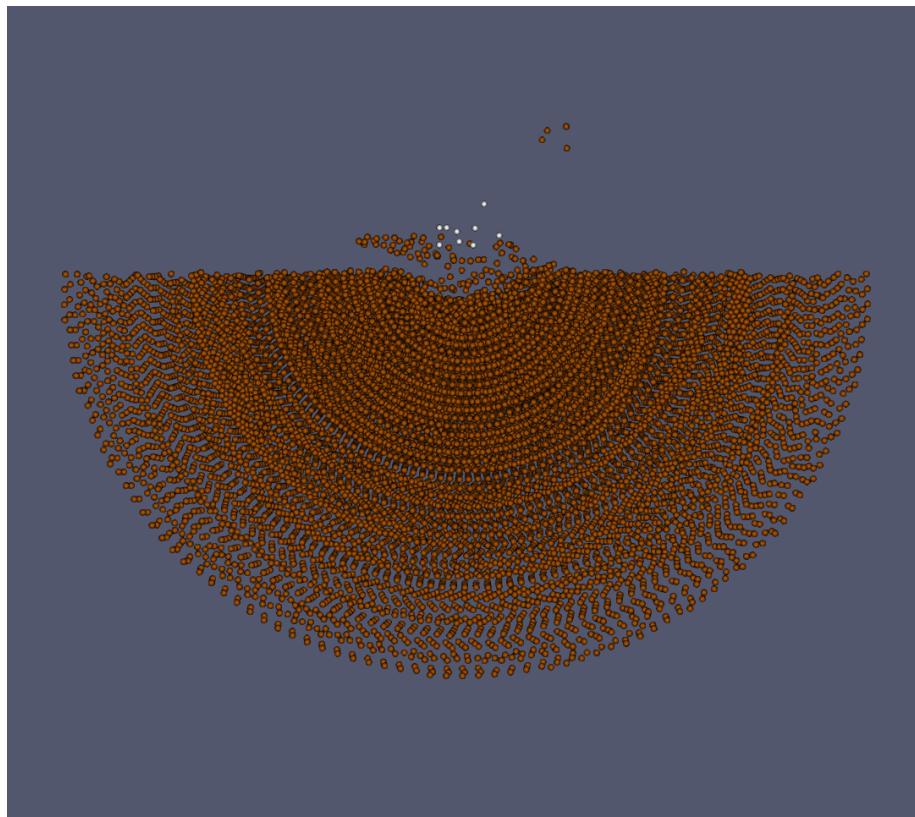


Figure 6: No porosity (0%), high strength ( $Y=1\text{MPa}$ )

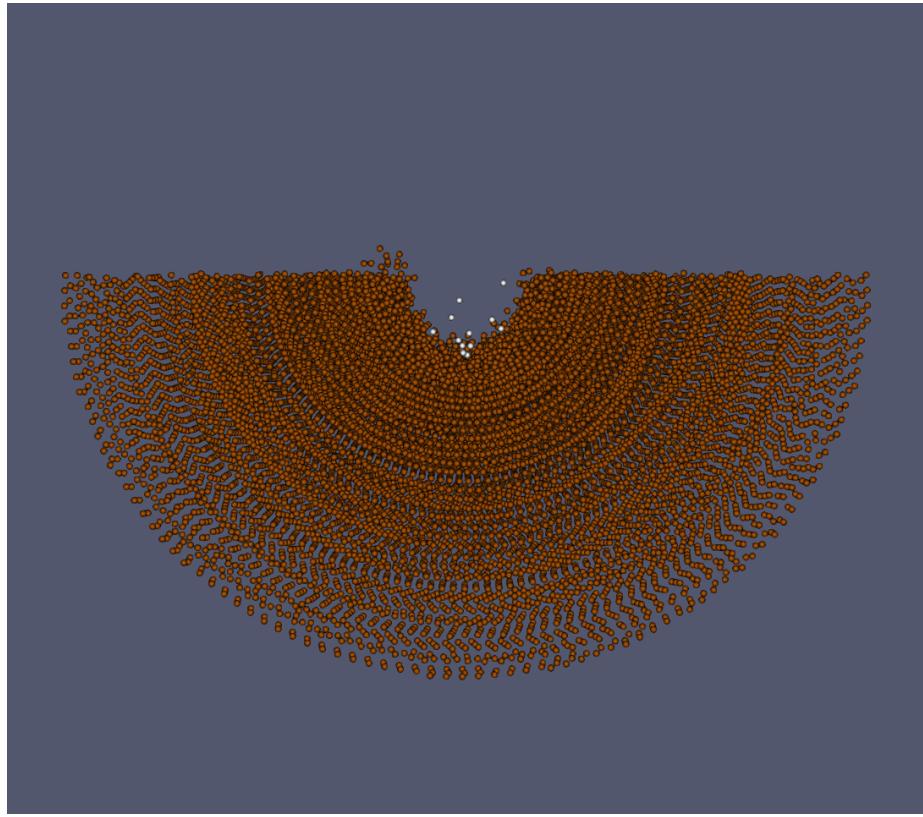


Figure 7: High porosity (50%), high strength ( $Y=1\text{ MPa}$ )

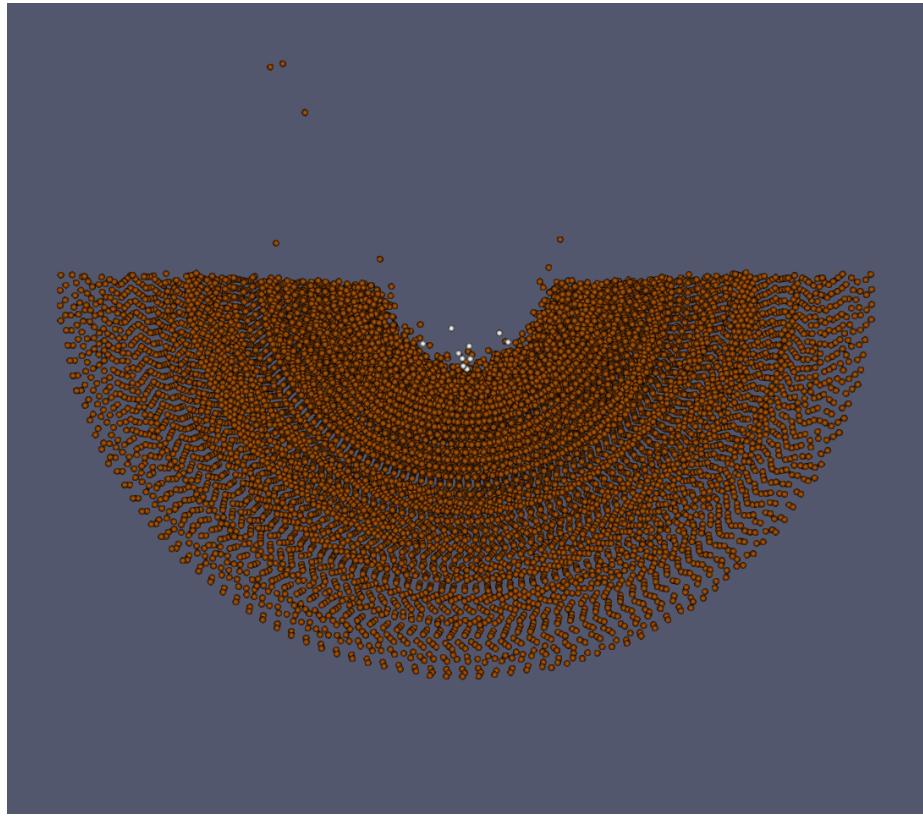


Figure 8: High porosity (50%), low strength ( $Y=1\text{kPa}$ )

- 45 degrees

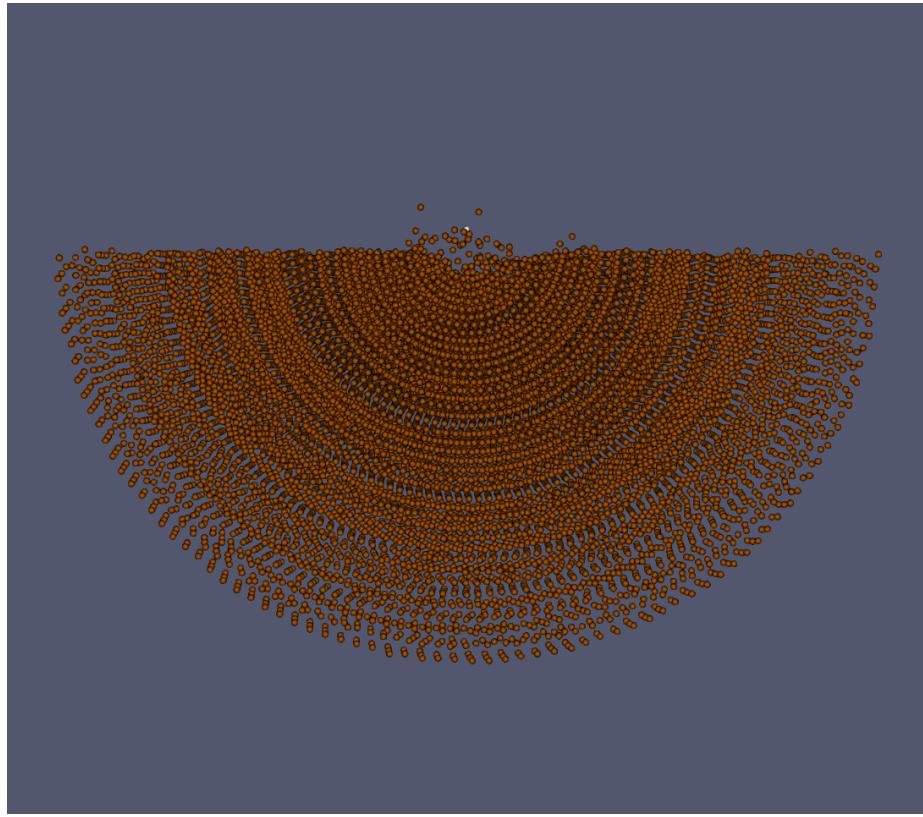


Figure 9: No porosity (0%), high strength ( $Y=1\text{MPa}$ ), 45 degree angle

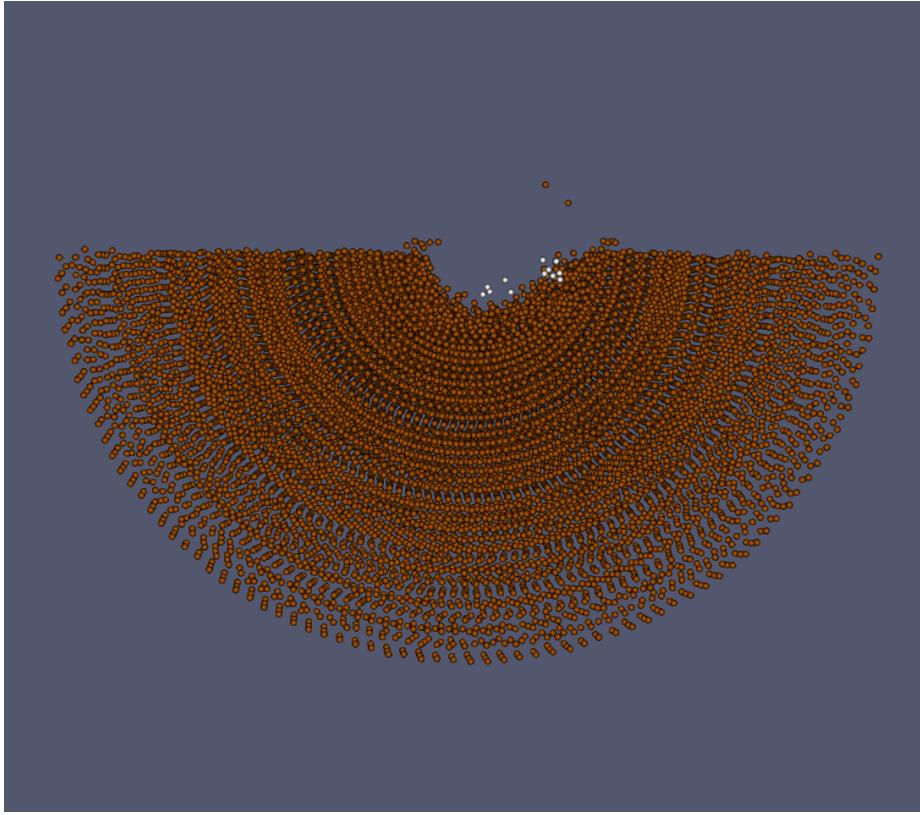


Figure 10: High porosity (50%), high strength ( $Y=1\text{ MPa}$ ), 45 degree angle

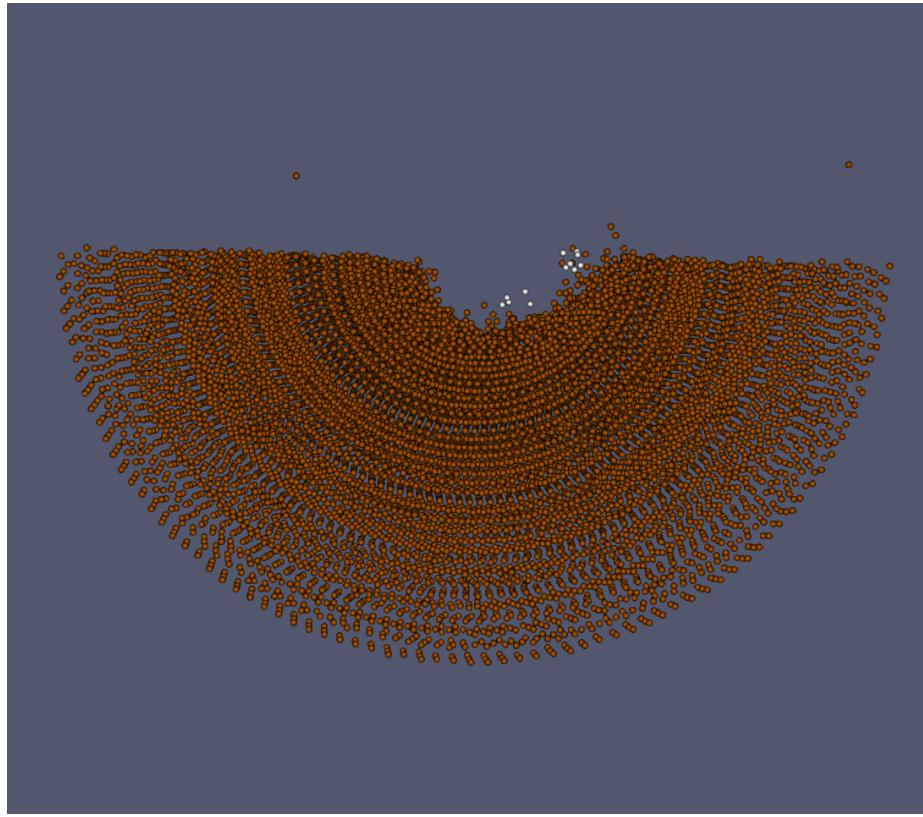


Figure 11: High porosity (50%), low strength ( $Y=1\text{kPa}$ ), 45 degree angle

#### 4.2 Beta factor

- explanation of beta factor

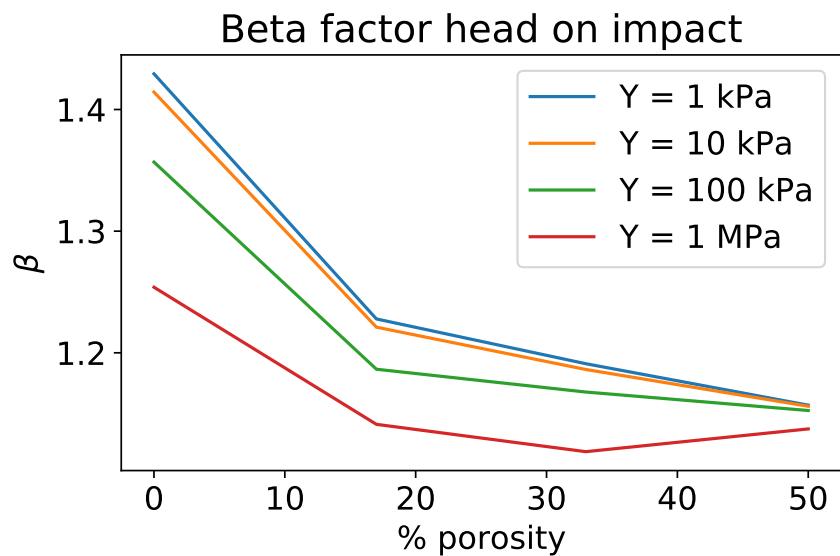


Figure 12: beta factor

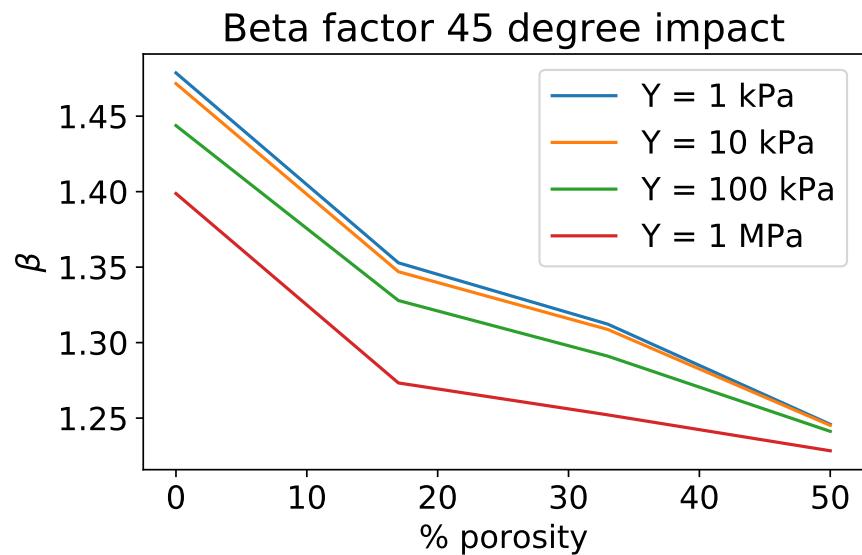


Figure 13: beta factor

- only particles with positive velocity along impact direction above escape velocity

$$v_{esc} = \sqrt{\left(\frac{2GM}{r_i}\right)} \quad (21)$$

where  $M = \frac{4}{3}\pi R^3 \rho_{asteroid}$  is the estimated mass of asteroid with  $R = 75\text{m}$  and  $\rho_{asteroid} = 2.8 \frac{\text{g}}{\text{cm}^3}$  and  $r_i$  distance of sph particle from center of estimated asteroid sphere - few particles (with highest velocities so the ones that are far away) account for most of the momentum

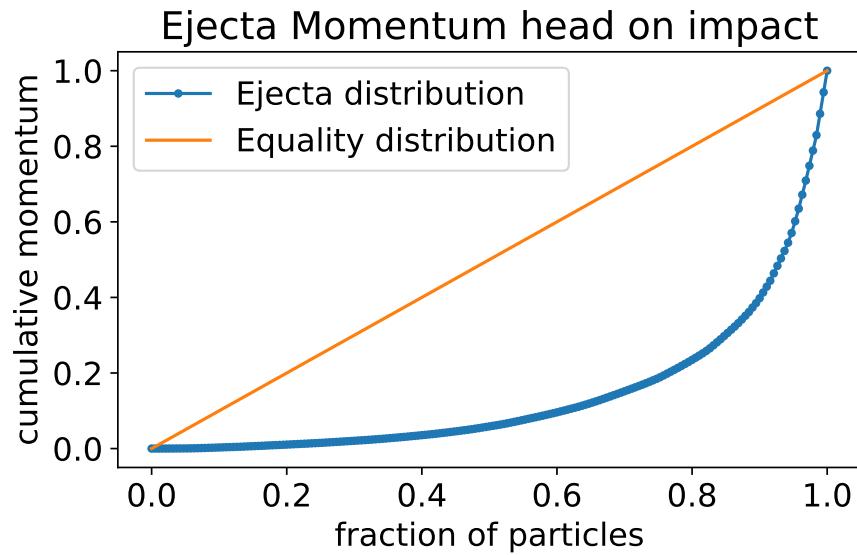


Figure 14: Lorenz curve for momentum distribution head on

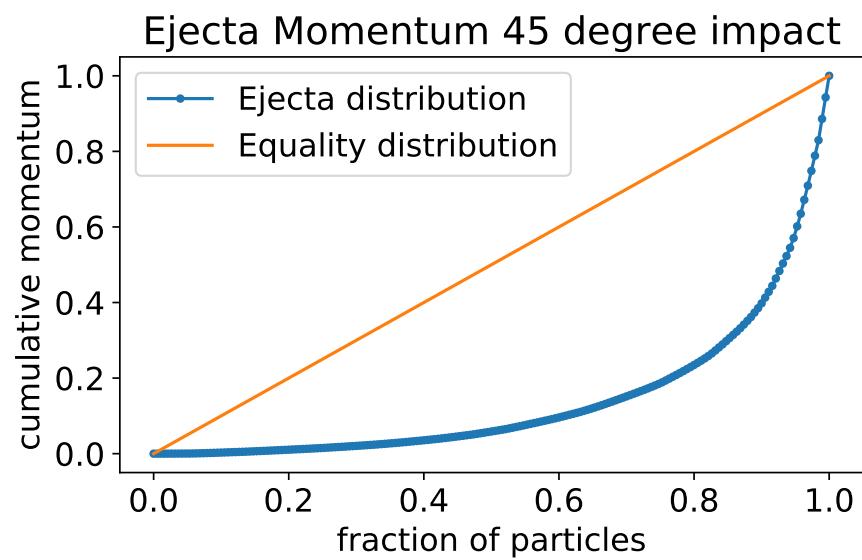


Figure 15: Lorenz curve for momentum distribution 45 degree

## 5 Discussion

- beta factor lower than Raducan grid based but comparable to Stickle SPH - upper limit beta below 2 because of momentum conservation?? – which inertial system is used for calculation of beta factor in other papers? - tensorial correction not implemented because of timestep getting to small - no real boundary conditions implemented - the whole target shows shift (should not be a problem since we are still in the inertial frame of rest of the target !?)

## References

- [1] J. H. Tillotson. “Metallic Equations of State For Hypervelocity Impact”. In: (July 1962), p. 3216.
- [2] R. A. Gingold and J. J. Monaghan. “Smoothed particle hydrodynamics: theory and application to non-spherical stars”. In: *Monthly Notices of the Royal Astronomical Society* 181.3 (Dec. 1977), pp. 375–389. ISSN: 0035-8711. DOI: 10.1093/mnras/181.3.375. eprint: <https://academic.oup.com/mnras/article-pdf/181/3/375/3104055/mnras181-0375.pdf>. URL: <https://doi.org/10.1093/mnras/181.3.375>.
- [3] Gareth Collins, Jay Melosh, and Boris Ivanov. “Modeling damage and deformation in impact simulations”. In: *Meteoritics & Planetary Science* 39 (Feb. 2004), pp. 217–231. DOI: 10.1111/j.1945-5100.2004.tb00337.x.
- [4] Martin Jutzi, Willy Benz, and Patrick Michel. “Numerical simulations of impacts involving porous bodies”. In: *Icarus* 198.1 (Nov. 2008), pp. 242–255. ISSN: 0019-1035. DOI: 10.1016/j.icarus.2008.06.013. URL: <http://dx.doi.org/10.1016/j.icarus.2008.06.013>.
- [5] Martin Jutzi. “SPH calculations of asteroid disruptions: The role of pressure dependent failure models”. In: *Planetary and Space Science* 107 (Mar. 2015), pp. 3–9. ISSN: 0032-0633. DOI: 10.1016/j.pss.2014.09.012. URL: <http://dx.doi.org/10.1016/j.pss.2014.09.012>.
- [6] C. Schaefer et al. “A smooth particle hydrodynamics code to model collisions between solid, self-gravitating objects”. In: *Astronomy and Astrophysics* 590 (Apr. 2016), A19. ISSN: 1432-0746. DOI: 10.1051/0004-6361/201528060. URL: <http://dx.doi.org/10.1051/0004-6361/201528060>.
- [7] A.M. Stickle et al. “Modeling impact outcomes for the Double Asteroid Redirection Test (DART) mission”. In: *Procedia Engineering* 204 (2017). 14th Hypervelocity Impact Symposium 2017, HVIS2017, 24-28 April 2017, Canterbury, Kent, UK, pp. 116–123. ISSN: 1877-7058. DOI: <https://doi.org/10.1016/j.proeng.2017.09.763>. URL: <http://www.sciencedirect.com/science/article/pii/S1877705817343217>.
- [8] Sabina Raducan et al. “The role of asteroid strength, porosity and internal friction in impact momentum transfer”. In: *Icarus* 329 (Apr. 2019), pp. 282–295. DOI: 10.1016/j.icarus.2019.03.040.
- [9] NASA/Johns Hopkins APL. [Online; accessed 22-September-2020]. 2020. URL: [https://dart.jhuapl.edu/Gallery/media/graphics/lg/DART-infographic\\_v4.jpg](https://dart.jhuapl.edu/Gallery/media/graphics/lg/DART-infographic_v4.jpg).
- [10] Wikimedia Commons. *File:SmallAsteroidImpacts-Frequency-Bolide-20141114.jpg* — Wikimedia Commons, the free media repository. [Online; accessed 20-September-2020]. 2020. URL: <https://commons.wikimedia.org/w/index.php?title=File:SmallAsteroidImpacts-Frequency-Bolide-20141114.jpg&oldid=447357435>.

- [11] Wikimedia Commons. *File:SPHInterpolationColorsVerbose.svg* — *Wikimedia Commons, the free media repository*. [Online; accessed 20-September-2020]. 2020. URL: <https://commons.wikimedia.org/w/index.php?title=File:SPHInterpolationColorsVerbose.svg&oldid=446219914>.
- [12] Jacob Kegerreis. *SEAGen*. <https://github.com/jkeger/seagen>. 2020.
- [13] C.M. Schäfer et al. “A versatile smoothed particle hydrodynamics code for graphic cards”. In: *Astronomy and Computing* 33 (2020), p. 100410. ISSN: 2213-1337. DOI: <https://doi.org/10.1016/j.ascom.2020.100410>. URL: <http://www.sciencedirect.com/science/article/pii/S2213133720300640>.