CONTINUOUS EPISTEMIC GEOMETRY: RIGOROUS CGCNF FORMALISM, DISJOINT GEOMETRIC COMPILATION, AND MEASURABLE BLACK-HOLE PHASE OBSERVABLES

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Abstract. We give a mathematically rigorous continuous extension of GCNF (cGCNF), correcting imprecisions that appear in informal drafts and separating proved statements from modeling assumptions. The framework is built on finite CNF syntax over continuous literals defined as preimages of open sets under continuous maps. We prove openness and robustness of model sets, formal forbidden-region semantics, measure-theoretic #SAT analogues via windowed volume, and a disjoint geometric compilation theorem in a tame Euclidean fragment. We also prove an exponential fragmentation lower bound. The formalism is then lifted to products of Riemannian manifolds. Finally, we construct a black-hole-relevant measurable layer on parameterized Einstein initial data using robust trapped-surface literals (strict negative null expansion margins), finite template banks, and auditable volume estimation with non-asymptotic concentration bounds. The resulting objects are directly compatible with the finite-verification and coherent-flow architecture developed in companion notes.

Contents

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1. Scope, Standard, and Corrections

This manuscript has one objective: strict formal correctness of the mathematical layer. Claims are tagged as [**Proved**] or [**Model**].

Auditability rules.

- **R1. Finite syntax.** Formulas are finite CNFs; no hidden infinitary disjunctions.
- **R2. Typed literals.** Every literal specifies domain, codomain, map regularity, and acceptance set.
- **R3. Topological rigor.** Openness, closure, and robustness statements are proved with explicit hypotheses.
- **R4. Measure rigor.** Every volume statement fixes a measure space and measurability assumptions.
- **R5. Physics scope separation.** Event horizons (global objects) are not conflated with local/quasilocal trapped-surface predicates.

Critical corrections to common informal formulations.

- **C1.** A literal of the form "$F = 0$" is generally not open; robust literals must use strict margins, e.g. $F < -\tau$.
- **C2.** "$\exists S$ such that trapped" is not finite syntax; finite audits require a finite template bank or a separate approximation theorem.
- **C3.** "Satisfiable implies positive volume" is false on a compact window boundary; the interior condition is required.
- **C4.** Parameter-openness claims require joint continuity in parameter and state variables.
- **C5.** Forbidden-region identities must distinguish finite and countable unions; finite CNF gives finite union.
- **C6.** Invariance claims under coordinate changes must specify literal transport precisely.
- **C7.** On manifolds, differentiability of distance at cut loci is irrelevant for openness; continuity suffices.
- **C8.** For black-hole observables, apparent-horizon/MOTS criteria are operational; event-horizon criteria are nonlocal in time.

2. Continuous cGCNF on Product Spaces [**Proved**]

2.1. Syntax and semantics.

Assumption 2.1 (Base product space). *Fix $n \in \mathbb{N}$, metric spaces (X_i, d_i), and*

$$X := \prod_{i=1}^{n} X_i$$

with product topology.

Definition 2.2 (Continuous literal, finite clause, finite formula). *A* continuous literal *is a tuple*

$$\ell = (I_\ell, f_\ell, Y_\ell, U_\ell),$$

where $I_\ell \subseteq \{1, \dots, n\}$ is nonempty and finite, Y_ℓ is a topological space, $f_\ell : \prod_{i \in I_\ell} X_i \to Y_\ell$ is continuous, and $U_\ell \subseteq Y_\ell$ is open. Its model set is

$$\mathrm{Mod}(\ell) := \{x \in X : \ f_\ell(\pi_{I_\ell} x) \in U_\ell\}.$$

A clause *is a finite disjunction $C = \ell_1 \vee \cdots \vee \ell_m$, with*

$$\mathrm{Mod}(C) := \bigcup_{t=1}^{m} \mathrm{Mod}(\ell_t).$$

A cGCNF formula *is a finite conjunction*

$$\Phi = C_1 \wedge \cdots \wedge C_p, \qquad \mathrm{Mod}(\Phi) := \bigcap_{j=1}^{p} \mathrm{Mod}(C_j).$$

Example 2.3 (Unary distance literal)**.** *For $X_i = \mathbb{R}$, center $a \in \mathbb{R}$, and open $A \subseteq [0,\infty)$,*

$$\ell(i,a,A): \quad |x_i - a| \in A$$

is Definition 2.2 with $I_\ell = \{i\}$, $f_\ell(x_i) = |x_i - a|$, $Y_\ell = [0,\infty)$, and $U_\ell = A$.

Theorem 2.4 (Openness of model sets)**.** *Every literal model set $\mathrm{Mod}(\ell)$ is open in X. Hence every clause model set and every finite cGCNF model set $\mathrm{Mod}(\Phi)$ are open.*

Proof. The map $x \mapsto \pi_{I_\ell}(x)$ is continuous, so

$$x \mapsto f_\ell(\pi_{I_\ell}(x))$$

is continuous. Therefore

$$\mathrm{Mod}(\ell) = \left(f_\ell \circ \pi_{I_\ell}\right)^{-1}(U_\ell)$$

is open. Finite unions and finite intersections preserve openness. □

Corollary 2.5 (Robust satisfiability)**.** *If $x^\star \in \mathrm{Mod}(\Phi)$, then there exists a neighborhood N of $x^\star$ in X such that $N \subseteq \mathrm{Mod}(\Phi)$. If X is metrized by any metric compatible with product topology, then $B(x^\star,\delta) \subseteq \mathrm{Mod}(\Phi)$ for some $\delta > 0$.*

Proof. Apply Theorem 2.4. □

2.2. **Forbidden-region semantics.**

Definition 2.6 (Clause falsification region and total forbidden region)**.** *For a clause $C = \ell_1 \vee \cdots \vee \ell_m$, define*

$$\mathrm{Forb}(C) := \bigcap_{t=1}^{m} (X \setminus \mathrm{Mod}(\ell_t)).$$

For $\Phi = \bigwedge_{j=1}^{p} C_j$, define

$$\mathcal{U}(\Phi) := \bigcup_{j=1}^{p} \mathrm{Forb}(C_j).$$

Proposition 2.7 (Exact complement identity)**.** *For every finite cGCNF formula Φ,*

$$\mathrm{Mod}(\Phi) = X \setminus \mathcal{U}(\Phi).$$

Each $\mathrm{Forb}(C_j)$ is closed, and $\mathcal{U}(\Phi)$ is a finite union of closed sets.

Proof.

$$x \notin \mathrm{Mod}(\Phi) \iff \exists j,\ x \notin \mathrm{Mod}(C_j) \iff \exists j,\ \forall t,\ x \notin \mathrm{Mod}(\ell_t) \iff x \in \bigcup_j \mathrm{Forb}(C_j).$$

Closedness follows from Theorem 2.4. □

2.3. Parameter-dependent formulas.

Assumption 2.8 (Jointly continuous parameterization). *Let $(\Theta, \mathrm{dist}_\Theta)$ be a metric parameter space. Fix a finite literal dictionary Λ. For each $\alpha \in \Lambda$, let Y_α be topological, $U_\alpha \subseteq Y_\alpha$ open, and*

$$F_\alpha : \Theta \times X \to Y_\alpha$$

continuous. For each $\theta \in \Theta$, literals are $L_\alpha(\theta, x) : \; F_\alpha(\theta, x) \in U_\alpha$. Each clause and formula uses finite subsets of Λ, independent of θ.

Theorem 2.9 (Openness in (θ, x) and in θ). *Under Assumption 2.8, the set*

$$\mathcal{S} := \{(\theta, x) \in \Theta \times X : \; x \models \Phi_\theta\}$$

is open. Consequently,

$$\mathcal{E} := \{\theta \in \Theta : \exists x \in X, \; x \models \Phi_\theta\}$$

is open in Θ.

Proof. Each literal satisfaction set

$$\{(\theta, x) : F_\alpha(\theta, x) \in U_\alpha\} = F_\alpha^{-1}(U_\alpha)$$

is open. Finite Boolean composition in CNF preserves openness, so $\mathcal{S}$ is open. For each fixed $x \in X$, the section

$$\mathcal{S}_x := \{\theta : (\theta, x) \in \mathcal{S}\}$$

is open as preimage of $\mathcal{S}$ under $\theta \mapsto (\theta, x)$. Then

$$\mathcal{E} = \bigcup_{x \in X} \mathcal{S}_x$$

is open. □

Corollary 2.10 (Windowed satisfiability in fixed K). *For any subset $K \subseteq X$,*

$$\mathcal{E}_K := \{\theta : \mathrm{Mod}(\Phi_\theta) \cap K \neq \emptyset\}$$

is open in Θ.

Proof.

$$\mathcal{E}_K = \bigcup_{x \in K} \mathcal{S}_x,$$

union of open sets. □

Remark 2.11 (Why strict inequalities matter). *Corollary 2.10 can fail if a literal uses a non-open acceptance set (for example $F = 0$ or $F \leq 0$). Robust margins $F < -\tau$ are mathematically cleaner and numerically stable.*

3. Measure Semantics and Continuous #SAT [Proved]

Assumption 3.1 (Measure layer). *Let $(X, \mathcal{B}(X), \mu)$ be a Borel measure space with μ Radon on X, finite on compact sets, and strictly positive on nonempty open sets. Fix compact $K \subset X$.*

Definition 3.2 (Windowed model volume). *For a finite cGCNF formula Φ, define*

$$V_K(\Phi) := \mu(\mathrm{Mod}(\Phi) \cap K).$$

This is the continuous #SAT analogue on the finite probe K.

Proposition 3.3 (Measurability and complement formula). *Under Assumption 3.1,*

$$V_K(\Phi) = \mu(K) - \mu(\mathcal{U}(\Phi) \cap K)$$

and both terms are well-defined.

Proof. By Theorem 2.4, $\mathrm{Mod}(\Phi)$ is Borel. By Proposition 2.7,

$$\mathrm{Mod}(\Phi) \cap K = K \setminus \big(\mathcal{U}(\Phi) \cap K\big),$$

and $\mu(K) < \infty$. □

Theorem 3.4 (Interior witness implies positive window volume)**.** *If* $\mathrm{Mod}(\Phi) \cap \mathrm{int}(K) \neq \emptyset$, *then* $V_K(\Phi) > 0$.

Proof. Take $x^\star \in \mathrm{Mod}(\Phi) \cap \mathrm{int}(K)$. Since both sets are open around $x^\star$, there is nonempty open $O \subseteq \mathrm{Mod}(\Phi) \cap K$. Assumption 3.1 gives $\mu(O) > 0$, so $V_K(\Phi) \geq \mu(O) > 0$. □

Proposition 3.5 (Monotonicity)**.** *If* $\Phi \Rightarrow \Psi$ *pointwise on* X, *then* $V_K(\Phi) \leq V_K(\Psi)$. *If* $K_1 \subseteq K_2$, *then* $V_{K_1}(\Phi) \leq V_{K_2}(\Phi)$.

Proof. Both claims follow from monotonicity of measure. □

Theorem 3.6 (Lower semicontinuity of parameter volume)**.** *Assume Assumptions 2.8 and 3.1. Define*

$$v_K(\theta) := \mu\big(\mathrm{Mod}(\Phi_\theta) \cap K\big).$$

Then $v_K : \Theta \to [0, \infty)$ *is lower semicontinuous.*

Proof. Let $\mathbf{1}_{\mathcal{S}}$ be the indicator of

$$\mathcal{S} = \{(\theta, x) : x \models \Phi_\theta\},$$

which is open by Theorem 2.9. Hence for each fixed x, the map

$$\theta \mapsto \mathbf{1}_{\mathcal{S}}(\theta, x)$$

is lower semicontinuous. Since

$$v_K(\theta) = \int_K \mathbf{1}_{\mathcal{S}}(\theta, x)\, d\mu(x),$$

Fatou's lemma yields lower semicontinuity. □

Theorem 3.7 (Auditable Monte Carlo estimator)**.** *Assume* $\mu(K) > 0$, *and sample* $\theta^{(1)}, \ldots, \theta^{(T)}$ *i.i.d. uniformly on* K *(or from known density with importance weights). Define*

$$I_t := \mathbf{1}_{\{\theta^{(t)} \in \mathrm{Mod}(\Phi)\}}, \qquad \widehat{p}_T := \frac{1}{T}\sum_{t=1}^{T} I_t, \qquad \widehat{V}_T := \mu(K)\widehat{p}_T.$$

Then $\widehat{V}_T$ *is unbiased for* $V_K(\Phi)$, *and for all* $\eta > 0$,

$$\Pr\Big(\Big|\widehat{V}_T - V_K(\Phi)\Big| \geq \eta\, \mu(K)\Big) \leq 2e^{-2T\eta^2}.$$

Proof. Unbiasedness is immediate from Bernoulli expectation. The concentration bound is Hoeffding's inequality applied to bounded i.i.d. variables $I_t \in [0, 1]$. □

4. Tame Euclidean Fragment and Disjoint Compilation [Proved]

4.1. Tame unary radial fragment.

Assumption 4.1 (Tame Euclidean setup)**.** *Let* $X = \mathbb{R}^n$, $K = \prod_{i=1}^{n} [L_i, U_i]$ *compact. Literals are unary and radial:*

$$\ell(i, a, A) : \ |x_i - a| \in A,$$

where each $A \subset [0, \infty)$ *is a finite union of open intervals.*

Lemma 4.2 (Finite interval decomposition per coordinate)**.** *Under Assumption 4.1, for each literal* $\ell(i, a, A)$*,*

$$\mathrm{Mod}(\ell) \cap K = \left(P_{i,\ell} \times \prod_{r \neq i} [L_r, U_r] \right),$$

where $P_{i,\ell} \subset [L_i, U_i]$ *is a finite union of open intervals. Its complement in* K *is*

$$([L_i, U_i] \setminus P_{i,\ell}) \times \prod_{r \neq i} [L_r, U_r],$$

with $[L_i, U_i] \setminus P_{i,\ell}$ *a finite union of closed intervals.*

Proof. Since A is finite union of open intervals, $\{x_i : |x_i - a| \in A\} \cap [L_i, U_i]$ is finite union of open intervals by explicit inversion of $x \mapsto |x - a|$ on bounded domains. The product form is immediate because the literal is unary. □

Proposition 4.3 (Clause forbidden set as finite union of axis-aligned boxes)**.** *Let* $C = \ell_1 \vee \cdots \vee \ell_m$ *be a clause in the tame fragment. Then* $\mathrm{Forb}(C) \cap K$ *is a finite union of closed axis-aligned boxes. If coordinate* i *contributes* q_i *closed intervals after intersecting all relevant complements, then*

$$\#\mathit{boxes} \leq \prod_{i=1}^{n} q_i.$$

Proof. By definition,

$$\mathrm{Forb}(C) \cap K = \bigcap_{t=1}^{m} \big(K \setminus (\mathrm{Mod}(\ell_t) \cap K)\big).$$

Each term is a coordinate cylinder from Lemma 4.2. Intersections factor coordinate-wise into finite unions of closed intervals, whose finite products are boxes. □

4.2. **Disjoint geometric compilation.**

Remark 4.4 (On boxes / Sobre cajas)**.** ***Español.*** *En este trabajo, una* caja alineada con los ejes *es cualquier subconjunto de* $\mathbb{R}^n$ *de la forma*

$$\prod_{i=1}^{n} I_i,$$

donde cada I_i *es un intervalo acotado (no necesariamente cerrado) de uno de los tipos* $[a, b]$*,* $[a, b)$*,* $(a, b]$*,* (a, b) *o un punto singular* $\{a\}$*. La medida de Lebesgue de una caja no depende de si se incluyen o no los puntos frontera, por lo que esta flexibilidad no afecta ninguno de los resultados de volumen ni las operaciones de compilación disjunta. En particular, la construcción lexicográfica del Lema 4.6 produce cajas que pueden tener extremos semiabiertos, garantizando la disjunción estricta sin pérdida de generalidad.*

English. *In this work, an* axis-aligned box *is any subset of* $\mathbb{R}^n$ *of the form*

$$\prod_{i=1}^{n} I_i,$$

where each I_i *is a bounded interval (not necessarily closed) of one of the types* $[a, b]$*,* $[a, b)$*,* $(a, b]$*,* (a, b) *or a singleton* $\{a\}$*. The Lebesgue measure of a box does not depend on whether boundary points are included; hence this flexibility does not affect any volume statements or the disjoint compilation procedures. In particular, the lexicographic slab construction in Lemma 4.6 yields boxes that may have half-open endpoints, ensuring strict disjointness without loss of generality.*

Definition 4.5 (Box difference operator)**.** *For closed boxes $P, R \subseteq K$,* $\mathrm{BoxDiff}(P,R)$ *is any finite disjoint family of axis-aligned boxes (allowing half-open endpoints) whose union equals $P \setminus R$.*

Lemma 4.6 (Size bound for box difference)**.** *For $P, R \subset \mathbb{R}^n$ closed boxes, there exists a disjoint decomposition*

$$P \setminus R = \bigsqcup_{t=1}^{M} B_t$$

with $M \leq 2n$. Moreover, one can choose the B_t to be axis-aligned boxes (allowing half-open endpoints) obtained by an explicit lexicographic slab construction.

Proof. Write

$$P = \prod_{i=1}^{n} [p_i^-, p_i^+], \qquad R = \prod_{i=1}^{n} [r_i^-, r_i^+].$$

If $P \cap R = \emptyset$, take $M = 1$ and $B_1 = P$. If $P \subseteq R$, take $M = 0$.

Assume now $\emptyset \neq P \cap R \neq P$. Define the clipped box

$$\widetilde{R} := P \cap R = \prod_{i=1}^{n} [\alpha_i, \beta_i], \qquad \alpha_i := \max(p_i^-, r_i^-), \quad \beta_i := \min(p_i^+, r_i^+).$$

Then $P \setminus R = P \setminus \widetilde{R}$. For each $i \in \{1, \dots, n\}$ define the *lexicographic slabs*

$$L_i := \Big(\prod_{j<i} [\alpha_j, \beta_j]\Big) \times [p_i^-, \alpha_i) \times \Big(\prod_{j>i} [p_j^-, p_j^+]\Big),$$

$$U_i := \Big(\prod_{j<i} [\alpha_j, \beta_j]\Big) \times (\beta_i, p_i^+] \times \Big(\prod_{j>i} [p_j^-, p_j^+]\Big).$$

Let

$$\mathcal{D} := \{L_i : \; L_i \neq \emptyset\} \;\cup\; \{U_i : \; U_i \neq \emptyset\}.$$

By construction $|\mathcal{D}| \leq 2n$.

Claim 1: $\bigcup_{B \in \mathcal{D}} B \subseteq P \setminus \widetilde{R}$. Indeed, if $x \in L_i$ then $x_i < \alpha_i$, hence $x \notin \widetilde{R}$; if $x \in U_i$ then $x_i > \beta_i$, hence $x \notin \widetilde{R}$. Also every slab lies in P by definition.

Claim 2: $P \setminus \widetilde{R} \subseteq \bigcup_{B \in \mathcal{D}} B$. Let $x \in P \setminus \widetilde{R}$. Choose the *lexicographically first* index

$$i^\star := \min\{\, i : \; x_i \notin [\alpha_i, \beta_i] \,\}.$$

Then $x_j \in [\alpha_j, \beta_j]$ for all $j < i^\star$, and either $x_{i^\star} < \alpha_{i^\star}$ or $x_{i^\star} > \beta_{i^\star}$. In the first case $x \in L_{i^\star}$; in the second, $x \in U_{i^\star}$.

Claim 3: The family $\mathcal{D}$ is pairwise disjoint. First, $L_i \cap U_i = \emptyset$ since $[p_i^-, \alpha_i) \cap (\beta_i, p_i^+] = \emptyset$. Next, if $i < j$, then every point of a slab of index j has its i-th coordinate in $[\alpha_i, \beta_i]$ (because i belongs to the frozen prefix), whereas every point of a slab of index i has its i-th coordinate outside $[\alpha_i, \beta_i]$ (either $< \alpha_i$ or $> \beta_i$). Hence no slab of index i can intersect a slab of index j.

Claims 1–3 yield the disjoint union identity

$$P \setminus \widetilde{R} = \bigsqcup_{B \in \mathcal{D}} B.$$

Enumerating $\mathcal{D}$ as $\{B_1, \dots, B_M\}$ completes the proof with $M \leq 2n$. □

Definition 4.7 (Incremental disjoint insertion)**.** *Let $\mathcal{U}$ be a finite disjoint family of boxes and Q a box. Define* $\mathrm{AddBox}(\mathcal{U}, Q)$*:*

(i) Set $\mathcal{R} := \{Q\}$.

(ii) For each $B \in \mathcal{U}$, replace every $P \in \mathcal{R}$ by $\mathrm{BoxDiff}(P, B)$*.*
(iii) Output $\mathcal{U} \cup \mathcal{R}$.

Proposition 4.8 (AddBox correctness)**.** *If $\mathcal{U}$ is disjoint,* $\mathrm{AddBox}(\mathcal{U}, Q)$ *is disjoint and*

$$\bigcup \mathrm{AddBox}(\mathcal{U}, Q) = \Big(\bigcup \mathcal{U}\Big) \cup Q.$$

Proof. At each subtraction step, pieces replacing P are disjoint and equal $P \setminus B$, so they avoid B. After processing all $B \in \mathcal{U}$, $\mathcal{R}$ equals $Q \setminus \cup\mathcal{U}$, hence is disjoint from $\mathcal{U}$. Union identity follows. □

Theorem 4.9 (Disjoint compilation of $\mathcal{U}(\Phi) \cap K$)**.** *Under Assumption 4.1, let*

$$\mathcal{U}(\Phi) \cap K = \bigcup_{j=1}^{M} Q_j$$

be any finite box decomposition (from Proposition 4.3 over all clauses). Define recursively

$$\mathcal{U}_0 := \emptyset, \qquad \mathcal{U}_t := \mathrm{AddBox}(\mathcal{U}_{t-1}, Q_t), \quad t = 1, \ldots, M.$$

Then $\mathcal{U}_M$ is disjoint and

$$\bigcup \mathcal{U}_M = \mathcal{U}(\Phi) \cap K.$$

Proof. Induct on t using AddBox correctness. □

Corollary 4.10 (Exact volume from disjoint pieces)**.** *Under the hypotheses of Theorem 4.9,*

$$\mu(\mathcal{U}(\Phi) \cap K) = \sum_{B \in \mathcal{U}_M} \mu(B),$$

hence

$$V_K(\Phi) = \mu(K) - \sum_{B \in \mathcal{U}_M} \mu(B).$$

4.3. Exponential fragmentation barrier.

Theorem 4.11 (A 2^n lower bound)**.** *In $K = [0,1]^n$, define*

$$\Phi_n := \bigwedge_{i=1}^{n} \Big((x_i \in (0, \tfrac{1}{3})) \vee (x_i \in (\tfrac{2}{3}, 1)) \Big).$$

Then $\mathrm{Mod}(\Phi_n)$ *has exactly 2^n connected components. Any exact disjoint axis-aligned box decomposition of* $\mathrm{Mod}(\Phi_n)$ *uses at least 2^n boxes.*

Proof. Directly,

$$\mathrm{Mod}(\Phi_n) = \prod_{i=1}^{n} \Big((0, \tfrac{1}{3}) \cup (\tfrac{2}{3}, 1) \Big),$$

which is the disjoint union of 2^n open boxes indexed by left/right choices in each coordinate. These are connected components. Every box is connected, hence contained in one component. To cover all components exactly, at least one box per component is required. □

5. Extension to Products of Riemannian Manifolds [Proved]

Assumption 5.1 (Riemannian product setting). *For each i, let (M_i, g_i) be a smooth connected Riemannian manifold with geodesic distance d_{g_i}. Set*

$$M := \prod_{i=1}^{n} M_i$$

with product topology and product Borel measure induced by Riemannian volume.

Definition 5.2 (Intrinsic distance literal). *For $a_i \in M_i$ and open $A \subset [0, \infty)$, define*

$$\ell(i, a_i, A) : \ d_{g_i}(x_i, a_i) \in A.$$

Theorem 5.3 (Intrinsic openness and robustness on M). *Under Assumption 5.1, all results of Section 2 remain valid. In particular,* $\mathrm{Mod}(\Phi) \subset M$ *is open and robust.*

Proof. The map $x_i \mapsto d_{g_i}(x_i, a_i)$ is continuous (in fact 1-Lipschitz), so Theorem 2.4 applies verbatim. □

Remark 5.4 (Cut locus is not an obstruction). *Distance may fail to be smooth at cut loci, but continuity is enough for topological statements (openness, robustness, Borel measurability).*

Theorem 5.5 (Positive volume on compact windows in manifolds). *Let $K \subset M$ be compact. If* $\mathrm{Mod}(\Phi) \cap \mathrm{int}(K) \neq \emptyset$, *then*

$$\mathrm{Vol}_g(\mathrm{Mod}(\Phi) \cap K) > 0.$$

Proof. Apply Theorem 3.4 with $\mu = \mathrm{Vol}_g$. □

Proposition 5.6 (Chart-wise approximation of measurable forbidden regions). *Let $A \subset K \subset M$ be Borel with $\mathrm{Vol}_g(\partial A) = 0$, K compact. For every $\eta > 0$, there exist finite unions $A_\eta^- \subseteq A \subseteq A_\eta^+$ such that:*

(i) each $A_\eta^\pm$ is a finite union of chart-box images,
(ii) $\mathrm{Vol}_g(A_\eta^+ \setminus A_\eta^-) < \eta$.

Proof. By regularity of Radon measure, pick compact $C \subseteq A \subseteq O$ open with $\mathrm{Vol}_g(O \setminus C) < \eta$. Cover K by finitely many charts. In each chart, approximate C from inside and O from outside by finite unions of Euclidean boxes. Push forward by chart maps and combine over the finite atlas. □

6. Black-Hole Measurable Layer [Proved core + Model interfaces]

6.1. Parameterization of initial data.

Assumption 6.1 (Finite-dimensional initial-data family). *Fix a smooth 3-manifold Σ, and a compact parameter set $\Theta \subset \mathbb{R}^d$. Let*

$$\theta \mapsto (h(\theta), K(\theta))$$

be a C^1 map from Θ into asymptotically flat Einstein initial data of regularity at least C^2 in space and continuous in θ in the corresponding norm.

Remark 6.2 (Operational scope). *This layer targets quasilocal trapped-surface observables in initial data. It does* not *define event horizons, which are global spacetime objects.*

6.2. **Robust trapped-surface literals.**

Definition 6.3 (Null expansion functional)**.** *Fix a closed embedded C^2 surface $S \subset \Sigma$ and a sign convention for the outer future null expansion $\theta_+(p; S, \theta)$. Define*

$$F_S(\theta) := \max_{p \in S} \theta_+(p; S, \theta), \qquad G_S(\theta) := \min_{p \in S} \theta_+(p; S, \theta).$$

Proposition 6.4 (Continuity of F_S, G_S)**.** *Under Assumption 6.1, F_S and G_S are continuous on Θ.*

Proof. The map $(\theta, p) \mapsto \theta_+(p; S, \theta)$ is continuous on compact S. Max/min over compact fibers preserve continuity. □

Definition 6.5 (Robust BH literals)**.** *For $\tau > 0$:*

$$\begin{aligned} L^{\mathrm{trap}}_{S,\tau}(\theta) : &\quad F_S(\theta) < -\tau, \\ L^{\mathrm{outer}+}_{S,\tau}(\theta) : &\quad G_S(\theta) > \tau. \end{aligned}$$

Corollary 6.6 (Openness of robust BH literals)**.** *Each set*

$$\{\theta : L^{\mathrm{trap}}_{S,\tau}(\theta)\}, \qquad \{\theta : L^{\mathrm{outer}+}_{S,\tau}(\theta)\}$$

is open in Θ.

Proof. Apply Proposition 6.4 and openness of strict inequalities. □

Remark 6.7 (Why $F_S = 0$ is excluded)**.** *The exact MOTS condition $F_S = 0$ defines a codimension-type boundary object in parameter space and is not open. For cGCNF robustness and numerical certification, strict margins are mandatory.*

6.3. **Finite-template BH cGCNF.**

Assumption 6.8 (Finite template banks)**.** *Fix finite banks:*

$$\mathcal{S}_+, \ \mathcal{S}_-, \ \mathcal{S}_c, \ \mathcal{S}_{\mathrm{out}}$$

of closed C^2 surfaces in Σ, intended respectively for the two individual horizons, common horizon candidates, and outer anti-common tests.

Definition 6.9 (Finite BH phase formulas)**.** *For fixed $\tau > 0$, define:*

$$\begin{aligned} C_+(\theta) &:= \bigvee_{S \in \mathcal{S}_+} L^{\mathrm{trap}}_{S,\tau}(\theta), \\ C_-(\theta) &:= \bigvee_{S \in \mathcal{S}_-} L^{\mathrm{trap}}_{S,\tau}(\theta), \\ C_c(\theta) &:= \bigvee_{S \in \mathcal{S}_c} L^{\mathrm{trap}}_{S,\tau}(\theta), \\ C_{\mathrm{out}}(\theta) &:= \bigvee_{S \in \mathcal{S}_{\mathrm{out}}} L^{\mathrm{outer}+}_{S,\tau}(\theta). \end{aligned}$$

Then

$$\begin{aligned} \Phi_{\mathrm{common}} &:= C_+ \wedge C_- \wedge C_c, \\ \Phi_{\mathrm{sep}} &:= C_+ \wedge C_- \wedge C_{\mathrm{out}}. \end{aligned}$$

Theorem 6.10 (Topological and measurable well-posedness of BH phases)**.** *Under Assumptions 6.1 and 6.8, both*

$$\mathrm{Mod}(\Phi_{\mathrm{common}}), \qquad \mathrm{Mod}(\Phi_{\mathrm{sep}})$$

are open in Θ, hence Borel measurable.

Proof. Finite OR/AND of open sets from robust literals. □

Definition 6.11 (Ideal existential trapped set [**Model**]). *Given an admissible (possibly infinite) class $\mathcal{S}_\infty$, define*

$$E_\tau := \left\{ \theta \in \Theta : \inf_{S \in \mathcal{S}_\infty} F_S(\theta) < -\tau \right\}.$$

Proposition 6.12 (Finite bank as certified inner approximation). *If $\mathcal{S}_N \subseteq \mathcal{S}_\infty$, then*

$$E_\tau^{(N)} := \left\{ \theta : \min_{S \in \mathcal{S}_N} F_S(\theta) < -\tau \right\} \subseteq E_\tau.$$

Proof. Immediate from $\mathcal{S}_N \subseteq \mathcal{S}_\infty$. □

Theorem 6.13 (Uniform-approximation transfer with margin). *Assume:*

(i) $\mathcal{S}_\infty$ *is compact in a surface metric* $\mathrm{dist}_{\mathcal{S}}$,
(ii) $F : \Theta \times \mathcal{S}_\infty \to \mathbb{R}$, $F(\theta, S) = F_S(\theta)$, *is jointly continuous,*
(iii) $\mathcal{S}_N$ *is an* ε_N*-net in* $\mathcal{S}_\infty$,
(iv) ω *is a modulus such that*

$$\mathrm{dist}_{\mathcal{S}}(S, S') \le r \Rightarrow |F(\theta, S) - F(\theta, S')| \le \omega(r) \ \forall \theta.$$

Then

$$E_{\tau+\omega(\varepsilon_N)} \subseteq E_\tau^{(N)} \subseteq E_\tau.$$

Proof. Right inclusion is Proposition 6.12. For left inclusion, take $\theta \in E_{\tau+\omega(\varepsilon_N)}$. Then some $S \in \mathcal{S}_\infty$ satisfies $F(\theta, S) < -\tau - \omega(\varepsilon_N)$. Choose $S_N \in \mathcal{S}_N$ with $\mathrm{dist}_{\mathcal{S}}(S, S_N) \le \varepsilon_N$. Then

$$F(\theta, S_N) \le F(\theta, S) + \omega(\varepsilon_N) < -\tau,$$

so $\theta \in E_\tau^{(N)}$. □

6.4. Measurable outputs and error bars.

Definition 6.14 (BH phase volume in a probe window). *For compact $K \subseteq \Theta$, define*

$$V_K^{\mathrm{common}} := \mu_\Theta\big(K \cap \mathrm{Mod}(\Phi_{\mathrm{common}})\big), \qquad V_K^{\mathrm{sep}} := \mu_\Theta\big(K \cap \mathrm{Mod}(\Phi_{\mathrm{sep}})\big).$$

Theorem 6.15 (Finite-sample confidence for BH phase volume). *Let $\widehat{V}_T$ be the Monte Carlo estimator from Theorem 3.7 applied to Φ_{common} (or Φ_{sep}). For confidence level $1 - \delta \in (0, 1)$,*

$$\left| \widehat{V}_T - V_K \right| \le \mu_\Theta(K) \sqrt{\frac{\log(2/\delta)}{2T}}$$

with probability at least $1 - \delta$.

Proof. Set $\eta = \sqrt{\log(2/\delta)/(2T)}$ in Theorem 3.7. □

Definition 6.16 (Gray zone near critical boundaries). *Define*

$$\mathcal{G} := K \setminus \big(\mathrm{Mod}(\Phi_{\mathrm{common}}) \cup \mathrm{Mod}(\Phi_{\mathrm{sep}})\big).$$

$\mathcal{G}$ is the certified uncertainty zone induced by finite banks and strict margins.

Remark 6.17 (Scientifically correct interpretation). *$\mathcal{G}$ is not a failure. It is the mathematically unavoidable interface between finite robust certification and sharp phase boundaries. Theorem 6.13 gives a controlled path to shrink $\mathcal{G}$ by denser template banks and smaller margins.*

6.5. Connection to coherent flow in finite theory space [Proved].

Definition 6.18 (Finite BH literal dictionary and free energy)**.** *Fix a finite dictionary $\mathcal{W}$ of robust literals (from banks and thresholds). For $T \subseteq \mathcal{W}$, let Φ_T be the conjunction of all literals in T (single-literal clauses). Define admissible theories*

$$\mathcal{K}_{\mathrm{BH}} := \{T \subseteq \mathcal{W} : \ V_K(\Phi_T) > 0\}.$$

For $\alpha, \beta, \gamma > 0$,

$$\mathcal{F}_{\mathrm{BH}}(T) := \alpha\, E_{\mathrm{ctr}}(T) + \beta|T| - \gamma \log V_K(\Phi_T), \qquad T \in \mathcal{K}_{\mathrm{BH}}.$$

Proposition 6.19 (Existence of coherent islands in BH literal space)**.** *If $\mathcal{K}_{\mathrm{BH}} \neq \emptyset$, $\mathcal{F}_{\mathrm{BH}}$ attains a global minimum on $\mathcal{K}_{\mathrm{BH}}$; every global minimizer is a coherent island with respect to Hamming-1 neighborhood.*

Proof. $\mathcal{W}$ is finite, hence $\mathcal{K}_{\mathrm{BH}}$ is finite. A real-valued function on a finite nonempty set attains a minimum. Minimality implies local minimality. □

7. Concise Formal Roadmap to Publishable BH Outputs

The previous sections imply the following complete pipeline:

- **P1.** Choose compact parameter window $K \subset \Theta$ and strict margin $\tau > 0$.
- **P2.** Build finite banks $\mathcal{S}_+, \mathcal{S}_-, \mathcal{S}_c, \mathcal{S}_{\mathrm{out}}$.
- **P3.** Define robust cGCNF phases $\Phi_{\mathrm{common}}, \Phi_{\mathrm{sep}}$.
- **P4.** Estimate $V_K^{\mathrm{common}}, V_K^{\mathrm{sep}}$ with Theorem 3.7.
- **P5.** Report confidence intervals and gray-zone volume $\mu_\Theta(\mathcal{G})$.
- **P6.** Refine banks and margins; use Theorem 6.13 to certify convergence.

Every step is finite, auditable, and mathematically typed.

8. Conclusion

The continuous formalism is now fully rigorous:

- **Q1.** cGCNF semantics is topologically intrinsic and robust by construction.
- **Q2.** Forbidden-region geometry admits exact disjoint compilation in tame fragments.
- **Q3.** Exponential fragmentation is a real structural barrier, not a numerical artifact.
- **Q4.** The manifold lift preserves the topological core.
- **Q5.** Black-hole-relevant observables become measurable cGCNF objects once strict trapped-surface margins and finite template banks are imposed.
- **Q6.** Volume-based outputs and uncertainty quantification are mathematically controlled and reproducible.

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