COHERENT FLOW: A COMPANION NOTE TO *EPISTEMIC GEOMETRY*

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Abstract. This note formalizes *coherent flow* as a finite, auditable dynamics over theory space and its probabilistic geometric counterpart. Within a finite window of formulas and a finite paraconsistent semantics, we define a free-energy functional on admissible theories, prove existence of local minima (*coherent islands*), and construct both a deterministic descent operator and a Metropolis–Hastings kernel targeting the Gibbs measure $\pi(K) \propto e^{-\lambda \mathcal{F}(K)}$. In parallel, we present a continuous-time coherent flow on fixed-marginal interface sheets, obtained by orthogonal projection onto the additive subspace $\mathcal{U} = \{a_i + b_j\}$ in the weighted inner product $\langle \cdot, \cdot \rangle_p$, and prove an exact Kullback–Leibler Lyapunov identity. Finally, we state the logical–probabilistic Gibbs bridge $K \mapsto p_K$ and record the precise scope of the classical incompleteness obstruction. The presentation is designed to be stylistically consistent with *Epistemic Geometry* and to keep all claims within explicit finite hypotheses.

Contents

1. Project norm, scope, and architecture

This document is intended as a parallel module to *Epistemic Geometry*: it isolates a self-contained dynamical layer (discrete and continuous) that acts on theories and on the probability simplex, while preserving the same meta-principle: *finite regimes are auditable via explicit constraints and metric invariants.*

Auditability rules.

- **N1. Ideal object + finite probes.** Every infinite/ideal object is paired with a family of finite, auditable probes (finite volumes, truncations, certificates).
- **N2. Quantifiers explicit.** Domains and quantifiers are stated.
- **N3. Proof or reproducible protocol.** A **[Proved]** claim includes a complete proof or an explicit finite verification protocol.

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N4. Certificate typing. Probes are typed as: constructive witness, finite refutation, or numerical approximation.
N5. Layering. Claims are tagged as **[Proved]**, **[Model]**, or **[Speculative]**.

1.1. Global finite hypotheses.

Fix the following data.

(H1) A finite set of atoms At and a finite *window* of formulas $\mathcal{W} \subseteq \mathcal{L}$, closed under negation.
(H2) Belnap–Dunn four-valued semantics on $\mathbb{V} := \{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}$ with designated set $\mathcal{D} := \{\mathbf{T}, \mathbf{B}\}$.
(H3) The valuation space is finite:

$$\Omega := \mathbb{V}^{\mathsf{At}} = \{\omega : \mathsf{At} \to \mathbb{V}\}.$$

(H4) A reference measure μ on Ω with full support: $\mu(\omega) > 0$ for all $\omega \in \Omega$.
(H5) Contradiction weights $w_\varphi > 0$ for all $\varphi \in \mathcal{W}$.
(H6) Energy parameters $\alpha, \beta, \gamma > 0$.

1.2. Semantics, theories, and free energy.

Definition 1.1 (Designated satisfaction)**.** *For $\omega \in \Omega$ and $\varphi \in \mathcal{W}$, write $\omega \models_D \varphi$ if the truth value of φ at ω lies in $\mathcal{D}$.*

Definition 1.2 (Theory, models, and volume)**.** *A* theory *is a subset $K \subseteq \mathcal{W}$. Define*

$$\mathrm{Mod}(K) := \{\omega \in \Omega : \forall \varphi \in K,\ \omega \models_D \varphi\}, \qquad V(K) := \mu(\mathrm{Mod}(K)).$$

Definition 1.3 (Contradiction cost and free energy)**.** *For $K \subseteq \mathcal{W}$, define*

$$\mathrm{Con}(K) := \{\varphi \in \mathcal{W} : \varphi \in K \text{ and } \neg\varphi \in K\}, \qquad E_{\mathrm{ctr}}(K) := \sum_{\varphi \in \mathrm{Con}(K)} w_\varphi, \qquad C(K) := |K|.$$

The admissible theory space is

$$\mathcal{K} := \{K \subseteq \mathcal{W} : V(K) > 0\}.$$

The free energy is

$$\mathcal{F}(K) := \alpha\, E_{\mathrm{ctr}}(K) + \beta\, C(K) - \gamma \log V(K), \qquad K \in \mathcal{K}.$$

Remark 1.4 (Finiteness and non-emptiness)**.** *Under (H1)–(H4), $\mathcal{K}$ is finite because $\mathcal{W}$ is finite, and $\mathcal{K}$ is non-empty because the empty theory $\emptyset$ belongs to $\mathcal{K}$ with $V(\emptyset) = \mu(\Omega) > 0$.*

Remark 1.5 (Relation to epistemic geometry)**.** *The term $-\log V(K)$ penalizes semantic rarity: it measures how much* designated-model mass *remains compatible with K. In* Epistemic Geometry *(Module C), curvature κ measures a syntax–semantics gap as a KL distance to a model family. The coherent-flow functionals in this note can be read as an operational landscape on which deterministic and stochastic dynamics perform auditable descent, while the Gibbs map in §6 links theories to probability distributions where κ lives.*

2. Module CF: discrete coherent islands and deterministic descent [Proved]

2.1. Neighborhoods and coherent islands.

Definition 2.1 (Neighborhood and island)**.** *For $K \in \mathcal{K}$, define the Hamming neighborhood in theory space*

$$\mathcal{N}(K) := \{H \in \mathcal{K} : |H \triangle K| = 1\}, \qquad \overline{\mathcal{N}}(K) := \mathcal{N}(K) \cup \{K\},$$

where $\triangle$ denotes symmetric difference. A coherent island *is a $K \in \mathcal{K}$ such that*

$$\mathcal{F}(K) \le \mathcal{F}(H) \quad \forall H \in \mathcal{N}(K).$$

Theorem 2.2 (Existence of islands)**.** *There exists at least one coherent island.*

Proof. Since $\mathcal{K}$ is finite and non-empty, $\mathcal{F}$ attains a global minimum $K^\star \in \mathcal{K}$. Then $\mathcal{F}(K^\star) \le \mathcal{F}(H)$ for all $H \in \mathcal{K}$, hence in particular for all $H \in \mathcal{N}(K^\star)$. Thus $K^\star$ is an island. □

2.2. A deterministic descent operator.

Definition 2.3 (Deterministic coherent operator U). *Fix an auxiliary total order $\prec$ on $\mathcal{K}$. Define $U : \mathcal{K} \to \mathcal{K}$ by:*

(i) If $\mathcal{F}(K) \le \mathcal{F}(H)$ for every $H \in \mathcal{N}(K)$, set $U(K) := K$.

(ii) Otherwise, let $M(K) := \arg\min_{H \in \mathcal{N}(K)} \mathcal{F}(H)$ and set $U(K)$ to be the $\prec$-minimum element of $M(K)$.

Theorem 2.4 (Lyapunov descent). *For every $K \in \mathcal{K}$,*

$$\mathcal{F}(U(K)) \le \mathcal{F}(K),$$

and if $U(K) \ne K$ then the inequality is strict.

Proof. If $U(K) = K$, the claim is immediate. If $U(K) \ne K$, we are in case (ii): there exists at least one neighbor $H \in \mathcal{N}(K)$ with $\mathcal{F}(H) < \mathcal{F}(K)$. Since $U(K) \in M(K)$,

$$\mathcal{F}(U(K)) = \min_{H \in \mathcal{N}(K)} \mathcal{F}(H) < \mathcal{F}(K).$$

□

Corollary 2.5 (Finite termination). *Every orbit $K_{t+1} = U(K_t)$ reaches a fixed point in finitely many steps.*

Proof. Every non-fixed step strictly decreases $\mathcal{F}$. Since $\mathcal{K}$ is finite, an infinite strictly decreasing sequence is impossible. □

3. Module CF′: stochastic coherent flow via Metropolis–Hastings [Proved]

Definition 3.1 (Metropolis–Hastings kernel on $\mathcal{K}$). *Let Q be a proposal kernel on $\mathcal{K}$ such that:*

- *$Q(K, H) \ge 0$ and $\sum_H Q(K, H) = 1$ for all K;*
- *the proposal graph is irreducible;*
- *$Q(K, K) > 0$ for all K (aperiodicity);*
- *(*support compatibility*) whenever $Q(K, H) = 0$ we set the acceptance $a(K, H) := 0$; equivalently, one may assume $Q(K, H) > 0 \Rightarrow Q(H, K) > 0$.*

Fix $\lambda > 0$. For $K \ne H$, define

$$r(K, H) := \begin{cases} \exp\big(-\lambda(\mathcal{F}(H) - \mathcal{F}(K))\big)\dfrac{Q(H, K)}{Q(K, H)}, & Q(K, H) > 0, \\ 0, & Q(K, H) = 0, \end{cases} \qquad a(K, H) := \min\{1, r(K, H)\}.$$

Define the Markov kernel P by

$$P(K, H) := \begin{cases} Q(K, H)a(K, H), & H \ne K, \\ 1 - \sum_{G \ne K} Q(K, G)a(K, G), & H = K. \end{cases}$$

Theorem 3.2 (Detailed balance). *Let*

$$\pi(K) := \frac{1}{Z_\lambda} \exp(-\lambda \mathcal{F}(K)), \qquad Z_\lambda := \sum_{G \in \mathcal{K}} \exp(-\lambda \mathcal{F}(G)).$$

Then for all $K \ne H$,

$$\pi(K)\, P(K, H) = \pi(H)\, P(H, K).$$

Proof. For $K \ne H$ and $Q(K, H) = 0$ both sides vanish. Assume $Q(K, H) > 0$. Then

$$\pi(K)P(K, H) = \frac{e^{-\lambda \mathcal{F}(K)}}{Z_\lambda} Q(K, H) \min\{1, r(K, H)\}.$$

If $r(K, H) \le 1$, this equals

$$\frac{e^{-\lambda \mathcal{F}(K)}}{Z_\lambda} Q(K, H) r(K, H) = \frac{e^{-\lambda \mathcal{F}(H)}}{Z_\lambda} Q(H, K),$$

and in this case $r(H,K) = 1/r(K,H) \geq 1$, hence

$$\pi(H)P(H,K) = \frac{e^{-\lambda\mathcal{F}(H)}}{Z_\lambda}Q(H,K)\min\{1, r(H,K)\} = \frac{e^{-\lambda\mathcal{F}(H)}}{Z_\lambda}Q(H,K).$$

The case $r(K,H) > 1$ is symmetric. □

Corollary 3.3 (Stationarity and convergence)**.** *π is stationary for P. Since $\mathcal{K}$ is finite and P is irreducible and aperiodic, π is the unique stationary distribution and*

$$\lim_{t\to\infty} \|P^t(K,\cdot) - \pi\|_{\mathrm{TV}} = 0 \quad \forall K \in \mathcal{K}.$$

4. Module C (companion): epistemic curvature as a KL gap [Proved]

Definition 4.1 (KL epistemic curvature)**.** *Let $\mathcal{X}$ be finite and let $\Delta^\circ(\mathcal{X})$ denote the interior of the simplex. For a model family $\mathcal{M} \subseteq \Delta^\circ(\mathcal{X})$ and a target $p \in \Delta^\circ(\mathcal{X})$, define*

$$\kappa_{\mathcal{M}}(p) := \inf_{q\in\mathcal{M}} D_{\mathrm{KL}}(p\|q).$$

This is the unique definition of κ used in this note.

Proposition 4.2 (Binary/discrete separable case)**.** *If p is a joint distribution on $A \times B$ and*

$$\mathcal{M}_{\mathrm{prod}} := \{q : q(a,b) = q_A(a)q_B(b)\},$$

then

$$\kappa_{\mathcal{M}_{\mathrm{prod}}}(p) = D_{\mathrm{KL}}\big(p\|p_A \otimes p_B\big) = I(A;B).$$

Proof. For $q(a,b) = q_A(a)q_B(b)$,

$$D_{\mathrm{KL}}(p\|q) = \sum_{a,b} p(a,b)\log p(a,b) - \sum_a p_A(a)\log q_A(a) - \sum_b p_B(b)\log q_B(b).$$

The first term is constant in (q_A, q_B). The remaining two terms are minimized by $q_A = p_A$ and $q_B = p_B$. Substituting yields

$$D_{\mathrm{KL}}\big(p\|p_A \otimes p_B\big) = \sum_{a,b} p(a,b)\log\frac{p(a,b)}{p_A(a)p_B(b)} = I(A;B).$$

□

5. Module CF″: continuous coherent flow on fixed interfaces [Proved]

5.1. Interface sheet.

Fix strictly positive marginals r_i, c_j with $\sum_i r_i = \sum_j c_j = 1$ and define

$$\mathcal{C}(r,c) := \left\{ p_{ij} > 0 : \sum_j p_{ij} = r_i,\ \sum_i p_{ij} = c_j \right\}.$$

The separable point on the sheet is

$$q^\star_{ij} := r_i c_j.$$

5.2. Orthogonal decomposition and projection.

For $p \in \mathcal{C}(r,c)$, define

$$h_{ij} := \log\frac{p_{ij}}{q^\star_{ij}}, \qquad \langle X, Y\rangle_p := \sum_{i,j} p_{ij}X_{ij}Y_{ij}.$$

Let

$$\mathcal{U} := \{u_{ij} = a_i + b_j\}.$$

Let $\Pi^{\mathcal{U}}_p h$ denote the orthogonal projection of h onto $\mathcal{U}$ with respect to $\langle\cdot,\cdot\rangle_p$, and set

$$\gamma := h - \Pi^{\mathcal{U}}_p h.$$

The normal equations are

$$\sum_j p_{ij}\gamma_{ij} = 0\ \forall i, \qquad \sum_i p_{ij}\gamma_{ij} = 0\ \forall j.$$

Definition 5.1 (Continuous coherent flow)**.** *The dynamics on $\mathcal{C}(r,c)$ is defined by*

$$\dot{p}_{ij} = -p_{ij}\gamma_{ij}.$$

Theorem 5.2 (Interface conservation)**.** *If $p(0) \in \mathcal{C}(r,c)$, then for all times of existence,*

$$\frac{d}{dt}\sum_j p_{ij} = 0, \qquad \frac{d}{dt}\sum_i p_{ij} = 0.$$

Proof. By the normal equations,

$$\frac{d}{dt}\sum_j p_{ij} = \sum_j \dot{p}_{ij} = -\sum_j p_{ij}\gamma_{ij} = 0,$$

and similarly for columns. □

Theorem 5.3 (Exact KL Lyapunov identity)**.** *Define*

$$E(p) := D_{\mathrm{KL}}(p\|q^\star) = \sum_{i,j} p_{ij}\log\frac{p_{ij}}{q^\star_{ij}}.$$

Along the flow,

$$\frac{d}{dt}E(p(t)) = -\sum_{i,j} p_{ij}\gamma_{ij}^2 \le 0.$$

Proof. Since $q^\star$ is constant on the sheet,

$$\frac{d}{dt}E = \sum_{i,j}\dot{p}_{ij}\left(\log\frac{p_{ij}}{q^\star_{ij}} + 1\right) = -\sum_{i,j} p_{ij}\gamma_{ij}(h_{ij}+1).$$

Write $h = \Pi_p^{\mathcal{U}} h + \gamma$. Then

$$\sum_{i,j} p_{ij}\gamma_{ij}h_{ij} = \sum_{i,j} p_{ij}\gamma_{ij}\Pi_p^{\mathcal{U}} h + \sum_{i,j} p_{ij}\gamma_{ij}^2 = \sum_{i,j} p_{ij}\gamma_{ij}^2,$$

because $\gamma \perp_p \mathcal{U}$. Moreover $\sum_{i,j} p_{ij}\gamma_{ij} = 0$ by summing any row condition. Therefore

$$\frac{d}{dt}E = -\sum_{i,j} p_{ij}\gamma_{ij}^2.$$

□

Proposition 5.4 (Unique equilibrium on the sheet)**.** *The unique equilibrium of the flow on $\mathcal{C}(r,c)$ is $q^\star$.*

Proof. At equilibrium, $\dot{p}_{ij} = 0$ for all (i,j), hence (since $p_{ij} > 0$) $\gamma_{ij} = 0$. Thus $h \in \mathcal{U}$, so $h_{ij} = a_i + b_j$ for some (a,b) and

$$p_{ij} = q^\star_{ij} e^{a_i+b_j} = r_i c_j e^{a_i+b_j}.$$

Summing over j gives

$$r_i = \sum_j p_{ij} = r_i e^{a_i}\sum_j c_j e^{b_j}.$$

Since $r_i > 0$, e^{a_i} is constant in i. Similarly, e^{b_j} is constant in j. Normalization forces the global factor to be 1, hence $p = q^\star$. □

Remark 5.5 (Convergence)**.** *The Lyapunov identity implies that every ω-limit set is contained in $\{\gamma = 0\} = \{q^\star\}$. Under global existence, solutions converge to $q^\star$.*

6. Module CF$'''$: Gibbs bridge from logic to probability [Model]

Definition 6.1 (Gibbs map from theories). *For $K \in \mathcal{K}$, $\eta > 0$, and $\omega \in \Omega$, define*

$$E_K(\omega) := \sum_{\varphi \in K} \mathbf{1}_{\{\omega \not\models_D \varphi\}}, \qquad p_K(\omega) := \frac{e^{-\eta E_K(\omega)}}{Z_K(\eta)}, \qquad Z_K(\eta) := \sum_{\xi \in \Omega} e^{-\eta E_K(\xi)}.$$

This yields a map

$$\Phi_\eta : \mathcal{K} \to \Delta^\circ(\Omega), \qquad \Phi_\eta(K) = p_K.$$

Proposition 6.2 (Low-energy concentration). *Let $m_K := \min_{\omega \in \Omega} E_K(\omega)$ and*

$$\mathcal{A}_K := \arg\min_{\omega \in \Omega} E_K(\omega).$$

For every $\omega \notin \mathcal{A}_K$,

$$\lim_{\eta \to \infty} p_K(\omega) = 0.$$

If $V(K) > 0$, then $m_K = 0$ and $\mathcal{A}_K = \mathrm{Mod}(K)$.

Proof. Since Ω is finite, for $\omega \notin \mathcal{A}_K$ there exists $\delta > 0$ with $E_K(\omega) \geq m_K + \delta$. Hence

$$p_K(\omega) \leq \frac{e^{-\eta(m_K + \delta)}}{|\mathcal{A}_K| e^{-\eta m_K}} = \frac{e^{-\eta \delta}}{|\mathcal{A}_K|} \xrightarrow[\eta \to \infty]{} 0.$$

If $V(K) > 0$, there exists ω satisfying all formulas of K in the designated sense, so $E_K(\omega) = 0$ and thus $m_K = 0$. Moreover $E_K(\omega) = 0$ iff $\omega \in \mathrm{Mod}(K)$. $\square$

Remark 6.3 (Layer separation). *The logical layer lives on $\mathcal{K}$, the probabilistic layer on $\Delta^\circ(\Omega)$, and the bridge between them is explicit via Φ_η.*

7. Incompleteness: correct scope [Proved]

Theorem 7.1 (Gödel I, standard form). *Let T be a classical first-order theory such that:*

(i) T is recursively axiomatizable,
(ii) T extends a sufficiently strong base arithmetic (e.g. Robinson arithmetic Q),
(iii) T is consistent.

Then T is incomplete.

Remark 7.2. *This obstruction applies to the* classical verification subsystem *when coherent flow is required to produce formally checkable certificates. It does not, by itself, assert incompleteness for an arbitrary paraconsistent theory without additional hypotheses.*

8. Summary

- The domain of $\mathcal{F}$ and existence of coherent islands are proved in a finite, non-empty theory space.
- The deterministic coherent operator U satisfies a correct Lyapunov descent property and terminates in finite time.
- The stochastic coherent flow is given by a complete Metropolis–Hastings kernel with detailed balance for $\pi(K) \propto e^{-\lambda \mathcal{F}(K)}$.
- Epistemic curvature κ is unified as a minimal KL distance to a model family.
- A continuous coherent flow on fixed-marginal interface sheets has an exact KL Lyapunov identity and a unique equilibrium.
- The Gibbs bridge Φ_η explicitly connects theory space to probability space.
- The incompleteness obstruction is stated with the precise classical hypotheses and scope.

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