Epistemic Closure Nets

Curvature, Holonomy, Certification, and Meta-Closure in an Expansive Network Formalism

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Abstract

This work unifies the SCE-IM closure framework—windowed semantics, curvature gaps, zipper signatures, and stability—with the closed epistemic kernel: internal certification, theory atlases, holonomy obstructions, and meta-closure towers. The unification is *network-expansive*: a typed diagram of nodes (syntax, semantics, certificates, resources, refinements, and experimental harnesses) connected by morphisms and compatibility constraints. Claims are tagged by evidence type; non-closures are isolated as explicit conjectures; experimental nodes are specified with auditable estimators, uncertainty quantification, and pre-registered falsification patterns.

Contents

1 Scope, contribution summary, and falsification discipline

1.1 Scope

This manuscript is *not* a completeness theorem for scientific knowledge. Its objective is an auditable *specification layer* that (i) separates soundness from completeness, (ii) makes finite-resource effects measurable, and (iii) isolates *non-closures* as typed obstructions.

1.2 Auditability rules

(A1) **Finite probes for ideal objects.** Every ideal object is paired with finite windows/banks used for certification.

(A2) **Explicit quantifiers.** Every existential/universal claim fixes its domain (windows, banks, templates, or protocols).

(A3) **Evidence typing.** Each claim is tagged [Proved], [Model], or [Conjecture].

(A4) **Error bars.** Every numerical or statistical claim comes with an auditable uncertainty contract.

(A5) **Pre-registered falsification.** For any empirical test node, the net (nodes/edges and decision rules) is frozen *before* sampling. Network expansion is allowed only as a *separate* post-mortem branch, never as a way to rescue a failed prediction.

1.3 Main contributions (within this paper)

(C1) A typed *closure net* connecting internal certification, windowed semantics, and refinement dynamics.

(C2) A clean separation between *index-trivial* holonomy and *program-level* (history-dependent) holonomy.

(C3) An atlas-groupoid layer whose triple-overlap cocycles obstruct global trivialization of predictions.

(C4) A finite-bank *transfer theorem* derived from explicit continuity moduli and covering-number control (with empirical estimators for both).

(C5) A suite of verifiable experimental nodes with auditable Monte Carlo estimators and pre-registered tests.

(C6) A minimal public-style *artifact node* proof-of-concept (Section 9.7.2) exhibiting the diagnostic regime $\widehat{G} \to 0$ while $\widehat{\Delta}_{\odot} \not\to 0$ under a controlled hidden-state ξ.

2 Definitions (nodes and edges)

2.1 Evidence contract

Definition 2.1 (Evidence tags)**.** *Statements are typed as*

$$[\text{Proved}], \quad [\text{Model}], \quad [\text{Conjecture}],$$

and the tag is part of the type system *of the manuscript: it adds no content, but fixes what kind of audit (proof, model contract, or open problem) is being asserted. Unless explicitly stated otherwise,* [Proved] *means derivable from the axioms in Section 3 relative to standard metatheory (e.g. ZFC).*

Remark 2.1 (Terminology: "curvature" and "holonomy")*.* In this paper, *curvature* means an *operational gap* (an infimum of an error functional under explicit resources), not Riemannian sectional curvature. Similarly, *holonomy* is used in the minimal sense of *path dependence under typed refinements* (protocol holonomy) or *triple-overlap cocycles* (atlas holonomy). Whenever a genuinely geometric structure (group action / cocycle class / conjugacy invariance) is present, it is stated explicitly.

2.2 Kernel (closure + certification layer)

Definition 2.2 (Closed epistemic kernel)**.** *A* closed epistemic kernel *at resource $r \in \mathbb{N}$ is the tuple*

$$K_r = (X, \sim, \mathcal{O}_r, F, \Sigma^*, \mathrm{supp}, \mathrm{Cert}_r, J_r),$$

where $X \neq \emptyset$, $F : X \to X$ is evolution/refinement, $\sim$ is gauge equivalence on X, $\mathcal{O}_r$ is a family of observables $o : X \to Y_o$ factoring through $X/{\sim}$, Σ^* is a token space, $\mathrm{supp} : X \to \mathcal{P}(\Sigma^*)$ *is available-token support,* $\mathrm{Cert}_r : X \times \Sigma^* \to \{0,1\}$ *is internal certification, and $J_r : \Sigma^* \rightharpoonup \mathcal{P}(X)$ is a partial semantics (soundness only).*

Definition 2.3 (Accessible observations)**.** *For $x \in X$, define*

$$\mathrm{Obs}_r(x) := \{(o, o(x)) : \ o \in \mathcal{O}_r\}.$$

Definition 2.4 (Meta-cine (certificate layer))**.** *The pair* $(\mathrm{supp}, \mathrm{Cert}_r)$ *is the* certification layer*: it produces and validates tokens. No completeness is assumed.*

2.3 SCE-IM / Epistemic Closure System with Metric Interface

Definition 2.5 (SCE-IM instance). *An SCE-IM instance is a tuple*

$$\mathcal{E} = (S, O, \Omega, \mathcal{T}, J, \mathrm{err}, \mu, \mathcal{K}, \Phi),$$

where S is a syntactic space, O an objective space, Ω a semantic space, $\mathcal{T}$ a set of teeth (constraints), $J : S \to \mathcal{P}(\Omega)$ a semantics map, $\mathrm{err} : S \times O \to [0, \infty]$ *an error functional, μ a measure on Ω, $\mathcal{K}$ windows, and Φ a dynamics on S.*

Definition 2.6 (Windowed volume). *For a window $K \in \mathcal{K}$ and $\sigma \in S$,*

$$\mathrm{Vol}_K(\sigma) := \mu(J(\sigma) \cap K).$$

Definition 2.7 (Resource-limited curvature). *Given a resource functional $\rho : S \to [0, \infty)$ and objective $o \in O$,*

$$\kappa_R(o) := \inf\{\mathrm{err}(\sigma, o) : \ \rho(\sigma) \leq R\}, \qquad \kappa(o) := \inf_{\sigma \in S} \mathrm{err}(\sigma, o).$$

2.4 Constraints as geometry (CNF / cGCNF nodes)

Definition 2.8 (Discrete CNF forbidden region). *On $\{0,1\}^n$, a CNF $F = \bigwedge_{j=1}^m C_j$ defines a forbidden region $U(F)$ as the union of clause-falsifying subcubes; the satisfying set is* $\mathrm{Mod}(F) = \{0,1\}^n \setminus U(F)$.

Definition 2.9 (Continuous GCNF (cGCNF)). *Let $\mathcal{X} = \prod_{i=1}^n X_i$ be a product of topological spaces. A literal is $\ell = (I, f, Y, U)$ with $I \subseteq [n]$, $f : \prod_{i \in I} X_i \to Y$ continuous, and $U \subseteq Y$ open. Define* $\mathrm{Mod}(\ell) = (f \circ \pi_I)^{-1}(U)$. *A clause is a finite union of literals; a formula is a finite intersection of clauses. For a clause $C = \bigvee_{t=1}^m \ell_t$, define*

$$\mathrm{Forb}(C) := \bigcap_{t=1}^m \big(\mathcal{X} \setminus \mathrm{Mod}(\ell_t)\big), \qquad U(\Phi) := \bigcup_j \mathrm{Forb}(C_j), \qquad \mathrm{Mod}(\Phi) := \mathcal{X} \setminus U(\Phi).$$

2.5 Finite-bank certification node (bank & margin)

Definition 2.10 (Finite-bank transfer data). *Let $(\Theta, \mathcal{A}, \mu_\Theta)$ be a measure space and $K \in \mathcal{A}$ a window. Fix a margin parameter $\tau > 0$ and bank size N. Let $E_\tau \in \mathcal{A}$ be an "ideal" robust region and $E_\tau^{(N)} \in \mathcal{A}$ the region certified by a finite bank. Let ε_N be the bank covering radius and $\omega : [0, \infty) \to [0, \infty)$ a modulus. Under the regularity hypothesis of Axiom 3.6, the inclusions*

$$E_{\tau+\omega(\varepsilon_N)} \subseteq E_\tau^{(N)} \subseteq E_\tau$$

are proved in Theorem 4.1.

2.6 Protocol holonomy (typed refinement square; no untyped inverses)

Definition 2.11 (Index category for refinements). *Let $\mathcal{I}$ be a small category whose objects are pairs (N, τ). For $N \leq N'$ there is a bank morphism*

$$B_{N \to N'} : (N, \tau) \to (N', \tau),$$

and for $\delta \geq 0$ there is a margin morphism

$$R_\delta : (N, \tau) \to (N, \tau + \delta).$$

Composition is defined by concatenation of refinements.

Definition 2.12 (Measured-set poset categories)**.** *Let* $\mathbf{Meas}^{\subseteq}(\Theta)$ *be the poset-category of measurable subsets of* Θ *with a unique morphism* $A \to B$ *iff* $A \subseteq B$*. Let* $\mathbf{Meas}^{\supseteq}(\Theta)$ *be the same objects with a unique morphism* $A \to B$ *iff* $A \supseteq B$ *(reverse inclusion).*

Definition 2.13 (Certified-set functor (variance-aware))**.** *A* certified-set functor *is a functor*

$$E : \mathcal{I} \to \mathbf{Meas}^{\supseteq}(\Theta), \qquad E(N,\tau) := E_{\tau}^{(N)},$$

so that refinement *morphisms (bank densification or margin tightening) point toward* more restrictive *certified sets. Equivalently: a morphism* $(N,\tau) \to (N',\tau')$ *in* $\mathcal{I}$ *implies*

$$E_{\tau}^{(N)} \supseteq E_{\tau'}^{(N')}.$$

Definition 2.14 (Index-level square comparison (trivial holonomy))**.** *Fix* $N \leq N'$ *and* $\delta \geq 0$*. There are two morphisms in* $\mathcal{I}$ *from* (N,τ) *to* $(N',\tau+\delta)$*:*

$$p_1 := B_{N\to N'} \circ R_\delta, \qquad p_2 := R_\delta \circ B_{N\to N'}.$$

Since E *lands in a* poset *(Definition 2.12), the composite morphisms* $E(p_1)$ *and* $E(p_2)$ *both identify the same endpoint object* $E(N',\tau+\delta)$*. Therefore any "holonomy" defined purely at the index level is forced to be trivial. We record this by defining, for any measurable window* $K \subseteq \Theta$*,*

$$\Delta_{\circlearrowleft}^{\text{idx}}(N,N',\tau,\delta;K) := 0.$$

Nontrivial protocol holonomy is defined at the program *level (Definition 9.5) where refinement semantics can be history-dependent.*

Remark 2.2 (Legacy shorthand). The earlier commutator notation $L := B^{-1}R_\delta^{-1}BR_\delta$ is retained as *informal shorthand* for "compare two refinement paths in a square". It becomes literal only after upgrading refinements to isomorphisms in a localized groupoid of *program states* (Definition 9.3), or after choosing non-canonical sections that provide partial inverses. At the index level, holonomy is declared trivial by Definition 2.14.

2.7 Theory atlas node

Definition 2.15 (Observable output assignment and probability functor)**.** *For each observable* $o \in \mathcal{O}_r$*, fix a measurable output space* $(Y_o, \mathcal{B}_o)$*. Regard* $\mathcal{O}_r$ *as a* discrete category $\mathcal{O}_r^{\text{disc}}$ *(objects are observables; only identity morphisms). Define the output assignment functor*

$$\mathcal{Y} : \mathcal{O}_r^{\text{disc}} \to \mathbf{Meas}, \qquad o \mapsto (Y_o, \mathcal{B}_o),$$

where **Meas** *is the category of measurable spaces and measurable maps. Let*

$$\mathcal{P} : \mathbf{Meas} \to \mathbf{Set}, \qquad (Y,\mathcal{B}) \mapsto \mathrm{Prob}(Y)$$

be the probability-measure functor into sets (here $\mathrm{Prob}(Y)$ *denotes probability measures on* $(Y,\mathcal{B})$*). Then* $(\mathcal{P}\circ\mathcal{Y})(o) = \mathrm{Prob}(Y_o)$*.*

Definition 2.16 (Prediction section (typed))**.** *A chart* T *determines a* prediction section

$$\mathrm{Pred}_T \in \Gamma(\mathcal{P}\circ\mathcal{Y}) := \prod_{o\in\mathcal{O}_r} \mathrm{Prob}(Y_o),$$

i.e. an assignment $o \mapsto \mathrm{Pred}_T(o) \in \mathrm{Prob}(Y_o)$*. When convenient, we also view* Pred_T *as a functor* $\mathcal{O}_r^{\text{disc}} \to \mathbf{Set}$ *sending* o *to* $\mathrm{Prob}(Y_o)$ *together with a chosen element in each fiber.*

Definition 2.17 (Chart validity region)**.** *A chart* T *has a validity region* $\Omega_r(T) \subseteq X$*.*

Definition 2.18 (Atlas of charts). *An atlas at resource r is a family $\{T_i\}_{i\in I}$ such that the accessible regime is covered: for the intended domain $\mathsf{A}_r \subseteq X$, $\mathsf{A}_r \subseteq \bigcup_i \Omega_r(T_i)$.*

Definition 2.19 (Transition operators (groupoid structure)). *On overlaps $\Omega_r(T_i) \cap \Omega_r(T_j)$ we assume a family of bijections*

$$g_{ij} = (g_{ij,o})_{o\in\mathcal{O}_r}, \qquad g_{ij,o} : \mathrm{Prob}(Y_o) \to \mathrm{Prob}(Y_o),$$

with pointwise inverses $g_{ji,o} = g_{ij,o}^{-1}$. These operators act on prediction sections by

$$(g_{ij} \cdot \mathrm{Pred})(o) := g_{ij,o}(\mathrm{Pred}(o)).$$

The family $\{T_i, g_{ij}\}$ defines a groupoid $\mathcal{G}_r$ (objects: charts; morphisms: transition operators). If $\mathcal{O}_r$ is upgraded from discrete to a category with nontrivial morphisms, impose the naturality constraint; in the default discrete case, naturality is vacuous.

Realization note: *each $g_{ij,o}$ may be induced by an invertible Markov kernel on Y_o; this links to the kernel pushforward action in Definition 9.23.*

Definition 2.20 (Atlas holonomy cocycle). *On triple overlaps define the cocycle operator*

$$h_{ijk} := g_{ij} \circ g_{jk} \circ g_{ki},$$

meaning componentwise, for each $o \in \mathcal{O}_r$,

$$h_{ijk,o} := g_{ij,o} \circ g_{jk,o} \circ g_{ki,o} : \mathrm{Prob}(Y_o) \to \mathrm{Prob}(Y_o).$$

Thus h_{ijk} is an automorphism of the section space $\Gamma(\mathcal{P} \circ \mathcal{Y})$ via pointwise action. If some $h_{ijk,o} \neq \mathrm{id}$ (in the chosen gauge quotient), the atlas has nontrivial holonomy.

Remark 2.3 (Gauge dependence and conjugacy invariance). Changing the representative transitions by a 0-cochain $\{a_i \in \mathsf{G}_r\}$ via $g'_{ij} := a_i\, g_{ij}\, a_j^{-1}$ conjugates the cocycles: $h'_{ijk} = a_i\, h_{ijk}\, a_i^{-1}$. Hence the *conjugacy class* of $\{h_{ijk}\}$ (and in standard settings its cohomology class) is the gauge-invariant holonomy datum.

2.8 The network object (expansive, non-sequential)

Definition 2.21 (Closure net as a typed diagram). *A* closure net $\mathcal{N}$ *is a typed directed multigraph (equivalently, a small category with decorations) whose nodes include:*

$$\{\textit{epistemic kernel } K_r,\ \textit{SCE-IM } \mathcal{E},\ \textit{CNF/cGCNF},\ \textit{Finite-bank functor } E,\ \textit{Atlas groupoid } \mathcal{G}_r\},$$

and whose edges are declared morphisms between them (semantics, volume, certification, refinements, transitions). A network extension *adds nodes/edges while preserving all previously declared typing and compatibility constraints.*

2.9 Gauge actions and rigid transport (geometric core)

Definition 2.22 (Net-derived gauge group on predictions). *Fix r and write the section space of predictions as*

$$\Gamma_r := \Gamma(\mathcal{P} \circ \mathcal{Y}) = \prod_{o\in\mathcal{O}_r} \mathrm{Prob}(Y_o).$$

Let $\mathrm{Aut}_{\mathrm{rig}}(\mathcal{N}_r)$ denote the group of rigid closure-net automorphisms at resource r (Definition 2.23 applied fiberwise on the observational family). Each $\varphi \in \mathrm{Aut}_{\mathrm{rig}}(\mathcal{N}_r)$ induces a componentwise pushforward

$$\rho_r(\varphi) : \Gamma_r \to \Gamma_r, \qquad (\nu_o)_{o\in\mathcal{O}_r} \mapsto (\varphi_{o\#}\nu_o)_{o\in\mathcal{O}_r}.$$

The canonical gauge group *is the image*

$$\mathsf{G}_r := \rho_r\big(\mathrm{Aut}_{\mathrm{rig}}(\mathcal{N}_r)\big) \le \mathrm{Aut}(\Gamma_r).$$

A choice of transition representatives $\{g_{ij}\}$ *is a* G_r*-valued* 1*-cochain on the nerve of the cover* $\{\Omega_r(T_i)\}$*.*

Proposition 2.1 ([Proved] Rigidity ⇒ gauge action)**.** *The assignment* $\rho_r : \mathrm{Aut}_{\mathrm{rig}}(\mathcal{N}_r) \to \mathrm{Aut}(\Gamma_r)$ *is a group homomorphism. Consequently, atlas-holonomy conjugacy classes computed from transition cochains are intrinsic to the rigid transport structure.*

Proof. Functoriality of pushforward under composition yields $\rho_r(\varphi \circ \psi) = \rho_r(\varphi) \circ \rho_r(\psi)$ and $\rho_r(\mathrm{id}) = \mathrm{id}$. Conjugacy invariance is the standard gauge change-of-trivialization calculation (Proposition 4.1). □

Definition 2.23 (Rigid isomorphism of closure nets)**.** *Let* $\mathcal{E}$ *and* $\mathcal{E}'$ *be two instantiations (possibly on different spaces). A* rigid isomorphism *is a triple* (h, g, u) *transporting state space, objective space, and micro-space, together with induced transports of windows, dynamics, and observables, such that error and windowed volumes are preserved. (When instantiated from SCE-IM, this matches the standard "rigid isomorphism" notion used to prove curvature invariance.)*

3 Axioms (consistency and minimal couplings)

Axiom 3.1 (Gauge invariance)**.** *All observables* $o \in \mathcal{O}_r$ *factor through* $X/\!\sim$*.*

Axiom 3.2 (Soundness-only internal certification)**.** *For each* r*, there exists a partial semantics* J_r *such that*

$$\mathrm{Cert}_r(x, s) = 1 \implies x \in J_r(s).$$

No completeness or totality is assumed.

Axiom 3.3 (Zipper refinement operator)**.** *There exists an operator* $\triangleleft : S \times \mathcal{T} \to S$ *and measurable sets* $D_\tau \subseteq \Omega$ *such that*

$$J(\sigma \triangleleft \tau) \subseteq J(\sigma) \cap D_\tau.$$

Axiom 3.4 (Lower semicontinuity of error)**.** *For each* $o \in O$*, the map* $\sigma \mapsto \mathrm{err}(\sigma, o)$ *is lower semicontinuous on* S*.*

Axiom 3.5 (Windowed measurability)**.** *For each* $\sigma \in S$ *and* $K \in \mathcal{K}$*,* $J(\sigma) \cap K$ *is measurable and* $\mu(K) < \infty$*.*

Axiom 3.6 (Bank regularity for transfer)**.** *There is a compact template space* $(\mathsf{S}_\infty, \mathrm{dist}_{\mathsf{S}})$ *and a score functional*

$$F : \Theta \times \mathsf{S}_\infty \to \mathbb{R}$$

such that for each $S \in \mathsf{S}_\infty$ *the map* $\theta \mapsto F(\theta, S)$ *is measurable, and for each* θ *the map* $S \mapsto F(\theta, S)$ *is uniformly continuous with modulus* ω *(independent of* θ*):*

$$|F(\theta, S) - F(\theta, S')| \le \omega(\mathrm{dist}_{\mathsf{S}}(S, S')).$$

A finite bank $\mathsf{S}_N \subset \mathsf{S}_\infty$ *is an* ε_N*-net: for every* $S \in \mathsf{S}_\infty$ *there exists* $S' \in \mathsf{S}_N$ *with* $\mathrm{dist}_{\mathsf{S}}(S, S') \le \varepsilon_N$*.*

Axiom 3.7 (Refinement typing)**.** *Refinements act as morphisms in the index category* $\mathcal{I}$ *(Definition 2.11) and induce monotone maps via the certified-set functor* E *(Definition 2.13).*

Axiom 3.8 (Atlas groupoid typing)**.** *Transitions form a groupoid of natural isomorphisms between prediction functors on overlaps (Definition 2.19). Atlas holonomy is defined by cocycles (Definition 2.20).*

Axiom 3.9 (Evidence separation)**.** *No axiom asserts: (i) global completeness of internal certification, (ii) that zipper signatures determine atlas/protocol holonomy without extra hypotheses, (iii) that holonomy must vanish.*

4 Lemmas and theorems (local compatibility and obstructions)

Lemma 4.1 (Monotonicity of resource-limited curvature)**.** *If $R \le R'$, then $\kappa_{R'}(o) \le \kappa_R(o)$ for all $o \in O$.*

Proof. The feasible set $\{\sigma : \rho(\sigma) \le R\}$ is contained in $\{\sigma : \rho(\sigma) \le R'\}$; taking infima yields the claim. □

Lemma 4.2 (Openness of cGCNF model sets)**.** *In cGCNF,* $\mathrm{Mod}(\Phi)$ *is open in $\mathcal{X}$.*

Proof. Each literal model set is open as a preimage of an open set under a continuous map. Finite unions/intersections preserve openness. □

Theorem 4.1 ([Proved] Finite-bank transfer from a modulus)**.** *Assume Axiom 3.6. Fix $\tau > 0$ and define the* ideal *robust existence set and its banked version by*

$$E_\tau := \{\theta \in \Theta : \exists S \in \mathsf{S}_\infty \text{ with } F(\theta, S) < -\tau\}, \qquad E_\tau^{(N)} := \{\theta \in \Theta : \exists S \in \mathsf{S}_N \text{ with } F(\theta, S) < -\tau\}.$$

Then the transfer inclusions hold:

$$E_{\tau+\omega(\varepsilon_N)} \subseteq E_\tau^{(N)} \subseteq E_\tau.$$

Proof. The inclusion $E_\tau^{(N)} \subseteq E_\tau$ is immediate since $\mathsf{S}_N \subseteq \mathsf{S}_\infty$. For the other inclusion, take $\theta \in E_{\tau+\omega(\varepsilon_N)}$. Then there exists $S \in \mathsf{S}_\infty$ with $F(\theta, S) < -(\tau + \omega(\varepsilon_N))$. By ε_N-net coverage, pick $S' \in \mathsf{S}_N$ with $\mathrm{dist}_{\mathsf{S}}(S, S') \le \varepsilon_N$. By the modulus inequality,

$$F(\theta, S') \le F(\theta, S) + \omega(\varepsilon_N) < -\tau,$$

so $\theta \in E_\tau^{(N)}$. □

Corollary 4.1 ([Proved] Lipschitz modulus)**.** *If, in addition, $S \mapsto F(\theta, S)$ is L-Lipschitz uniformly in θ, then one may take $\omega(\varepsilon) = L\varepsilon$.*

Proof. Immediate from the Lipschitz bound. □

Theorem 4.2 ([Proved] Non-contradiction of epistemic kernel and SCE-IM)**.** *Under Axioms 3.1–3.9, the combined system is consistent relative to the metatheory: a product model exists in which the epistemic kernel and SCE-IM components coexist without forcing contradictions.*

Proof. The axioms constrain disjoint sorts: $(X, \Sigma^*, \mathrm{Cert}_r, J_r, \mathcal{O}_r)$ on one side and $(S, \Omega, J, \mu, \mathrm{err}, \triangleleft)$ on the other. No axiom identifies these sorts or asserts completeness. Interpret each component independently and take a product/disjoint union model. □

Theorem 4.3 ([Proved] Protocol holonomy: index-triviality and program non-commutativity)**.** ***(Index level)*** *Under Definitions 2.12–2.14, the index-level quantity $\Delta_\circlearrowleft^{\mathrm{idx}}(N, N', \tau, \delta; K)$ is identically* 0.

(Program level) *Fix a start program state c and two protocols $p_1, p_2 : c \to c_1, c_2$ in $\mathcal{I}^{\mathrm{prog}}$ (Definition 9.3), and a window $K \subseteq \Theta$. Let $\Delta_\circlearrowleft(p_1, p_2; K)$ be as in Definition 9.5. Then $\Delta_\circlearrowleft(p_1, p_2; K) = 0$ iff the produced certified sets agree μ_Θ-a.e. on K, and $\Delta_\circlearrowleft(p_1, p_2; K) > 0$ implies they differ on a μ_Θ-non-null subset of K.*

Proof. Immediate from Definition 2.14: equality of images implies symmetric difference measure 0, and positive symmetric difference implies inequality on positive measure. □

Theorem 4.4 ([Proved] Atlas holonomy obstructs global trivialization)**.** *Under Axiom 3.8, if some cocycle $h_{ijk} \ne \mathrm{id}$ in the gauge quotient, then there is no single prediction functor* $\mathrm{Pred}_\star$ *and natural isomorphisms* $\mathrm{Pred}_{T_i} \Rightarrow \mathrm{Pred}_\star$ *that simultaneously trivialize all transitions on the covered regime.*

Proof. A global trivialization yields a choice of gauge in which all transitions are identities, forcing all triple cocycles to be identities. Thus a nontrivial cocycle obstructs such a choice. □

Proposition 4.1 ([Proved] Gauge invariance of atlas holonomy)**.** *Under Definition 2.22 and the gauge update* $g'_{ij} := a_i g_{ij} a_j^{-1}$*, the cocycles transform by conjugation and the holonomy class (conjugacy/cohomology class) is invariant.*

Proof. This is the computation recorded in the remark after Definition 2.20. □

Proposition 4.2 ([Proved] Curvature invariance under rigid transport)**.** *Let* (h, g, u) *be a rigid isomorphism (Definition 2.23) transporting error as* $\mathrm{err}'(h(\sigma), g(o)) = \mathrm{err}(\sigma, o)$*. Then* $\kappa'(g(o)) = \kappa(o)$ *for all objectives* o*.*

Proof. Take infima over σ and use bijectivity of h. □

Theorem 4.5 ([Proved] Diagonal obstruction requires arithmetized self-reference)**.** *Soundness-only internal certification (Axiom 3.2) does not by itself imply a Gödel diagonal obstruction. Such an obstruction requires an additional arithmetization layer that (i) encodes sentences into tokens and (ii) ties token-certification to an internal reflection principle.*

Proof. Diagonalization requires a fixed-point lemma in an internal language plus a mapping from sentences to tokens and a reflection principle that connects Cert_r to semantic truth/representability. These are not present in Axiom 3.2. □

5 Invariants (what remains under equivalence)

Definition 5.1 (Zipper signature)**.** *Fix an interval* $I = [\varepsilon_0, \varepsilon_1]$ *of sublevel thresholds. The zipper signature is*

$$\Sigma_{\mathrm{zip}}(\mathcal{E}, o) := \big(\kappa(o), \mathrm{MT}_I(o), \tau^o\big),$$

where $\mathrm{MT}_I(o)$ *is a merge tree and* τ^o *are hitting-time observables.*

Definition 5.2 (Atlas holonomy invariant)**.** *The gauge-equivalence class of cocycles* $\{h_{ijk}\}$ *defines* $\mathrm{Hol}_{\mathrm{atlas}}$*.*

Definition 5.3 (Protocol holonomy invariant)**.** *The family of square-comparison magnitudes* $\Delta_{\circlearrowleft}(N, N', \tau, \delta; K)$ *defines* $\mathrm{Hol}_{\mathrm{prot}}$*.*

Proposition 5.1 ([Proved] Independence under current axioms)**.** *Under Axioms 3.1–3.9,* Σ_{zip} *and* $(\mathrm{Hol}_{\mathrm{atlas}}, \mathrm{Hol}_{\mathrm{prot}})$ *are independent invariants: no implication between them is provable without extra coupling axioms.*

Proof. No axiom links merge-tree/hitting-time data to atlas groupoid cocycles or refinement-square comparisons. □

6 Structural predictions (modeled, falsifiable)

Proposition 6.1 ([Model] gray-zone scaling with bank density)**.** *Assume* $\omega(\varepsilon) \approx L\varepsilon$ *and* $\varepsilon_N \asymp N^{-1/d}$ *for an effective dimension* d*. Then the bank-induced uncertainty thickness scales as* $O(N^{-1/d})$ *(for fixed margin* τ *and window* K*).*

Proposition 6.2 ([Model] holonomy scaling with transfer modulus)**.** *Assume that non-commutativity arises only through the bank-transfer gap measured by* $\omega(\varepsilon_N)$*. Then* $\Delta_{\circlearrowleft}(N, N', \tau, \delta; K) = O(\omega(\varepsilon_N))$ *as* $N \to \infty$ *for fixed* (N', τ, δ) *and* K*.*

Proposition 6.3 ([Model] atlas obstructions manifest as path dependence). *If* $\mathrm{Hol}_{\mathrm{atlas}} \neq 0$*, then prediction translation along different chart-paths around an overlap cycle yields path-dependent outputs on some observable family.*

Definition 6.1 (Asymptotic trivialization in the resource limit). *Let* $r \mapsto c_r$ *be a directed resource schedule (refinements in* $\mathcal{I}^{\mathrm{prog}}$*). Protocol holonomy is* asymptotically trivial *on window* K *if for every fixed loop type,*

$$\lim_{r\to\infty} \Delta_{\odot}(c_r; K) = 0.$$

Proposition 6.4 ([Model] Sufficient condition for $\Delta_{\odot} \to 0$). *If (i) bank transfer moduli satisfy* $\omega(\varepsilon_r) \to 0$*, (ii) the auxiliary state stabilizes (*$\xi_r = \xi_\infty$ *eventually), and (iii) atlas cocycle classes vanish on the relevant loop family, then* $\Delta_{\odot}(c_r; K) \to 0$ *as* $r \to \infty$*.*

Proposition 6.5 ([Model] Persistent holonomy decomposition). *In the same setting, observing* $\widehat{G} \to 0$ *while* $\widehat{\Delta}_{\odot} \not\to 0$ *supports one of: (a) non-stabilization of* ξ *(history-dependent semantics), (b) nontrivial atlas holonomy class (groupoid obstruction), (c) misspecified transfer modulus (failure of the tested regularity hypothesis). The artifact node in Section 9.7.2 realizes case (a) via a window-choice state.*

7 Declared holes (non-closure) and network expansion

Definition 7.1 (Hole vs contradiction). *A* contradiction *is a derivation of* $\perp$ *from the axioms. A* hole *is a desired coupling statement not derivable from the axioms; holes are promoted to conjectures or to new explicit axioms.*

Remark 7.1 (Current explicit holes). (H1) Zipper-to-holonomy: Σ_{zip} determines $\mathrm{Hol}_{\mathrm{prot}}$ (or forces $\mathrm{Hol}_{\mathrm{prot}} = 0$) under a natural rigidity hypothesis.

(H2) Zipper algebra: conditions under which $\triangleleft$ generates a monoid/semigroup action on S and induces functorial actions on invariants.

(H3) Atlas-to-zipper: $\mathrm{Hol}_{\mathrm{atlas}}$ is encoded by merge-tree data on I.

(H4) Meta-closure tower: explicit endofunctor M on charts capturing "closing the gap" while preserving soundness, and its (non-)fixed-point structure.

(H5) Arithmetization layer: minimal conditions under which internal certification implies a diagonal obstruction and hence $\kappa_R > 0$ for some targets.

(H6) Interaction of bank-functor E and prediction-functor Pred_T (a functorial bridge from certified parameter sets to prediction kernels).

7.1 Conjectures as safe network expansions

Conjecture 7.1 (Zipper $\Rightarrow$ trivial protocol holonomy under rigidity). *Assume a rigidity axiom identifying bank/margin refinements with homotopies that preserve the sublevel filtration on* I *and preserve* MT_I*. Then constancy of* Σ_{zip} *on* I *implies* $\Delta_{\circlearrowleft}(N, N', \tau, \delta; K) = 0$ *for the induced window semantics.*

Conjecture 7.2 (Zipper monoid and induced actions). *Assume a set of teeth* $\mathcal{T}$ *equipped with a composition law* $*$ *such that applying* τ_1 *then* τ_2 *is equivalent (up to* $\sim$*) to applying* $\tau_1 * \tau_2$*. Then* $\triangleleft$ *defines a (partial) right action of a monoid* $(\mathcal{T}, *)$ *on* S*, and any functorial invariant of the induced filtrations is constant along action-orbits.*

Conjecture 7.3 (Atlas holonomy as groupoid cohomology class). *There exists a cohomology theory on the nerve* $\mathrm{Nerve}(\mathcal{G}_r)$ *in which* $\mathrm{Hol}_{\mathrm{atlas}}$ *is represented by a class of the cocycle* $\{h_{ijk}\}$*.*

Conjecture 7.4 (Meta-closure tower has no sound complete fixed point)**.** *Assume a meta-operator* M *that extends a chart by adding internal certification power while preserving soundness. If* M *is sufficiently expressive to encode its own certification predicate, then there is no chart* T *such that* $\mathsf{M}(T) \cong T$ *and certification becomes sound* and *complete at fixed finite resources.*

Conjecture 7.5 (Zipper monoid action and complexity proxy)**.** *Let* $\triangleleft$ *be the zipper refinement operator (Axiom 3.3). There exists a monoid* $(\mathcal{T}, *)$ *and an action* $\mathcal{T} \curvearrowright \mathsf{S}$ *such that each elementary refinement corresponds to multiplication by a generator in* $\mathcal{T}$*. Define the* zipper complexity *of reaching a target syntax token* $s^\star$ *from* s *as the minimal word length*

$$\mathrm{Comp}_\triangleleft(s \to s^\star) := \min\{\ell : \exists t_1 * \cdots * t_\ell \in \mathcal{T} \text{ with } (t_1 * \cdots * t_\ell) \cdot s = s^\star\}.$$

In systems where refinements model coarse-graining, this length behaves analogously to circuit depth.

Remark 7.2 (Renormalization-group analogy). In QFT-style settings, a zipper step $\triangleleft$ admits an interpretation as integrating out degrees of freedom and producing an effective kernel. Non-commutativity of refinement programs corresponds to non-commutativity of coarse-graining steps under different scheme/order choices, with holonomy providing a quantitative invariant of the mismatch.

8 Network diagram (expansive)

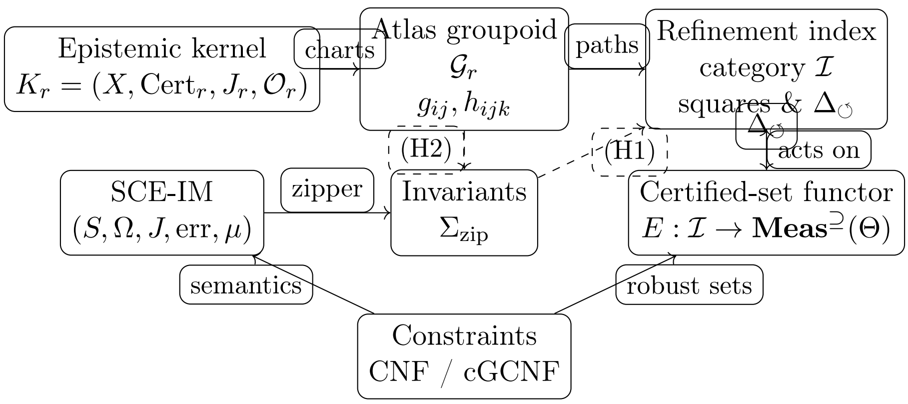

Figure 1: **Closure net schematic.** Solid arrows are typed morphisms already defined in Sections 1–3 (charts, refinements, and certified-set functors). Dashed arrows labeled (H1)–(H2) are *declared coupling holes*: they require additional hypotheses to connect zipper signatures to holonomy data. The diagram is intended as a navigation aid rather than a completeness claim.

9 Verifiable Extensions: Experimental Nodes and Protocols

9.1 Experimental contract (auditable measurement)

Definition 9.1 (Verifiable extension)**.** *A* verifiable extension *adds a node* D *(data/harness) and a finite family of measurable observables* $\mathcal{M} = \{M_1, \ldots, M_k\}$ *together with:*

(V1) explicit domains (datasets or instance families) and windows K*,*

(V2) estimators $\widehat{M_i}$ *computable from finite samples,*

(V3) auditable error bars (concentration bounds or bootstrap intervals),

(V4) falsifiable predictions (inequalities or scaling laws with declared regimes).

Definition 9.2 (Window sampling oracle)**.** *An* oracle *for window sampling is a procedure returning i.i.d. samples $\theta_t \sim \mu_\Theta(\cdot \mid K)$ or MCMC samples with effective sample size* ESS(T)*. In this section, i.i.d. is treated as* [Proved] *for concentration bounds; MCMC bounds are treated as* [Model]*.*

9.2 Typed protocol holonomy: deterministic vs history-dependent semantics

Remark 9.1 (Deterministic indexed sets imply trivial holonomy). If $E_\tau^{(N)}$ is defined as a function *only* of the terminal indices (N, τ), then any two paths $(N, \tau) \to (N', \tau + \delta)$ with the same endpoint produce the same set and holonomy is forced to vanish. Therefore nontrivial protocol holonomy requires *history-dependent* refinement semantics.

Definition 9.3 (Refinement program category)**.** *Let $\mathcal{I}^{\mathrm{prog}}$ be a category whose objects are* certificate states

$$c = (\mathcal{S}, \tau, \xi),$$

where $\mathcal{S}$ is a concrete bank instance (not just its size), τ is a margin parameter, and ξ is auxiliary state (random seed, optimizer state, stopping rule, etc.). Morphisms are program steps *acting on c (bank augmentation, margin tightening, recomputation). A protocol is a path $p : c \to c'$ in $\mathcal{I}^{\mathrm{prog}}$.*

Definition 9.4 (Program-evaluated certified sets)**.** *A* program-evaluated certification functor *is a functor*

$$\mathcal{E}^{\mathrm{prog}} : \mathcal{I}^{\mathrm{prog}} \to \mathbf{Meas}^{\supseteq}(\Theta), \qquad \mathcal{E}^{\mathrm{prog}}(c) =: E_c \subseteq \Theta.$$

Two protocols $p_1, p_2 : c \to c'$ may yield distinct outputs even if c' matches in index values (e.g. same $(N', \tau + \delta)$), because the terminal bank instance $\mathcal{S}$ or auxiliary state ξ can differ.

Definition 9.5 (Protocol holonomy observable (program semantics))**.** *Fix a start state c and two protocols $p_1, p_2 : c \to c'$. For a window $K \subseteq \Theta$, define*

$$\Delta_{\circlearrowleft}(p_1, p_2; K) := \mu_\Theta\big((E_{p_1} \cap K) \triangle (E_{p_2} \cap K)\big),$$

where $E_{p_i} := \mathcal{E}^{\mathrm{prog}}(p_i)(c)$ denotes the output certified set produced by executing p_i from c. Definition 2.14 is recovered as a special case when certificate states are restricted to (N, τ) and programs are canonical.

Definition 9.6 (Normalized protocol holonomy)**.** *If $\mu_\Theta(K) > 0$, define the* normalized *magnitude*

$$\widetilde{\Delta}_{\circlearrowleft}(p_1, p_2; K) := \frac{\Delta_{\circlearrowleft}(p_1, p_2; K)}{\mu_\Theta(K)} \in [0, 1].$$

This removes dependence on the absolute scale of μ_Θ when only relative window fractions are desired.

9.3 Monte Carlo estimators for window-measures

Definition 9.7 (Indicator estimator)**.** *Let $A \subseteq \Theta$ be measurable and let $\theta_t \sim \mu_\Theta(\cdot \mid K)$ i.i.d. Define*

$$\widehat{\mu}_T(A \cap K) := \mu_\Theta(K) \cdot \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}_A(\theta_t).$$

Theorem 9.1 ([Proved] Hoeffding bound for window-measures)**.** *Under i.i.d. sampling, for any $\epsilon > 0$,*

$$\Pr(|\widehat{\mu}_T(A \cap K) - \mu_\Theta(A \cap K)| \geq \epsilon) \leq 2 \exp\left(-\frac{2T\epsilon^2}{\mu_\Theta(K)^2}\right).$$

Proof. Apply Hoeffding to bounded random variables $X_t = \mu_\Theta(K)\mathbf{1}_A(\theta_t) \in [0, \mu_\Theta(K)]$. $\square$

Remark 9.2 ([Model] MCMC replacement). If only MCMC samples are available, replace T in Theorem 9.1 by $\mathrm{ESS}(T)$ (using a declared estimator of integrated autocorrelation time).

9.4 Normalized window probabilities (avoiding explicit $\mu_\Theta(K)$)

Definition 9.8 (Conditional window measure). *Assume $0 < \mu_\Theta(K) < \infty$. Define the conditional window probability measure*

$$\mu_\Theta^K(A) := \Pr_{\theta \sim \mu_\Theta(\cdot|K)}[\theta \in A] = \frac{\mu_\Theta(A \cap K)}{\mu_\Theta(K)}.$$

Definition 9.9 (Normalized indicator estimator). *Let $\theta_t \sim \mu_\Theta(\cdot \mid K)$ i.i.d. Define the normalized estimator*

$$\widehat{\mu^K}_T(A) := \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}_A(\theta_t),$$

which estimates $\mu_\Theta^K(A)$ and does not *require knowledge of $\mu_\Theta(K)$.*

Theorem 9.2 ([Proved] Hoeffding bound for normalized window probabilities). *Under i.i.d. sampling, for any $\epsilon > 0$,*

$$\Pr\Big(\Big|\widehat{\mu^K}_T(A) - \mu_\Theta^K(A)\Big| \geq \epsilon\Big) \leq 2\exp\big(-2T\epsilon^2\big).$$

Proof. Apply Hoeffding to Bernoulli random variables $\mathbf{1}_A(\theta_t) \in [0, 1]$. $\square$

Definition 9.10 (Normalized gray-zone and protocol-holonomy observables). *For certified sets $E_1, E_2 \subseteq \Theta$, define the normalized protocol holonomy in window K:*

$$\widetilde{\Delta}_\circlearrowleft(E_1, E_2 \mid K) := \mu_\Theta^K(E_1 \triangle E_2).$$

For an ideal target region E_τ and a bank-certified region $E(\mathcal{S}, \tau)$, define the normalized gray-zone:

$$\widetilde{G}(\mathcal{S}, \tau \mid K) := \mu_\Theta^K\big(E_\tau \setminus E(\mathcal{S}, \tau)\big).$$

Both are estimable by Definition 9.9 with error bars from Theorem 9.2.

Remark 9.3 (Absolute vs normalized measures). The absolute quantities $\Delta_\circlearrowleft$ and G are recovered by multiplying by $\mu_\Theta(K)$. Normalized observables $\widetilde{\Delta}_\circlearrowleft$ and $\widetilde{G}$ are preferable when $\mu_\Theta(K)$ is unknown or when only conditional sampling is available.

9.5 Pre-registration: sample size planning and decision thresholds

Definition 9.11 (([Proved]) Sample size for normalized indicator tests). *Fix accuracy $\epsilon \in (0, 1)$ and risk level $\eta \in (0, 1)$. For the normalized estimator in Definition 9.9, define*

$$T(\epsilon, \eta) := \left\lceil \frac{1}{2\epsilon^2} \log \frac{2}{\eta} \right\rceil.$$

Proposition 9.1 ([Proved] Guaranteed confidence interval width). *If $T \geq T(\epsilon, \eta)$, then with probability at least $1 - \eta$,*

$$\Big|\widehat{\mu^K}_T(A) - \mu_\Theta^K(A)\Big| \leq \epsilon.$$

Equivalently, the confidence interval $[\widehat{\mu^K}_T(A) - \epsilon, \widehat{\mu^K}_T(A) + \epsilon]$ is valid at level $1 - \eta$.

Proof. Apply Theorem 9.2 and solve $2\exp(-2T\epsilon^2) \le \eta$ for T. $\square$

Definition 9.12 (Decision rule schema). *Given a pre-registered tolerance $\varepsilon_{\mathrm{hol}} > 0$ and risk η, declare*

$$\text{“holonomy detected”} \quad \Longleftrightarrow \quad \widehat{\widehat{\Delta}}_T - \epsilon \ge \varepsilon_{\mathrm{hol}},$$

with ϵ chosen and $T \ge T(\epsilon, \eta)$. Analogous rules apply for gray-zone collapse tests $\widetilde{G} \to 0$ by substituting $\widehat{\widetilde{G}}_T$.

9.6 Distances between predicted distributions (for atlas probes)

Definition 9.13 (Total variation and Wasserstein). *For probability measures P, Q on $(Y, \mathcal{B})$:*

$$d_{\mathrm{TV}}(P,Q) := \sup_{A \in \mathcal{B}} |P(A) - Q(A)|.$$

If $Y \subseteq \mathbb{R}^m$ with metric $\|\cdot\|$, define the 1-Wasserstein distance

$$W_1(P,Q) := \sup_{\mathrm{Lip}(f) \le 1} \left| \int f\, dP - \int f\, dQ \right|.$$

Definition 9.14 (Maximum mean discrepancy (MMD)). *Fix a bounded positive definite kernel k on Y. Define*

$$\mathrm{MMD}_k^2(P,Q) := \mathbb{E}\, k(X,X') + \mathbb{E}\, k(Y,Y') - 2\mathbb{E}\, k(X,Y),$$

with $X, X' \sim P$ i.i.d. and $Y, Y' \sim Q$ i.i.d. An unbiased empirical estimator is obtained by replacing expectations with sample averages over independent batches.

9.7 Experimental Nodes (P1–P5) as network expansions

9.7.1 P1: Refinement-square holonomy experiment (finite-bank)

Definition 9.15 (Template-pool instantiation). *Fix a large finite pool $\mathcal{S}_{\mathrm{pool}}$, serving as an empirical stand-in for an idealized infinite bank. For each template $s \in \mathcal{S}_{\mathrm{pool}}$, fix a measurable score function $F_s : \Theta \to \mathbb{R}$. For a concrete bank instance $\mathcal{S} \subseteq \mathcal{S}_{\mathrm{pool}}$ and margin $\tau > 0$, define the certified set*

$$E(\mathcal{S}, \tau) := \{\theta \in \Theta : \max_{s \in \mathcal{S}} F_s(\theta) < -\tau\}.$$

Lemma 9.1 ([Model] Pool Lipschitz–net transfer inclusions). *Assume the template pool $\mathcal{S}_{\mathrm{pool}}$ is equipped with a metric $d_{\mathcal{S}}$ and that, for all $\theta \in K$, the score map is Lipschitz in the template index:*

$$|F_s(\theta) - F_{s'}(\theta)| \le L\, d_{\mathcal{S}}(s, s') \qquad \forall s, s' \in \mathcal{S}_{\mathrm{pool}}.$$

If $\mathcal{S} \subseteq \mathcal{S}_{\mathrm{pool}}$ is an ε-net of $\mathcal{S}_{\mathrm{pool}}$ (every s is within ε of some $s' \in \mathcal{S}$), then for all $\tau > 0$,

$$E(\mathcal{S}_{\mathrm{pool}}, \tau + L\varepsilon) \subseteq E(\mathcal{S}, \tau) \subseteq E(\mathcal{S}_{\mathrm{pool}}, \tau).$$

Thus the abstract transfer axiom $E_{\tau+\omega(\varepsilon)} \subseteq E_\tau^{(N)} \subseteq E_\tau$ holds in the pool model with $\omega(\varepsilon) = L\varepsilon$.

9.7.2 P1⋆: Proof-of-concept artifact node (cut-state holonomy)

This node instantiates the panel's “uncertainty $\to 0$ while $\Delta_\odot \not\to 0$” regime in a controlled, stylized lattice-extrapolation harness designed to mirror common workflow choices (fit windows / cuts) in numerical field theory.

n_{cfg}	N	$\widehat{U}$	s.e.	$\widehat{\Delta}_{\odot}$	s.e.
30	360	0.821	0.00869	0.3252	0.00995
120	1440	0.4165	0.00461	0.1618	0.00496
480	5760	0.2069	0.0023	0.08881	0.00265
1920	23040	0.1034	0.00117	0.05624	0.00158
7680	92160	0.05135	0.000565	0.04504	0.000944
30720	368640	0.02583	0.000293	0.04541	0.000514

Table 1: Same artifact node as Figure 2. Here $N = 12\, n_{\mathrm{cfg}}$ (twelve design points). The column $\widehat{U}$ is the within-protocol half-width proxy (Definition 9.18).

Definition 9.16 (Lattice-extrapolation harness (stylized but auditable)). *Let (a, m) range over a finite design grid (lattice spacing a and mass m). Observations are scalar summaries*

$$y(a, m) = \theta_0 + c\, a^2 + f\, a^4 + d\, m + e\, a^2 m + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2/n_{\mathrm{cfg}}),$$

with unknown target θ_0 and finite-configuration noise controlled by n_{cfg}. Define the prediction set for θ_0 as an interval $I_p(N)$ produced by a protocol p (below) with an auditable half-width estimator $\widehat{w}_p(N)$.

Definition 9.17 (Two protocols distinguished by a cut-state ξ). *Fix a design grid with four a-values and three m-values. Protocol p_{fine} uses only the two finest a-values in the $a^2 \to 0$ regression stage (whereas p_{full} uses all four). Both complete the same terminal extrapolation to $(a, m) = (0, 0)$, but with different internal state*

$$\xi \in \{\text{fine cut}, \text{full fit}\}.$$

Let $I_{\mathrm{fine}}(N)$ and $I_{\mathrm{full}}(N)$ denote the resulting intervals.

Definition 9.18 (Uncertainty-width and holonomy estimators). *Define a within-protocol uncertainty proxy (not the global gray-zone set-measure in Definition 9.21) by*

$$\widehat{U}(N) := \max\{\widehat{w}_{\mathrm{fine}}(N), \widehat{w}_{\mathrm{full}}(N)\}, \qquad \widehat{\Delta}_{\odot}(N) := \mathrm{dist}_H\big(I_{\mathrm{fine}}(N), I_{\mathrm{full}}(N)\big),$$

with dist_H the Hausdorff distance on intervals (equal to the absolute midpoint difference when widths match).

Proposition 9.2 ([Model] Persistent cut-holonomy diagnostic). *For the harness in Definition 9.16, if $f \neq 0$ and the design includes non-negligible coarse-a points, then the two protocols generally converge to different limits as $n_{\mathrm{cfg}} \to \infty$:*

$$\lim_{N\to\infty} \widehat{\Delta}_{\odot}(N) = \Delta_{\infty}(\xi) \neq 0, \qquad \lim_{N\to\infty} \widehat{U}(N) = 0.$$

Thus $\widehat{U} \to 0$ while $\widehat{\Delta}_{\odot} \not\to 0$ flags a hidden-state dependence (here: ξ encodes the fit window).

Remark 9.4 (Concrete numbers). For a fixed synthetic design grid (four a values, three m values) and Gaussian noise, Monte Carlo evaluation produces the trend in Figure 2 and Table 1. In this instance, $\widehat{U}$ decays approximately as $N^{-1/2}$ while $\widehat{\Delta}_{\odot}$ approaches a small nonzero constant.

Remark 9.5 (Reproducibility). The full source code that generates Figure 2 and Table 1 (including random seeds and audit logs) is provided as an *Artifact Bundle* in the companion ZIP: `artifact/P1star_cut_state/` (script `p1star_cut_state.py`, notebook `p1star_cut_state.ipynb`).

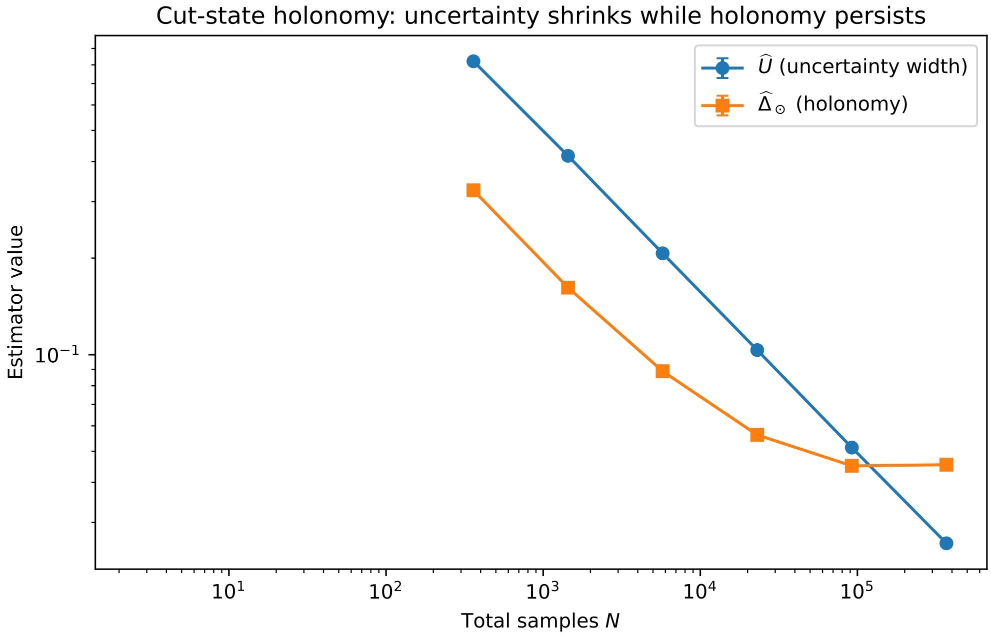

Figure 2: Artifact node: estimated uncertainty width $\widehat{U}$ and protocol-holonomy $\widehat{\Delta}_{\odot}$ vs total samples N (log–log axes). The design uses two protocol states $\xi \in \{\text{fine cut}, \text{full fit}\}$; the limiting separation reflects model curvature (fa^4) and window choice.

9.7.3 P1**: Public-data instantiation (GW150914 skeleton)

/Protocol/ This optional node upgrades P1* from a stylized harness to a public-data instantiation, aligning with Case Study C (Definition 10.7). We pre-register two analysis states $\xi \in \{\xi_{\text{bandpass}}, \xi_{\text{whiten}}\}$ (detrending/whitening/PSD windowing) and define a square by swapping the order of whitening and bandpass filtering. The artifact bundle includes a runnable script that downloads calibrated strain segments from GWOSC and computes an auditable parameter estimate (e.g. a chirp-mass proxy) under both protocol orders, together with the resulting holonomy estimator $\widehat{\Delta}_{\odot}$; see `artifact/P2_gw150914_skeleton/`.

Definition 9.19 (Adaptive bank augmentation operator)**.** *Fix a window K and an exploration distribution ν supported on K. Given $(\mathcal{S}, \tau)$, define an augmentation step* Aug *by sampling $\theta \sim \nu$ and selecting*

$$s^{\star} \in \arg \max_{s \in \mathcal{S}_{\text{pool}} \setminus \mathcal{S}} F_s(\theta), \quad \textit{then setting} \quad \mathcal{S} \leftarrow \mathcal{S} \cup \{s^{\star}\}.$$

This makes bank refinement history-dependent*, enabling nontrivial holonomy (Remark 9.1).*

Definition 9.20 (Program square (two orders))**.** *Define two protocols from the same start state $c_0 = (\mathcal{S}, \tau, \xi)$:*

$$p_1 := \mathsf{Aug}^m \circ \mathsf{Tighten}_\delta, \qquad p_2 := \mathsf{Tighten}_\delta \circ \mathsf{Aug}^m,$$

where Aug^m denotes m augmentation steps (targeting bank size increase) and $\mathsf{Tighten}_\delta$ updates $\tau \leftarrow \tau + \delta$ (possibly also re-evaluating certificates). Define $\Delta_{\circlearrowleft}(p_1, p_2; K)$ by Definition 9.5.

Remark 9.6 (Measurable test target). In the pool instantiation, $\Delta_{\circlearrowleft}(p_1, p_2; K)$ reduces to the window measure of a symmetric difference between two explicit sets $E(\mathcal{S}_1, \tau_1)$ and $E(\mathcal{S}_2, \tau_2)$, and is estimable via Theorem 9.1.

Estimator pseudocode (auditable).

```
Input: start state c0=(S,tau,xi), protocols p1,p2, window K, sample size T
1) Execute p1 from c0 to obtain certified set E1 (membership oracle: theta -> 1_{E1}(theta))
2) Execute p2 from c0 to obtain certified set E2 (membership oracle)
3) Sample theta_1,...,theta_T ~ mu_Theta(. | K) (i.i.d. or MCMC with ESS)
4) Return mu(K) * (1/T) * sum_t 1[ (theta_t in E1) XOR (theta_t in E2) ]
Output: \hat{Delta}_loop and Hoeffding error bar (or ESS-adjusted bar)
```

9.7.4 P2: Gray-zone scaling (estimating effective dimension)

Definition 9.21 (Gray-zone observable). *Assume an ideal region E_τ is available in the pool model, for instance $E_\tau := E(\mathcal{S}_{\text{pool}}, \tau)$. For a finite bank instance $\mathcal{S}$ (size N), define the gray-zone in window K as*

$$G(\mathcal{S}, \tau; K) := \mu_\Theta\big((E_\tau \setminus E(\mathcal{S}, \tau)) \cap K\big).$$

Definition 9.22 (Empirical covering dimension). *Let $(\mathcal{S}_{\text{pool}}, d_{\mathcal{S}})$ be the template pool metric space (Definition 9.15). For $\varepsilon > 0$ define the covering number*

$$\mathcal{N}(\varepsilon) := \min\{m : \exists s_1, \ldots, s_m \in \mathcal{S}_{\text{pool}} \text{ with } \mathcal{S}_{\text{pool}} \subseteq \cup_{j=1}^m B_\varepsilon(s_j)\}.$$

Define the effective covering dimension

$$d_{\text{eff}} := \limsup_{\varepsilon \downarrow 0} \frac{\log \mathcal{N}(\varepsilon)}{\log(1/\varepsilon)}.$$

Operationally, a two-scale estimator uses $\varepsilon_1 < \varepsilon_2$ and $\widehat{d} = \frac{\log \mathcal{N}(\varepsilon_1) - \log \mathcal{N}(\varepsilon_2)}{\log(\varepsilon_2/\varepsilon_1)}$.

Proposition 9.3 ([Model] log–log slope test). *Assume a Hölder-type transfer modulus $\omega(\varepsilon) \approx L\varepsilon^\alpha$ on window K and covering control $\varepsilon_N \asymp N^{-1/d_{\text{eff}}}$ where d_{eff} is estimated from the pool metric (Definition 9.22). Then, in an asymptotic regime,*

$$G(N, \tau; K) \asymp N^{-\alpha/d_{\text{eff}}}.$$

The slope α/d_{eff} is estimable by linear regression on $\log \widehat{G}$ vs $\log N$ with uncertainty from Theorem 9.1, together with a separate estimate of d_{eff} from pool covering numbers.

9.7.5 P3: Atlas holonomy with probes (groupoid on kernels)

Definition 9.23 (Kernel action on measures). *Let $(Y, \mathcal{B})$ be measurable and let K be a Markov kernel on Y. Let* **Prob** *denote the category whose objects are measurable spaces and whose morphisms are Markov kernels. For a probability measure $\nu \in \text{Prob}(Y)$ define its pushforward by K as*

$$K_\# \nu(A) := \int_Y K(A \mid y)\, d\nu(y), \qquad A \in \mathcal{B}.$$

If K is an isomorphism in **Prob**, *then $K_\# : \text{Prob}(Y) \to \text{Prob}(Y)$ is bijective.*

Definition 9.24 (Probe-set holonomy magnitude (section + operator tests)). *Fix a finite probe set $\{o_\ell\}_{\ell=1}^L \subseteq \mathcal{O}_r$ and a distance d on $\text{Prob}(Y_{o_\ell})$ (e.g. d_{TV}, W_1, or MMD). Let h_{ijk,o_ℓ} denote the component of the atlas cocycle acting on $\text{Prob}(Y_{o_\ell})$ and interpret its action by pushforward (Definition 9.23).*

(Section-based magnitude) *Define*

$$\Delta^{\text{sec}}_{ijk} := \sum_{\ell=1}^L d(\text{Pred}_{T_i}(o_\ell),\ (h_{ijk,o_\ell})_\# \,\text{Pred}_{T_i}(o_\ell)).$$

(Operator-detecting magnitude) *Fix, for each ℓ, a finite family of test measures $\{\nu_{\ell,m}\}_{m=1}^{M_\ell} \subseteq \mathrm{Prob}(Y_{o_\ell})$ ("probe measures"). Define*

$$\Delta_{ijk}^{\mathrm{op}} := \sum_{\ell=1}^{L} \sum_{m=1}^{M_\ell} d(\nu_{\ell,m},\ (h_{ijk,o_\ell})_{\#}\nu_{\ell,m}).$$

Finally set $\Delta_{ijk}^{\mathrm{atlas}} := \Delta_{ijk}^{\mathrm{sec}} + \Delta_{ijk}^{\mathrm{op}}$. This detects holonomy even when the particular predicted section $\mathrm{Pred}_{T_i}(o_\ell)$ is fixed by h_{ijk,o_ℓ}.

Empirical estimator (samples). For each probe o_ℓ, obtain sample batches from $\mathrm{Pred}_{T_i}(o_\ell)$ and from $(h_{ijk}\cdot\mathrm{Pred}_{T_i})(o_\ell)$ by simulating kernels. Estimate d by empirical TV/Wasserstein/MMD; attach bootstrap confidence intervals.

9.7.6 P4: Zipper vs protocol holonomy correlation test (bridge H1 as an experiment)

Definition 9.25 (Operational correlation observable)**.** *Fix a family of resource settings producing zipper signatures $\Sigma_{\mathrm{zip}}(\mathcal{E}, o)$ and protocol holonomy magnitudes $\Delta_{\circlearrowleft}$. Define the correlation functional*

$$\mathsf{Corr} := \mathrm{corr}\big(\mathbf{1}[\Sigma_{\mathrm{zip}} \textit{ stable on } I],\ \Delta_{\circlearrowleft}\big),$$

where stability is defined by a declared tolerance in κ, merge-tree edit distance on MT_I, and hitting-time deviation.

Remark 9.7 (Falsifiable bridge). Conjecture 7.1 becomes an experimentally checkable implication: "stability of Σ_{zip}" $\Rightarrow$ "$\Delta_{\circlearrowleft} \approx 0$" in the tested regime.

9.7.7 P5: CNF-to-volume calibration node

Definition 9.26 (Discrete-to-continuous embedding calibration)**.** *For a discrete CNF F on $\{0,1\}^n$, embed assignments into $[0,1]^n$ by mapping each vertex to its unit cube cell. Let $K = [0,1]^n$ and let μ be Lebesgue measure. Then $\mathrm{Vol}_K(\mathrm{Mod}(F))$ equals $\#\mathrm{SAT}(F)\cdot 2^{-n}$ exactly in this embedding.*

Remark 9.8 (Purpose). This node provides a sanity check for Monte Carlo volume estimators and connects the CNF/cGCNF node to the window-volume node.

9.8 Pre-registered falsification patterns (diagnostics)

9.9 Pre-registered protocols (decision rules with error bars)

Definition 9.27 (Pre-registration schema)**.** *For each experiment* $\mathrm{P} \in \{\mathrm{P1}, \ldots, \mathrm{P5}\}$*, a* pre-registration *is the tuple*

$$\Pi_{\mathrm{P}} = (\mathsf{Input}, \mathsf{Output}, \widehat{\mathsf{Est}}, \mathsf{ErrBar}, \mathsf{Null}, \mathsf{Alt}, \mathsf{Decision}),$$

where:

- Input *fixes the harness (window K, distribution μ_Θ, pool/banks, probes),*
- Output *names the target observable(s) (G, $\Delta_{\circlearrowleft}$, Δ^{atlas}),*
- $\widehat{\mathsf{Est}}$ *specifies the estimator (e.g. Theorem 9.1),*
- ErrBar *specifies an auditable error bar (Hoeffding or ESS-adjusted),*
- Null *and* Alt *are mutually exclusive inequalities to test,*

- Decision *is a threshold rule with declared confidence.*

Definition 9.28 (Example: P1 commutation test)**.** *Fix start state c_0, two protocols p_1, p_2 (Definition 9.20), window K, and sample size T. Let $\widehat{\Delta}_T$ be the Monte Carlo estimator of $\Delta_{\circlearrowleft}(p_1, p_2; K)$. Declare error bar ϵ_T such that*

$$\Pr(|\widehat{\Delta}_T - \Delta_{\circlearrowleft}| \geq \epsilon_T) \leq \eta.$$

A concrete choice under i.i.d. is $\epsilon_T := \mu_\Theta(K)\sqrt{\frac{\log(2/\eta)}{2T}}$ (Theorem 9.1). Define:

$$\textsf{Null}: \ \Delta_{\circlearrowleft}(p_1, p_2; K) = 0, \qquad \textsf{Alt}: \ \Delta_{\circlearrowleft}(p_1, p_2; K) \geq \Delta_{\min}.$$

Decision rule: accept Alt *if $\widehat{\Delta}_T - \epsilon_T \geq \Delta_{\min}$; otherwise do not reject Null.*

Remark 9.9 (Additional pre-registrations)*.* Analogous pre-registrations are defined for P2 (log–log slope test), P3 (atlas operator test using probe measures $\nu_{\ell,m}$), P4 (correlation threshold), and P5 (CNF-to-volume exactness check).

Definition 9.29 (Holonomy–gap diagnostic)**.** *In the finite-bank harness, measure both G and $\Delta_{\circlearrowleft}$ along a bank-sweep $N \uparrow$. Declare:*

(D1) $\quad G \to 0 \wedge \Delta_{\circlearrowleft} \not\to 0 \quad \Longrightarrow \quad$ *additional non-commutativity source (new node/edge required).*

Remark 9.10 (Network-expansive response rule)*.* When (D1) occurs, extend the net by adding an explicit node capturing the extra source (e.g. optimizer state, discretization, numerical tolerances), and re-type refinements as morphisms acting on that node. This preserves consistency while expanding explanatory power.

10 Physics/Tech Case Studies (end-to-end harness instantiations)

Remark 10.1 (Case studies are instantiation templates)*.* The case studies in this section are *schemas* mapping standard scientific pipelines into the closure-net types. They are not presented as new physics claims; their role is to specify exactly what would be measured (certified sets, gray zones, holonomy magnitudes) under a frozen, pre-registered experimental node.

10.1 Case Study A: Materials stability under competing phases

Definition 10.1 (Materials harness)**.** *Let Θ be a measurable parameter space of material candidates (e.g. compositions, structures, thermodynamic settings), with window $K \subseteq \Theta$ and sampling measure $\mu_\Theta(\cdot \mid K)$. Let $\mathcal{S}_{\text{pool}}$ be a large pool of* competing phases *or* competing decompositions. *For each $s \in \mathcal{S}_{\text{pool}}$, define a measurable violation score*

$$F_s(\theta) := \Delta E_s(\theta),$$

interpreted as an energy-above-competitor quantity (lower is better). Fix a margin $\tau > 0$ (stability slack) and define the ideal stability region

$$E_\tau := \{\theta \in \Theta : \sup_{s \in \mathcal{S}_{\text{pool}}} F_s(\theta) \leq -\tau\}.$$

For a finite competitor-bank instance $\mathcal{S} \subseteq \mathcal{S}_{\text{pool}}$, define the finite-bank certified region

$$E(\mathcal{S}, \tau) := \{\theta \in \Theta : \sup_{s \in \mathcal{S}} F_s(\theta) \leq -\tau\}.$$

Definition 10.2 (Materials refinement programs)**.** *A certificate state is $c = (\mathcal{S}, \tau, \xi)$ where ξ records the selection rule, random seeds, surrogate-model state, and stopping criteria. Define:*

(A1) Tighten step $\mathsf{Tighten}_\delta : (\mathcal{S}, \tau, \xi) \mapsto (\mathcal{S}, \tau + \delta, \xi)$.

(A2) Augment step $\mathsf{Aug} : (\mathcal{S}, \tau, \xi) \mapsto (\mathcal{S} \cup \{s^\star\}, \tau, \xi')$ *where*

$$s^\star \in \arg \max_{s \in \mathcal{S}_{\text{pool}} \setminus \mathcal{S}} \mathbb{E}_{\theta \sim \mu_\Theta(\cdot|K)} \big[\max(F_s(\theta) + \tau, 0)\big],$$

and ξ' updates the selection trace.

The augment rule is history-dependent through ξ and through the empirical objective approximated from finite samples.

Definition 10.3 (Pre-registered material protocols)**.** *Fix $m \in \mathbb{N}$ augment steps and $\delta > 0$. From a common start $c_0 = (\mathcal{S}_0, \tau_0, \xi_0)$ define two protocols:*

$$p_1 := \mathsf{Aug}^m \circ \mathsf{Tighten}_\delta, \qquad p_2 := \mathsf{Tighten}_\delta \circ \mathsf{Aug}^m.$$

Let E_{p_i} be the resulting certified set in Θ produced by executing p_i. Define normalized holonomy and gray-zone observables using Definition 9.10:

$$\widetilde{\Delta}_\circlearrowleft(p_1, p_2 \mid K) := \mu_\Theta^K(E_{p_1} \triangle E_{p_2}), \qquad \widetilde{G}(\mathcal{S}, \tau \mid K) := \mu_\Theta^K(E_\tau \setminus E(\mathcal{S}, \tau)).$$

Proposition 10.1 ([Model] Materials holonomy–gap diagnostic)**.** *In the materials harness, perform a bank sweep (increasing $|\mathcal{S}|$) while recording $\widetilde{G}$ and $\widetilde{\Delta}_\circlearrowleft$. If $\widetilde{G} \to 0$ but $\widetilde{\Delta}_\circlearrowleft \not\to 0$, then the observed non-commutativity is not explained by finite competitor coverage alone and must be attributed to additional program state ξ (e.g. surrogate bias, adaptive sampling, numerical tolerances), which is represented by a new node/edge in the closure net.*

10.2 Case Study B: Collider analysis pipeline

Definition 10.4 (Collider harness)**.** *Let D denote a dataset of binned observables (histograms) and let Θ be a measurable parameter space of interest (signal strengths, EFT coefficients, nuisance parameters). Fix a window $K \subseteq \Theta$ and sampling measure $\mu_\Theta(\cdot \mid K)$. Let $\mathcal{S}_{\text{pool}}$ be a pool of systematic variations and simulation configurations. For each $s \in \mathcal{S}_{\text{pool}}$ and analysis configuration ξ (cuts, calibrations, unfolding choices), define a measurable discrepancy score*

$$F_s(\theta; \xi) := \chi^2\big(D,\ \widehat{D}_s(\theta; \xi)\big),$$

where $\widehat{D}_s(\theta; \xi)$ is the predicted binned distribution under variation s. For a finite bank $\mathcal{S} \subseteq \mathcal{S}_{\text{pool}}$ and tolerance $\tau > 0$, define the certified region

$$E(\mathcal{S}, \tau, \xi) := \{\theta \in \Theta : \sup_{s \in \mathcal{S}} F_s(\theta; \xi) \le \tau\}.$$

Definition 10.5 (Pipeline refinements as programs)**.** *A certificate state is $c = (\mathcal{S}, \tau, \xi)$ where ξ includes: cut thresholds, calibration constants, unfolding regularization, and random seeds. Define:*

(B1) Tighten step $\mathsf{Tighten}_\delta : (\mathcal{S}, \tau, \xi) \mapsto (\mathcal{S}, \tau - \delta, \xi)$ *(stricter agreement), or alternatively $\xi \mapsto \xi'$ as stricter cuts.*

(B2) Augment step $\mathsf{Aug} : (\mathcal{S}, \tau, \xi) \mapsto (\mathcal{S} \cup \{s^\star\}, \tau, \xi')$ *selecting*

$$s^\star \in \arg \max_{s \in \mathcal{S}_{\text{pool}} \setminus \mathcal{S}} \mathbb{E}_{\theta \sim \mu_\Theta(\cdot|K)} \big[\max(F_s(\theta; \xi) - \tau, 0)\big].$$

Non-commutativity arises because $F_s(\cdot; \xi)$ depends on ξ, and ξ changes under tightening.

Definition 10.6 (Pre-registered collider holonomy test)**.** *Fix a start state $c_0 = (\mathcal{S}_0, \tau_0, \xi_0)$ and two protocols p_1, p_2 (tighten-then-augment vs augment-then-tighten). Estimate $\widetilde{\Delta}_{\circlearrowleft}(p_1, p_2 \mid K)$ by sampling $\theta_t \sim \mu_\Theta(\cdot \mid K)$ and computing*

$$\widehat{\widetilde{\Delta}}_T = \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}[\theta_t \in E_{p_1} \triangle E_{p_2}] \,,$$

with Hoeffding error bars from Theorem 9.2. Declare holonomy-detection at confidence $1 - \eta$ if the lower confidence bound exceeds a pre-registered tolerance $\varepsilon_{\text{hol}} > 0$.

10.3 Case Study C: Gravitational-wave coherence across detectors

Definition 10.7 (GW multi-detector coherence harness)**.** *Let $D = \{d_\alpha\}_{\alpha \in \{\text{H1,L1,V1,...}\}}$ be calibrated strain time series from multiple detectors. Let Θ parameterize waveform families and nuisance degrees of freedom; fix window $K \subseteq \Theta$ and sampling $\mu_\Theta(\cdot \mid K)$. Let $\mathcal{S}_{\text{pool}}$ be a pool of* stress tests *(glitch models, noise nonstationarity segments, calibration perturbations). For $s \in \mathcal{S}_{\text{pool}}$ and analysis state ξ (PSD estimation, windowing, vetoes), define*

$$F_s(\theta; \xi) := \max_{\alpha} \ \text{Res}\big(d_\alpha, \ h_\alpha(\cdot; \theta); \ s, \xi\big),$$

a measurable residual statistic (e.g. normalized energy in the whitened residual, or negative log-likelihood gap). Define the certified region

$$E(\mathcal{S}, \tau, \xi) := \{\theta : \ \sup_{s \in \mathcal{S}} F_s(\theta; \xi) \le \tau\}.$$

Bank augmentation adds stress tests; tightening increases coherence requirements by decreasing τ or strengthening vetoes in ξ. The holonomy observable $\widetilde{\Delta}_{\circlearrowleft}(p_1, p_2 \mid K)$ is defined as in Definition 9.10.

Remark 10.2 (Utility). Case Studies A–C instantiate the same abstract observables $(\widetilde{G}, \widetilde{\Delta}_{\circlearrowleft})$ in three domains. They operationalize "network expansion" as a diagnostic tool: persistent holonomy after gray-zone collapse forces an explicit new node/edge for the missing mechanism.

10.4 Reproducibility bundle (artifact node)

Definition 10.8 (Artifact bundle)**.** *A* reproducibility artifact bundle *for a case study is a tuple*

$$\mathsf{Art} = (\mathsf{DataID}, \ \mathsf{CodeID}, \ \mathsf{SeedLog}, \ \mathsf{Params}, \ \mathsf{Report}),$$

where:

(R1) DataID *identifies the dataset snapshot (DOI/run list/hash) and preprocessing recipe.*

(R2) CodeID *identifies code version (commit hash/container digest) implementing membership oracles for $E(\cdot)$, protocols, and estimators.*

(R3) SeedLog *records all randomness: seeds, initial banks $\mathcal{S}_0$, and augmentation traces.*

(R4) Params *records $(K, \mu_\Theta(\cdot \mid K), \tau_0, \delta, m, \varepsilon_{\text{hol}}, \eta)$ and the distance choice for atlas probes.*

(R5) Report *contains $\widehat{\widetilde{G}}$ and $\widehat{\widetilde{\Delta}}$ with confidence intervals and the pre-registered decisions (Definition 9.12).*

Remark 10.3 (Minimal public data anchors). The case studies admit concrete public anchors without changing the formalism:

(A1) Materials: a public DFT repository snapshot plus a fixed competitor-pool extraction rule for $\mathcal{S}_{\mathrm{pool}}$.

(A2) Collider: a public HEP open-data release plus a fixed histogramming + simulation/systematics pool definition.

(A3) GW: a public calibrated strain release plus a fixed residual statistic and a fixed stress-test pool definition.

The manuscript treats these anchors as *parameters* of Art; theorems depend only on measurability and the sampling contract.

.1 Case Study D: Lattice QCD systematics as protocol holonomy (instantiation template)

Let Θ parameterize a lattice ensemble family (lattice spacing a, box size L, quark masses, and algorithmic tolerances). Let a "bank" S_N be a finite set of gauge configurations / solver initializations / multistart heuristics used to certify a target observable, e.g. a hadron mass estimator $M_p(\theta)$ with a robustness margin τ on residuals or fit quality.

Two refinement protocols. A typical non-commuting square is:

$$p_1 : \text{increase statistics/bank} \Rightarrow \text{continuum extrapolation} \Rightarrow \text{chiral extrapolation}, \qquad p_2 : \text{increase statistics/}$$

Execute both as program paths in $\mathcal{I}^{\mathrm{prog}}$ with fixed, pre-registered auxiliary state ξ (seeds, fit priors, stopping rules), and compare the resulting certified parameter sets or certified output intervals via $\widetilde{\Delta}_{\circlearrowleft}$. A nonzero value quantifies systematic path dependence under finite resources, in the same operational sense as in the finite-bank black-hole setting.

Falsifiable prediction skeleton. Under a transfer modulus $\omega(\varepsilon_N)$ and stable extrapolation maps, one expects $\widetilde{\Delta}_{\circlearrowleft} \to 0$ as $N \to \infty$, with a rate controlled by $\omega(\varepsilon_N)$ and the effective dimension of the banked configuration/template space (Section 5).

A Provenance and scope

- Added typed refinement category $\mathcal{I}$ and functor E; replaced untyped inverse-commutator by square-comparison holonomy (Defs. 2.11–2.14).
- Added prediction functors into the Markov-kernel category and typed atlas transitions as a groupoid of natural isomorphisms (Defs. 2.16–2.20).
- Resolved notation collision by renaming the cGCNF ambient product space to $\mathcal{X}$ (no change in meaning).
- Extended holes list to include the missing bridge between bank-certified parameter sets and prediction kernels.

B Bibliography (minimal pointers)

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