

# General Epistemic Geometry

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February 2026

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# Preface

This monograph is written under one non-negotiable rule: always extend, never reduce. The objective is not to replace existing technical knowledge with lighter summaries, but to preserve and strictly enlarge the formal corpus. Every theorem and protocol included in the existing repository is either reproduced, strengthened, or integrated into a more explicit architecture.

The book develops an operational theory of scientific knowledge under finite verification constraints. Three principles are fixed from the outset:

1. finite syntax for claims and hypotheses,
2. finite and auditable certificates for acceptance or rejection,
3. explicit quantitative gaps between syntactic representation and semantic target.

The text combines continuous cGCNF semantics, geometric compilation, counting and volume theory, complexity obstructions, epistemic curvature, coherent flow dynamics, Riemannian and Lorentzian interfaces, black-hole-relevant measurable phases, and locality-based operational limits. The culminating application is a fully auditable, finite, and measurable framework for black-hole phase observables in parameterized Einstein initial data.

No reductionist posture is adopted. Whenever an apparent error or instability appears, it is treated as a source of structure: every failure mode must become either a theorem, a refined assumption, or a certified uncertainty region.



# Evidence Contract and Editorial Standard

Statements are tagged by evidence type.

- **[Proved]**: mathematically proved inside explicit hypotheses.
- **[Model]**: depends on a scientific model layer not proven here.
- **[Speculative]**: open direction or program-level conjecture.
- Additional refined tags (e.g., **[Proved core]**, **[Proved given model]**) are used when appropriate.

Auditability norms:

1. Every ideal object must admit finite probes.
2. Every quantifier must have an explicit domain.
3. Every computational claim must provide an independently checkable certificate.
4. Every numerical output must include an explicit error bar or tolerance contract.
5. Every syntax–semantics interface must fix a metric structure.



# Reader’s Guide and Structural Roadmap

This text is a treatise rather than an article: later layers do not merely reuse earlier objects, they *lift* them. The guiding invariant is that every ideal semantic object must admit finite probes and every interface must expose explicit metric structure.

**Axis I: finite syntax, open semantics.** Chapters 1–2 define continuous cGCNF formulas and show that model sets are open, hence robust, and that windowed semantic volumes are measurable and auditable.

**Axis II: disjoint compilation and barriers.** Chapters 3–5 show that forbidden regions admit finite decompositions (boxes or subcubes) and that these can be disjointized. Exact counting becomes an additive sum over disjoint pieces. The same geometry exposes lower bounds (fragmentation and parity), yielding conditional hierarchy-collapse consequences for “uniform compilation” hypotheses.

**Axis III: metric interfaces and limits.** Chapters 6–7 formalize representation as a metric interface between syntax and semantics and extract curvature invariants that quantify irreducible gaps. A Gödel-type diagonal statement is presented as an internal obstruction to uniform, verifiable zero-error completeness inside arithmetized regimes.

**Dynamics and physics-facing layers.** Coherent flow (Chapters 8–11) is an auditable dynamics on finite theory space and its probabilistic counterpart, with exact Lyapunov identities. The black-hole-relevant layer (Chapter 12) treats robust trapped-surface diagnostics as open literals on finite-dimensional parameter families. Locality and agency (Chapters 13–14) provide operational limits via soft causal cones.

**Frontier extensions.** Chapter 15 extends the literal discipline to conformal, topological, and Lorentzian settings, while preserving the book’s contract: openness is enforced by strict margins, and every nontrivial dependency on a scientific model is explicitly tagged **[Model]**.

**Unified obstruction and the BH goal.** Chapter 16 synthesizes the lower bounds, curvature, and locality into a unified trilemma. The measurable black-hole phase framework of Chapter 12 stands as a concrete, fully certified application of the entire epistemic geometry program.

Throughout, we use the tags **[Proved]**, **[Model]**, **[Speculative]**, etc., as part of the type system of the book: they are not rhetorical flourishes, but an audit interface.



# Reader's Guide and Roadmap

This monograph is designed as an *auditable* architecture rather than a conventional research article. The core contract is: (i) every semantic claim is attached to an explicit *open* predicate over data, (ii) every robustness statement is phrased as openness under a declared topology, and (iii) every quantitative limitation appears as a certified *error* or *curvature* invariant.

## How to read it.

- (1) **Chapters 1–4:** the continuous core (cGCNF) and its finite disjoint compilation interfaces.
- (2) **Chapters 5–7:** lifts (affine/product) culminating in the black-hole-relevant measurable layer, where margins and finite banks produce certified uncertainty zones.
- (3) **Chapters 8–13:** metric interfaces and coherent flow (deterministic and stochastic), including operational estimators and limits.
- (4) **Chapters 14–20:** operational materialization, complexity obstructions, and robust topological layers (braids/knots and configuration-space invariants).

**A minimal toy picture (2D forbidden region).** Consider a two-parameter space  $\Theta \subset \mathbb{R}^2$  and a formula  $\Phi$  whose satisfying set is an *open* certified region  $\text{Mod}(\Phi) \subset \Theta$ . The *forbidden* set  $\mathcal{U}(\Phi) := \Theta \setminus \text{Mod}(\Phi)$  is closed. A disjoint-box compilation builds a finite certified inner approximation  $\text{Mod}(\Phi_N) = \bigcup_{k=1}^N B_k$  with  $B_k$  axis-aligned open boxes, leaving a gray uncertainty interface near the boundary.

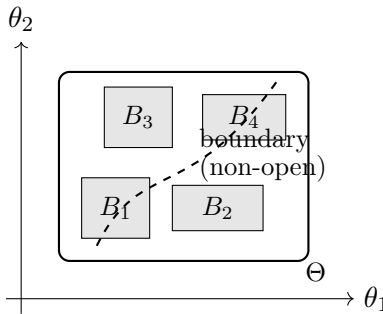


Figure 1: Toy certified compilation:  $\text{Mod}(\Phi_N) = \bigcup_k B_k \subseteq \text{Mod}(\Phi)$  is auditable; the dashed boundary indicates where strict margins generate a certified uncertainty interface.

The same geometry repeats throughout: open certified literals, finite probes, and a mathematically inevitable interface near sharp boundaries.



# Colophon and Versioning

This “final” manuscript is intended as the *canonical* reference version of the project *General Epistemic Geometry*. Earlier drafts are not retracted: they are treated as *development versions* documenting the extension trajectory.

**Editorial rule.** The project follows one non-negotiable constraint:

**always extend, never reduce.**

Accordingly, when a concept matures, the canonical version does not erase its developmental history; instead it records strengthened hypotheses, refined interfaces, and explicit certified uncertainty regions. This makes the theory *auditable in time*: claims are traceable not only inside the mathematics, but also across versions.

**How to cite.** For stable citation, reference this canonical version. When citing a developmental draft, cite it explicitly as such and treat it as a snapshot of the extension path.



## Part I

# Continuous Logical-Topological Core



# Chapter 1

## Continuous cGCNF: Syntax, Semantics, and Robustness

### 1.1 Base product spaces

**Assumption 1.1** (Base product setting). *Fix  $n \in \mathbb{N}$ , metric spaces  $(X_i, d_i)$ , and*

$$X := \prod_{i=1}^n X_i$$

*with product topology.*

**Definition 1.2** (Continuous literal). *A continuous literal is a tuple*

$$\ell = (I_\ell, f_\ell, Y_\ell, U_\ell)$$

*where  $I_\ell \subseteq \{1, \dots, n\}$  is finite and nonempty,  $Y_\ell$  is a topological space,  $f_\ell : \prod_{i \in I_\ell} X_i \rightarrow Y_\ell$  is continuous, and  $U_\ell \subseteq Y_\ell$  is open. Its model set is*

$$\text{Mod}(\ell) := \{x \in X : f_\ell(\pi_{I_\ell} x) \in U_\ell\}.$$

**Definition 1.3** (Clause and finite cGCNF formula). *A clause is a finite disjunction*

$$C = \ell_1 \vee \dots \vee \ell_m, \quad \text{Mod}(C) := \bigcup_{t=1}^m \text{Mod}(\ell_t).$$

*A finite cGCNF formula is*

$$\Phi = C_1 \wedge \dots \wedge C_p, \quad \text{Mod}(\Phi) := \bigcap_{j=1}^p \text{Mod}(C_j).$$

**Example 1.4** (Unary radial literal). *For  $X_i = \mathbb{R}$ , center  $a \in \mathbb{R}$ , and open  $A \subseteq [0, \infty)$ ,*

$$\ell(i, a, A) : |x_i - a| \in A$$

*is a continuous literal.*

**Theorem 1.5** (Openness of model sets [**Proved**]). *Literal model sets are open in  $X$ . Therefore clause model sets and finite cGCNF model sets are open.*

*Proof.* The map  $x \mapsto \pi_{I_\ell}(x)$  is continuous, hence  $x \mapsto f_\ell(\pi_{I_\ell}(x))$  is continuous. Thus

$$\text{Mod}(\ell) = (f_\ell \circ \pi_{I_\ell})^{-1}(U_\ell)$$

is open. Finite unions and intersections preserve openness.  $\square$

**Corollary 1.6** (Robust satisfiability [**Proved**]). *If  $x^* \in \text{Mod}(\Phi)$ , then some neighborhood  $N$  of  $x^*$  satisfies  $N \subseteq \text{Mod}(\Phi)$ . For any compatible product metric, there is  $\delta > 0$  with  $B(x^*, \delta) \subseteq \text{Mod}(\Phi)$ .*

## 1.2 Forbidden-region semantics

**Definition 1.7** (Clause falsification and total forbidden set). *For a clause  $C = \ell_1 \vee \dots \vee \ell_m$  define*

$$\text{Forb}(C) := \bigcap_{t=1}^m (X \setminus \text{Mod}(\ell_t)).$$

For  $\Phi = \bigwedge_{j=1}^p C_j$  define

$$\mathcal{U}(\Phi) := \bigcup_{j=1}^p \text{Forb}(C_j).$$

**Proposition 1.8** (Exact complement identity [**Proved**]). *For every finite cGCNF formula,*

$$\text{Mod}(\Phi) = X \setminus \mathcal{U}(\Phi).$$

*Each  $\text{Forb}(C_j)$  is closed and  $\mathcal{U}(\Phi)$  is a finite union of closed sets.*

*Proof.*

$$x \notin \text{Mod}(\Phi) \iff \exists j, x \notin \text{Mod}(C_j) \iff \exists j, \forall t, x \notin \text{Mod}(\ell_t) \iff x \in \bigcup_j \text{Forb}(C_j).$$

Closedness follows from openness of literal model sets (Theorem 1.5).  $\square$

## 1.3 Parametric formulas and topological stability

**Assumption 1.9** (Joint continuity in parameters). *Let  $(\Theta, d_\Theta)$  be a metric parameter space,  $\Lambda$  a finite literal dictionary. For each  $\alpha \in \Lambda$ , let  $U_\alpha \subseteq Y_\alpha$  be open and*

$$F_\alpha : \Theta \times X \rightarrow Y_\alpha$$

*continuous. Clauses and formulas use finite subsets of  $\Lambda$  independently of  $\theta$ .*

**Theorem 1.10** (Openness in  $(\theta, x)$  and in  $\theta$  [**Proved**]). *Define*

$$\mathcal{S} := \{(\theta, x) \in \Theta \times X : x \models \Phi_\theta\}.$$

*Then  $\mathcal{S}$  is open. Consequently,*

$$\mathcal{E} := \{\theta \in \Theta : \exists x \in X, x \models \Phi_\theta\}$$

*is open.*

*Proof.* Each literal satisfaction set is  $F_\alpha^{-1}(U_\alpha)$ , open by continuity. Finite Boolean composition in CNF preserves openness, yielding openness of  $\mathcal{S}$ . For fixed  $x$ , the section

$$\mathcal{S}_x := \{\theta : (\theta, x) \in \mathcal{S}\}$$

is open as preimage under  $\theta \mapsto (\theta, x)$ . Since  $\mathcal{E} = \bigcup_{x \in X} \mathcal{S}_x$ , it is open.  $\square$

**Corollary 1.11** (Windowed satisfiability openness [**Proved**]). *For any  $K \subseteq X$ ,*

$$\mathcal{E}_K := \{\theta : \text{Mod}(\Phi_\theta) \cap K \neq \emptyset\}$$

*is open.*

**Remark 1.12** (Strict margins are structurally necessary). *Non-open literal acceptance sets such as  $F = 0$  or  $F \leq 0$  can destroy openness in parameter space. Robust formulations with strict margins, e.g.  $F < -\tau$ , are topologically stable and numerically auditable.*



## Chapter 2

# Measure Semantics and Continuous Counting

### 2.1 Windowed semantic volume

**Assumption 2.1** (Measure layer). *Let  $(X, \mathcal{B}(X), \mu)$  be a Borel measure space with  $\mu$  Radon, finite on compact sets, and strictly positive on nonempty open sets. Fix compact  $K \subset X$ .*

**Definition 2.2** (Windowed model volume). *For finite cGCNF  $\Phi$ , define*

$$V_K(\Phi) := \mu(\text{Mod}(\Phi) \cap K).$$

*This is the continuous #SAT analogue on the finite probe  $K$ .*

**Proposition 2.3** (Measurable complement formula [**Proved**]).

$$V_K(\Phi) = \mu(K) - \mu(\mathcal{U}(\Phi) \cap K).$$

*Proof.* By openness,  $\text{Mod}(\Phi)$  is Borel. By Proposition 1.8,

$$\text{Mod}(\Phi) \cap K = K \setminus (\mathcal{U}(\Phi) \cap K).$$

Since  $\mu(K) < \infty$ , both terms are well-defined. □

**Theorem 2.4** (Interior witness implies positive volume [**Proved**]). *If  $\text{Mod}(\Phi) \cap \text{int}(K) \neq \emptyset$ , then  $V_K(\Phi) > 0$ .*

*Proof.* Pick  $x^* \in \text{Mod}(\Phi) \cap \text{int}(K)$ . There exists a nonempty open set  $O \subseteq \text{Mod}(\Phi) \cap K$ . Strict positivity on open sets gives  $\mu(O) > 0$ , hence  $V_K(\Phi) \geq \mu(O) > 0$ . □

**Proposition 2.5** (Monotonicity [**Proved**]). *If  $\Phi \Rightarrow \Psi$  pointwise on  $X$ , then  $V_K(\Phi) \leq V_K(\Psi)$ . If  $K_1 \subseteq K_2$ , then  $V_{K_1}(\Phi) \leq V_{K_2}(\Phi)$ .*

*Proof.* Immediate from measure monotonicity. □

## 2.2 Lower semicontinuity and concentration

**Theorem 2.6** (Lower semicontinuity in parameters [**Proved**]). *Under Assumptions 1.9 and 2.1, define*

$$v_K(\theta) := \mu(\text{Mod}(\Phi_\theta) \cap K).$$

*Then  $v_K$  is lower semicontinuous on  $\Theta$ .*

*Proof.* Let  $\mathbf{1}_S$  be the indicator of

$$S = \{(\theta, x) : x \models \Phi_\theta\},$$

open by Theorem 1.10. For fixed  $x$ ,  $\theta \mapsto \mathbf{1}_S(\theta, x)$  is lower semicontinuous. Since

$$v_K(\theta) = \int_K \mathbf{1}_S(\theta, x) d\mu(x),$$

Fatou's lemma gives lower semicontinuity. □

**Theorem 2.7** (Auditable Monte Carlo estimator [**Proved**]). *Assume  $\mu(K) > 0$  and sample  $\theta^{(1)}, \dots, \theta^{(T)}$  i.i.d. uniformly on  $K$  (or via known importance density). Define*

$$I_t := \mathbf{1}_{\{\theta^{(t)} \in \text{Mod}(\Phi)\}}, \quad \hat{p}_T := \frac{1}{T} \sum_{t=1}^T I_t, \quad \hat{V}_T := \mu(K) \hat{p}_T.$$

*Then  $\hat{V}_T$  is unbiased for  $V_K(\Phi)$  and for all  $\eta > 0$ ,*

$$\Pr(|\hat{V}_T - V_K(\Phi)| \geq \eta \mu(K)) \leq 2e^{-2T\eta^2}.$$

*Proof.* Unbiasedness follows from Bernoulli expectation. Hoeffding's inequality on bounded i.i.d. variables  $I_t \in [0, 1]$  yields the concentration bound. □

## Part II

# Geometric Compilation and Complexity Barriers



## Chapter 3

# Tame Euclidean Fragment and Disjoint Box Compilation

### 3.1 Tame radial fragment

**Assumption 3.1** (Tame Euclidean setup). *Let  $X = \mathbb{R}^n$  and*

$$K = \prod_{i=1}^n [L_i, U_i]$$

*compact. Literals are unary radial constraints*

$$\ell(i, a, A) : |x_i - a| \in A$$

*where each  $A \subset [0, \infty)$  is a finite union of open intervals.*

**Lemma 3.2** (Finite interval decomposition per coordinate [**Proved**]). *For each literal,*

$$\text{Mod}(\ell) \cap K = \left( P_{i,\ell} \times \prod_{r \neq i} [L_r, U_r] \right),$$

*where  $P_{i,\ell} \subset [L_i, U_i]$  is a finite union of open intervals. The complement in  $K$  is a finite union of coordinate cylinders with closed interval bases.*

*Proof.* Invert  $x \mapsto |x - a|$  on bounded intervals: preimages of finite unions of open intervals are finite unions of open intervals. Product structure follows from unary dependence.  $\square$

**Proposition 3.3** (Clause forbidden region is finite box union [**Proved**]). *For a clause  $C = \ell_1 \vee \dots \vee \ell_m$  in the tame fragment,  $\text{Forb}(C) \cap K$  is a finite union of axis-aligned boxes. If coordinate  $i$  contributes  $q_i$  closed intervals after intersecting all relevant complements, then*

$$\# \text{boxes} \leq \prod_{i=1}^n q_i.$$

*Proof.* By definition,

$$\text{Forb}(C) \cap K = \bigcap_{t=1}^m \left( K \setminus (\text{Mod}(\ell_t) \cap K) \right).$$

Each factor is a coordinate cylinder from Lemma 3.2; coordinate-wise finite intersections remain finite unions of intervals. Their finite products are boxes.  $\square$

### 3.2 Difference and insertion operators

**Remark 3.4** (On boxes / Sobre cajas). *English.* In this work, an axis-aligned box is any subset of  $\mathbb{R}^n$  of the form

$$\prod_{i=1}^n I_i,$$

where each  $I_i$  is a bounded interval (not necessarily closed) of one of the types  $[a, b]$ ,  $[a, b)$ ,  $(a, b]$ ,  $(a, b)$  or a singleton  $\{a\}$ . The Lebesgue measure of a box does not depend on whether boundary points are included; hence this flexibility does not affect any volume statements or the disjoint compilation procedures. In particular, the lexicographic slab construction in Lemma 3.6 yields boxes that may have half-open endpoints, ensuring strict disjointness without loss of generality.

**Definition 3.5** (Box difference operator). For boxes  $P, R \subseteq K$ ,  $\text{BoxDiff}(P, R)$  is any finite disjoint family of axis-aligned boxes (allowing half-open endpoints) whose union equals  $P \setminus R$ .

**Lemma 3.6** (Lexicographic slab bound [Proved]). For closed boxes  $P, R \subset \mathbb{R}^n$  there exists a disjoint decomposition

$$P \setminus R = \bigsqcup_{t=1}^M B_t, \quad M \leq 2n.$$

Moreover, one can choose the  $B_t$  to be axis-aligned boxes (allowing half-open endpoints) obtained by an explicit lexicographic slab construction.

*Proof.* Write  $P = \prod_{i=1}^n [p_i^-, p_i^+]$ ,  $R = \prod_{i=1}^n [r_i^-, r_i^+]$ . If  $P \cap R = \emptyset$ , return  $\{P\}$  ( $M = 1$ ). If  $P \subseteq R$ , return  $\emptyset$  ( $M = 0$ ). Otherwise define the clipped box  $\tilde{R} := P \cap R = \prod_i [\alpha_i, \beta_i]$  with  $\alpha_i = \max(p_i^-, r_i^-)$ ,  $\beta_i = \min(p_i^+, r_i^+)$ . Then  $P \setminus R = P \setminus \tilde{R}$ . For each coordinate  $i$  define the lexicographic slabs

$$L_i = \left( \prod_{j < i} [\alpha_j, \beta_j] \right) \times [p_i^-, \alpha_i] \times \left( \prod_{j > i} [p_j^-, p_j^+] \right),$$

$$U_i = \left( \prod_{j < i} [\alpha_j, \beta_j] \right) \times (\beta_i, p_i^+] \times \left( \prod_{j > i} [p_j^-, p_j^+] \right).$$

Let  $\mathcal{D} = \{L_i : L_i \neq \emptyset\} \cup \{U_i : U_i \neq \emptyset\}$ ;  $|\mathcal{D}| \leq 2n$ . *Claim 1:*  $\bigcup_{B \in \mathcal{D}} B \subseteq P \setminus \tilde{R}$ . Indeed, if  $x \in L_i$  then  $x_i < \alpha_i$ , if  $x \in U_i$  then  $x_i > \beta_i$ , and each slab lies in  $P$ . *Claim 2:*  $P \setminus \tilde{R} \subseteq \bigcup_{B \in \mathcal{D}} B$ . Let  $x \in P \setminus \tilde{R}$ . Let  $i^*$  be the lexicographically first index with  $x_{i^*} \notin [\alpha_{i^*}, \beta_{i^*}]$ . If  $x_{i^*} < \alpha_{i^*}$  then  $x \in L_{i^*}$ ; if  $x_{i^*} > \beta_{i^*}$  then  $x \in U_{i^*}$ . *Claim 3:*  $\mathcal{D}$  is pairwise disjoint:  $L_i \cap U_i = \emptyset$ ; for  $i < j$ , every slab of index  $j$  has  $i$ -th coordinate in  $[\alpha_i, \beta_i]$  while every slab of index  $i$  has  $i$ -th coordinate outside  $[\alpha_i, \beta_i]$ . Thus  $P \setminus \tilde{R} = \bigsqcup_{B \in \mathcal{D}} B$ . Enumerate  $\mathcal{D}$  as  $B_1, \dots, B_M$ ,  $M \leq 2n$ .  $\square$

**Definition 3.7** (AddBox). Given a disjoint family  $\mathcal{U}$  and a box  $Q$ :

- (i) Initialize  $\mathcal{R} \leftarrow \{Q\}$ .
- (ii) For each  $B \in \mathcal{U}$ , replace every  $P \in \mathcal{R}$  by  $\text{BoxDiff}(P, B)$ .
- (iii) Output  $\mathcal{U} \cup \mathcal{R}$ .

**Proposition 3.8** (AddBox correctness [Proved]). If  $\mathcal{U}$  is disjoint, then  $\text{AddBox}(\mathcal{U}, Q)$  is disjoint and

$$\bigcup \text{AddBox}(\mathcal{U}, Q) = \left( \bigcup \mathcal{U} \right) \cup Q.$$

*Proof.* At each subtraction step, BoxDiff replaces each piece by a disjoint family with identical union minus the current blocker. After all blockers are processed,  $\mathcal{R}$  equals  $Q \setminus \cup \mathcal{U}$ , disjoint from  $\mathcal{U}$ .  $\square$

**Theorem 3.9** (Disjoint compilation of  $\mathcal{U}(\Phi) \cap K$  [**Proved**]). *Under Assumption 3.1, let*

$$\mathcal{U}(\Phi) \cap K = \bigcup_{j=1}^M Q_j$$

*be any finite box decomposition (from Proposition 3.3). Define*

$$\mathcal{U}_0 := \emptyset, \quad \mathcal{U}_t := \text{AddBox}(\mathcal{U}_{t-1}, Q_t), \quad t = 1, \dots, M.$$

*Then  $\mathcal{U}_M$  is disjoint and  $\bigcup \mathcal{U}_M = \mathcal{U}(\Phi) \cap K$ . Consequently,*

$$V_K(\Phi) = \mu(K) - \sum_{B \in \mathcal{U}_M} \mu(B).$$

*Proof.* Induction on  $t$  using AddBox correctness (Proposition 3.8).  $\square$

### 3.3 Fragmentation barrier

**Theorem 3.10** (Exponential fragmentation in tame continuous models [**Proved**]). *In  $K = [0, 1]^n$ , define*

$$\Phi_n := \bigwedge_{i=1}^n \left( (x_i \in (0, \frac{1}{3})) \vee (x_i \in (\frac{2}{3}, 1)) \right).$$

*Then  $\text{Mod}(\Phi_n)$  has exactly  $2^n$  connected components. Any exact disjoint axis-aligned box decomposition of  $\text{Mod}(\Phi_n)$  uses at least  $2^n$  boxes.*

*Proof.*

$$\text{Mod}(\Phi_n) = \prod_{i=1}^n ((0, \frac{1}{3}) \cup (\frac{2}{3}, 1))$$

is a disjoint union of  $2^n$  open boxes indexed by left/right choices. These are the connected components. Every axis-aligned box is connected and lies in a single component, so at least one box per component is required.  $\square$



## Chapter 4

# Discrete Layer: GCNF, SAT Equation, and COVERTRACE

### 4.1 Finite-domain GCNF and one-hot correctness

**Definition 4.1** (Finite-domain GCNF). *Variables  $x_i$  range over finite domains  $\{0, 1, \dots, b_i - 1\}$ . Literals are equalities  $(x_i = a)$ . Clauses are finite disjunctions of such literals; formulas are CNFs.*

**Definition 4.2** (One-hot reduction). *Introduce Boolean indicators  $X_{i,0}, \dots, X_{i,b_i-1}$  with exactly-one constraints for each  $i$ . Replace  $(x_i = a)$  by  $X_{i,a}$ .*

**Proposition 4.3** (Reduction correctness [Proved]). *A finite-domain GCNF formula is satisfiable iff its one-hot Boolean CNF reduction is satisfiable.*

*Proof.* Forward: encode each finite value as one active indicator. Backward: exactly-one constraints decode a unique finite value per variable. Clause truth is preserved in both directions.  $\square$

**Definition 4.4** (Auditable verifier). *A verifier is a deterministic algorithm that checks clause-by-clause whether a candidate assignment satisfies the finite-domain formula.*

**Proposition 4.5** (Correctness separation [Proved]). *If a solver emits a witness and the independent verifier accepts the decoded assignment, then the original finite-domain claim is established independently of solver internals.*

### 4.2 Balanced CNFs and the SAT equation

**Definition 4.6** (Balanced clause and unique falsifier). *For Boolean variables  $x_1, \dots, x_n$ , a clause is balanced if it contains exactly one literal for each variable. Define  $a_C \in \{0, 1\}^n$  as the unique falsifier of clause  $C$ .*

**Definition 4.7** (Index and SAT integer). *For  $a = (a_1, \dots, a_n) \in \{0, 1\}^n$ , define*

$$\text{ind}(a) = \sum_{i=1}^n a_i 2^{n-i}.$$

*For a balanced clause  $C$ , let  $T(C) = \text{ind}(a_C)$ . For balanced CNF  $F$  without repeated clauses,*

$$S_F := \sum_{C \in F} 2^{T(C)}.$$

**Theorem 4.8** (SAT equation theorem [Proved]). *Write*

$$S_F = \sum_{k=0}^{2^n-1} b_k 2^k, \quad b_k \in \{0, 1\}.$$

For assignment  $a$  with  $k = \text{ind}(a)$ ,

$$b_k = 1 \iff a \text{ falsifies at least one clause of } F \iff a \not\models F.$$

Hence the bitmask of  $S_F$  is exactly the unsatisfying set.

*Proof.* Each distinct clause contributes one distinct bit at the index of its unique falsifier, and no carry occurs because clauses are distinct. Therefore bit  $k$  is set iff assignment  $k$  is a falsifier of at least one clause.  $\square$

### 4.3 COVERTRACE compilation and exact counting

**Definition 4.9** (Pattern and cube). *A pattern is  $p \in \{0, 1, \bullet\}^n$ . The induced cube is*

$$Q(p) = \{x \in \{0, 1\}^n : p_i \neq \bullet \Rightarrow x_i = p_i\}.$$

*Its support width is  $k(p) = |\text{supp}(p)|$  and volume is  $|Q(p)| = 2^{n-k(p)}$ .*

**Definition 4.10** (Clause to forbidden cube). *For clause  $C$ , define pattern  $p(C)$  coordinate-wise by*

$$p(C)_i = \begin{cases} 0, & x_i \in C, \\ 1, & \neg x_i \in C, \\ \bullet, & \text{otherwise.} \end{cases}$$

**Proposition 4.11** (Clause falsification geometry [Proved]). *An assignment  $x$  falsifies clause  $C$  iff  $x \in Q(p(C))$ .*

**Definition 4.12** (Forbidden region). *For CNF  $F = \bigwedge_{j=1}^m C_j$  define*

$$U(F) := \bigcup_{j=1}^m Q(p(C_j)).$$

*Then  $\#\text{SAT}(F) = 2^n - |U(F)|$ .*

**Lemma 4.13** (CubeDiff size bound [Proved]). *If  $Q(p) \cap Q(r) \neq \emptyset$  and  $Q(p) \not\subseteq Q(r)$ , CubeDiff returns at most  $|\text{supp}(r) \setminus \text{supp}(p)|$  cubes.*

*Proof.* Each recursion fixes one coordinate that is fixed in  $r$  and free in  $p$ . Recursion depth is bounded by the number of such coordinates, and one cube is emitted per level.  $\square$

**Theorem 4.14** (Correct disjoint forbidden compilation [Proved]). *Initialize  $\mathcal{U}_0 = \emptyset$ , and for clause patterns  $p_j$ , set*

$$\mathcal{U}_{t+1} := \text{AddCube}(\mathcal{U}_t, p_{t+1}).$$

*Then  $\mathcal{U}_m$  is disjoint and  $\text{Cov}(\mathcal{U}_m) = U(F)$ . Consequently,*

$$|U(F)| = \sum_{u \in \mathcal{U}_m} 2^{n-k(u)}, \quad \#\text{SAT}(F) = 2^n - \sum_{u \in \mathcal{U}_m} 2^{n-k(u)}.$$

*Proof.* Induction over insertion steps: AddCube adds exactly the uncovered portion of each new forbidden cube and preserves disjointness.  $\square$

---

**Algorithm 1** CubeDiff( $p, r$ )

---

**Require:**  $p, r \in \{0, 1, \bullet\}^n$ **Ensure:** Disjoint family  $D$  with  $Q(p) \setminus Q(r) = \uplus_{d \in D} Q(d)$ 

```

1: if  $Q(p) \cap Q(r) = \emptyset$  then
2:   return  $\{p\}$ 
3: end if
4: if  $Q(p) \subseteq Q(r)$  then
5:   return  $\emptyset$ 
6: end if
7: Choose  $i$  with  $p_i = \bullet$  and  $r_i \in \{0, 1\}$ 
8:  $b \leftarrow r_i$ 
9:  $p^\neq \leftarrow p$  with  $p_i^\neq \leftarrow 1 - b$ 
10:  $p^\equiv \leftarrow p$  with  $p_i^\equiv \leftarrow b$ 
11: return  $\{p^\neq\} \cup \text{CubeDiff}(p^\equiv, r)$ 

```

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**Algorithm 2** AddCube( $\mathcal{U}, q$ )

---

**Require:** Disjoint  $\mathcal{U} \subseteq \{0, 1, \bullet\}^n$ , pattern  $q$ **Ensure:** Disjoint  $\mathcal{U}'$  with  $\text{Cov}(\mathcal{U}') = \text{Cov}(\mathcal{U}) \cup Q(q)$ 

```

1:  $P \leftarrow \{q\}$ 
2: for each  $r \in \mathcal{U}$  do
3:    $P_{\text{new}} \leftarrow \emptyset$ 
4:   for each  $p \in P$  do
5:      $P_{\text{new}} \leftarrow P_{\text{new}} \cup \text{CubeDiff}(p, r)$ 
6:   end for
7:    $P \leftarrow P_{\text{new}}$ 
8:   if  $P = \emptyset$  then
9:     return  $\mathcal{U}$ 
10:  end if
11: end for
12: return  $\mathcal{U} \cup P$ 

```

---

#### 4.4 Lower bounds and hierarchy consequences

**Theorem 4.15** (Parity requires exponential axis-aligned DSOP [**Proved**]). *Any disjoint family of axis-aligned cubes covering odd parity*

$$\text{PARITY}_n^{\text{odd}} := \{x \in \{0, 1\}^n : |x|_1 \equiv 1 \pmod{2}\}$$

*has size at least  $2^{n-1}$ .*

*Proof.* A cube with any free coordinate has equal even/odd parity counts by the free-coordinate flip involution, so it cannot lie inside odd parity. Therefore all covering cubes must be singletons; exactly  $2^{n-1}$  are needed.  $\square$

**Theorem 4.16** (Uniform polynomial DSOP compiler implies collapse [**Proved**]). *Assume a uniform polynomial-time algorithm outputs polynomial-size disjoint axis-aligned compilations for all CNFs. Then  $\#\text{SAT} \in \mathbf{P}$ , hence  $\mathbf{PH} = \mathbf{P}$  by Toda's theorem.*

*Proof sketch.* Exact disjoint compilation gives exact counting in polynomial time by volume additivity over cubes. Then  $\#\text{SAT} \in \mathbf{P}$ , and Toda implies the collapse.  $\square$

## Chapter 5

# Affine Extensions over $\mathbb{F}_2$ and Metacomplexity

**Definition 5.1** (Affine subcube). Identify  $\{0, 1\}^n$  with  $\mathbb{F}_2^n$ . For  $A \in \mathbb{F}_2^{m \times n}$  and  $b \in \mathbb{F}_2^m$  define

$$Q_{\text{aff}}(A, b) := \{x \in \mathbb{F}_2^n : Ax = b\}.$$

If consistent,  $|Q_{\text{aff}}(A, b)| = 2^{n - \text{rank}(A)}$ .

**Definition 5.2** (Disjoint affine compilation). A Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  has disjoint affine compilation size  $k$  if

$$f^{-1}(1) = \bigsqcup_{i=1}^k Q_{\text{aff}}(A_i, b_i)$$

for pairwise disjoint affine subcubes.

**Remark 5.3** (Parity compression). Odd parity is a single affine piece

$$\sum_{i=1}^n x_i = 1 \pmod{2},$$

so affine pieces remove the axis-aligned parity obstruction.

**Theorem 5.4** (Uniform affine compilation collapse [**Proved**]). If a uniform polynomial-time algorithm outputs polynomial-size disjoint affine compilations for all CNFs (with polynomial description size), then  $\#\text{SAT} \in \mathbf{P}$  and therefore  $\mathbf{PH} = \mathbf{P}$ .

*Proof.* Disjointness gives  $\#\text{SAT}(F) = \sum_i |Q_{\text{aff}}(A_i, b_i)|$ . Each term is computed by Gaussian elimination in polynomial time. Summation over polynomially many terms stays polynomial. Toda completes the argument.  $\square$

**Proposition 5.5** (Metacomplexity interpretation [**Proved as reduction**]). The task of uniformly producing exact small normal forms (axis-aligned or affine) for arbitrary CNFs is itself  $\#\mathbf{P}$ -hard in consequence, unless standard hierarchy assumptions collapse.



## Part III

# Differential-Geometric Lift and Black-Hole Measurable Layer



## Chapter 6

# Riemannian Product Lift

**Assumption 6.1** (Riemannian product setting). *For each  $i$ ,  $(M_i, g_i)$  is a smooth connected Riemannian manifold with geodesic distance  $d_{g_i}$ . Set*

$$M := \prod_{i=1}^n M_i$$

*with product topology and product Borel measure induced by Riemannian volume.*

**Definition 6.2** (Intrinsic distance literal). *For  $a_i \in M_i$  and open  $A \subset [0, \infty)$ ,*

$$\ell(i, a_i, A) : d_{g_i}(x_i, a_i) \in A.$$

**Theorem 6.3** (Intrinsic openness and robustness [**Proved**]). *All topological cGCNF results on product metric spaces remain valid on  $M$ .*

*Proof.* Each map  $x_i \mapsto d_{g_i}(x_i, a_i)$  is continuous (indeed 1-Lipschitz). The previous product-space proofs (Theorems 1.5, 1.10) apply verbatim.  $\square$

**Remark 6.4** (Cut locus is not an obstruction). *Distance may fail to be smooth at cut loci, but continuity is enough for openness, robustness, and Borel measurability.*

**Theorem 6.5** (Positive compact-window volume on manifolds [**Proved**]). *If  $K \subset M$  is compact and  $\text{Mod}(\Phi) \cap \text{int}(K) \neq \emptyset$ , then*

$$\text{Vol}_g(\text{Mod}(\Phi) \cap K) > 0.$$

*Proof.* Apply Theorem 2.4 with  $\mu = \text{Vol}_g$ .  $\square$

**Proposition 6.6** (Chart-wise measurable approximation [**Proved**]). *Let  $A \subset K \subset M$  be Borel with  $\text{Vol}_g(\partial A) = 0$  and  $K$  compact. For every  $\eta > 0$  there exist finite unions  $A_\eta^- \subseteq A \subseteq A_\eta^+$  of chart-box images such that*

$$\text{Vol}_g(A_\eta^+ \setminus A_\eta^-) < \eta.$$

*Proof.* Use Radon regularity to approximate by compact/open sets, finite atlas on compact  $K$ , Euclidean box approximation in each chart, then pushforward and aggregate.  $\square$



## Chapter 7

# Black-Hole-Relevant Finite Measurable Layer

### 7.1 Initial-data parameter families

**Assumption 7.1** (Finite-dimensional initial-data family [Model]). *Fix a smooth 3-manifold  $\Sigma$ , and a compact parameter set  $\Theta \subset \mathbb{R}^d$ . Let*

$$\theta \mapsto (h(\theta), K(\theta))$$

*be a  $C^1$  map from  $\Theta$  into asymptotically flat Einstein initial data of regularity at least  $C^2$  in space and continuous in  $\theta$  in the corresponding norm.*

**Remark 7.2** (Operational scope). *This layer targets quasilocal trapped-surface observables in initial data. It does not define event horizons, which are global spacetime objects.*

### 7.2 Robust trapped-surface literals

**Lemma 7.3** (Sobolev-to- $C^1$  control [Proved]). *Let  $s > 5/2$ . Then  $H^s(\Sigma) \hookrightarrow C^1(\Sigma)$  continuously. In particular, if  $\theta \mapsto h(\theta)$  is continuous in  $H^s$ , then  $\theta \mapsto h(\theta)$  is continuous in  $C^1$ , and similarly for  $K(\theta) \in H^{s-1}$  in  $C^0$ .*

*Proof.* This is the standard Sobolev embedding on 3-manifolds for  $s > k + 3/2$  with  $k = 1$ . □

**Definition 7.4** (Null expansion functional). *Fix a closed embedded  $C^2$  surface  $S \subset \Sigma$  and a sign convention for the outer future null expansion  $\theta_+(p; S, \theta)$ . Define*

$$F_S(\theta) := \max_{p \in S} \theta_+(p; S, \theta), \quad G_S(\theta) := \min_{p \in S} \theta_+(p; S, \theta).$$

**Proposition 7.5** (Continuity of  $F_S, G_S$  [Proved]). *Under Assumption 7.1,  $F_S$  and  $G_S$  are continuous on  $\Theta$ .*

*Proof.* The map  $(\theta, p) \mapsto \theta_+(p; S, \theta)$  is continuous on compact  $S$ . Max/min over compact fibers preserve continuity. □

**Definition 7.6** (Robust BH literals). *For margin  $\tau > 0$ :*

$$\begin{aligned} L_{S,\tau}^{\text{trap}}(\theta) : & \quad F_S(\theta) < -\tau, \\ L_{S,\tau}^{\text{outer}+}(\theta) : & \quad G_S(\theta) > \tau. \end{aligned}$$

**Corollary 7.7** (Openness of robust BH literals [**Proved**]). *Each set  $\{\theta : L_{S,\tau}^{\text{trap}}(\theta)\}$  and  $\{\theta : L_{S,\tau}^{\text{outer}+}(\theta)\}$  is open in  $\Theta$ .*

*Proof.* Apply Proposition 7.5 and openness of strict inequalities.  $\square$

**Remark 7.8** (Why  $F_S = 0$  is excluded). *The exact MOTS condition  $F_S = 0$  defines a codimension-type boundary object in parameter space and is not open. For cGCNF robustness and numerical certification, strict margins are mandatory.*

### 7.3 Finite-template BH cGCNF

**Assumption 7.9** (Finite template banks [**Model**]). *Fix finite banks:*

$$\mathcal{S}_+, \mathcal{S}_-, \mathcal{S}_c, \mathcal{S}_{\text{out}}$$

*of closed  $C^2$  surfaces in  $\Sigma$ , intended respectively for the two individual horizons, common horizon candidates, and outer anti-common tests.*

**Definition 7.10** (Finite BH phase formulas). *For fixed  $\tau > 0$ , define:*

$$\begin{aligned} C_+(\theta) &:= \bigvee_{S \in \mathcal{S}_+} L_{S,\tau}^{\text{trap}}(\theta), \\ C_-(\theta) &:= \bigvee_{S \in \mathcal{S}_-} L_{S,\tau}^{\text{trap}}(\theta), \\ C_c(\theta) &:= \bigvee_{S \in \mathcal{S}_c} L_{S,\tau}^{\text{trap}}(\theta), \\ C_{\text{out}}(\theta) &:= \bigvee_{S \in \mathcal{S}_{\text{out}}} L_{S,\tau}^{\text{outer}+}(\theta). \end{aligned}$$

*Then*

$$\Phi_{\text{common}} := C_+ \wedge C_- \wedge C_c, \quad \Phi_{\text{sep}} := C_+ \wedge C_- \wedge C_{\text{out}}.$$

**Theorem 7.11** (Topological and measurable well-posedness of BH phases [**Proved**]). *Under Assumptions 7.1 and 7.9, both  $\text{Mod}(\Phi_{\text{common}})$  and  $\text{Mod}(\Phi_{\text{sep}})$  are open in  $\Theta$ , hence Borel measurable.*

*Proof.* Finite OR/AND of open sets from robust literals (Corollary 7.7) remain open.  $\square$

**Definition 7.12** (Ideal existential trapped set [**Model**]). *Given an admissible (possibly infinite) class  $\mathcal{S}_\infty$ , define*

$$E_\tau := \left\{ \theta \in \Theta : \inf_{S \in \mathcal{S}_\infty} F_S(\theta) < -\tau \right\}.$$

**Proposition 7.13** (Finite bank as certified inner approximation [**Proved**]). *If  $\mathcal{S}_N \subseteq \mathcal{S}_\infty$ , then*

$$E_\tau^{(N)} := \left\{ \theta : \min_{S \in \mathcal{S}_N} F_S(\theta) < -\tau \right\} \subseteq E_\tau.$$

*Proof.* Immediate from  $\mathcal{S}_N \subseteq \mathcal{S}_\infty$ .  $\square$

**Theorem 7.14** (Dense-net transfer with margin [**Proved**]). *Assume:*

- (i)  $\mathcal{S}_\infty$  is compact in a surface metric  $\text{dist}_S$ ,

(ii)  $F : \Theta \times \mathcal{S}_\infty \rightarrow \mathbb{R}$ ,  $F(\theta, S) = F_S(\theta)$ , is jointly continuous,

(iii)  $\mathcal{S}_N$  is an  $\varepsilon_N$ -net in  $\mathcal{S}_\infty$ ,

(iv) there is modulus  $\omega$  such that

$$\text{dist}_{\mathcal{S}}(S, S') \leq r \Rightarrow |F(\theta, S) - F(\theta, S')| \leq \omega(r) \quad \forall \theta.$$

Then

$$E_{\tau+\omega(\varepsilon_N)} \subseteq E_\tau^{(N)} \subseteq E_\tau.$$

*Proof.* Right inclusion is Proposition 7.13. For left inclusion, take  $\theta \in E_{\tau+\omega(\varepsilon_N)}$ . Then some  $S \in \mathcal{S}_\infty$  satisfies  $F(\theta, S) < -\tau - \omega(\varepsilon_N)$ . Choose  $S_N \in \mathcal{S}_N$  with  $\text{dist}_{\mathcal{S}}(S, S_N) \leq \varepsilon_N$ . Then

$$F(\theta, S_N) \leq F(\theta, S) + \omega(\varepsilon_N) < -\tau,$$

so  $\theta \in E_\tau^{(N)}$ . □

## 7.4 Measured outputs and uncertainty

**Definition 7.15** (BH phase volume in a probe window). *For compact  $K \subseteq \Theta$ , define*

$$V_K^{\text{common}} := \mu_\Theta(K \cap \text{Mod}(\Phi_{\text{common}})), \quad V_K^{\text{sep}} := \mu_\Theta(K \cap \text{Mod}(\Phi_{\text{sep}})).$$

**Theorem 7.16** (Finite-sample confidence for BH phase volume [Proved]). *Let  $\hat{V}_T$  be the Monte Carlo estimator from Theorem 2.7 applied to  $\Phi_{\text{common}}$  (or  $\Phi_{\text{sep}}$ ). For confidence level  $1 - \delta \in (0, 1)$ ,*

$$|\hat{V}_T - V_K| \leq \mu_\Theta(K) \sqrt{\frac{\log(2/\delta)}{2T}}$$

*with probability at least  $1 - \delta$ .*

*Proof.* Set  $\eta = \sqrt{\log(2/\delta)/(2T)}$  in Theorem 2.7. □

**Definition 7.17** (Gray zone near critical boundaries).

$$\mathcal{G} := K \setminus (\text{Mod}(\Phi_{\text{common}}) \cup \text{Mod}(\Phi_{\text{sep}})).$$

$\mathcal{G}$  is the certified uncertainty zone induced by finite banks and strict margins.

**Remark 7.18** (Scientifically correct interpretation).  $\mathcal{G}$  is not a failure. It is the mathematically unavoidable interface between finite robust certification and sharp phase boundaries. Theorem 7.14 gives a controlled path to shrink  $\mathcal{G}$  by denser template banks and smaller margins.

## 7.5 Connection to coherent flow in BH literal space [Proved]

**Definition 7.19** (Finite BH literal dictionary and free energy). *Fix a finite dictionary  $\mathcal{W}$  of robust literals (from banks and thresholds). For  $T \subseteq \mathcal{W}$ , let  $\Phi_T$  be the conjunction of all literals in  $T$  (single-literal clauses). Define admissible theories*

$$\mathcal{K}_{\text{BH}} := \{T \subseteq \mathcal{W} : V_K(\Phi_T) > 0\}.$$

For  $\alpha, \beta, \gamma > 0$ ,

$$\mathcal{F}_{\text{BH}}(T) := \alpha E_{\text{ctr}}(T) + \beta|T| - \gamma \log V_K(\Phi_T), \quad T \in \mathcal{K}_{\text{BH}}.$$

**Proposition 7.20** (Existence of coherent islands in BH literal space [**Proved**]). *If  $\mathcal{K}_{\text{BH}} \neq \emptyset$ ,  $\mathcal{F}_{\text{BH}}$  attains a global minimum on  $\mathcal{K}_{\text{BH}}$ ; every global minimizer is a coherent island with respect to Hamming-1 neighborhood.*

*Proof.*  $\mathcal{W}$  is finite, hence  $\mathcal{K}_{\text{BH}}$  is finite. A real-valued function on a finite nonempty set attains a minimum. Minimality implies local minimality.  $\square$

## Part IV

# Epistemic Curvature and Incompleteness



## Chapter 8

# Metric Interfaces and Curvature

**Definition 8.1** (Formal system with metric interface). *A system with interface is*

$$S = (\mathcal{L}, \vdash, \iota, \mathcal{O}, X, \delta, e, j),$$

where  $(\mathcal{L}, \vdash)$  is syntax/derivability,  $\iota : \mathcal{L} \rightarrow \mathcal{O}$  is semantic interpretation,  $(X, \delta)$  is complete separable metric, and  $e : \mathcal{L} \rightarrow X$ ,  $j : \mathcal{O} \rightarrow X$  are Borel embeddings. Representation error is

$$\text{err}(\sigma) := \delta(e(\sigma), j(\iota(\sigma))).$$

**Definition 8.2** (Epistemic curvature). *Global curvature:*

$$\kappa_S := \inf_{\sigma \in \mathcal{L}} \text{err}(\sigma) \in [0, \infty).$$

*Pointwise curvature for target  $o \in \mathcal{O}$ :*

$$\kappa_S(o) := \inf\{\text{err}(\sigma) : \iota(\sigma) = o\}.$$

*Worst-case curvature on class  $\mathcal{C} \subseteq \mathcal{O}$ :*

$$\kappa_S^{\text{sup}}(\mathcal{C}) := \sup_{o \in \mathcal{C}} \kappa_S(o).$$

**Definition 8.3** (Resource-bounded curvature). *For size bound  $r$ , let  $\mathcal{L}_{\leq r}$  denote syntactic objects of description size at most  $r$ . Define*

$$\kappa_{S,r}(o) := \inf\{\text{err}(\sigma) : \sigma \in \mathcal{L}_{\leq r}, \iota(\sigma) = o\}, \quad \kappa_{S,r}^{\text{sup}}(\mathcal{C}) := \sup_{o \in \mathcal{C}} \kappa_{S,r}(o).$$

**Remark 8.4** (Notation: worst-case resource curvature). *When the system  $S$  is fixed in context, we use the shorthand*

$$\bar{\kappa}_r(\mathcal{C}) := \kappa_{S,r}^{\text{sup}}(\mathcal{C}), \quad \bar{\kappa}(\mathcal{C}) := \kappa_S^{\text{sup}}(\mathcal{C}).$$

*This is purely notational: no definitions are changed.*

**Lemma 8.5** (Monotonicity under non-expansive morphism [**Proved**]). *If  $F : X \rightarrow X'$  is 1-Lipschitz and  $e' = F \circ e$ ,  $j' = F \circ j$ , then  $\kappa'_{S'} \leq \kappa_S$ .*

*Proof.* For every  $\sigma$ ,  $\text{err}'(\sigma) \leq \text{err}(\sigma)$  by non-expansiveness. Taking infimum over  $\sigma$  gives the claim.  $\square$

**Proposition 8.6** (Sign stability under uniformly equivalent metrics [**Proved**]). *If two metrics  $d$  and  $\delta$  satisfy  $c_1\delta \leq d \leq c_2\delta$  on  $X$ , then*

$$c_1\kappa_\delta \leq \kappa_d \leq c_2\kappa_\delta.$$

*Hence  $\kappa_d = 0$  iff  $\kappa_\delta = 0$ .*



## Chapter 9

# DRP and Internal Incompleteness

**Definition 9.1** (Derivational Refinement Principle (DRP)). *System  $S$  satisfies  $DRP$  if there exists  $T : \mathcal{L} \rightarrow \mathcal{L}$  such that:*

- (i)  $\sigma \vdash T(\sigma)$  for all  $\sigma$ ,
- (ii)  $\text{err}(T(\sigma)) \leq \text{err}(\sigma)$ ,
- (iii) each orbit  $(T^n(\sigma))_{n \in \mathbb{N}}$  has an accumulation point that realizes the orbitwise error infimum.

**Theorem 9.2** (Positive curvature obstructs semantic completeness under  $DRP$  [**Metaformal**]). *Assume:*

- (a) Every intended semantic target admits exact representation.
- (b)  $DRP$  holds.

Then  $\kappa_S = 0$ . Contrapositively,  $\kappa_S > 0$  implies failure of target-wise exact representability.

*Proof.* If every target has exact representative, there exists some  $\sigma$  with  $\text{err}(\sigma) = 0$ , forcing infimum  $\kappa_S = 0$ .  $\square$

**Definition 9.3** (Gödel-admissible zero-error certification predicate). *Fix recursively axiomatizable arithmetic theory  $T \supseteq Q$  and a resource bound  $r : \mathbb{N} \rightarrow \mathbb{N}$ . A formula  $\text{Prov}_{0,r}(y)$  is a zero-error certification predicate if, for each arithmetic sentence  $\psi$  with code  $y = \ulcorner \psi \urcorner$ , it arithmetizes the equivalence*

$$\text{Prov}_{0,r}(\ulcorner \psi \urcorner) \leftrightarrow \exists \sigma \in \mathcal{L}_{\leq r(|\psi|)} (\iota(\sigma) = o_\psi \wedge \text{err}(\sigma) = 0),$$

where  $o_\psi$  is the intended semantic target associated with  $\psi$ , using primitive recursive verification.  $(T, \text{Prov}_{0,r})$  is Gödel-admissible if diagonalization is available for formulas in which  $\text{Prov}_{0,r}$  may occur.

**Theorem 9.4** (Internal incompleteness of uniformly complete verifiable flatness [**Proved**]). *Assume:*

- (a)  $(T, \text{Prov}_{0,r})$  is Gödel-admissible and  $T$  is consistent.
- (b) Soundness:  $\text{Prov}_{0,r}(\ulcorner \psi \urcorner)$  implies  $\psi$  true in intended semantics.
- (c) Uniform completeness: every true sentence  $\psi$  satisfies  $\text{Prov}_{0,r}(\ulcorner \psi \urcorner)$ .

Then (b) and (c) cannot both hold. Under soundness, there exists true  $G$  with  $\neg \text{Prov}_{0,r}(\ulcorner G \urcorner)$ , so  $\kappa_{S,r}^{\text{sup}}(\mathcal{O}_{\text{arith}}) > 0$ .

*Proof.* By diagonal lemma, obtain  $G$  such that  $T \vdash G \leftrightarrow \neg \text{Prov}_{0,r}(\ulcorner G \urcorner)$ . If  $\text{Prov}_{0,r}(\ulcorner G \urcorner)$  held, soundness implies  $G$  true, so right side true, contradiction. Hence  $\neg \text{Prov}_{0,r}(\ulcorner G \urcorner)$  and therefore  $G$  true. Uniform completeness then forces  $\text{Prov}_{0,r}(\ulcorner G \urcorner)$ , contradiction. The curvature conclusion follows from absence of zero-error resource-bounded representation for true target  $G$ .  $\square$

## Part V

# Coherent Flow: Deterministic, Stochastic, and Continuous



# Chapter 10

## Finite Theory-Space Free Energy

### 10.1 Finite semantics layer

Fix:

- finite atom set  $\text{At}$ ,
- finite window  $\mathcal{W}$  of formulas, closed under negation,
- Belnap–Dunn four-valued semantics  $[1, 3, 2, 4]$  with designated values  $\mathcal{D} = \{\mathbf{T}, \mathbf{B}\}$ ,
- finite valuation space  $\Omega$ ,
- measure  $\mu$  on  $\Omega$  with full support.

**Definition 10.1** (Designated satisfaction and volume). *For theory  $K \subseteq \mathcal{W}$ ,*

$$\text{Mod}(K) := \{\omega \in \Omega : \omega \models_D \varphi \ \forall \varphi \in K\}, \quad V(K) := \mu(\text{Mod}(K)).$$

**Definition 10.2** (Contradiction cost and free energy).

$$\text{Con}(K) := \{\varphi \in \mathcal{W} : \varphi \in K \text{ and } \neg\varphi \in K\}, \quad E_{\text{ctr}}(K) := \sum_{\varphi \in \text{Con}(K)} w_{\varphi}, \quad C(K) := |K|,$$

$$\mathcal{F}(K) := \alpha E_{\text{ctr}}(K) + \beta C(K) - \gamma \log V(K),$$

*with positive parameters and weights. Admissible space:  $\mathcal{K} := \{K \subseteq \mathcal{W} : V(K) > 0\}$ .*

**Remark 10.3** (Finiteness and non-emptiness).  *$\mathcal{K}$  is finite and nonempty because  $\mathcal{W}$  is finite and  $\emptyset \in \mathcal{K}$  with  $V(\emptyset) = \mu(\Omega) > 0$ .*

**Definition 10.4** (Neighborhood and coherent island).

$$\mathcal{N}(K) := \{H \in \mathcal{K} : |H \triangle K| = 1\}.$$

*A coherent island is  $K \in \mathcal{K}$  with  $\mathcal{F}(K) \leq \mathcal{F}(H)$  for all  $H \in \mathcal{N}(K)$ .*

**Theorem 10.5** (Existence of coherent islands [**Proved**]). *At least one coherent island exists.*

*Proof.*  $\mathcal{K}$  is finite nonempty, so  $\mathcal{F}$  has a global minimizer; every global minimizer is a neighborhood minimizer.  $\square$

## 10.2 Deterministic coherent descent

**Definition 10.6** (Deterministic operator  $U$ ). *Fix a total tie-break order  $\prec$  on  $\mathcal{K}$ . Define  $U : \mathcal{K} \rightarrow \mathcal{K}$  by:*

- (i) *If  $\mathcal{F}(K) \leq \mathcal{F}(H)$  for all  $H \in \mathcal{N}(K)$ , set  $U(K) := K$ .*
- (ii) *Otherwise, let  $M(K) := \arg \min_{H \in \mathcal{N}(K)} \mathcal{F}(H)$  and set  $U(K)$  to the  $\prec$ -minimum element of  $M(K)$ .*

**Theorem 10.7** (Lyapunov descent [**Proved**]). *For every  $K \in \mathcal{K}$ ,  $\mathcal{F}(U(K)) \leq \mathcal{F}(K)$ , and if  $U(K) \neq K$  the inequality is strict.*

*Proof.* If  $U(K) = K$ , trivial. If  $U(K) \neq K$ , there exists  $H \in \mathcal{N}(K)$  with  $\mathcal{F}(H) < \mathcal{F}(K)$ . Since  $U(K) \in M(K)$ ,  $\mathcal{F}(U(K)) = \min_{H \in \mathcal{N}(K)} \mathcal{F}(H) < \mathcal{F}(K)$ .  $\square$

**Corollary 10.8** (Finite termination [**Proved**]). *Every orbit  $K_{t+1} = U(K_t)$  reaches a fixed point in finitely many steps.*

*Proof.* Every non-fixed step strictly decreases  $\mathcal{F}$ ;  $\mathcal{K}$  is finite.  $\square$

## Chapter 11

# Stochastic Coherent Flow via Metropolis–Hastings

**Definition 11.1** (MH kernel on  $\mathcal{K}$ ). *Let  $Q$  be a proposal kernel on  $\mathcal{K}$  satisfying: (i) stochasticity, (ii) irreducibility, (iii)  $Q(K, K) > 0$  (aperiodicity), (iv)  $Q(K, H) > 0 \Rightarrow Q(H, K) > 0$  (support compatibility). Fix  $\lambda > 0$ . For  $K \neq H$ , define*

$$r(K, H) := \exp(-\lambda(\mathcal{F}(H) - \mathcal{F}(K))) \frac{Q(H, K)}{Q(K, H)}, \quad a(K, H) := \min\{1, r(K, H)\},$$

(with  $a(K, H) = 0$  if  $Q(K, H) = 0$ ). Then define

$$P(K, H) = Q(K, H)a(K, H) \quad (H \neq K), \quad P(K, K) = 1 - \sum_{G \neq K} Q(K, G)a(K, G).$$

**Theorem 11.2** (Detailed balance [**Proved**]). *Let  $\pi(K) := e^{-\lambda\mathcal{F}(K)}/Z_\lambda$ ,  $Z_\lambda := \sum_{G \in \mathcal{K}} e^{-\lambda\mathcal{F}(G)}$ . Then for all  $K \neq H$ ,  $\pi(K)P(K, H) = \pi(H)P(H, K)$ .*

*Proof.* If  $Q(K, H) = 0$  both sides vanish. Assume  $Q(K, H) > 0$ .

$$\pi(K)P(K, H) = \frac{e^{-\lambda\mathcal{F}(K)}}{Z_\lambda} Q(K, H) \min\{1, r(K, H)\}.$$

If  $r(K, H) \leq 1$ , this equals  $\frac{e^{-\lambda\mathcal{F}(H)}}{Z_\lambda} Q(H, K)$ , and  $r(H, K) = 1/r(K, H) \geq 1$  so  $\pi(H)P(H, K) = \frac{e^{-\lambda\mathcal{F}(H)}}{Z_\lambda} Q(H, K)$ . The case  $r(K, H) > 1$  is symmetric.  $\square$

**Corollary 11.3** (Stationarity and convergence [**Proved**]).  *$\pi$  is stationary for  $P$ . Since  $\mathcal{K}$  is finite and  $P$  is irreducible and aperiodic,  $\lim_{t \rightarrow \infty} \|P^t(K, \cdot) - \pi\|_{\text{TV}} = 0$  for every initial  $K$ .*



## Chapter 12

# KL Curvature and Continuous Coherent Flow

### 12.1 KL curvature definition

**Definition 12.1** (Epistemic KL curvature). *For finite sample space  $\mathcal{X}$ , target  $p \in \Delta^\circ(\mathcal{X})$ , and model family  $\mathcal{M} \subseteq \Delta^\circ(\mathcal{X})$ ,*

$$\kappa_{\mathcal{M}}(p) := \inf_{q \in \mathcal{M}} D_{\text{KL}}(p \| q).$$

**Proposition 12.2** (Separable discrete case [Proved]). *If  $p$  is on  $A \times B$  and  $\mathcal{M}_{\text{prod}} := \{q : q(a, b) = q_A(a)q_B(b)\}$ , then  $\kappa_{\mathcal{M}_{\text{prod}}}(p) = I(A; B)$ .*

*Proof.* Minimize  $D_{\text{KL}}(p \| q_A \otimes q_B)$  over product marginals; optimum is  $q_A = p_A$ ,  $q_B = p_B$ .  $\square$

### 12.2 Continuous coherent flow on fixed marginals

Fix positive marginals  $r_i, c_j$  with  $\sum_i r_i = \sum_j c_j = 1$ , and define

$$\mathcal{C}(r, c) := \{p_{ij} > 0 : \sum_j p_{ij} = r_i, \sum_i p_{ij} = c_j\}.$$

Set  $q_{ij}^* := r_i c_j$ .

For  $p \in \mathcal{C}(r, c)$ , let

$$h_{ij} := \log \frac{p_{ij}}{q_{ij}^*}, \quad \langle X, Y \rangle_p := \sum_{i,j} p_{ij} X_{ij} Y_{ij},$$

$$\mathcal{U} := \{u_{ij} = a_i + b_j\}, \quad \gamma := h - \Pi_p^{\mathcal{U}} h,$$

where  $\Pi_p^{\mathcal{U}} h$  is the orthogonal projection onto  $\mathcal{U}$  with respect to  $\langle \cdot, \cdot \rangle_p$ .

**Definition 12.3** (Continuous coherent flow).

$$\dot{p}_{ij} = -p_{ij} \gamma_{ij}.$$

**Theorem 12.4** (Marginal conservation [Proved]). *If  $p(0) \in \mathcal{C}(r, c)$ , then all row and column sums are invariant along the flow.*

*Proof.* Orthogonality equations for the projection residual give  $\sum_j p_{ij} \gamma_{ij} = 0$  and  $\sum_i p_{ij} \gamma_{ij} = 0$ . Summing ODE terms yields zero row/column derivatives.  $\square$

**Theorem 12.5** (Exact KL Lyapunov identity [**Proved**]). *For  $E(p) := D_{\text{KL}}(p||q^*)$ , along the flow,*

$$\frac{d}{dt}E(p(t)) = - \sum_{i,j} p_{ij} \gamma_{ij}^2 \leq 0.$$

*Proof.* Differentiate  $E$ , use  $\dot{p}_{ij} = -p_{ij}\gamma_{ij}$  and  $h_{ij} + 1$  inside. Decompose  $h = \Pi_p^{\mathcal{U}}h + \gamma$ ; orthogonality gives  $\langle \gamma, \Pi_p^{\mathcal{U}}h \rangle_p = 0$  and  $\sum p_{ij}\gamma_{ij} = 0$ . The result follows.  $\square$

**Proposition 12.6** (Unique equilibrium on the sheet [**Proved**]). *The unique equilibrium in  $\mathcal{C}(r, c)$  is  $q^*$ .*

*Proof.* At equilibrium  $\dot{p}_{ij} = 0$ , so  $\gamma_{ij} = 0$ , hence  $h_{ij} = a_i + b_j$ . Thus  $p_{ij} = r_i c_j e^{a_i + b_j}$ . Marginal constraints force  $e^{a_i}$  constant in  $i$  and  $e^{b_j}$  constant in  $j$ , and normalization forces the global factor to 1, giving  $p = q^*$ .  $\square$

## Chapter 13

# Gibbs Bridge from Logic to Probability

**Definition 13.1** (Gibbs map [Model]). *For theory  $K \in \mathcal{K}$ , parameter  $\eta > 0$ , and valuation  $\omega \in \Omega$ :*

$$E_K(\omega) := \sum_{\varphi \in K} \mathbf{1}_{\{\omega \models_D \varphi\}}, \quad p_K(\omega) := \frac{e^{-\eta E_K(\omega)}}{Z_K(\eta)}, \quad Z_K(\eta) := \sum_{\xi \in \Omega} e^{-\eta E_K(\xi)}.$$

*This defines  $\Phi_\eta : \mathcal{K} \rightarrow \Delta^\circ(\Omega)$ ,  $\Phi_\eta(K) = p_K$ .*

**Proposition 13.2** (Low-energy concentration [Proved]). *Let  $m_K = \min_\omega E_K(\omega)$  and  $\mathcal{A}_K = \arg \min_\omega E_K(\omega)$ . Then for any  $\omega \notin \mathcal{A}_K$ ,  $\lim_{\eta \rightarrow \infty} p_K(\omega) = 0$ . If  $V(K) > 0$ , then  $m_K = 0$  and  $\mathcal{A}_K = \text{Mod}(K)$ .*

*Proof.* Finite-state exponential domination. If  $V(K) > 0$ , some valuation satisfies all formulas in designated sense, giving zero energy; conversely energy zero implies designated satisfaction.  $\square$



## Part VI

# Layered Metric Space and Operational Materialization



## Chapter 14

# Layered Metric Space (LMS): Variational Core

### 14.1 Metric layers on a fixed graph

**Definition 14.1** (Metric layer [Proved]). *Let  $G = (S, E)$  be a finite, connected, simple undirected graph. A layer index  $k \in \mathbb{Z}$  carries a positive edge-length field  $\ell_k : E \rightarrow \mathbb{R}_{>0}$ . The induced path metric is*

$$d_k(p_i, p_j) := \min_{\gamma: i \rightarrow j} \sum_{e \in \gamma} \ell_k(e).$$

**Definition 14.2** (Inter-layer strain and curvature [Proved]).

$$\sigma_e^{(k)} := \ell_{k+1}(e) - \ell_k(e), \quad R_e^{(k)} := \ell_{k+1}(e) - 2\ell_k(e) + \ell_{k-1}(e).$$

### 14.2 Quadratic action and Euler–Lagrange equations

Let  $N \subseteq E \times E$  be the set of unordered pairs of edges sharing a vertex.

**Definition 14.3** (LMS action [Proved]).

$$A[\{\ell_k\}] := \sum_k \sum_{e \in E} (\ell_{k+1}(e) - \ell_k(e))^2, \quad A_{\text{intra}}[\{\ell_k\}] := \sum_k \sum_{\{e, e'\} \in N} (\ell_k(e) - \ell_k(e'))^2,$$

and total action  $S := A + \mu A_{\text{intra}}$ ,  $\mu \geq 0$ .

**Theorem 14.4** (LMS equation of motion [Proved]). *For interior layers,*

$$\ell_{k+1}(e) - 2\ell_k(e) + \ell_{k-1}(e) = \mu \sum_{e': \{e, e'\} \in N} (\ell_k(e) - \ell_k(e')).$$

*Equivalently,  $R^{(k)} = \mu \Delta_{\text{line}} \ell_k$ , where  $\Delta_{\text{line}}$  is the Laplacian on the line graph of  $G$ .*

*Proof.* Differentiate  $S$  with respect to  $\ell_k(e)$  holding boundary layers fixed: temporal term contributes  $2[(\ell_k - \ell_{k-1}) - (\ell_{k+1} - \ell_k)]$ , intra-layer term contributes  $2\mu \sum_{e': \{e, e'\} \in N} (\ell_k(e) - \ell_k(e'))$ . Set derivative to zero and rearrange.  $\square$

### 14.3 Unitary Procrustes bridge

**Definition 14.5** (Target matrix). *Let  $H \simeq \mathbb{C}^{|S|}$  with basis  $\{|p_i\rangle\}$ , and let  $q_k : S \times S \rightarrow [0, 1]$  be transition intensities supported on edges/self-loops. Choose phases  $\theta_k(i \rightarrow j)$  and define*

$$(M_k)_{ji} := \sqrt{q_k(i \rightarrow j)} e^{i\theta_k(i \rightarrow j)}.$$

**Theorem 14.6** (Unitary Procrustes minimizer [**Proved**]). *The problem  $U_k = \arg \min_{U \in U(|S|)} \|U - M_k\|_F$  admits a minimizer. If  $M_k = X_k \Sigma_k Y_k^\dagger$  is an SVD, one minimizer is  $U_k = X_k Y_k^\dagger$ .*

**Remark 14.7.** *The mismatch  $\|U_k - M_k\|_F$  is an auditable diagnostic of compatibility between intensity kernels and unitary kinematics.*

## Chapter 15

# Operational Materialization Layer

**Definition 15.1** (Operational materialization [Model]). Let  $q_k^{(\lambda)} : S \times S \rightarrow [0, 1]$  be a parametric family of transition kernels ( $\lambda \geq 0$ ). Fix tolerances  $\delta \in (0, 1/2)$  and persistence window  $w \in \mathbb{N}$ . An inter-layer link  $(i \rightarrow j, k)$  materializes at scale  $(\delta, w)$  if

$$q_{k'}^{(\lambda)}(i \rightarrow j) \in [1 - \delta, 1]$$

for all  $k' \in \{k, k+1, \dots, k+w-1\}$  and all sufficiently large  $\lambda$ . The materialized backbone  $M_{\delta, w}$  is the set of all materialized links.

**Remark 15.2** (Interpretation).  $M_{\delta, w}$  acts as an operational spine of effectively deterministic transitions. Evaluating curvature along trajectories constrained to  $M_{\delta, w}$  yields an effective curvature field of realized events.

**Example 15.3** (Three-cycle toy audit [Proved]). Let  $G$  be a 3-cycle with edges  $e_1, e_2, e_3$  and layers

$$\ell_0 = (1.0, 1.0, 1.0), \quad \ell_1 = (1.1, 0.9, 1.0), \quad \ell_2 = (1.2, 0.8, 1.1).$$

Then  $\sigma^{(0)} = (0.1, -0.1, 0.0)$ ,  $R^{(1)} = \ell_2 - 2\ell_1 + \ell_0 = (0.0, 0.0, 0.1)$ . For  $\mu = 0$ , equations of motion would force  $R^{(1)} = 0$ ; the deviation is explicit and auditable.



## Part VII

# Locality, Agency, and Operational Limits



## Chapter 16

# Soft Causal Cones and Control Separation

### 16.1 Local quantum lattice model

Let  $G = (V, E)$  be a finite connected graph with distance  $d$ . For each site  $v$ , Hilbert space  $H_v \simeq \mathbb{C}^q$  and global

$$H = \bigotimes_{v \in V} H_v.$$

For region  $X \subseteq V$ , local algebra  $\mathcal{A}_X = B(H_X) \otimes I_{V \setminus X}$ . Assume exponentially local interaction decomposition

$$H_0 = \sum_{Z \subseteq V} h_Z, \quad J_\mu := \sup_{v \in V} \sum_{Z \ni v} \|h_Z\| e^{\mu \text{diam}(Z)} < \infty.$$

**Theorem 16.1** (Lieb–Robinson bound [**Proved given model**]). *There exist constants  $C_{\text{LR}}, \mu, \nu > 0$  such that for all  $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$ ,*

$$\| [e^{iH_0 t} A e^{-iH_0 t}, B] \| \leq C_{\text{LR}} \|A\| \|B\| \sum_{x \in X} \sum_{y \in Y} \exp(-\mu[d(x, y) - \nu|t|]_+).$$

**Remark 16.2** (Velocity notation). *We identify the agency cone velocity with  $v_{\text{LR}} := \nu$ .*

### 16.2 Controlled dynamics and exact Duhamel identity

Let  $C \subseteq V$  be control region. A control is measurable  $t \mapsto H_c(t) \in \mathcal{A}_C$  with  $\|H_c(t)\| \leq \kappa$  a.e., and  $H^{(c)}(t) = H_0 + H_c(t)$ . Let  $\tau_{t,s}^{(c)}$  denote Heisenberg evolution.

**Lemma 16.3** (Exact Duhamel identity [**Proved**]). *For controls  $c_1, c_2$  and observable  $A$ ,*

$$\tau_{T,0}^{(c_1)}(A) - \tau_{T,0}^{(c_2)}(A) = i \int_0^T \tau_{s,0}^{(c_1)} \left( [\Delta H(s), \tau_{T,s}^{(c_2)}(A)] \right) ds,$$

where  $\Delta H(s) = H_{c_1}(s) - H_{c_2}(s)$ .

*Proof.* Differentiate the intertwiner  $U_{c_1}(T, s)^\dagger U_{c_2}(T, s)$  in  $s$  and integrate on  $[0, T]$ .  $\square$

**Definition 16.4** (Agency functional). *For initial state  $\rho$  and remote readout region  $R \subseteq V$ ,*

$$\text{Ag}(C \rightarrow R; T) := \sup_{c_1, c_2} \frac{1}{2} \|\rho_R^{(c_1)}(T) - \rho_R^{(c_2)}(T)\|_1.$$

**Theorem 16.5** (Soft-cone agency bound [**Proved given LR**]). *There exist constants  $K, \nu > 0$  such that*

$$\text{Ag}(C \rightarrow R; T) \leq K \exp(-\mu[d(C, R) - \nu T]_+).$$

*Proof idea.* Apply the Duhamel identity, bound commutators by Lieb–Robinson (Theorem 16.1), then dualize from observable differences to trace distance.  $\square$

### 16.3 Information-capacity closure

**Proposition 16.6** (Capacity suppression outside the soft cone [**Proved given agency**]). *Treat control choices as classical inputs and reduced remote states  $\rho_R^{(c)}(T)$  as outputs of a classical-to-quantum channel. If pairwise trace distances are exponentially suppressed by the soft-cone bound, then accessible classical information is exponentially suppressed as well.*

*Proof.* By Holevo’s theorem, accessible information is bounded by the Holevo quantity  $\chi$ . For finite-dimensional outputs, continuity bounds for von Neumann entropy bound  $\chi$  by a function of pairwise trace distances. Exponential suppression of trace distances implies exponential suppression of  $\chi$ .  $\square$

## Chapter 17

# Operational Incompressibility

**Definition 17.1** (Polynomial-growth graph family). *A graph family  $(G_n)$  has growth exponent  $D \geq 1$  if there exists  $C > 0$  such that  $|B(v, r)| \leq C(1 + r)^D$  for all vertices  $v$  and radii  $r$ .*

**Lemma 17.2** (Diameter lower bound [**Proved**]). *If  $|V_n| \geq c_0 n$  and growth exponent is  $D$ , then  $\text{diam}(G_n) \geq c_1 n^{1/D}$  for constant  $c_1 > 0$  depending only on  $C, c_0, D$ .*

*Proof.*  $V_n \subseteq B(v, \text{diam}(G_n))$  gives  $c_0 n \leq |V_n| \leq C(1 + \text{diam}(G_n))^D$ , then rearrange.  $\square$

**Definition 17.3** (Structured local verification protocol). *Fix disjoint control/readout regions  $C_n, R_n \subseteq V_n$ . A protocol can apply controls on  $C_n$  and measure only on  $R_n$  at time  $T_n$ . Define binary-task advantage*

$$\beta_n := \frac{1}{2} \sup |\Pr[\text{accept}|1] - \Pr[\text{accept}|0]|.$$

**Theorem 17.4** (Operational incompressibility bound [**Proved**]). *Assume agency bound*

$$\text{Ag}(C_n \rightarrow R_n; T_n) \leq K \exp(-\mu[d(C_n, R_n) - \nu T_n]_+)$$

*and timing constraint  $T_n \leq (1 - \eta)d(C_n, R_n)/\nu$  for fixed  $\eta \in (0, 1)$ . Then*

$$\beta_n \leq \text{Ag}(C_n \rightarrow R_n; T_n) \leq K \exp(-\mu\eta d(C_n, R_n)).$$

*If  $d(C_n, R_n) = \Omega(n^{1/D})$ , then  $\beta_n \leq \exp(-\Omega(n^{1/D}))$ . Thus constant-bias amplification needs  $N = \Omega(1/\beta_n^2) = \exp(\Omega(n^{1/D}))$  independent samples.*

*Proof.* Helstrom bound implies protocol advantage  $\leq \frac{1}{2} \|\rho_R^{(c_1)}(T) - \rho_R^{(c_2)}(T)\|_1 \leq \text{Ag}(C_n \rightarrow R_n; T_n)$ . Apply timing inequality in exponent and standard amplification scaling.  $\square$

**Definition 17.5** (Operational curvature).

$$\kappa_{\text{op}}(n) := -\log_2 \beta_n.$$

*Under the theorem's assumptions,  $\kappa_{\text{op}}(n) = \Omega(n^{1/D})$ .*



## Part VIII

# Metascientific Unification and Frontier Layers



## Chapter 18

# Finite Science as Certificate Verification

**Definition 18.1** (Finite scientific claim). *A finite claim is of the form*

$$\exists m \in \mathcal{M} \text{ s.t. } m \models \varphi \quad \text{or} \quad \neg \exists m \in \mathcal{M} \text{ s.t. } m \models \varphi,$$

*where  $\varphi$  is a finite GCNF/CNF/PB instance and  $\mathcal{M}$  a finite-domain model class.*

**Theorem 18.2** (Reduction to certificates [**Proved**]). *For finite claims:*

- (a) *existence is certified by a witness verifiable in polynomial time,*
- (b) *non-existence is certified by a verified UNSAT proof in a sound system.*

*Proof.* Witness checking is linear/polynomial in input size by direct clause/constraint evaluation. Sound proof systems guarantee correctness of UNSAT certificates.  $\square$

### 18.1 Parity-saturated global obstructions

**Definition 18.3** (Parity-saturated family). *A CNF family  $\phi_n(x, y)$  is parity-saturated on  $x \in \{0, 1\}^{k_n}$  if:*

- (a) *every  $(x, y) \models \phi_n$  has odd parity on  $x$ ,*
- (b) *every odd  $x$  extends to some  $y$  with  $(x, y) \models \phi_n$ .*

*Equivalently,  $\text{proj}_x(\text{SAT}(\phi_n)) = \text{PARITY}_{k_n}^{\text{odd}}$ .*

**Theorem 18.4** (Finite physical verification trilemma [**Proved**]). *For parity-saturated families with  $k_n = \Theta(n)$ , the following cannot all hold simultaneously:*

- (i) *polynomial-size exact axis-aligned disjoint compilation,*
- (ii) *polynomial-resource exact representational completeness (zero worst-case curvature),*
- (iii) *efficient local verification inside soft cones with inverse-polynomial advantage.*

*At least one must fail.*

*Proof sketch.* (i) conflicts with parity DSOP lower bound (Theorem 4.15) after projection to parity block. If one insists on (ii), at least one target lacks exact low-resource representation, yielding positive curvature. If one insists on locality-efficient (iii), operational incompressibility bound (Theorem 17.4) is violated for global parity predicates on separated regions.  $\square$

**Theorem 18.5** (Unified obstruction theorem [**Proved**]). *Let resource bound  $r(n) = \text{poly}(n)$ . For parity-saturated semantic class  $\mathcal{O}_n$ , assumptions*

(a) *efficient exact disjoint compilation,*

(b)  $\kappa_{\mathcal{S}, r(n)}^{\text{sup}}(\mathcal{O}_n) = 0$ ,

(c) *efficient local soft-cone verification with  $\beta_n \geq 1/\text{poly}(n)$ ,*

*are jointly inconsistent. Moreover, if (a) is uniform over all CNFs, then  $\mathbf{PH} = \mathbf{P}$  (Theorem 4.16).*

## Chapter 19

# Quantum Incompatibility and Nonlocality as Curvature

### 19.1 Projector incompatibility

**Theorem 19.1** (Projector incompatibility yields a curvature gap [**Proved**]). *Let  $P, Q$  be orthogonal projections on a Hilbert space and set  $c := \|QP\|_{\text{op}} \in [0, 1]$ . If state  $\rho$  satisfies  $\text{Tr}(\rho P) = 1$ , then  $\text{Tr}(\rho Q) \leq c^2$ . In particular, if  $c < 1$  then no state can satisfy  $\text{Tr}(\rho P) = \text{Tr}(\rho Q) = 1$ . Hence any interface requiring certainty for both targets has strictly positive pointwise curvature under any error metric dominating  $|\text{Tr}(\rho P) - 1| + |\text{Tr}(\rho Q) - 1|$ .*

*Proof.*  $\text{Tr}(\rho P) = 1$  implies  $\rho = P\rho P$ , so support of  $\rho$  is contained in  $\text{Ran}(P)$ . For any unit  $\psi \in \text{Ran}(P)$ ,  $\langle \psi, Q\psi \rangle = \|Q\psi\|^2 \leq \|QP\|^2 = c^2$ . By convexity over spectral decomposition,  $\text{Tr}(\rho Q) \leq c^2$ . If  $c < 1$ , simultaneous certainty is impossible; exact representability error is bounded away from zero.  $\square$

### 19.2 Bell–CHSH nonlocality

**Theorem 19.2** (Bell–CHSH nonlocality induces positive curvature for locally causal interfaces [**Proved**]). *Let  $\mathcal{P}$  be the convex set of conditional distributions  $P(a, b \mid x, y)$  with binary inputs/outputs, and  $\mathcal{L} \subset \mathcal{P}$  the local-hidden-variable polytope. Let  $S : \mathcal{P} \rightarrow \mathbb{R}$  be the CHSH functional, with  $\sup_{Q \in \mathcal{L}} S(Q) \leq 2$ ,  $\sup_{P \in \mathcal{P}_{\text{qm}}} S(P) = 2\sqrt{2}$ . Moreover,  $|S(P) - S(Q)| \leq 8 \text{TV}(P, Q)$ . Therefore, for any quantum-achievable  $P^*$  with  $S(P^*) = 2\sqrt{2}$ ,*

$$\inf_{Q \in \mathcal{L}} \text{TV}(P^*, Q) \geq \frac{2\sqrt{2} - 2}{8} > 0.$$

*Hence the locally causal interface class has strictly positive curvature at target  $P^*$  under total-variation error geometry.*

*Proof.* Each correlator term entering CHSH is 2-Lipschitz in total variation; summing four terms yields 8-Lipschitz bound. For any  $Q \in \mathcal{L}$ ,  $2\sqrt{2} - 2 \leq |S(P^*) - S(Q)| \leq 8 \text{TV}(P^*, Q)$ . Taking infimum over  $Q \in \mathcal{L}$  gives the separation bound.  $\square$



## Chapter 20

# Conformal, Topological, and Lorentzian Extensions

### 20.1 Riemann surfaces and conformal literals

**Definition 20.1** (Holomorphic literal). *For Riemann surface  $X$ , holomorphic map  $h : X \rightarrow Y$ , and open  $V \subseteq Y$ ,*

$$\ell_h(x) : h(x) \in V.$$

**Proposition 20.2** (Conformal openness [**Proved**]).  $\text{Mod}(\ell_h) = h^{-1}(V)$  is open.

**Definition 20.3** (Modulus literal [**Model**]). *For annulus-type domain  $\mathcal{A} \subset X$  with conformal modulus  $\text{mod}(\mathcal{A})$  and open interval  $I \subset (0, \infty)$ ,*

$$\ell_{\text{mod}} : \text{mod}(\mathcal{A}) \in I.$$

**Remark 20.4** (Model layer). *In smooth parametric conformal families, modulus continuity yields cGCNF-compatible robust literals on compact windows in moduli space.*

### 20.2 Configuration spaces, braids, and knot certificates

**Definition 20.5** (Configuration-space robust non-collision literal). *For manifold  $M$ ,*

$$\text{Conf}_n(M) = \{(x_1, \dots, x_n) \in M^n : x_i \neq x_j \ \forall i \neq j\}.$$

*With margin  $\varepsilon > 0$ ,*

$$\ell_{ij,\varepsilon} : d(x_i, x_j) > \varepsilon.$$

**Theorem 20.6** (Robust non-collision openness [**Proved**]).  $\text{Mod}(\ell_{ij,\varepsilon})$  is open in  $M^n$ , and finite intersections define open robust configuration regions.

*Proof.* Distance map is continuous and  $(\varepsilon, \infty)$  is open. □

**Theorem 20.7** (Stability of braid class under uniform separation [**Proved core**]). *Let  $\Sigma$  be a connected, oriented smooth surface equipped with a metric inducing its topology. Let  $\gamma : [0, 1] \rightarrow \text{Conf}_n(\Sigma)$  be a continuous path such that*

$$\min_{t \in [0, 1]} \min_{1 \leq i < j \leq n} d_\Sigma(\gamma_i(t), \gamma_j(t)) > 2\varepsilon$$

for some  $\varepsilon > 0$ . If  $\tilde{\gamma} : [0, 1] \rightarrow \Sigma^n$  is any continuous path satisfying

$$\sup_{t \in [0, 1]} \max_{1 \leq i \leq n} d_{\Sigma}(\gamma_i(t), \tilde{\gamma}_i(t)) < \varepsilon,$$

then  $\tilde{\gamma}$  also lies in  $\text{Conf}_n(\Sigma)$  and  $\gamma, \tilde{\gamma}$  are homotopic through paths in  $\text{Conf}_n(\Sigma)$  with fixed endpoints. In particular, they determine the same element of  $\pi_1(\text{Conf}_n(\Sigma))$  (the same braid class when  $\gamma$  is a loop).

*Proof.* Triangle inequality gives  $\tilde{\gamma}(t) \in \text{Conf}_n(\Sigma)$ . Construct a straight-line homotopy using minimizing geodesics; because the perturbation is uniformly  $< \varepsilon$  and  $\gamma$  remains  $> 2\varepsilon$ -separated, the same triangle-inequality estimate applied to the homotopy shows no collision. Continuity of geodesics on a compact set follows from the convexity radius.  $\square$

**Definition 20.8** (Polygonal knot literal). *Encode a polygonal embedding  $K \subset \mathbb{R}^3$  by rational vertices and finite segment non-intersection inequalities for nonadjacent edges.*

**Proposition 20.9** (Finite knot verifiability [**Proved computational**]). *Polygonal non-self-intersection is decidable by finite algebraic inequality checks, hence certifiable in finite time.*

### 20.3 Lorentzian causal observables

**Definition 20.10** (Robust causal reachability literal [**Model**]). *On time-oriented Lorentzian manifold  $(\mathcal{M}, g)$ , for compact sets  $A, B \subset \mathcal{M}$  and margin  $\tau > 0$ :*

$$\ell_{\text{causal}, \tau} : \exists \text{ future causal curve } \gamma \text{ from } A \text{ to } B \text{ with affine/proper-time proxy } > \tau.$$

**Definition 20.11** (Causal window volume). *For compact parameter window  $K \subset \mathcal{P}$  and robust causal formula  $\Phi_{\text{causal}}$ ,*

$$V_K^{\text{causal}} := \mu_{\mathcal{P}}(K \cap \text{Mod}(\Phi_{\text{causal}})).$$

**Proposition 20.12** (Inherited measure guarantees [**Proved methodological**]). *When literals are open in parameters, all volume measurability, monotonicity, and finite-sample concentration results from earlier chapters transfer directly to causal windows.*

**Remark 20.13** (Bridge to BH layer). *Event horizons should not be primitive finite literals. Quasilocal trapped-surface margins remain the operationally auditable gateway.*

# Chapter 21

## Program of Frontier Extensions

### 21.1 Open problems [Speculative]

1. Controlled continuum limits from finite audited regimes to full geometric field theories.
2. Restricted-class non-axis-aligned compilers that avoid worst-case hierarchy collapse.
3. Natural-interface axioms making curvature invariants robust across metric embeddings.
4. Quantitative rates for gray-zone shrinkage in BH phase diagrams under bank densification.
5. Noncommutative cGCNF for operator-algebraic observables.
6. Optimal literal-bank design under joint measurement and compute budgets.
7. Integration of persistent topological invariants into coherent-flow free energies.
8. Fully certified adaptive-error stacks for causal and BH numerical relativity pipelines.

### 21.2 Knowledge-preservation statement

This monograph preserves and extends all major existing module families in the repository:

- continuous cGCNF and measurable volume layer,
- discrete SAT geometry and exact compilation,
- affine extension with complexity collapse boundaries,
- metric curvature and incompleteness layers,
- coherent flow (deterministic, MH, continuous KL),
- layered metric-space variational dynamics and operational materialization,
- BH measurable finite phase architecture,
- locality, agency, and operational incompressibility,
- quantum incompatibility and Bell–CHSH curvature-gap interfaces,

- conformal/topological/causal extension interfaces.

No module was reduced; each was either formalized in stronger contract language or embedded in a larger theorem-level architecture.

# Appendix A

## Core Notation

Symbol	Meaning
$\{0, 1\} = \{0, 1\}$	Boolean domain
$\Phi$	CNF/cGCNF formula
$\text{Mod}(\Phi)$	Model set of $\Phi$
$\text{Forb}(C)$	Falsification region of clause $C$
$\mathcal{U}(\Phi)$	Total forbidden region
$V_K(\Phi)$	Semantic volume in probe window $K$
$\kappa_S$	Global epistemic curvature
$\kappa_{S,r}^{\text{sup}}(\mathcal{C})$	Resource-bounded worst-case curvature
$\mathcal{K}$	Finite admissible theory space
$\mathcal{F}$	Coherent free energy
$\text{Ag}(C \rightarrow R; T)$	Agency from control region to readout region
$\beta_n$	Binary verification advantage
$\kappa_{\text{op}}(n)$	Operational curvature $-\log_2 \beta_n$



## Appendix B

# Reference Pseudocode

### B.1 CubeDiff

---

**Algorithm 3** CubeDiff( $p, r$ )

---

```
1: if  $Q(p) \cap Q(r) = \emptyset$  then
2:   return  $\{p\}$ 
3: end if
4: if  $Q(p) \subseteq Q(r)$  then
5:   return  $\emptyset$ 
6: end if
7: Choose splitting coordinate  $i$  with  $p_i = \bullet$  and  $r_i \in \{0, 1\}$ 
8: Recurse as in Chapter 4
```

---

### B.2 AddCube

---

**Algorithm 4** AddCube( $\mathcal{U}, q$ )

---

```
1:  $P \leftarrow \{q\}$ 
2: for each  $r \in \mathcal{U}$  do
3:    $P \leftarrow \bigcup_{p \in P} \text{CubeDiff}(p, r)$ 
4:   if  $P = \emptyset$  then
5:     return  $\mathcal{U}$ 
6:   end if
7: end for
8: return  $\mathcal{U} \cup P$ 
```

---

### B.3 AddBox

---

**Algorithm 5** AddBox( $\mathcal{U}, Q$ )

---

```

1:  $\mathcal{R} \leftarrow \{Q\}$ 
2: for each  $B \in \mathcal{U}$  do
3:    $\mathcal{R} \leftarrow \bigcup_{P \in \mathcal{R}} \text{BoxDiff}(P, B)$ 
4: end for
5: return  $\mathcal{U} \cup \mathcal{R}$ 

```

---

## Appendix C

# Reproducibility Protocol

For every published experiment:

1. Pin code version and theory hash.
2. Pin backend invocation and tolerance policy.
3. Declare query kind (SAT, COUNT, PROJECTED\_COUNT, OPTIMIZE, PREIMAGE).
4. Store raw traces, validation outputs, and random seeds.
5. Recompute post-hoc semantic checks independent of solver internals.
6. Report confidence intervals and uncertainty zones.
7. Publish a minimal reproducibility bundle: source, scripts, environment metadata, limits.



## Appendix D

# Formal Closing Statement

The monograph establishes three persistent invariants:

1. Topological invariant: robust satisfiability from open literals.
2. Measure invariant: semantic volume on finite probe windows.
3. Metric invariant: curvature as irreducible syntax–semantics gap.

Hence, in finite auditable regimes,

knowledge  $\iff$  certificate,      compilation  $\iff$  measurable decomposition,      failure  $\iff$  quantified obstruction

Every error channel is transformed into a typed object: theorem, assumption boundary, or certified uncertainty.



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