Finite-Bank Certification in Epistemic Geometry: Gray-Zone Scaling in Binary Black-Hole Initial Data and an "Epistemic Perihelion"

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Abstract

We package a finite-bank, auditable methodology to certify phase structure in binary-black-hole (BBH) initial data: "common apparent horizon" vs "separate horizons" vs a certified gray zone induced by finite resources. The construction treats surface templates as a compact search space equipped with an explicit metric, enabling ε-net banks and a transfer theorem that quantifies the gap between an ideal (infinite-bank) semantics and finite-bank certification via a modulus $\omega(\varepsilon)$. We provide (i) a reproducible toy study using published critical-separation data as an external benchmark boundary, (ii) explicit Monte Carlo error bars for phase volumes, (iii) a practical procedure to estimate ω as an effective Lipschitz modulus, and (iv) a protocol-holonomy observable—the "epistemic perihelion"—which measures non-closure under finite-resource refinement loops. All source code, figures, and data are included.

1 Scope and positioning

We work at the level of *quasi-local* trapped-surface certification in BBH *initial data* $(h(\theta), K(\theta))$ with parameters $\theta \in \Theta$. Our focus is not event horizons (global objects), but auditable finite-time predicates built from robust inequalities on null expansions. This aligns with numerical relativity practice where apparent-horizon finding is an elliptic problem and depends on initial guesses. For concreteness we anchor the benchmark boundary to the published critical-separation curve $d_{\mathrm{crit}}(q)$ for time-symmetric two-BH data [Jaramillo and Lousto, 2010].

2 Robust literals and finite banks

Let $S \subset \Sigma$ be a smooth closed two-surface in a Cauchy slice of initial data, and let $\theta_+(p; S, \theta)$ denote the outgoing null expansion at $p \in S$ for parameters θ. Define the functionals

$$F_S(\theta) := \max_{p\in S} \theta_+(p; S, \theta), \tag{1}$$

$$G_S(\theta) := \min_{p\in S} \theta_+(p; S, \theta). \tag{2}$$

For a margin parameter $\tau > 0$, define the robust literals

$$L^{\mathrm{trap}}_{S,\tau}(\theta): \quad F_S(\theta) < -\tau, \tag{3}$$

$$L^{\mathrm{outer}+}_{S,\tau}(\theta): \quad G_S(\theta) > \tau. \tag{4}$$

The strict inequalities define open sets in parameter space and are stable under small perturbations.

A *finite bank* is a finite set of surface templates $S_N = \{S_1, \dots, S_N\}$. Phase certification is expressed as disjunctions over banked literals, e.g.

$$\Phi_{\text{common}}(\theta) \;:=\; \bigvee_{S \in S_N^c} L_{S,\tau}^{\text{trap}}(\theta), \tag{5}$$

with analogous formulas for separate horizons using banks S_N^+, S_N^- and an "anti-common" outer bank S_N^{out}. In practice, S_N corresponds to multi-start initial guesses for an apparent-horizon finder.

3 Surface parameterization, compactness, and metrics

We parameterize a compact family of star-shaped surfaces around a center c by spherical-harmonic coefficients:

$$r(\theta, \phi) = r_0\Big(1 + \sum_{1 \le l \le L} \sum_{|m| \le l} a_{lm} Y_{lm}(\theta, \phi)\Big), \tag{6}$$

with bounds $r_0 \in [r_{\min}, r_{\max}]$ and $|a_{lm}| \le A_l$. The coefficient set is compact; equip it with the metric

$$\text{dist}_S(S, S') := \Big(|r_0 - r_0'|^2 + \sum_{l,m} w_l |a_{lm} - a_{lm}'|^2\Big)^{1/2}, \tag{7}$$

making ε-net construction explicit.

4 Transfer theorem and the modulus ω

Let E_τ denote the ideal (infinite-bank) "existence" set:

$$E_\tau := \{\theta \in \Theta : \exists S \in S_\infty \text{ such that } F_S(\theta) < -\tau\}. \tag{8}$$

Let $E_\tau^{(N)}$ denote the certified set under the finite bank S_N. Under continuity of $(\theta, S) \mapsto F_S(\theta)$ and ε_N-net coverage of S_∞, there exists a modulus ω such that

$$E_{\tau + \omega(\varepsilon_N)} \subseteq E_\tau^{(N)} \subseteq E_\tau. \tag{9}$$

Equation (9) is the key device: it turns "finite bank" into a quantitative statement about certified semantics.

5 Gray zone and volume estimation with finite error bars

Fix a compact parameter window $K \subset \Theta$ with measure μ_Θ. The *certified gray zone* for a given (N, τ) is

$$G_{N,\tau}(K) := (E_\tau \cap K) \setminus (E_{\tau + \omega(\varepsilon_N)} \cap K), \tag{10}$$

and the main observable is

$$V_{\text{gray}}(N, \tau) := \mu_\Theta\big(G_{N,\tau}(K)\big). \tag{11}$$

Estimate V_{gray} by Monte Carlo sampling $\theta^{(i)} \sim \text{Unif}(K)$. For T samples and confidence $1 - \delta$, Hoeffding gives the auditable bound

$$|\widehat{V}_T - V_K| \le \mu_\Theta(K) \sqrt{\frac{\log(2/\delta)}{2T}}. \tag{12}$$

6 Estimating ω as an effective Lipschitz modulus

The modulus ω is not a philosophical parameter; it can be estimated empirically. Fix a grid of radii r_j in surface space. For each sampled $\theta^{(i)}$ and each r_j, draw pairs (S, S') with $\mathrm{dist}_S(S, S') \approx r_j$ and compute $\Delta F_i(r_j) := |F_S(\theta^{(i)}) - F_{S'}(\theta^{(i)})|$. Define a robust modulus estimate by a high quantile (e.g. 0.99):

$$\widehat{\omega}(r_j) := \mathrm{Quantile}_{0.99}\big(\{\Delta F_i(r_j)\}\big). \tag{13}$$

If $\widehat{\omega}(r) \approx Lr$ over a regime, report L as an effective sensitivity of the literal to surface perturbations.

7 Scaling predictions

Two falsifiable scaling laws follow.

Margin scaling. Near a smooth critical boundary, the gray zone induced by a strict margin behaves like a band of thickness $\sim 2\tau$, hence $V_{\mathrm{gray}}(\infty, \tau)$ grows approximately linearly in τ.

Bank scaling. If $\widehat{\omega}(r) \sim Lr$ and the bank satisfies $\varepsilon_N \sim N^{-1/d_S}$ for an effective surface-space dimension d_S, then the bank-induced gray excess obeys

$$V_{\mathrm{gray}}(N, \tau) - V_{\mathrm{gray}}(\infty, \tau) = O(N^{-1/d_S}). \tag{14}$$

8 Epistemic perihelion: protocol holonomy as a curvature observable

Define two refinement operations:

- Bank refinement B: $S_N \mapsto S_{N'}$ (reducing ε_N),
- Margin refinement R_δ: $\tau \mapsto \tau + \delta$.

The loop (commutator) is $\mathcal{L} := B^{-1} R_\delta^{-1} B R_\delta$. The *epistemic perihelion* is the non-closure measured as a set-distance in parameter space, e.g.

$$\Delta_{\circlearrowleft}(N, \tau) := \mu_\Theta\Big((E_\tau^{(N)} \cap K)\ \Delta\ (\mathcal{L}(E_\tau^{(N)}) \cap K)\Big), \tag{15}$$

where Δ denotes symmetric difference. In "flat" operational regimes, $\Delta_{\circlearrowleft} \approx 0$. In curved regimes induced by finite resources, $\Delta_{\circlearrowleft} > 0$ and should scale with $\omega(\varepsilon_N)$.

9 Toy study with published BBH critical separations

To provide a fully reproducible sanity check, we use the published normalized critical separations $d_{\mathrm{crit}}/(M_1+M_2)$ as a benchmark boundary [Jaramillo and Lousto, 2010]. Figure 1 plots $d_{\mathrm{crit}}/(M_1+M_2)$ vs mass ratio q from Table 1 in [Jaramillo and Lousto, 2010]. Using a parameter window $K = \{(q, d) : q \in [0.1, 1], d \in [0.60, 0.80]\}$ and a margin τ, we estimate the gray-band fraction by Monte Carlo, producing Figure 2. Finally, we emulate finite-bank boundary resolution by subsampling the boundary with N points and using the resulting sup-norm boundary error to predict gray fractions (Figure 3).

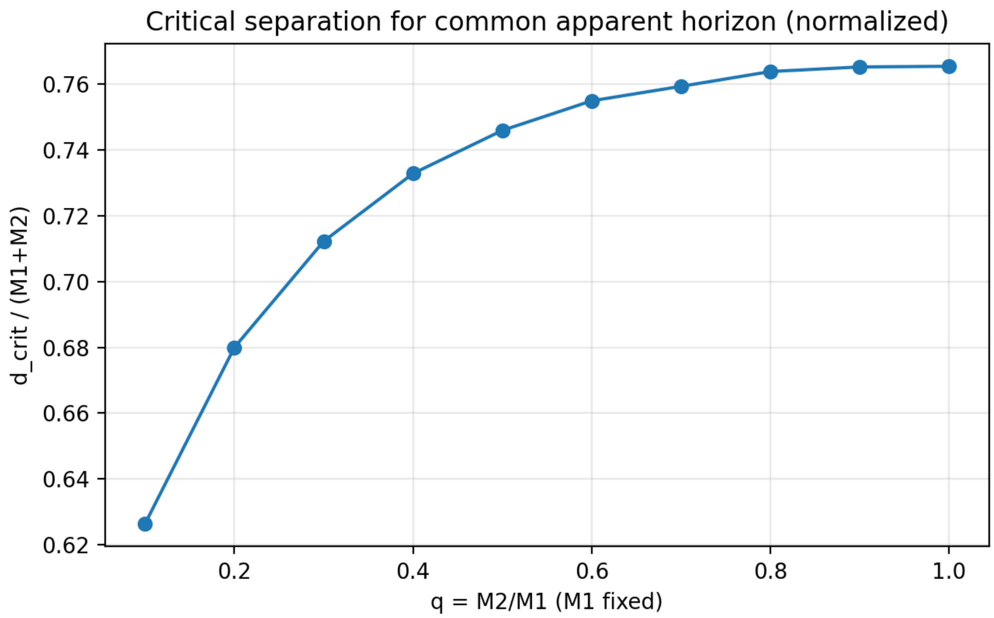

Figure 1: Normalized critical separation for common apparent-horizon formation in time-symmetric two-BH data (values from Table 1 of Jaramillo and Lousto, 2010).

10 Implementation sketch in the Einstein Toolkit

An implementable pipeline:

1. Generate time-symmetric two-BH initial data with `IDAnalyticBH` [ET, b, Ein].
2. For each parameter θ, run `AHFinderDirect` with multi-start over the bank S_N [ET, a].
3. Certify literals (3)–(4) with a strict margin τ.
4. Compute phase volumes and gray-zone volume in any chosen window K with the bound (12).
5. Estimate $\widehat{\omega}$ via controlled perturbations in surface coefficients, and test the scaling laws.

This turns the finite-guess dependency of practical horizon finding into a controlled, reportable finite-bank effect.

A Mercury perihelion benchmark (analogy)

For context, the GR perihelion precession for Mercury is $\Delta\varphi \approx \frac{6\pi G M_\odot}{a(1-e^2)c^2}$ [Pri]. A script reproducing the standard $42.98''$/century benchmark is included in `code/mercury_perihelion.py`. (Constants: AU per IAU 2012 [IAU].)

References

Einstein toolkit thorn guide: Ahfinderdirect. `https://www.einsteintoolkit.org/thornguide/EinsteinAnalysis/AHFinderDirect/documentation.html`, a. Accessed: 2026-02-13.

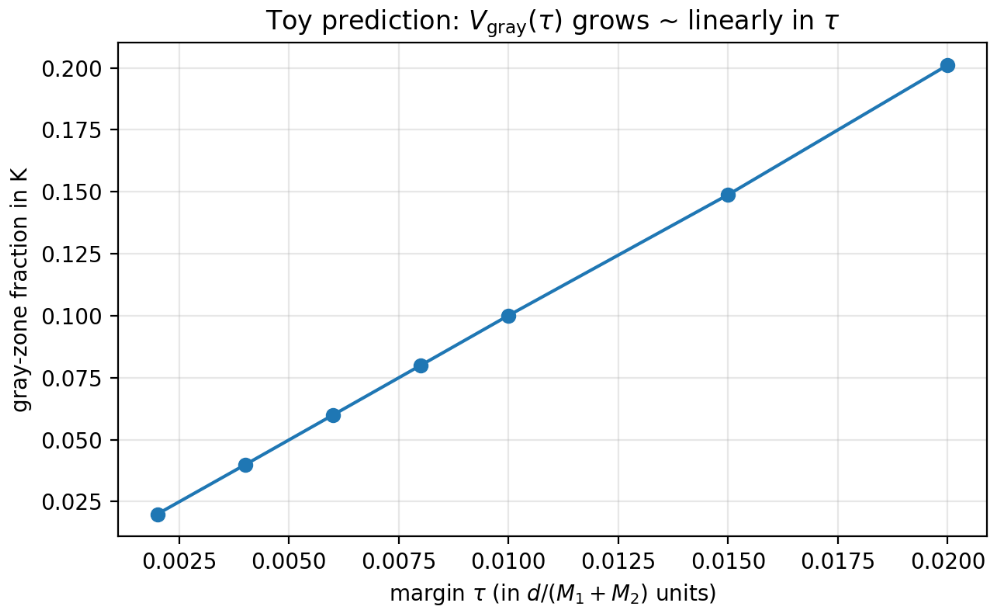

Figure 2: Toy gray-zone fraction in K vs margin τ, computed as a band around the benchmark boundary.

Einstein toolkit thorn guide: Idanalyticbh. `https://einsteintoolkit.org/thornguide/EinsteinInitialData/IDAnalyticBH/documentation.html`, b. Accessed: 2026-02-13.

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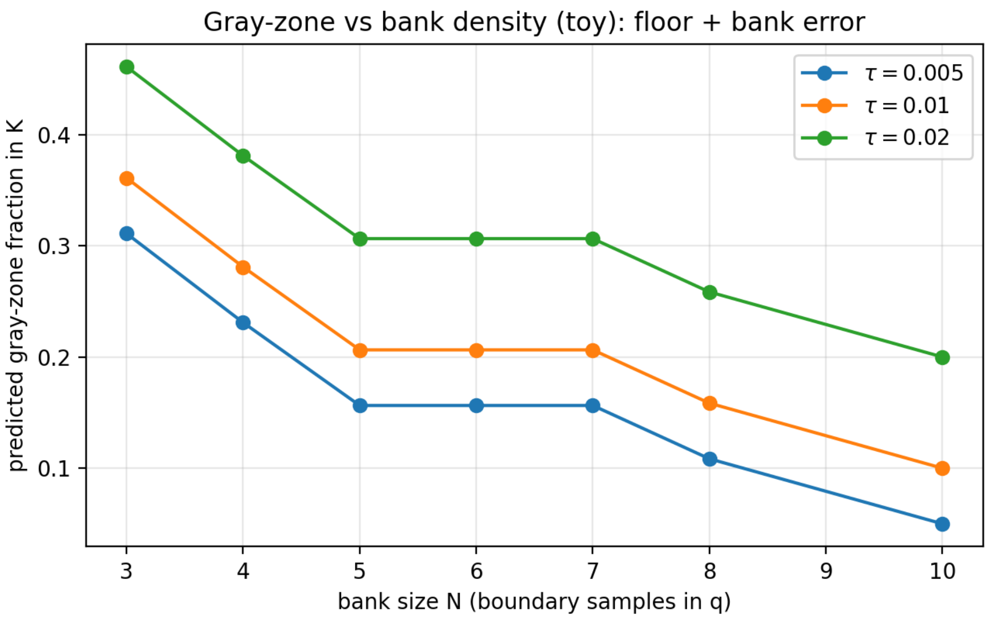

Figure 3: Toy gray-zone fraction vs bank density (boundary subsampling). The curve approaches the margin-imposed floor.