Epistemic Geometry of Closure

SCE-IM, Coherent Flow, Stability, and Operational Completeness

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Abstract

This manuscript formalizes and unifies an *epistemic closure* framework (SCE-IM) with three mutually compatible layers: (i) a semantic layer (windowed volume / continuous #SAT), (ii) a geometric–metric layer (curvature as an operational *gap*), and (iii) a thermodynamic dynamic layer (*coherent flow* as Lyapunov descent and Gibbs/MH exploration). We introduce operational invariants (the zipper signature), stability results, and an operational completeness theorem (with and without resources) in classes where the physics of closure collapses to a tree-like structure (merge tree).

Contents

Chapter 1

Rigor contract and auditable objects

1.1 Conventions and typing of statements

We work with ideal objects (potentially infinite spaces) and their auditable *probes* (finite/compact windows). When an ideal object is not globally computable or certifiable, we explicitly declare the class of windows $\mathcal{K}$ and formulate invariants in terms of windowed measurements.

Every statement is implicitly tagged according to the contract:

- **[Proved]**: provable in the given formal system (definitions + stated hypotheses).
- **[Model]**: valid under class hypotheses (tameness, topological/mechanical zipper, etc.).
- **[Speculative]**: conjectural or an open program.

Remark 1.1 (Formal framework for **[Proved]**). Unless explicitly indicated otherwise, statements tagged **[Proved]**are understood as formalizable within a standard mathematical framework (for instance, ZFC or an equivalent set theory), starting from the definitions and the declared hypotheses. The tag is part of the evidence contract: it adds no content; it only makes explicit the type of audited guarantee.

1.2 Minimal structure of spaces

In what follows, S denotes the state space, O the objective space, and Ω the space of microconfigurations. Depending on context, S and O will be topological or metric; Ω will be measurable and equipped with a reference measure μ.

In particular, to avoid ambiguity throughout the manuscript, we assume by default that S and O are *topological* spaces (usually metrizable). Whenever notions of distance are used (for example, in stability, semicontinuity, or quasi-isometric equivalences), we fix a metric compatible with the topology and explicitly state the required control.

Chapter 2

SCE-IM: Epistemic Closure System with Metric Interface

2.1 Definition of SCE-IM

Definition 2.1 (SCE-IM)**.** An *Epistemic Closure System with Metric Interface* is a 10-tuple

$$\mathcal{E} = (S, O, \Omega, \mathcal{T}, J, \mathrm{err}, \mu, \mathcal{K}, F, \Phi)$$

where:

1. S is the state space and O the objective space.
2. Ω is a measurable space of microconfigurations. Concretely, we fix a σ-algebra Σ_Ω and write $\Omega = (\Omega, \Sigma_\Omega)$.
3. $\mathcal{T}$ is the set of *teeth* (elementary constraints). Moreover, each $\tau \in \mathcal{T}$ is equipped with a *semantic domain* $D_\tau \in \Sigma_\Omega$ (the set of microconfigurations that satisfy the tooth), so that the "content" of τ is expressed in Ω.
4. $J : S \to \mathcal{P}(\Omega)$ assigns to each state its *compatible micro-set*; we always assume that $J(\sigma) \in \Sigma_\Omega$ for every $\sigma \in S$.
5. $\mathrm{err} : S \times O \to [0, \infty]$ is the epistemic error.
6. μ is a reference measure on Ω.
7. $\mathcal{K}$ is a family of auditable windows $K \subseteq \Omega$ (typically finite or compact).
8. $F : S \to \mathbb{R} \cup \{+\infty\}$ is a free energy / potential.
9. Φ is a dynamics: either a discrete map $\Phi : S \to S$ or a semiflow $(\Phi_t)_{t \geq 0}$.

For $K \in \mathcal{K}$ we define the *windowed volume*

$$V_K(\sigma) := \mu(J(\sigma) \cap K).$$

Definition 2.2 (Teeth semantics and measurability)**.** We assume fixed an application

$$D : \mathcal{T} \to \Sigma_\Omega, \qquad \tau \mapsto D_\tau,$$

which interprets each tooth as a measurable set of admissible microconfigurations. We say that a state σ *satisfies* the tooth τ in the window $K \in \mathcal{K}$ if

$$\mu(J(\sigma) \cap D_\tau \cap K) = \mu(J(\sigma) \cap K),$$

that is, D_τ does not cut volume inside $J(\sigma) \cap K$. Moreover, in order for $V_K(\sigma) = \mu(J(\sigma) \cap K)$ to be auditable, we assume (when required) measurability of the correspondence J in the standard sense of measurable selection, so that $\sigma \mapsto V_K(\sigma)$ is Borel for each $K \in \mathcal{K}$.

2.2 Minimal axioms

Axiom 2.3 (Semantic zipper)**.** There exists an operator $\triangleleft : S \times \mathcal{T} \to S$ such that for every $\sigma \in S$, $\tau \in \mathcal{T}$,

$$J(\sigma \triangleleft \tau) \subseteq J(\sigma) \cap D_\tau.$$

Axiom 2.4 (Error regularity)**.** For each $o \in O$, the map $\sigma \mapsto \mathrm{err}(\sigma, o)$ is lower semicontinuous on S.

Definition 2.5 (Minimal curvature)**.** Given $o \in O$, the *curvature* (operational gap) is defined as

$$\kappa(o) := \inf_{\sigma \in S} \mathrm{err}(\sigma, o).$$

Axiom 2.6 (Lyapunov mechanics)**.** The dynamics decreases the energy:

- (discrete) $F(\Phi(\sigma)) \le F(\sigma)$ for all $\sigma \in S$;
- (continuous) for all $\sigma \in S$, assuming $t \mapsto F(\Phi_t(\sigma))$ is differentiable for $t \ge 0$, we have $\frac{\mathrm{d}}{\mathrm{d}t} F(\Phi_t(\sigma)) \le 0$, i.e., $\mathrm{d}F(\Phi_t(\sigma))/\mathrm{d}t \le 0$.

Axiom 2.7 (Energy–compression barrier)**.** There exists a nondecreasing function $\mathcal{B} : [0, \infty) \to [0, \infty]$ (typically convex) such that, for every $K \in \mathcal{K}$,

$$F(\sigma) \ge \mathcal{B}(-\log V_K(\sigma)), \qquad \text{whenever } V_K(\sigma) > 0.$$

Remark 2.8. The barrier $\mathcal{B}$ encodes the principle "compression costs energy": if $V_K(\sigma)$ is very small (high compression in the window), then the potential $F(\sigma)$ must grow at least as some function of $-\log V_K(\sigma)$. In applications, $\mathcal{B}(s) = cs + b$ or $\mathcal{B}(s) = c\, s^p$ (with $p \ge 1$) are typical choices. When one wants to connect energy explicitly with accuracy, one may additionally impose an inequality of the form

$$\mathrm{err}(\sigma, o) \;\ge\; G(-\log V_K(\sigma))$$

for some function G, but this linkage is optional and model-dependent (hence it is not included in the minimal axioms).

2.3 Sublevel filtration

For a fixed objective o, define the landscape $f(\sigma) := \mathrm{err}(\sigma, o)$ and the sublevel filtration

$$L_\varepsilon(o) := \{\sigma \in S : \; f(\sigma) \le \varepsilon\}, \qquad \varepsilon \ge 0.$$

Chapter 3

Rigid morphisms and invariants

3.1 Rigid isomorphism

Definition 3.1 (Rigid isomorphism)**.** Let $\mathcal{E}$ and $\mathcal{E}'$ be two SCE-IM systems. A *rigid isomorphism* is a triple (h, g, u) where $h : S \to S'$ is a homeomorphism, $g : O \to O'$ is a homeomorphism, and $u : \Omega \to \Omega'$ is a measurable isomorphism (typically the identity when Ω is fixed), such that for every $\sigma \in S$, $o \in O$,

$$\mathrm{err}'(h(\sigma), g(o)) = \mathrm{err}(\sigma, o), \qquad F'(h(\sigma)) = F(\sigma),$$

and the dynamics is conjugated: $h \circ \Phi_t = \Phi'_t \circ h$ (or $h \circ \Phi = \Phi' \circ h$ in discrete time). In addition, there is a transport of windows and teeth, namely:

1. **Windows:** for each $K \in \mathcal{K}$ we fix $K' := u(K) \in \mathcal{K}'$, and the windowed volume is preserved:
$$\mu(J(\sigma) \cap K) = \mu'(J'(h(\sigma)) \cap K') \qquad (\forall\, \sigma \in S).$$
2. **Teeth:** there exists $\theta : \mathcal{T} \to \mathcal{T}'$ such that $u(D_\tau) = D'_{\theta(\tau)}$ and the zipper transports:
$$h(\sigma \triangleleft \tau) = h(\sigma) \triangleleft \theta(\tau) \qquad (\forall\, \sigma \in S,\ \tau \in \mathcal{T}).$$

3.2 Immediate invariants

Proposition 3.2 (Curvature invariance)**.** *If (h, g, u) is a rigid isomorphism, then $\kappa'(g(o)) = \kappa(o)$ for every $o \in O$.*

Proof. By definition, $\mathrm{err}'(h(\sigma), g(o)) = \mathrm{err}(\sigma, o)$. Taking infima over σ and using bijectivity of h yields the claim. □

Proposition 3.3 (Transport of sublevels)**.** *Under a rigid isomorphism, $h(L_\varepsilon(o)) = L'_\varepsilon(g(o))$ for every $\varepsilon \geq 0$.*

Proof. Immediate from $\mathrm{err}'(h(\sigma), g(o)) = \mathrm{err}(\sigma, o)$. □

Chapter 4

Resource-bounded curvature and phases

Remark 4.1 (Extended structure with a resource). From this point on we consider, when needed, the *extended* structure

$$\mathcal{E}^{\rho} := (\mathcal{E}, \rho),$$

where $\rho : S \to [0, \infty)$ is a resource/budget functional (size, computational cost, energy, etc.). It is not included in the minimal 10-tuple because many statements only use err; when κ_R appears, ρ is assumed given and auditable.

4.1 Curvature under a budget

Definition 4.2 (Resource-bounded curvature)**.** Let $\rho : S \to [0, \infty)$ be a resource function. For each objective $o \in O$ we define

$$\kappa_R(o) := \inf\{\mathrm{err}(\sigma, o) : \ \rho(\sigma) \leq R\}.$$

Remark 4.3 (Explicit dependence on ρ and on the objective). The notation $\kappa_R(o)$ implicitly fixes the system S and the resource function ρ. When convenient, we make the dependence explicit by writing

$$\kappa_{S,\rho}(o; R) := \inf\{\mathrm{err}(\sigma, o) : \ \sigma \in S, \ \rho(\sigma) \leq R\}.$$

In particular, $o \in O$ is a semantic objective, whereas $\rho : S \to [0, \infty)$ is a syntactic functional (description length, computation time, experimental budget, etc.).

Lemma 4.4 (Monotonicity and budget limit)**.** *For every $o \in O$, the function $R \mapsto \kappa_R(o)$ is monotone nonincreasing: if $0 \leq R \leq R'$ then $\kappa_{R'}(o) \leq \kappa_R(o)$. Moreover, if $\rho(\sigma) < \infty$ for all $\sigma \in S$, then*

$$\inf_{R \geq 0} \kappa_R(o) = \kappa(o).$$

Proof. If $R \leq R'$, the feasible set $\{\sigma : \rho(\sigma) \leq R\}$ is contained in $\{\sigma : \rho(\sigma) \leq R'\}$; taking infima of the same function $\sigma \mapsto \mathrm{err}(\sigma, o)$ over nested sets yields $\kappa_{R'}(o) \leq \kappa_R(o)$.

For the infimum identity, first note that $\kappa_R(o) \geq \kappa(o)$ for all R, since we minimize over a subset of S. On the other hand, given $\sigma \in S$, take $R := \rho(\sigma)$; then σ is feasible and $\kappa_R(o) \leq \mathrm{err}(\sigma, o)$. Taking the infimum over σ gives $\inf_{R \geq 0} \kappa_R(o) \leq \kappa(o)$, closing the equality. □

4.2 Phase classification

Definition 4.5 (Phases A/B/C)**.** For an objective $\mathsf{o} \in O$ (letter "o", not the digit 0):

- **Phase A (finitely closable):** $\exists R < \infty$ such that $\kappa_R(\mathsf{o}) = 0$.
- **Phase B (asymptotically closable):** $\kappa(\mathsf{o}) = 0$ but $\kappa_R(\mathsf{o}) > 0$ for all $R < \infty$.
- **Phase C (residual):** $\kappa(\mathsf{o}) > 0$.

Chapter 5

Bridge to Coherent Flow

Remark 5.1 (Explicit dependence on the GEGB). This layer replicates, in SCE-IM notation, the core of *coherent flow* developed in the *General Epistemic Geometry Book* (GEGB), mainly in the chapters on deterministic and stochastic dynamics. Definitions are rewritten here to keep the text self-contained; when useful, direct correspondences with GEGB definitions and theorems will be indicated.

5.1 Theories as states

Let W be a finite universe of formulas and $\mathbb{K} \subseteq \mathcal{P}(W)$ a class of admissible theories. Let Ω be a finite space of valuations and μ a measure with full support.

Definition 5.2 (Models and volume)**.** For $K \in \mathbb{K}$, define

$$\mathrm{Mod}(K) := \{\omega \in \Omega : \ \forall \varphi \in K, \ \omega \models \varphi\}, \qquad V(K) := \mu(\mathrm{Mod}(K)).$$

5.2 Free energy and dynamics

Definition 5.3 (Coherent-flow free energy)**.** Let $\mathbb{K}$ be a (finite) class of theories $K \subseteq \mathcal{T}$. For each theory K, define its *semantic set*

$$J_K := \bigcap_{\tau \in K} D_\tau \ \in \Sigma_\Omega, \qquad \text{and its volume} \qquad V(K) := \mu(J_K).$$

Let $E_{\mathrm{ctr}}(K) \geq 0$ be a contradiction cost and $C(K) = |K|$ a complexity. Define the free energy

$$F(K) := \alpha E_{\mathrm{ctr}}(K) + \beta C(K) - \gamma \log V(K),$$

for parameters $\alpha, \beta, \gamma > 0$ and theories with $V(K) > 0$.

Definition 5.4 (Minimal model for contradiction cost)**.** A minimal (and auditable) model for E_{ctr} consists of fixing a distinguished subset $\mathcal{T}_{\mathrm{ctr}} \subseteq \mathcal{T}$ of “witness teeth” for contradiction and a weight $w : \mathcal{T}_{\mathrm{ctr}} \to [0, \infty)$, and defining

$$E_{\mathrm{ctr}}(K) := \sum_{\tau \in K \cap \mathcal{T}_{\mathrm{ctr}}} w(\tau).$$

Other models (for example, penalties for incompatible pairs $\mu(D_{\tau_1} \cap D_{\tau_2}) = 0$) are treated as **[Model]** variants.

Definition 5.5 (Deterministic coherent-descent operator)**.** Let $\mathbb{K} \subseteq \mathcal{P}(W)$ be the finite space of admissible theories (for example, $\mathbb{K} := \{K \subseteq W : \; V(K) > 0\}$). Fix a total tie-breaking order $\prec$ on $\mathbb{K}$. For $K \in \mathbb{K}$, define the Hamming–1 neighborhood

$$N(K) := \{H \in \mathbb{K} : \; |H \triangle K| = 1\}.$$

Define $U : \mathbb{K} \to \mathbb{K}$ by:

1. if $F(K) \leq F(H)$ for all $H \in N(K)$, then $U(K) := K$;
2. otherwise, let $M(K) := \arg\min_{H \in N(K)} F(H)$ and set $U(K)$ to be the $\prec$-minimum element of $M(K)$.

At the formal level, this operator coincides with the deterministic descent operator U in *coherent flow* (GEGB, Def. 10.6), but it is rewritten here to avoid mandatory external references.

Proposition 5.6 (Deterministic descent)**.** *In the finite case, taking U as in Definition 5.5, we have $F(U(K)) \leq F(K)$, with strict inequality if $U(K) \neq K$. In particular, every orbit reaches a fixed point in finitely many steps.*

Proof. The set of theories is finite and U does not increase F, so there can be no infinite strictly decreasing chains. The Lyapunov property follows directly from the definition of U. □

5.3 Gibbs bridge and KL curvature

Let $p \in \Delta^\circ(\Omega)$ be a probabilistic objective. For $\eta > 0$, define a smooth energy

$$E_K(\omega) := \sum_{\varphi \in K} \mathbb{1}\{\omega \not\models \varphi\}, \qquad p_K(\omega) := \frac{e^{-\eta E_K(\omega)}}{Z_K(\eta)}.$$

This defines a map $\Phi_\eta : K \mapsto p_K$.

Definition 5.7 (Induced KL curvature (dependence on η))**.** For each $\eta > 0$ define the curvature

$$\kappa_{\mathrm{KL},\eta}(p) := \inf_{K \in \mathbb{K}} \mathrm{KL}\big(p \,\|\, \Phi_\eta(K)\big),$$

and its resource-bounded version

$$\kappa_{\mathrm{KL},\eta,R}(p) := \inf\{\mathrm{KL}(p \| \Phi_\eta(K)) : \; \rho(K) \leq R\}.$$

When η is fixed by convention (e.g. in an auditable "temperature" regime), we write $\kappa_{\mathrm{KL}}(p)$ and $\kappa_{\mathrm{KL},R}(p)$ omitting the subscript.

Chapter 6

Stability of invariants

6.1 Comparison distance

Definition 6.1 (d_∞ via reparametrization)**.** For pairs $(\mathcal{E}, o)$ and $(\mathcal{E}', o')$, define

$$d_\infty\big((\mathcal{E}, o), (\mathcal{E}', o')\big) := \inf_{h \in \mathrm{Homeo}(S,S')} \sup_{\sigma \in S} \big|\mathrm{err}(\sigma, o) - \mathrm{err}'(h(\sigma), o')\big|.$$

6.2 Stability of curvature

Theorem 6.2 (Lipschitz stability of κ)**.** *If $d_\infty\big((\mathcal{E}, o), (\mathcal{E}', o')\big) \le \delta$, then*

$$|\kappa(o) - \kappa'(o')| \le \delta.$$

Proof. Let $\varepsilon > 0$. By the definition of d_∞, there exists a homeomorphism $h_\varepsilon \in \mathrm{Homeo}(S, S')$ such that

$$\sup_{\sigma \in S} \big|\mathrm{err}(\sigma, o) - \mathrm{err}'(h_\varepsilon(\sigma), o')\big| \le d_\infty\big((\mathcal{E}, o), (\mathcal{E}', o')\big) + \varepsilon \le \delta + \varepsilon.$$

In particular, for every $\sigma \in S$, $\mathrm{err}(\sigma, o) \ge \mathrm{err}'(h_\varepsilon(\sigma), o') - (\delta + \varepsilon)$. Taking infima over $\sigma \in S$ and using that h_ε is bijective, we obtain

$$\kappa(o) \;\ge\; \kappa'(o') - (\delta + \varepsilon).$$

Applying the same argument to the pair $(\mathcal{E}', o')$ vs. $(\mathcal{E}, o)$ (or equivalently to the inverse h_ε^{-1}), we get $\kappa'(o') \ge \kappa(o) - (\delta + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we conclude $|\kappa(o) - \kappa'(o')| \le \delta$. $\square$

6.3 Stability under metric equivalence

Definition 6.3 (Error induced by an embedding and a metric)**.** Let (X, d) be a metric space and let $e : S \to X$, $j : O \to X$ be embeddings (or measurable maps, depending on context). Define the error induced by d as

$$\mathrm{err}_d(\sigma, o) := d\big(e(\sigma), j(o)\big),$$

and the induced curvature as $\kappa_d(o) := \inf_{\sigma \in S} \mathrm{err}_d(\sigma, o)$. When an abstract err already exists, one can view err_d as a particular *model* of metric interface.

Proposition 6.4 (Sign invariance under equivalent metrics)**.** *Let d and δ be two metrics on the same interface space X and suppose they are uniformly equivalent, i.e., there exist $c_1, c_2 > 0$ such that $c_1\,\delta \le d \le c_2\,\delta$ pointwise on $X \times X$. Then, for each objective o,*

$$c_1\,\kappa_\delta(o) \;\le\; \kappa_d(o) \;\le\; c_2\,\kappa_\delta(o).$$

In particular, $\kappa_d(o) = 0$ if and only if $\kappa_\delta(o) = 0$.

Proof. By Definition 6.3, for every σ we have $\mathrm{err}_d(\sigma, o) = d(e(\sigma), j(o)) \le c_2\,\delta(e(\sigma), j(o)) = c_2\,\mathrm{err}_\delta(\sigma, o)$. Taking the infimum over σ gives $\kappa_d(o) \le c_2\,\kappa_\delta(o)$. The lower inequality follows analogously from $c_1\,\delta \le d$. □

6.4 Stability of persistence and merge trees

In *tame* classes, the stability of barcodes/merge trees under sup perturbations of the landscape is a standard result in TDA. In this manuscript it is used as a class hypothesis (**[Model]**) whenever full formalization inside the same system is required.

Chapter 7

Operational completeness

7.1 Windowed filtered–dynamic equivalence

Let $I = [\varepsilon_0, \varepsilon_1]$.

Definition 7.1 (Filtered–dynamic equivalence)**.** We say that $(\mathcal{E}, o)$ and $(\mathcal{E}', o')$ are equivalent in the physics of closure on I if there exists a homeomorphism $h : S \to S'$ such that:

1. $h(L_\varepsilon(o)) = L'_\varepsilon(o')$ for all $\varepsilon \in I$;
2. $h \circ \Phi_t = \Phi'_t \circ h$ on $f^{-1}(I)$.

7.2 Zipper class: topological and mechanical hypotheses

Assumption 7.2 (Topological zipper (ZT))**.** On I, the sublevels L_ε do not exhibit relevant persistent holes and components only merge as ε increases. More formally, we assume a *tameness* condition for the landscape $f(\sigma) = \mathrm{err}(\sigma, o)$ restricted to $f^{-1}(I)$: there is a finite set of critical values $c_1 < \cdots < c_m$ in I such that, for every $\varepsilon \in I \setminus \{c_i\}$, there exists $\delta > 0$ with

$$L_{\varepsilon'}(o) \;\simeq\; L_\varepsilon(o) \qquad \forall\, \varepsilon' \in (\varepsilon - \delta, \varepsilon + \delta) \cap I,$$

and the only changes when crossing a critical value in I are *merges* of connected components (no births of new components inside I). Consequently, the filtration on I determines a *merge tree* $\mathsf{MT}_I(o)$ (equivalently PD_0 on I).

Assumption 7.3 (Mechanical zipper (ZM))**.** The flow is monotonically closing: $t \mapsto f(\Phi_t(\sigma))$ does not increase for every σ. Moreover, there is no effective internal recirculation inside each sublevel component, in the operational sense that the dynamics collapses (at auditable scale) toward attractors per component.

In order for the "hitting time" to reduce in a well-defined way to the merge tree (cf. Definition 7.5), we assume in particular the following constancy per component: for every $\varepsilon \in I$, every connected component C of $L_\varepsilon(o)$, and every $\varepsilon' < \varepsilon$ with $\varepsilon' \in I$, the function

$$\sigma \;\longmapsto\; \boldsymbol{\tau}_o(\sigma, \varepsilon') := \inf\{t \geq 0 :\; f(\Phi_t(\sigma)) \leq \varepsilon'\}$$

is constant on C. Equivalently: $\boldsymbol{\tau}_o(\cdot, \varepsilon')$ factors through the projection $\pi_I : f^{-1}(I) \to \mathsf{MT}_I(o)$.

7.3 Operational zipper signature

Definition 7.4 (Zipper signature)**.** The operational zipper signature on the window I is defined as

$$\Sigma_{\text{zip}}(\mathcal{E}, o) := \big(\kappa(o),\ \mathsf{MT}_I(o),\ \boldsymbol{\tau}_o\big),$$

where $\boldsymbol{\tau}_o(\sigma, \varepsilon) := \inf\{t \geq 0 :\ f(\Phi_t(\sigma)) \leq \varepsilon\}$, with the convention $\inf \emptyset := +\infty$.

Definition 7.5 (Hitting time reduced to the merge tree)**.** Let $\pi_I : f^{-1}(I) \to \mathsf{MT}_I(o)$ be the canonical projection that maps σ to the merge-tree point representing the connected component of $L_{f(\sigma)}(o)$ that contains σ. Under hypothesis ZM, the hitting time is (operationally) constant on each component, so we define a well-defined function

$$\widehat{\tau}_o(x, \varepsilon) := \boldsymbol{\tau}_o(\sigma, \varepsilon) \quad \text{for any } \sigma \in f^{-1}(I) \text{ with } \pi_I(\sigma) = x.$$

Definition 7.6 (Equality of zipper signatures)**.** We say that $\Sigma_{\text{zip}}(\mathcal{E}, o) = \Sigma_{\text{zip}}(\mathcal{E}', o')$ on the window I if:

1. $\kappa(o) = \kappa'(o')$;
2. there exists a merge-tree isomorphism $\psi : \mathsf{MT}_I(o) \to \mathsf{MT}'_I(o')$ preserving height (ε-values);
3. the reduced hitting times agree under ψ: $\widehat{\tau}'_{o'}(\psi(x), \varepsilon) = \widehat{\tau}_o(x, \varepsilon)$ for all x and all $\varepsilon \in I$.

Remark 7.7. A merge tree (equivalently PD_0) only encodes connectivity information (degree-0 homology). Therefore, without additional hypotheses, it does not determine the full topology of S nor of the sublevels. The "strong" conclusions (homeomorphisms of S or global conjugacy) require an additional realizability/rigidity hypothesis.

Assumption 7.8 (Rigid realizability (RZ))**.** On the window I, assume that the merge tree, together with the projection π_I, is *rigidly realizable* by a state homeomorphism: if $\psi : \mathsf{MT}_I(o) \to \mathsf{MT}'_I(o')$ is a merge-tree isomorphism preserving height, then there exists a homeomorphism $h : S \to S'$ such that $h(L_\varepsilon(o)) = L'_\varepsilon(o')$ for all $\varepsilon \in I$ and $\pi'_I \circ h = \psi \circ \pi_I$ on $f^{-1}(I)$. (This hypothesis holds, for example, in "tree-like" classes where sublevels are obtained only by merges of components and there is no relevant homology above degree 0 on I.)

7.4 Operational completeness theorem

Theorem 7.9 (Zipper operational completeness)**.** *[Model] Under hypotheses Theorems 7.2 and 7.3 (and tameness), if*

$$\Sigma_{\text{zip}}(\mathcal{E}, o) = \Sigma_{\text{zip}}(\mathcal{E}', o')$$

in the sense of Definition 7.6, then:

1. *(Quotient equivalence) the filtered data and hitting times agree under the merge-tree isomorphism ψ;*
2. *(Strong form) if in addition the rigid realizability hypothesis Theorem 7.8 holds, then $(\mathcal{E}, o)$ and $(\mathcal{E}', o')$ are filtered–dynamically equivalent on the window I (Def. 7.1).*

Remark 7.10. The result is *constructive*: equality of merge trees fixes the (branch, height) coordinate and equality of τ fixes the dynamic parametrization within each component.

7.5 Corollary for coherent flow

Corollary 7.11 (CF as a special case)**.** ***[Model]*** *In coherent flow over theories $K \in \mathbb{K}$, if the filtration induced by an objective error $f(K)$ satisfies ZT and the dynamics U (or MH cooling) satisfies ZM on a window I, then the zipper signature classifies the physics of closure on I.*

Chapter 8

Operational completeness with resources

8.1 Bi-filtration (ε, R)

Definition 8.1 (Bi-filtration by error and resource)**.** Given $\rho : S \to [0, \infty)$, define

$$L_{\varepsilon,R} := \{\sigma \in S : \ \mathrm{err}(\sigma, o) \le \varepsilon, \ \rho(\sigma) \le R\}.$$

8.2 Resource-aware zipper signature

Definition 8.2 (Resource-aware zipper signature)**.** Let $\mathcal{R} \subset (0, \infty)$ be a finite set of audited budgets. Define

$$\Sigma^{(2)}_{\mathrm{zip}}(\mathcal{E}, o) := \Big((\kappa_R(o))_{R\in\mathcal{R}}, \ (\mathsf{MT}_{I,R}(o))_{R\in\mathcal{R}}, \ (\boldsymbol{\tau}_{o,R})_{R\in\mathcal{R}} \Big),$$

where $\mathsf{MT}_{I,R}(o)$ is the merge tree of $\varepsilon \mapsto L_{\varepsilon,R}$ restricted to I, and $\boldsymbol{\tau}_{o,R}$ is the hitting time to sublevels within $S_{\le R}$.

8.3 Completeness theorem with resources

Theorem 8.3 (Operational completeness with resources)**.** ***[Model]*** *Suppose that for every $R \in \mathcal{R}$ the zipper hypotheses (ZT, ZM) hold for the filtration $\varepsilon \mapsto L_{\varepsilon,R}$. If*

$$\Sigma^{(2)}_{\mathrm{zip}}(\mathcal{E}, o) = \Sigma^{(2)}_{\mathrm{zip}}(\mathcal{E}', o'),$$

then the systems are filtered–dynamically equivalent within each budget $R \in \mathcal{R}$ on the window I.

8.4 Stability with resources

Theorem 8.4 (Stability of κ_R)**.** *Assume there exists $h : S \to S'$ such that, for all $\sigma \in S$, we have $\rho'(h(\sigma)) \le \rho(\sigma)$, and moreover*

$$\sup_{\rho(\sigma)\le R} \big|\mathrm{err}(\sigma, o) - \mathrm{err}'(h(\sigma), o')\big| \le \delta.$$

Then we have the one-sided bound

$$\kappa'_R(o') \ \leq \ \kappa_R(o) + \delta,$$

equivalently $\kappa_R(o) \geq \kappa'_R(o') - \delta$*. Moreover, if there also exists* $g : S' \to S$ *with* $\rho(g(\sigma')) \leq \rho'(\sigma')$ *and* $\sup_{\rho'(\sigma') \leq R} |\mathrm{err}'(\sigma', o') - \mathrm{err}(g(\sigma'), o)| \leq \delta'$*, then*

$$|\kappa_R(o) - \kappa'_R(o')| \leq \max\{\delta, \delta'\}.$$

Proof. For the first claim, fix $\epsilon > 0$ and choose $\sigma \in S$ with $\rho(\sigma) \leq R$ such that $\mathrm{err}(\sigma, o) \leq \kappa_R(o) + \epsilon$. Then $\rho'(h(\sigma)) \leq R$ and, by the sup bound, $\mathrm{err}'(h(\sigma), o') \leq \mathrm{err}(\sigma, o) + \delta \leq \kappa_R(o) + \epsilon + \delta$. Taking the infimum over $S'_{\leq R}$ yields $\kappa'_R(o') \leq \kappa_R(o) + \epsilon + \delta$; letting $\epsilon \to 0$ gives $\kappa'_R(o') \leq \kappa_R(o) + \delta$.

The second claim follows by applying the first in both directions (with h and with g) and combining the inequalities. □

Remark 8.5. In a previous version, a symmetric bound $|\kappa_R - \kappa'_R| \leq \delta$ was stated assuming only a map $h : S \to S'$. That form is false in general if $h(S_{\leq R})$ is not surjective onto $S'_{\leq R}$: the infimum defining κ'_R could be attained outside the image. The formulation above explicitly separates the one-sided bound (always valid) and symmetry (which requires control in both directions).

Chapter 9

Appendix: auditability, volume estimation, and verifiability

9.1 Windows and windowed volume

In continuous problems, $V_K(\sigma) = \mu(J(\sigma) \cap K)$ can be estimated via Monte Carlo sampling on K when μ is samplable. When $J(\sigma) \cap K$ is definable by constraints (e.g. cGCNF), one obtains unbiased estimators and concentration bounds (Hoeffding/Chernoff) *under* i.i.d. sampling from μ (or μ restricted and normalized to K) and bounded indicator variables (for instance, $\mathbb{1}\{\omega \in J(\sigma)\} \in [0, 1]$).

9.2 Traces and certificates

The zipper signature Σ_{zip} can be certified on finite windows:

- κ via lower bounds + witnesses (or impossibility certificates in decidable classes).
- MT via exploration of the connectivity graph of sublevels (at audited tolerances).
- τ via verifiable traces of the dynamics and evaluation of the error along the trajectory.

Minimal bibliography (template)

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