Formal Moduli Problems

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Formal Moduli Problems

Deformation Context

Definition

A deformation context is a pair $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$, where $\mathcal{A}$ is a presentable ∞-category with finite limits and E is a set of objects of $\mathrm{Stab}(\mathcal{A})$.

1. A morphism in $\mathcal{A}$ is elementary if it is a pull-back of $* \to \Omega^{\infty-n} E_\alpha$.
2. A morphism in $\mathcal{A}$ is small if it can be written as a finite sequence of elementary morphisms.
3. An object A is artinian(small) if the morphism $A \to *$ is small.

Example

If $C = D(k)$, which is already stable, in this context , we can consider the spectrum object $E = (k[n+1])_{n \in \mathbb{Z}}$.

Formal Moduli Problems

Definition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context. A formal moduli problem is a functor $X : \mathcal{A}^{\mathrm{art}} \to \mathcal{S}$ satisfying the following pair of conditions:

1. The space $X(*)$ is contractible.
2. Let σ

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

   be a pullback diagram in $\mathcal{A}^{\mathrm{art}}$ such that ϕ is small, then $X(\sigma)$ is pullback diagram in $\mathcal{S}$.

Example

Let $B \in \mathcal{A}^{\mathrm{art}}$,

$$\mathrm{Spf}(B) : \mathcal{A}^{art} \to \mathcal{S}, \quad A \mapsto \mathrm{Map}_{\mathcal{A}}(B, A)$$

Tangent Complex

Definition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, $Y : \mathcal{A}^{\mathrm{art}} \to \mathcal{S}$ be a formal moduli problem. For each $\alpha \in T$, the tangent complex of Y at α is the following composite functor

$$\mathcal{S}_*^{fin} \xrightarrow{E_\alpha} \mathcal{A}^{art} \xrightarrow{Y} \mathcal{S}.$$

Proposition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be deformation context and let $u : X \to Y$ be a map of formal moduli problems. Suppose that u induces an equivalence of tangent complexes

$$X(E_\alpha) \to Y(E_\alpha)$$

for each $\alpha \in T$. Then u is an equivalence.

Weak Deformation Theory

Definition

A weak deformation theory for a deformation context $(\mathcal{A}, \{E_\alpha\})$ is a functor $\mathcal{D} : \mathcal{A}^{op} \to \mathcal{B}$ satisfying the following conditions

1. The ∞-category is presentable.
2. The functor admits a left adjoint $D' : \mathcal{B} \to \mathcal{A}^{op}$.
3. There exists a full subcategory $\mathcal{B}_0 \subset \mathcal{B}$ satisfying the following conditions:
   - For every $K \in \mathcal{B}_0$, the unit map $K \to \mathcal{D}\mathcal{D}' K$ is an equivalence.
   - $\mathcal{B}_0$ contains the initial object $\emptyset \in \mathcal{B}$.
   - For every $\alpha \in T$ and every $n \geq 1$, there exists an object $K_{\alpha,n} \in \mathcal{B}_0$ and an equivalence $\Omega^{\infty-n} E_\alpha \simeq \mathcal{D}' K_{\alpha,n}$.
   - For every pushout diagram

$$\begin{tikzcd} K_\alpha \arrow[r] \arrow[d] & k \arrow[d] \\ \emptyset \arrow[r] & K' \end{tikzcd}$$

     If K belongs to $\mathcal{B}_0$, them K' also belongs to $\mathcal{B}_0$.

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context, $\mathcal{D} : \mathcal{A}^{op} \to \mathcal{B}$ a weak deformation theory, and $j : \mathcal{B} \to \mathrm{Fun}(\mathcal{B}^{op}, \mathcal{S})$ be the Yoneda embedding. Then

1. For every $B \in \mathcal{B}$, the composition

$$\mathcal{A}^{\mathrm{art}} \subset \mathcal{A} \xrightarrow{\mathcal{D}} \mathcal{B}^{op} \xrightarrow{j(B)} \mathcal{S}$$

   is a formal moduli problem.
2. The construction $B \mapsto (j(B) \circ \mathcal{D})|_{\mathcal{A}^{\mathrm{art}}}$ determine a functor

$$\Psi : \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$$

3. The diagram

$$\begin{tikzcd} \mathcal{A}^{op} \arrow[rr, "\mathrm{Spf}"] \arrow[dr, "\mathcal{D}"'] & & \mathrm{Moduli}^{\mathcal{A}} \\ & \mathcal{B} \arrow[ur, "\Psi"'] & \end{tikzcd}$$

   commutes up to homotopy.

Weak Deformation Theory

Proposition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $\mathcal{D} : \mathcal{A}^{op} \to \mathcal{B}$ a weak deformation theory. $\mathcal{B}_0 \subset \mathcal{B}$ be a full subcategory satisfying the condition above, then

1. $\mathcal{D}$ carries final objects of $\mathcal{A}$ to initial objects of $\mathcal{B}$.
2. If $A = \mathcal{D}'(K)$ for some $K \in \mathcal{B}_0$.Then the unit map $A \to \mathcal{D}'\mathcal{D}(A)$ is an equivalence in $\mathcal{A}$.
3. If $A \in \mathcal{A}^{\mathrm{art}}$, $\mathcal{D}(A) \in \mathcal{B}_0$ and the unit map $A \to \mathcal{D}'\mathcal{D}(A)$ is an equivalence in $\mathcal{A}$.
4. If we have a pullback diagram σ

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

in $\mathcal{A}$ where $A, B \in \mathcal{A}^{\mathrm{art}}$ and the morphism ϕ is small. Then $\mathcal{D}(\sigma)$ is a pushout diagram in $\mathcal{B}$.

Deformation Theory

Lemma

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha\in T})$ be a deformation context and $\mathcal{D} : \mathcal{A}^{op} \to \mathcal{B}$ a weak deformation theory. For each $\alpha \in T$ and each $K \in B$, the composite map

$$\mathcal{S}_*^{fin} \xrightarrow{E_\alpha} \mathcal{A} \xrightarrow{\mathcal{D}} \mathcal{B}^{op} \xrightarrow{j(K)} \mathcal{S}$$

is reduced and excisive and therefore can be identified with a spectrum which we will denote by $e_\alpha(K)$. This determines a functor $e_\alpha : \mathcal{B} \to \mathrm{Sp}$.

Definition

A deformation theory for $(\mathcal{A}, \{E_\alpha\}_{\alpha\in T})$ is a weak deformation theory $\mathcal{D} : \mathcal{A}^{op} \to \mathcal{B}$ satisfying the following condition: For each $\alpha \in T$, the functor $e_\alpha : \mathcal{B} \to \mathrm{Sp}$ preserves small sifted colimits. Morevever, a morphism f in B is an equivalence if and only each $e_\alpha(f)$ is an equivalence of spectra.

Formal Moduli Problems

Main Theorem

Given a deformation context $(\mathcal{A}^{op}, \{E_\alpha\}_{\alpha \in T})$ and a deformation theory (Koszul duality context)

$$\mathfrak{D} : \mathcal{A}^{op} \leftrightarrows \mathcal{B} : \mathfrak{D}',$$

Then the functor

$$\Psi : \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$$

is an equivalence of ∞-category.

Sketch of Proof

Lemma

Let $(A, \{E_\alpha\}_{\alpha\in T})$ be a deformation context and let $\mathcal{D} : \mathcal{A}^{op} \to \mathcal{B}$ be a deformation theory. For every Artinian object $A \in \mathcal{A}^{art}$, $\mathcal{D}(A)$ is a compact object of the ∞-category $\mathcal{B}$.

The functor $\Psi : \mathcal{B} \to \mathrm{Moduli}^A \subset \mathrm{Fun}(\mathcal{A}^{\mathrm{art}}, \mathcal{S})$ is defined by

$$\Psi(K)(A) = \mathrm{Map}_{\mathcal{B}}(\mathcal{D}(A), K)$$

Ψ preserves small limits. And Ψ preserves filtered colimits and is therefore accessible. So by the ∞-categorical adjoint functor theorem, Ψ admits a left adjoint Φ. To prove that Ψ is an equivalence, it will suffice to show that

1. The functor Ψ is conservative.
2. The unit transformation $u : \mathrm{Id} \to \Psi \circ \Phi$ is an equivalence.

Proof of Conservative

Let $f : K \to K'$ in $\mathcal{B}$, such that $\Psi(f)$ is an equivalence.

$$\mathrm{Map}_{\mathcal{B}}(\mathcal{D}(\Omega^{\infty-n}E_\alpha), K) \simeq \Psi(K)\mathcal{D}(\Omega^{\infty-n}E_\alpha)$$
$$\mathrm{Map}_{\mathcal{B}}(\mathcal{D}(\Omega^{\infty-n}E_\alpha), K') \simeq \Psi(K')\mathcal{D}(\Omega^{\infty-n}E_\alpha)$$

It follows that $e_\alpha(K) \simeq e_\alpha(K')$. Since the functors are jointly conservative, we conclude that f is an equivalence.

Proof of Equivalence

To prove that $X \to \Psi \circ \Phi(X)$ is an equivalence, by the proposition of tangent complex. it suffice to show that for each $\alpha \in T$, the induced map

$$\theta : X(E_\alpha) \to (\Psi \circ \Phi)(X)(E_\alpha) \simeq e_\alpha(\Phi X)$$

is equivalence of spectra.
Every formal moduli problems admits a smooth hypercovering by "affine" objects.

Proposition

Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and $X : \mathcal{A}^{\mathrm{art}} \to \mathcal{S}$ be a formal moduli problem.Then there exists a simplicial objects $X_\bullet$ in $\mathrm{Moduli}^{A}_{/X}$ with the following properties:

1. Each X_n is prorepresentable.
2. For each $n \geq 0$, let $M_n(X_\bullet)$ denote the matching object of the simplicial object $X_\bullet$.Then the canonical map $X_n \to M_n(X_\bullet)$ is smooth.

In particular, X is equivalent to the geometric realization $|X_\bullet|$ in $\mathrm{Fun}(A^{\mathrm{art}}, \mathcal{S})$.

$$\theta : X(E_\alpha) \to (\Psi \circ \Phi)(X)(E_\alpha) \simeq e_\alpha(\Phi X)$$

Choose a simplicial object $X_\bullet$ of $\mathrm{Moduli}^{A}_{/X}$ satisfying the above proposition. For each $a \in A^{\mathrm{art}}$.

1. $X_\bullet(a)$ is an hypercovering of $X(A)$, $|X_\bullet(A)| \to X(A)$ is a homotopy equivalence.
2. X is a colimit of the diagram $X_\bullet$ in the ∞-category $\mathrm{Fun}(A^{\mathrm{art}}, \mathcal{S})$.
3. Similarly, $X(E_\alpha)$ is equivalent to the geometric realization $|X_\bullet(E_\alpha)|$.
4. Since Φ preserves small colimits and e_α preserves sifted colimits.

$$e_\alpha(\Phi(X)) \simeq e_\alpha(\Phi|X_\bullet|) \simeq |e_\alpha(\Phi X_\bullet)|.$$

5. It follows that θ is a geometric realization of a simplicial morphism $\theta_\bullet : X_\bullet(E_\alpha) \to e_\alpha(\Phi X_\bullet)$.
6. It suffices to prove that each θ_n is an equivalence.
7. Equivalent to prove that $X_n \to (\Psi \circ \Phi)(X_n)$ is an equivalence.

When X is prorepresentable, since Φ and Ψ both commutes with filtered colimits. We may reduce to the case $X = \mathrm{Spf}(A)$ for some $A \in \mathcal{A}^{art}$. But $\Phi(\mathrm{Spf}(A)) = \mathcal{D}(A)$, it is equivalent to prove that for each $B \in \mathcal{A}^{\mathrm{art}}$, the map

$$\mathrm{Map}_{\mathcal{A}(A,B)} \to \mathrm{Map}_{\mathcal{B}}(\mathcal{D}(B), \mathcal{D}(A)) \simeq \mathrm{Map}_{\mathcal{A}}(A, \mathcal{D}'\mathcal{D}(B)).$$

This a consequence form above proposition.

Applications

Formal Moduli Problems in Different Graded Algebras

1. Cdga_k^{aug} is the ∞-category of augmented commutative differential graded algebras.
2. A morphism in Cdga_k^{aug} is called elementary if it is a pullback of $k \to k \oplus k[n]$ for some $n \geq 1$, where $k \to k \oplus k[n]$ is the square zero extension of k by k[n].
3. A morphism in Cdga_k^{aug} is called small if it is a finite composition of elementary morphisms.
4. An object in Cdga_k^{aug} is called small if the augmentation morphism $\epsilon : A \to k$ is small.

Proposition

An object Cdga_k^{aug} is small if and only if the following conditions hold:

1. $H^n(A) = \{0\}$ for n positive and for n sufficiently negative.
2. All cohomology groups $H^n(A)$ are finite dimensional over k.
3. $H^0(A)$ is a local ring with maximal ideal m, and the morphism $H^0(A)/m \to k$ is an isomorphism.

Definition

A formal moduli problem is an ∞-functor $X : (Cdga)_k^{sm} \to \mathcal{S}$ satisfying the following two conditions:

1. X(k) is contractible.
2. X perserves pull-back along small morphisms.

The second condition means that given a Cartesian diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & A \arrow[d] \\ M \arrow[r] & B \end{tikzcd}$$

in Cdga where $A \to B$ is small, then

$$\begin{tikzcd} X(N) \arrow[r] \arrow[d] & X(A) \arrow[d] \\ X(M) \arrow[r] & X(B) \end{tikzcd}$$

is Cartesian.

The second condition is stable under composition and pullback. We can replace small morphism in the condition by $k \to k \oplus k[n]$.

Theorem

There is a equivalence of ∞-categories $\mathrm{dgLie}_k \to \mathrm{Moduli}_k$.

Chevalley-Eilenberg Complex

For any differential graded Lie algebra $\mathfrak{g}$, we can construct the homological and cohomological Chevalley-Eilenberg complex $CE_\bullet$

1. As vector space $CE_\bullet = S(\mathfrak{g}[1])$ is the graded symmetric algebra of $\mathfrak{g}[1]$. The differential is obtained by extending, as a degree graded coderivation. The complex $CE_\bullet$ is actually counital, conilpotent cocommutative coalgebra object in the category of complexes.
2. $CE^\bullet$ is the linear dual of $CE_\bullet(\mathfrak{g})$, it is an augmented cdga.

$$CE_\bullet(\mathfrak{g}) \simeq k \overset{L}{\otimes}_{U(\mathfrak{g})} k \simeq \mathrm{Tor}_\bullet^{U(\mathfrak{g})}(k, k)$$

$$CE^\bullet(\mathfrak{g}) \simeq R\mathrm{Hom}_{U(\mathfrak{g})}(k, k) \simeq \mathrm{Ext}^\bullet_{U(\mathfrak{g})}(k, k)$$

The Chevalley-Eilenberg construction preserves weak equivalence, thus defining an functor

$$CE^{\bullet} : \mathrm{Lie}_k^{op} \to \mathrm{CAlg}_k^{aug}$$

This functor commutes with small colimits.
The ∞-category Lie_k is presentable, so $CE^{\bullet}$ admits a left adjoint. We denote this adjoint $\mathcal{D}$.

$$\mathcal{D} : \mathrm{CAlg}_k^{aug} \leftrightarrows \mathrm{Lie}_k^{op} : CE^{\bullet}$$

We define an ∞-functor form Lie_k to $\mathrm{Fun}(\mathrm{CAlg}_k^{aug}, \mathcal{S})$

$$\Delta(\mathfrak{g}) = \mathrm{Hom}_{\mathrm{Lie}_k^{op}}(\mathfrak{g}, \mathcal{D}(-)) = \mathrm{Hom}_{\mathrm{Lie}_k}(\mathcal{D}(-), \mathfrak{g})$$

A differential graded Lie algebra L is good if there exists a finite chain $0 = L_0 \to L_1 \to \cdots \to L_n = L$ such that each of these morphism appears in the pushout diagram

$$\begin{tikzcd}
\mathrm{free} k[-n_i - 1] \arrow[r] \arrow[d] & L_i \arrow[d] \\
\{0\} \arrow[r] & L_{i+1}
\end{tikzcd}$$

is

Lemma

If $\mathfrak{g}$ is good, the counit morphism $\mathcal{D}CE^{\bullet}(\mathfrak{g})$ in Lie_k^{op} is an equivalence.

If we have a cartesian diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & k \arrow[d] \\ M \arrow[r] & k \oplus k[n] \end{tikzcd}$$

where N and M are small, then

$$\begin{tikzcd} \mathcal{D}(N) \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \mathcal{D}(M) \arrow[r] & \mathcal{D}(k \oplus k[n]) \end{tikzcd}$$

is also cartesian in Lie_k^{op} and therefore

$$\begin{tikzcd}
\Delta(\mathfrak{g})(N) \arrow[rr] \arrow[d] & & \star \arrow[d] \\
\Delta(\mathfrak{g})(M) \arrow[r] & \Delta(\mathfrak{g})(k \oplus k[n]) &
\end{tikzcd}$$

is also cartesian in sSet. So Δ is an object of FMP_l. Hence Δ factor through the category FMP_k.

Formal Moduli Problem for Associative Algebras

Assume that k is a field, $X : \mathrm{Alg}_k^{art} \to \mathcal{S}$ be a functor. We will say that X is a formal E_1-moduli problem if it satisfies the following conditions:

1. $X(k)$ is contractible.
2. For every pullback diagram σ

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

   in $\mathrm{Alg}_k^{\mathrm{art}}$ where the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. Then $X(\sigma)$ is a pull back square.

Theorem

Let k be a field. Then there is an equivalence of ∞-categories

$$\mathrm{Alg}_k^{aug} \to \mathrm{Moduli}_k^{(1)}.$$

Moduli Problem for E_n algebras

There is a diagram

$$\cdots \to \mathrm{Alg}_k^{(3)} \to \mathrm{Alg}_k^{(2)} \to \mathrm{Alg}_k^{(1)} \simeq \mathrm{Alg}_k,$$

where $\mathrm{Alg}^{(n)}$ denote the ∞-category of E_n algebras over k.

We say that $A \in \mathrm{Alg}_k^{(n)}$ is Artinian if its image in Alg_k is Artinian.

$X : \mathrm{Alg}_k^{(n),art} \to \mathcal{S}$ be a functor. We will say that X is a formal E_n-moduli problem if it satisfies the following conditions:

1. $X(k)$ is contractible.
2. For every pullback diagram σ

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_0 \arrow[d] \\ R_1 \arrow[r] & R_{01} \end{tikzcd}$$

in $\mathrm{Alg}_k^{(n),\mathrm{art}}$ where the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. Then $X(\sigma)$ is a pull back square.

Theorem

Let k be a field. Then there is an equivalence of ∞-categories

$$\mathrm{Alg}_k^{(n),aug} \to \mathrm{Moduli}_k^{(n)}.$$

Morever, the diagram

$$\begin{tikzcd}
\mathrm{Alg}_k^{(n),aug} \arrow[r, "\Psi"] \arrow[d, "m_A"] & \mathrm{Moduli}_k^{n} \arrow[d, "\Sigma^{-n}T"] \\
\mathrm{Mod}_k \arrow[r] & \mathrm{Sp}
\end{tikzcd}$$

commutes up to homotopy.

Deformation as Formal Moduli Problems

Given a smooth scheme Z over k, then formal deformation theory of Z deal with the equivalence classes of Cartesian diagrams

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

where A is a local artinian algebra with residue field k. This define a deformation functor Def_Z from the category of local artinian algebra to sets.

Theorem

When $A = k[t]/t^2$. There is a bijection between the isomorphism class of X over $\mathrm{Spec}(k[t]/t^2)$ and the cohomology $H^1(Z, T_Z)$.

Let $f : X \to S$ be a scheme, and $t : S \to S'$ be a square zero infinitesimal thickening, which is morphism of scheme with the kernel

$$\mathcal{I} = \ker(\mathcal{O}_{S'} \to \mathcal{O}_S)$$

satisfying $\mathcal{I}^2 = 0$. Given a $\mathcal{O}_X$-module $\mathcal{G}$, and a morphism $\mathcal{I} \to \mathcal{G}$ of $\mathcal{O}_X$ module. We ask whether we can find a $\mathcal{M}$ fitting into the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{M}(?) \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[u] & \mathcal{O}_{S'} \arrow[r] \arrow[u] & \mathcal{O}_S \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{1}$$

and what situation the solution is unique?

Theorem

In the situation above we have

1. There is a canonical element $\zeta \in \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to the above diagram.
2. If there exists a solution, then the set of isomorphism classes of solution is principal homogeneous under $\mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.
3. Given a solution X', the set of automorphisms of X' fitting into the diagram is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$

Deformation Theory in the Higher Categorical Case

Let k be field, $\mathcal{C}$ be a stable k-linear ∞-category, and $E \in \mathcal{C}$

$$\mathrm{Def} : \mathrm{Alg}^{art} \to S$$

$$B \mapsto (R\mathrm{Mod}_B(\mathcal{C}) \times_{\mathcal{C}} E)^{\simeq}$$

Theorem

Let k be field, $\mathcal{C}$ be a stable k-linear ∞-category, and $E \in \mathcal{C}$. Let $\Psi : \mathrm{Alg}_k^{aug} \to \mathrm{Moduli}_k^{(1)}$ be the equivalence of ∞-category of formal moduli problem. Then there is an equivalence of formal $\mathbb{E}_1$-moduli problems

$$\mathrm{Def}_E \simeq \Psi(k \oplus \mathrm{End}(E)).$$

PD Operads and Partition Lie Algebra

Partition Lie Algebra

Definition

The Monad $\mathrm{Lie}^{\pi}_{k,\Delta}$ is defined by the following properties

1. If V is a finite dimensional k-vector space, then $\mathrm{Lie}^{\pi}_{k,\Delta}(V)$ is the linear dual of the algebraic cotangent fiber of $k \oplus V^{\vee}$, the trivial square-zero extension of k by $V^{\vee}$.
2. If $V \simeq \mathrm{Tot}(V^{\bullet})$ is represented by a cosimplicial k-vector space $V^{\bullet}$, then

$$\mathrm{Lie}^{\pi}_{k,\Delta}(V) = \bigoplus_{n} \mathrm{Tot}(\widetilde{C}^{\bullet}(\Sigma|\Pi_n|^{\diamond}, k) \otimes (V^{\bullet})^{\otimes n})^{\Sigma_n}.$$

   Here $\widetilde{C}^{\bullet}(\Sigma|\Pi_n|^{\diamond}, k)$ denote the k-valued cosimplices on the space $\Sigma|\Pi_n|^{\diamond}$, the functor $(-)^{\Sigma}$ takes the strict fixed points, and the tensor product is computed in cosimplicial k-modules.
3. The functor $\mathrm{Lie}^{\pi}_{k,\Delta}$ commuted with filtered colimits and geometric realisations.
4. The tangent fiber T_X of any $X \in \mathrm{Moduli}_{k,\Delta}$ has the structure of a $\mathrm{Lie}^{\pi}_{k,\Delta}$-algebra.

Theorem (Brantner-Mathew, 2019)

If k is a field, there is an equivalence of ∞-categories

$$\mathrm{Moduli}_{k,\Delta} \simeq \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k,\Delta}}$$

between formal moduli problems and partition Lie algebra k. k. It sends a formal moduli problem $X \in \mathrm{Moduli}_{k,\Delta}$ to its tangent fibre T_X equipped with a suitable partition Lie algebra structure.