Derived Level Structures

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- Moduli Spaces and Spectral Algebraic Geometry
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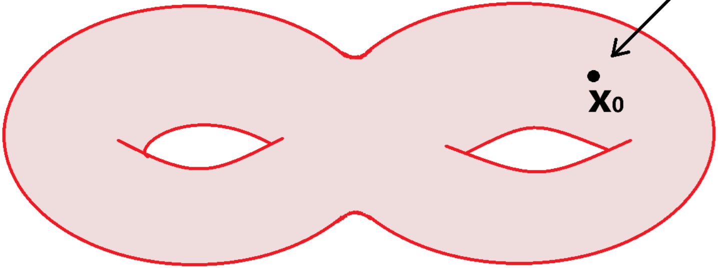
I

Moduli Spaces and Spectral Algebraic Geometry

Moduli Spaces



Each point in this space represents an object in a certain category C

Geometric objects: Vector Spaces, Topological Spaces, Manifolds, Varieties, Schemes, Stacks, Derived Stacks

Examples

1. The Teichmüller space parametrizes complex structures of a surface up to isotopy.
2. Hilbert Schemes, relative Cartier divisors, Chow schemes
3. $\mathcal{M}_{ell}$ of elliptic curves, $\mathcal{M}_g$ of genus g algebraic curves for $g \geq 2$.
4. $\mathcal{M}_{CY}$ of polarized Calabi-Yau varieties, $\mathcal{M}_{Fano}^{K}$ of K-stable Fano varieties.
5. $\mathrm{QCoh}_{r,d}(C)$ and $\mathrm{Bun}_{r,d}(C)$ for C be a smooth, connected, and projective curve over a field k.
6. The moduli space of G-bundles with flat connections over a Riemann surface (phase spaces of G-Chern-Simons theory).
7. The Hitchin moduli space of Higgs bundles over an algebraic curve.
8. The moduli space of monopoles.

Derived Moduli Spaces

- The hidden smoothness principle refers to the conjectural picture envisioned in 1980s by Deligne, Drinfeld, Beilinson, Kontsevich that moduli spaces in algebraic geometry which are often singular, should be just truncations of a moduli spaces in some derived sense.
- These moduli spaces should be smooth, and this property is lost due to truncation. The derived moduli spaces were realized in derived algebraic geometry.

Algebraic Geometry	Derived Algebraic Geometry
Commutative rings	Simplicial commutative rings, $\mathbb{E}_\infty$-rings, CDGA
Schemes, Stacks	Derived Schemes, Derived Stacks
$\mathrm{Hom}(X, Y) \in \mathrm{Set}$	$\mathrm{Map}(\mathcal{X}, \mathcal{Y}) \in \mathcal{S}$
$\mathrm{Sch} \in \mathrm{Cat}$	$d\mathrm{Sch} \in \mathrm{Cat}_\infty$

Higher Algebra

A stable homotopy theory is a presentable symmetric monoidal stable ∞-category $(\mathcal{C}, \otimes, \mathbb{I})$ such that the tensor product commutes with all colimits. (simplicial rings, Sp, $D(R)$)

1. $\mathrm{Map}(X, Y) \in \mathcal{S}$
2. $\mathrm{Ho}(\mathcal{C})$ is a symmetric monoidal triangulated category.
3. There is an equivalence
$$\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega.$$
4. We can define homotopy groups
$$\pi_n E := [\Sigma^n \mathbb{I}, E].$$
5. $\mathrm{CAlg}(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$ consists of those M, such that $\{M(\rho^i) : M(\langle n \rangle) \to M\langle 1 \rangle\}_{1 \leq i \leq n}$ determines an equivalence $M(\langle n \rangle) \to M(\langle 1 \rangle)^n$.
6. We will say that $M \in \mathrm{Mod}_A$ is flat (étale, finite flat) if the following conditions holds
   - $\pi_0 M$ is flat(étale, finite flat) over $\pi_0 A$.
   - $\pi_n A \otimes_{\pi_0 A} \pi_0 M \cong \pi_n M$

Derived Stacks

1. The category of derived affine schemes over k is

$$\mathrm{Sch}^{\mathrm{affine}} := (\mathrm{CAlg}^{\mathrm{cn}})^{\mathrm{op}}.$$

2. The category of derived prestack is

$$\mathrm{PrStk} := \mathrm{Fun}(((\mathrm{Sch})^{\mathrm{affine}})^{\mathrm{op}}, \mathcal{S}).$$

3. The category of derived stack is

$$\mathrm{Shv}_{\mathcal{S}}(\mathrm{Sch}^{\mathrm{affine}})$$

4. $f : \mathcal{X}_1 \to \mathcal{X}_2$ in Stk is k-representable if for any $S \in \mathrm{Sch}^{\mathrm{affine}}$ and $S \to \mathcal{X}_2$, $S \times_{\mathcal{X}_2} \mathcal{X}_1$ is representable by a $(k-1)$-Artin stack.
5. $\mathrm{Stk}^{k-\mathrm{Artin}}$ is the category consists of those $\mathcal{X}$ satisfies
   1. The diagonal map $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is (k-1)-representable.
   2. There exists $\mathcal{Z} \in \mathrm{Stk}^{(k-1)-\mathrm{Artin}}$ and a map $f : \mathcal{Z} \to \mathcal{X}$ which is $(k-1)$-representable, which is smooth and projective.
6. $\mathrm{Stack}^{0-\mathrm{Artin}} \subset \mathrm{Sch} \subset \mathrm{Stack}^{1-\mathrm{Artin}} \subset \cdots$.

Spectral Stacks

Definition

A nonconnective spectral Deligne-Mumford stack is a spectrally ringed ∞-topos $X = (\mathcal{X}, \mathcal{O}_X)$ which locally likes $\mathrm{Sp\acute{e}t} A$, for an E_∞ ring A. We say X is a spectral Deligne-Mumford stack, if all such A is connective.

1. We say $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a n-truncated Deligne-Mumford stack if the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is n-truncated.
2. We say $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford n-stack if $X(R_0)$ is n-truncated for R_0 a commutative ring. A spectral algebraic space is a Deligne-Mumford 0-stack.

Recognition Criterion

Theorem

A spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack if and only if it satisfying following conditions:

1. The underlying ringed topos $(\mathcal{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathcal{X}})$ is a classical Deligne-Mumford stack.
2. The canoncial geometric morphism $\phi_* : \mathcal{X} \to \mathrm{Shv}_{\mathcal{S}}(\mathcal{X}^{\heartsuit})$ is étale.
3. The homotopy group $\pi_n\mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf on $(\mathcal{X}^{\heartsuit}, \pi_0\mathcal{O}_{\mathcal{X}})$.
4. $\mathcal{O}_{\mathcal{X}}$ is a hypercomplete sheaf.

Spectral Varieties and Spectral p-Divisible Groups

Definition

A spectral variety X over an E_∞-ring R is a nonconnective spectral Deligne-Mumford stack X, such that $\tau_{\geq 0}X \to \mathrm{Spet}\tau_{\geq 0}R$ is flat, proper, locally almost of finite presentation, geometrically reduced and geometrically connected.

- Abelian varieties over R : commutative monoidal objects of $\mathrm{Var}(R)$.
- Spectral elliptic curves over R: spectral abelian varieties of dimension 1 over R.

Definition

A height h p-divisible group over A is a functor $X : (\mathrm{Ab}^{p}_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{FFG}(A)$ with the following conditions

1. $X(0)$ is trivial.
2. X send exact sequence to fiber sequence.
3. X(M) has degree $|M|^h$ over A for a finite p-group M.

II

Derived Level Structures

Derived Relative Cartier Divisors

For a spectral Deligne-Mumford stack X/S, a derived relative Cartier divisor is a morphism $D \to X$ such that $D \to X$ is a closed immersion, the ideal sheaf of D is a line bundle over X, and the morphism $D \to S$ is flat, proper and locally almost of finite presentation.

Theorem (Xuecai Ma, 2024)

Suppose that E is a spectral algebraic space over a connective $\mathbb{E}_\infty$-ring R, such that $E \to R$ is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. Then the functor

$$\begin{aligned} \mathrm{CDiv}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{cn} \to \mathcal{S} \\ & R' \mapsto \mathrm{CDiv}(E_{R'}/R') \end{aligned}$$

is representable by a spectral algebraic space which is locally almost of finite presentation over R.

Derived Level Structures of Spectral Elliptic Curves

For A a finite abelian group, a derived A-level structure of a spectral elliptic curve E/R is a relative Cartier divisor $D \to E$ satisfying its restriction to the heart comes from an ordinary A-level structure.

Theorem (Xuecai Ma， 2024)

For a spectral elliptic curve E over a connective $\mathbb{E}_\infty$-ring R, the functor

$$\begin{aligned} \mathrm{Level}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \\ & R' \mapsto \mathrm{Level}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is representable by an affine spectral Deligne-Mumford stack.

Derived Level Structures of Spectral p-Divisible Groups

Let G/R be a height h spectral p-divisible group, a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure

$$\phi : D \to G[p^k]$$

of $G[p^k]$. We let $\mathrm{Level}(k, G/R)$ denote the ∞-groupoid of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structures of G/R.

Theorem (Xuecai Ma， 2024)

Suppose G is a spectral p-divisible group of height h over a connective $\mathbb{E}_\infty$-ring R. Then the functor

$$\mathrm{Level}^k_{G/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}; \quad R' \to \mathrm{Level}(k, G_{R'}/R')$$

is representable by an affine spectral Deligne-Mumford stack $S(k) = \mathrm{Sp\acute{e}t}\,\mathcal{P}^k_{G/R}$.

Formal Moduli Problems

A formal moduli problem is a functor $X : (\mathrm{CAlg})_k^{\mathrm{Artin}} \to \mathcal{S}$ satisfying the following two conditions:

1. X(k) is contractible.
2. X perserves pull-back along small morphisms.

Theorem

- (Pridham-2010, Lurie-2011) If k is a field of characteristic zero, there is an equivalence of ∞-categories

$$\mathrm{dgLie}_k \to \mathrm{Moduli}_k.$$

- (Brantner-Mathew, 2019) If k is a field of positive characteristic, there is an equivalence of ∞-categories

$$\mathrm{Moduli}_{k,\Delta} \simeq \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k,\Delta}}$$

between formal moduli problems and partition Lie algebra k.

Representability Theorem

Spectral Artin Representability Theorem (Lurie， 2004-2018)

Let $M : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor and R is a Noetherian $\mathbb{E}_\infty$-ring such that $\pi_0 R$ is a Grothendieck ring. If $f : M \to \mathrm{Spec} R$ is a natural transformation. If we have

1. $M(R_0)$ is n-truncated for any discrete commutative ring R_0.
2. M is an étale sheaf.
3. M admits a connective cotangent complex L_M.
4. M is nilcomplete, integrable and infinitesimally cohesive.
5. f is locally almost of finite presentation.

Then M is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R.

Cohesive

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is

- **cohesive**, if for every pull-back diagram on the left in $\mathrm{CAlg}^{\mathrm{cn}}$ such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the induced diagram on the right is a pullback square in $\mathcal{S}$.

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ B' \arrow[r, "g"] & B \end{tikzcd} \qquad \begin{tikzcd} X(A') \arrow[r] \arrow[d] & X(A) \arrow[d, "X(f)"] \\ X(B') \arrow[r, "X(g)"] & X(B) \end{tikzcd}$$

- **infinitesimally cohesive**, if for every pull-back diagram on the left in $\mathrm{CAlg}^{\mathrm{cn}}$ such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective whose kernel are nilpotent ideals in $\pi_0 A$ and $\pi_0 B'$, the induced diagram is a pull-back square in $\mathcal{S}$.

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ B' \arrow[r, "g"] & B \end{tikzcd} \qquad \begin{tikzcd} X(A') \arrow[r] \arrow[d] & X(A) \arrow[d, "X(f)"] \\ X(B') \arrow[r, "X(g)"] & X(B) \end{tikzcd}$$

Nilcomplete and Integrable

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is

1. **nilcomplete** if for every connective E_∞-ring R, the canonical map

$$X(R) \to \lim_{\leftarrow} X(\tau_{\leq n} R)$$

   is a homotopy equivalence.

2. **integrable** if for a local Noetherian E_∞-ring which is complete with respect to its maximal ideal $m \subset \pi_0 A$, the inclusion of functors $\mathrm{Spf} A \to \mathrm{Spec} A$ induces a homotopy equivalence

$$X(A) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spec} A, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf} A, X).$$

   It can be prove that this is equivalent to say that the canonical map

$$X(A) \to \lim_{\leftarrow n} X(A/m^n)$$

   is a homotopy equivalence.

Higher Étale Sheaves

Let $\mathcal{C}$ be a ∞-category equipped with a Grothendieck topology $\mathcal{T}$ for the details of Grothendieck topology on an ∞-category), and $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}$ be a functor, we say $\mathcal{F}$ is an $\mathcal{T}$-sheaf if for any object $C \in \mathcal{C}$, and a $\mathcal{T}$ cover sieve $\{U_i \to C\}$, $\mathcal{F}(C)$ is the limit of the diagram

$$\mathrm{Tot} : \Delta^{op} \to \mathcal{S}, \quad [n] \mapsto \coprod \mathcal{F}(U_{i_1, i_n})$$

The following theorem gives a relation between an étale sheaf and its restriction to discrete case.

Proposition

Let $X : \mathrm{CAlg}^{cn} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a contangent complex. Then the following conditions are equivalent:

1. The functor X is a sheaf with respect to the étale topology,
2. The functor $X|_{\mathrm{CAlg}^{\heartsuit}}$ is a sheaf with respect to the étale topology.

Proof

Suppose that we already know that $X|_{\mathrm{CAlg}^{\heartsuit}}$ is a sheaf with respect to the étale topology. We wish to prove that $X : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is an étale sheaf.

- étale is a local condition, so we only need $X|_{\mathrm{CAlg}_R^{ét}}$ is an étale sheaf.
- nilcomplete sheaf, so we only need $X_{\tau_{\leq n}R} : \mathrm{CAlg}^{et}_{\tau_{\leq n}R} \to \mathcal{S}, A \mapsto X(\tau_{\leq n}A)$ is étale.
- The case $n = 0$ follows from the assumption, now assume it is true for $n - 1$.
- R is a square-zero extension of $R' = \tau_{\leq n-1}R$ by $M = \Sigma^n(\pi_n R)$,

$$\begin{array}{ccc} \tau_{\leq n}R & \longrightarrow & \tau_{\leq n-1}R \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}R & \longrightarrow & \tau_{\leq n-1}R \oplus \Sigma M \end{array}$$

We define two functors $Y_{\tau_{\leq n-1}R}, Z_{\tau_{\leq n-1}R} : \mathrm{CAlg}^{ét}_{\tau_{\leq n}R} \to \mathcal{S}$ by the formula

$$Y_{\tau_{\leq n-1}R}(A) = X(A \otimes_{\tau_{\leq n}R} \tau_{\leq n-1}R) = X(\tau_{\leq n-1}A)$$

$$Z_{\tau_{\leq n-1}R}(A) = X(A \otimes_{\tau_{\leq n}R} (\tau_{\leq n-1}R \oplus \Sigma M)) = X(\tau_{\leq n-1}A \oplus (A \otimes_{\tau_{\leq n}R} M)).$$

- By the infinitessimally cohesiveness of X, we then have a pullback diagram of functors

$$\begin{tikzcd} X_{\tau_{\leq n}R} \arrow[r] \arrow[d] & Y_{\tau_{\leq n-1}R} \arrow[d] \\ Y_{\tau_{\leq n-1}R} \arrow[r] & Z_{\tau_{\leq n-1}R} \end{tikzcd}$$

$Y_{\tau_{\leq n-1}R}$ is an étale sheaf, so it is enough to prove that $Z_{\tau_{\leq n-1}R}$ is an étale sheaf.

- We consider the nature projection $Z_{\tau_{\leq n-1}R} \to Y_{\tau_{\leq n-1}R}$, by the fiber principle, it is enough to prove that each fiber of this functor is an étale sheaf. This is equivalent to say that:

  (*) For every étale $\tau_{\leq n}R$-algebra A, and every point $\eta \in X(\tau_{\leq n-1}A)$, the functor $\mathcal{F} : \mathrm{CAlg}_A^{\acute{e}t} \to \mathcal{S}$ defined by

$$B \mapsto \mathrm{fib}(X(\tau_{\leq n-1}B \oplus (A \otimes_{\tau_{\leq n}R} M)) \to X(\tau_{\leq n-1}B))$$

is an étale sheaf. But by the definition of cotangent complex of L_X, we find that

$$\mathcal{F}(B) = \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq n-1}A}}(\eta^* L_X, B \otimes_R M).$$

It then follows from that Hom and $\otimes$ satisfying étale descent.

III

Applications in Chromatic Homotopy Theory

Local-to-Global Principle

F or any chain complex $M \in \mathcal{D}_{\mathbb{Z}}$, we have

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \prod_p M_p^{\wedge} \arrow[d] \\ \mathbb{Q} \otimes M \arrow[r] & \mathbb{Q} \otimes_p \prod_p M_p^{\wedge} \end{tikzcd}$$

which is a homotopy pullback square, where $M_p^{\wedge}$ denote the derived p-completion (p-local and $\mathrm{Ext}^i(\mathbb{Q}, M_p^{\wedge}) = 0$, for $i = 0, 1$).

$\mathcal{D}_{\mathbb{Q}}$: The derived category of $\mathbb{Q}$-vector spaces.

$(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$: The category of derived p-complete complexes of abelian groups.

- $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$ is compactly generated by $\mathbb{Z}/p$.
- The only proper localizing subcategory of$(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$ is (0).
- The irreducible blocks of $\mathcal{D}_{\mathbb{Z}}$: $\{\mathcal{D}_{\mathbb{Q}}$ and $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$ for p prime$\}$.

There is a map $\phi : \mathbb{S} \to H\mathbb{Z}$,

$$\mathrm{Sp} \simeq \mathrm{Mod}_{S^0}(\mathrm{Sp}) \xrightarrow{\phi^*} \mathrm{Mod}_{H\mathbb{Z}}(\mathrm{Sp}) \simeq \mathcal{D}_{\mathbb{Z}}$$

Question: What is the inverse image of the irreducible building block $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$?

Morava E-theories and Morava K-theories

Let G_0 be a formal group over a perfect field k with characteristic p, then a deformation of G_0 to R is a triple (G, i, Ψ), where G is a formal group over R, $i : k \to R/m$, $\Psi : \pi^* G \cong i^* G_0$ is an isomorphism of formal groups over R/m.

Theorem (Lubin-Tate, 1966)

There is a universal formal group G over $R_{LT} = W(k)[[v_1, \cdots, v_n - 1]]$ in the following sense: for every infinitesimal thickening A of k, there is a bijection

$$\mathrm{Hom}_{/k}(R_{LT}, A) \to \mathrm{Def}(A).$$

There is a spectrum $E(n)$ called **Morava E-theory**, whose homotopy group is

$$\pi_* E(n) = W(k)[\![v_1, \cdots, v_{n-1}]\!][\beta^{\pm 1}],$$

This is a even spectrum $K(n)$ called **Morava K-theory**, whose homotopy groups is

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}]/(t_0, t_1, \cdots t_{p^n-2}, t_{p^n}, \cdots) \cong \mathbb{F}_p[v_n^{\pm 1}]$$

Thick Subcategories

We say that $\mathcal{C} \subset \mathrm{Sp}$ is **thick** if it contains 0, closed under fibers and cofibers, and every retract of a spectrum belongs to $\mathcal{C}$ also belongs to $\mathcal{C}$.
Thick subcategories of Sp (Hopkins-Smith， 1988-1996)

$$
\begin{array}{cccc}
\mathcal{P}_{2,\infty} & \mathcal{P}_{3,\infty} & \cdots & \mathcal{P}_{3,\infty}\cdots \\
\vdots & \vdots & & \vdots \\
\mathcal{P}_{2,n+1} & \mathcal{P}_{3,n+1} & \cdots & \mathcal{P}_{p,n+1}\cdots \\
\mathcal{P}_{2,n} & \mathcal{P}_{3,n} & \cdots & \mathcal{P}_{p,n}\cdots \\
\vdots & \vdots & & \vdots \\
\mathcal{P}_{2,2} & \mathcal{P}_{3,2} & \cdots & \mathcal{P}_{p,2}\cdots \\
 & & \mathcal{P}_{0,1} &
\end{array}
$$

- $\mathcal{P}_{0,1} = \ker(SH^c \to SH^c_{\mathbb{Q}} \cong D^b(\mathbb{Q}))$, $\mathcal{P}_{n.\infty} = \ker(SH^c \to SH^c_{(p)})$.
- $\mathcal{P}_{p,n} = \ker(SH^c \to SH^c_{(p)} \to \mathrm{Mod}_{\mathbb{F}_p[v_{n-1}^{\pm 1}]})$ of localization at p and $K_{p,n-1}$.

Spectral Deformations of p-Divisible Groups

G_0 be a p-divisible group over R_0. A deformation of G_0 along $\rho_A : A \to R_0$ is a pair (G, α), where G is a spectral p-divisible group over A and $\alpha : G_0 \simeq \rho_A^* G$.

Theorem (Lurie '18)

There exists a connective E_∞-ring $R_{G_0}^{un}$ with a morphism $\rho : R_{G_0}^{un} \to R_0$, and a deformation G of G_0 with the following properties:

- $R_{G_0}^{un}$ is Noetherian, $\pi_0(\rho) : \pi_0(R_{G_0}^{un}) \to R_0$ is surjective, and $R_{G_0}^{un}$ is complete with respect to the ideal $\ker(\pi_0(\rho))$.
- For other $\rho_A : A \to R_0$. The extension of scalars induces an equivalence of ∞-categories

$$\mathrm{Map}_{\mathrm{CAlg}_{/R_0}}(R_{G_0}^{un}, A) \to \mathrm{Def}_{G_0}(A, \rho_A).$$

We refer to $R_{G_0}^{un}$ as the spectral deformation ring of the p-divisible group G_0.

Orientations

Definition

A preorientation of an 1-dimensional formal group G over a E_∞-ring R is a map

$$e : S^2 \to \Omega^\infty G(\tau_{\geq 0} R)$$

of based spaces, where the based points goes to the identity of the group structure.

- An orientation of a formal group is a preorientation e whose Bott map is an equivalence.

Theorem (Lurie '18)

Then there exists an E_∞-ring $\mathcal{D}_G$ and $e \in \mathrm{Or}(X_{\mathcal{D}_G})$, such that for other $R' \in \mathrm{CAlg}_R$

$$\mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{D}_G, R') \to \mathrm{Or}(G_{R'}).$$

We refer to $\mathcal{D}_X$ as the orientation classifer.

Orientation Deformation Rings

Let G_0 be a nonstationary p-divisible group over a Noetherian $\mathbb{F}_p$-algebra. Let G be the universal deformation of G_0, and $R^{or}_{G_0}$ denote the orientation classifier for the underlying formal group G°, we refer $R^{or}_{G_0}$ as the orientation deformation ring.

Theorem (Lurie , 18)

1. The homotopy groups of $R^{or}_{G_0}$ concentrated in even degrees, and $R_{LT} = R^{cl}_{G_0} \cong \pi_0(R^{or}_{G_0})$.
2. For any adic $\mathbb{E}_\infty$-ring A, the mapping space

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{or}_{G_0}, A) = \mathrm{Def}^{or}_{G_0}(A)$$

classifying triples (G, α, e), where G is a deformation of G_0 to A, α is an equivalence class of G_0-taggings of A , and e is an orientation of the identity component of G°.

Lurie's Theorem

Theorem (Lurie， 2010-2018)

Let M_{pd}^n denote the moduli stack of one dimensional height n p-divisible groups, then there is a sheaf of E_∞-rings $\mathcal{O}^{\mathrm{Top}}$ on the étale site, such that for any

$$E := \mathcal{O}^{\mathrm{Top}}(\mathrm{Spec}R \xrightarrow{G} M_{pd}^n)$$

we have

$$\mathrm{Spf}\pi_0 E^{\mathbb{C}P^\infty} = G^0$$

where G^0 is the formal part of the p-divisible group G.

1. $\hat{G}_0$ is a formal group over k, viewed as an identity component of a connected p-divisible group G_0, then $E_{G_0} = L_{K_n} R_{G_0}^{or}$ is just Morava E-theory.
2. Topological automorphic forms: $\mathcal{O}^{\mathrm{Top}}$ on PEL Shimura stacks (moduli stacks of abelian varieties with the extra structure of polarization, endomorphisms, and level structures) which associated to a rational form of the unitary group $U(1, n-1)$.

Elliptic Cohomology

An elliptic cohomology consists of triples (E, C, ϕ), where E is an even periodic spectrum, C is an elliptic curve C over $\pi_0 E$, $\phi : G_E \cong \hat{C}$.

Theorem(Goerss-Hopkins-Miller-Lurie)

There is a sheaf $\mathcal{O}_{tmf}$ of E_∞-ring spectra over the stack $\overline{\mathcal{M}}_{ell}$ for the étale topology. For any étale morphism $f : \mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$, there is a natural structure of elliptic spectrum $(\mathcal{O}_{tmf}(f), C_f, \phi)$, satisfying $\pi_0 \mathcal{O}_{tmf}(f) = R$, and C_f is a generalized elliptic curve over R classified by f.

- There exists a nonconnective spectral Deligne-Mumford stack $\mathcal{M}^{or}_{ell}$ such that

$$\mathrm{Map}_{\mathrm{SpDM}^{nc}}(\mathrm{Spét} R, \mathcal{M}^{or}_{ell}) \cong \mathrm{Ell}^{or}(R)^{\simeq}$$

- The étale topos $\mathcal{U}$ of M_{ell} is the full subcategory of the underlying topos $\mathcal{X}$ of $\mathcal{M}^{s}_{ell}$. We have a map $\phi : \mathcal{M}^{or}_{ell} \to \mathcal{M}^{s}_{ell}$, we consider the direct image sheaf $\phi_* \mathcal{O}_{\mathcal{M}^{or}_{ell}}$, which is sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. So we get a functor $\mathcal{O}^{Top}_{\mathcal{M}_{ell}} : \mathcal{U}^{op} \to \mathrm{CAlg}$. This construction can be viewed as a construction of elliptic cohomology.

Moduli Stack of Spectral Elliptic Curves with Derived Level Structures

Theorem (Xuecai Ma， 2024)

$$\begin{aligned} \mathcal{M}_{ell}(\mathcal{A}) \quad : \quad & \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}_{ell}(\mathcal{A})(R) = \mathrm{Ell}(\mathcal{A})(R) \end{aligned}$$

is representable by a spectral Deligne-Mumford stack.

Proof: Let $\{R \to U_i\}$ be an étale cover of R, and $U_\bullet$ be the associate check simplicial object. We consider the following diagram

$$\begin{tikzcd} \mathrm{Ell}(\mathcal{A})(R) \arrow[r, "f"] \arrow[d, "p"] & \lim_\Delta \mathrm{Ell}(\mathcal{A})(U_\bullet) \arrow[d, "q"] \\ \mathrm{Ell}(R) \arrow[r, "g"] & \lim_\Delta \mathrm{Ell}(U_\bullet). \end{tikzcd}$$

p is a left fibration between Kan complex, so is a Kan fibration. And the right vertical map is a pointwise Kan fibration. By picking a suitable model for the homotopy limit we may assume that q is a Kan fibration as well. We have g is an equivalence .

To prove that f is a equivalence. We only need to prove that for every $E \in \mathrm{Ell}(R)$, the map

$$p^{-1}E \simeq \mathrm{Level}(\mathcal{A}, E/R) \to \lim_{\Delta} \mathrm{Level}(\mathcal{A}, E \times_R U_\bullet / U_\bullet) \simeq q^{-1}g(E)$$

is an equivalence. But this is true due to étaleness of derived level structures.

Higher Categorical Lubin-Tate Towers

We let $\mathrm{Level}(k, G/R)$ denote the space of derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of a height h spectral p-divisible group G. We consider the following functor

$$\begin{aligned} \mathcal{M}_k \quad &: \quad \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ &\quad\;\; R \to \mathrm{DefLevel}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}(G_0, R, k)$ is the ∞-category whose objects are triples (G, ρ, η)

1. G is a spectral p-divisible group over R.
2. ρ is an equivalence of G_0 taggings of R.
3. $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G.

Theorem (Xuecai Ma， 2024)

The functor $\mathcal{M}_k$ is corepresentable by an $\mathbb{E}_\infty$-ring which is finite over the unoriented spectral deformation ring of G_0.

We let $E_{univ}/R^{\mathrm{un}}_{G_0}$ denote the universal spectral deformation of G_0/R_0. Suppose that G is a spectral deformation G_0 to R. We have the following equivalence

$$\mathrm{Level}(k, G/R) \simeq \mathrm{Level}(k, E_{univ} \times_{R^{un}_{G_0}} R) \simeq \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R),$$

where $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is the universal object of derived level structure functor associated with the p-divisible group $E_{univ}/R^{un}_{G_0}$. Then we consider the following moduli problem

$$\mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}, \quad R \mapsto \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R).$$

The space $\mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R)$ can viewed the ∞-categories of pairs (α, f), where

$$\alpha : R^{un}_{G_0} \to R$$

classifies map a spectral p-divisible group G, which is a deformation of G_0, that is $\alpha = (G, \rho)$, and $f \in \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R) = \mathrm{Level}(k, E_{univ} \times_{R^{un}_{G_0}} R)$ is a derived level structure of G/R. So we have $\mathcal{P}_{E_{univ}/R^{un}_{G_0}}$ is the desired spectrum.

Suppose G_0 is a p-divisible group of height h over a perfect F_p-algebra R_0. We consider the following functor

$$\begin{aligned} \mathcal{M}_k^{\mathrm{or}} \quad : \quad & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{\mathrm{or}}(G_0, R, k) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, k)$ is the ∞-category spanned by those quaternions (G, ρ, e, η)

1. G is a spectral p-divisible group over R.
2. ρ is an equivalence class of G_0-taggings of R.
3. e is an orientation of the identity component of G.
4. $\eta : D \to G$ is a derived $(\mathbb{Z}/p^k\mathbb{Z})^h$-level structure of G/R.

Theorem (Xuecai Ma, 2024)

The functor $\mathcal{M}_k^{or}$ is corepresentable by an $\mathbb{E}_\infty$-ring $\mathcal{JL}_k$, where $\mathcal{JL}_k$ is a finite $R_{G_0}^{or}$-algebra, $R_{G_0}^{or}$ is the orientation deformation ring of G_0.

We call this spectrum $\mathcal{JL}_k$ the Jacquet-Langlands spectrum. It is easy to see that this $\mathcal{JL}_k$ admit an action of $GL_h(\mathbb{Z}/p^k\mathbb{Z}) \times \mathrm{Aut}(G_0)$. When k varies, we have a tower

$$\begin{array}{c} \mathrm{Sp\acute{e}t}\,\mathcal{JL}_k \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathcal{JL}_{k-1} \\ \downarrow \\ \cdots \\ \downarrow \\ \mathrm{Sp\acute{e}t}\,\mathcal{JL}_0. \end{array}$$

We call this tower higher categorical Lubin-Tate tower.

Topological Lifts of Power Operation Rings of Morava E-theories

Theorem (Strickland， 1997-1998)

- Deformations of Frobenius is equivalent to deformations with fixed order subgroups.
- There exists a universal object $A[r]$ of height r deformations of Frobenius.
- $A[r] \cong E^0(B\Sigma_{p^r})/I$, where I is the transfer ideal.

For the power operation $R^k(X) \to R^k(X \times B\Sigma_m)$. When $x = *$, we have

$$\pi_0 R \to E^0(B\Sigma_{p^r})/I \otimes \pi_0 R = A[r] \otimes \pi_0 R$$

This make $\pi_0 R$ becomes a Γ-module, where Γ is the dual of $A[r]$ (Rezk, 11).

$A[r]$ corepresents the following moduli problem

$$\begin{aligned} \mathcal{M}_{0,r} \quad : \quad & \mathrm{CAlg}_k^{\heartsuit} \to \mathcal{S} \\ & R \to \mathrm{Def}(G_0, R, p^r) \end{aligned}$$

where $\mathrm{Def}(G_0, R, p^r)$ consists of pairs (G, H), where G is an deformation G_0 to R, and H is a rank p^r subgroup of G.

Proposition (Xuecai Ma , 2024)

For every integer $r \geq 1$, there exists a E_∞-ring $E_{n,r}$, such that $\pi_0 E_{n,r} = A_r$.

For the formal group G_0 over a field k of characteristic p. We just consider the functor $\mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S}$ by sending an E_∞-ring R to quadruples (G, ρ, e, η), where (G, ρ) is spectral deformation of G_0 to R. e is an orientation of G°, the identity component G, and $\eta \in \mathrm{Level}_0(k, G/R)$ is a derived level structure. Using the same argument in full level structure and the fact $\mathrm{Level}_{G/R}^{0,k}$ is representable.

Further Problems

1. Computations of homotopy groups of higher categorical Lubin-Tate towers.
2. Computations of $\mathcal{JL}$-theory on certain spaces, such as $B\Sigma_n$.
3. The relation between $\mathcal{JL}$ and the choosing moduli problems.
4. Representation theory in Spectral Algebraic Geometry.
5. Let $\mathcal{JL}$ be the ℓ-adic complete Jacquet-Langlands spectrum, and X be a spectrum with an action of $\mathbb{G}_n$. We conjecture that the function spectrum $F(X, \mathcal{JL})$ admits an action of $GL_n(\mathbb{Z}_p)$.
6. Find certain resolutions by using the above correspondence.

Thanks for Listening !