# Formal Moduli Problems

Xuecai Ma

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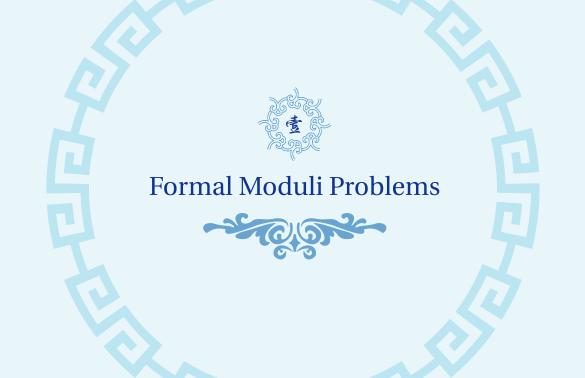


Formal Moduli Problems

Applications

PD Operads and Partition Lie Algebra





## **Deformation Context**

#### Definition

A deformation context is a pair  $(A, \{E_{\alpha}\}_{{\alpha} \in T})$ , where A is a presentable  $\infty$ -category with finite limits and E is a set of objects of  $\operatorname{Stab}(A)$ .

- 1. A morphism in  $\mathcal A$  is elementary if it is a pull-back of  $* \to \Omega^{\infty n} E_{\alpha}$  .
- 2. A morphism in A is small if it can be written as a finite sequence of elementary morphisms.
- 3. An object *A* is artinian(small) if the morphism  $A \rightarrow *$  is small.

## Example

If C=D(k), which is already stable, in this context , we can consider the spectrum object  $E=(k[n+1])_{n\in\mathbb{Z}}$ .

## Formal Moduli Problems

### Definition

Let  $(A, \{E_{\alpha}\}_{{\alpha} \in T})$  be a deformation context. A formal moduli problem is a functor  $X : A^{\operatorname{art}} \to S$  satisfying the following pair of conditions:

- 1. The space X(\*) is contractible.
- 2. Let  $\sigma$

$$A' \longrightarrow B'$$

$$\downarrow \phi$$

$$A \longrightarrow B$$

be a pullback diagram in  $\mathcal{A}^{\mathrm{art}}$  such that  $\phi$  is small, then  $X(\sigma)$  is pullback diagram in  $\mathcal{S}$ .

## Example

Let  $B \in \mathcal{A}^{\operatorname{art}}$ ,

$$\operatorname{Spf}(B): \mathcal{A}^{art} \to \mathcal{S}, \quad A \mapsto \operatorname{Map}_{\mathcal{A}}(B, A)$$



## **Tangent Complex**

### Definition

Let  $(\mathcal{A}, \{E_{\alpha}\}_{\alpha \in T})$  be a deformation context,  $Y: \mathcal{A}^{\operatorname{art}} \to \mathcal{S}$  be a formal moduli problem. For each  $\alpha \in T$ , the tangent complex of Y at  $\alpha$  is the following composite functor

$$\mathcal{S}_*^{fin} \stackrel{E_{\alpha}}{\to} \mathcal{A}^{art} \stackrel{Y}{ o} \mathcal{S}.$$

## Proposition

Let  $(A, \{E_{\alpha}\}_{{\alpha} \in T})$  be deformation context and let  $u: X \to Y$  be a map of formal moduli problems. Suppose that u induces an equivalence of tangent complexes

$$X(E_{\alpha}) \to Y(E_{\alpha})$$

for each  $\alpha \in T$ . Then u is an equivalence.



# Weak Deformation Theory

#### Definition

A weak deformation theory for a deformation context  $(\mathcal{A}, \{E_{\alpha}\})$  is a functor  $\mathcal{D}: \mathcal{A}^{op} \to \mathcal{B}$  satisfying the following conditions

- 1. The  $\infty$ -category is presentable.
- 2. The functor admits a left adjoint  $D': \mathcal{B} \to \mathcal{A}^{op}$ .
- 3. There exists a full subcategory  $\mathcal{B}_0 \subset \mathcal{B}$  satisfying the following conditions:
  - **□** For every  $K \in \mathcal{B}_0$ , the unit map  $K \to \mathcal{DD}'K$  is an equivalence.
  - $\blacksquare$   $\mathcal{B}_0$  contains the initial object  $\emptyset \in \mathcal{B}$ .
  - For every  $\alpha \in T$  and every  $n \geq 1$ , there exists an object  $K_{\alpha,n} \in \mathcal{B}_0$  and an equivalence  $\Omega^{\infty n} E_{\alpha} \simeq \mathcal{D}' K_{\alpha,n}$ .
  - For every pushout diagram



If K belongs to  $\mathcal{B}_0$ , them K' also belongs to  $\mathcal{B}_0$ .



Let  $(A, \{E_{\alpha}\}_{{\alpha} \in T})$  be a deformation context,  $\mathcal{D} : A^{op} \to \mathcal{B}$  a weak deformation theory, and  $j : \mathcal{B} \to \operatorname{Fun}(\mathcal{B}^{op}, \mathcal{S})$  be the Yoneda embedding. Then

1. For every  $B \in \mathcal{B}$ , the composition

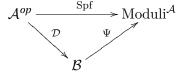
$$\mathcal{A}^{\operatorname{art}} \subset \mathcal{A} \stackrel{\mathcal{D}}{
ightarrow} \mathcal{B}^{op} \stackrel{j(B)}{
ightarrow} \mathcal{S}$$

is a formal moduli problem.

2. The construction  $B\mapsto (j(B)\circ \mathcal{D})|_{\mathcal{A}^{\mathrm{art}}}$  determine a functor

$$\Psi: \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$$

3. The diagram



commutes up to homotopy.



## Weak Deformation Theory

## Proposition

Let  $(A, \{E_{\alpha}\}_{{\alpha} \in T})$  be a deformation context and  $\mathcal{D}: \mathcal{A}^{op} \to \mathcal{B}$  a weak deformation theory.  $\mathcal{B}_0 \subset \mathcal{B}$  be a full subcategory satisfying the condition above, then

- 1.  $\mathcal{D}$  carries final objects of  $\mathcal{A}$  to initial objects of  $\mathcal{B}$ .
- 2. If  $A = \mathcal{D}'(K)$  for some  $K \in \mathcal{B}_0$ . Then the unit map  $A \to \mathcal{D}'\mathcal{D}(A)$  is an equivalence in  $\mathcal{A}$ .
- 3. If  $A \in \mathcal{A}^{\mathrm{art}}$ ,  $\mathcal{D}(A) \in \mathcal{B}_0$  and the unit map  $A \to \mathcal{D}'\mathcal{D}(A)$  is an equivalence in  $\mathcal{A}$ .
- 4. If we have a pullback diagram  $\sigma$



in  $\mathcal A$  where  $A,B\in\mathcal A^{\mathrm{art}}$  and the morphism  $\phi$  is small. Then  $\mathcal D(\sigma)$  is a pushout diagram in  $\mathcal B$ .



# **Deformation Theory**

#### Lemma

Let  $(A, \{E_{\alpha}\}_{\alpha \in T})$  be a deformation context and  $\mathcal{D}: A^{op} \to \mathcal{B}$  a weak deformation theory. For each  $\alpha \in T$  and each  $K \in \mathcal{B}$ , the composite map

$$\mathcal{S}_*^{\mathit{fin}} \overset{E_{lpha}}{ o} \mathcal{A} \overset{\mathcal{D}}{ o} \mathcal{B}^{\mathit{op}} \overset{\mathit{j}(\mathit{K})}{ o} \mathcal{S}$$

is reduced and excisive and therefore can be identified with a spectrum which we will denote by  $e_{\alpha}(K)$ . This determines a functor  $e_{\alpha}: \mathcal{B} \to \operatorname{Sp}$ .

### Definition

A deformation theory for  $(\mathcal{A}, \{E_{\alpha}\}_{\alpha \in T})$  is a weak deformation theory  $\mathcal{D}: \mathcal{A}^{op} \to \mathcal{B}$  satisfying the following condition: For each  $\alpha \in T$ , the functor  $e_{\alpha}: \mathcal{B} \to \operatorname{Sp}$  preserves small sifted colimits. Morevever, a morphism f in B is an equivalence if and only each  $e_{\alpha}(f)$  is an equivalence of spectra.

## Formal Moduli Problems

#### Main Theorem

Given a deformation context  $(\mathcal{A}^{op}, \{E_{\alpha}\}_{\alpha \in T})$  and a deformation theory (Koszul duality context )

$$\mathfrak{D}:\mathcal{A}^{op}\leftrightarrows\mathcal{B}:\mathfrak{D}',$$

Then the functor

$$\Psi: \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$$

is an equivalence of  $\infty$ -category.



## Sketch of Proof

#### Lemma

Let  $(A, \{E_{\alpha}\}_{\alpha \in T})$  be a deformation context and let  $\mathcal{D}: \mathcal{A}^{op} \to \mathcal{B}$  be a deformation theory. For every Artinian object  $A \in \mathcal{A}^{art}$ ,  $\mathcal{D}(A)$  is a compact object of the  $\infty$ -category  $\mathcal{B}$ .

The functor  $\Psi : \mathcal{B} \to \mathrm{Moduli}^A \subset \mathrm{Fun}(\mathcal{A}^{\mathrm{art}}, \mathcal{S})$  is defined by

$$\Psi(K)(A) = \operatorname{Map}_{\mathcal{B}}(\mathcal{D}(A), K)$$

 $\Psi$  preserves small limits. And  $\Psi$  preserves filtered colimits and is therefore accessible. So by the  $\infty$ -categorical adjoint functor theorem,  $\Psi$  admits a left adjoint  $\Phi$ . To prove that  $\Psi$  is an equivalence, it will suffice to show that

- 1. The functor  $\Psi$  is conservative.
- 2. The unit transformation  $u : \mathrm{Id} \to \Psi \circ \Phi$  is an equivalence.



## **Proof of Conservative**

Let  $f: K \to K'$  in  $\mathcal{B}$ , such that  $\Psi(f)$  is an equivalence.

$$\operatorname{Map}_{\mathcal{B}}(\mathcal{D}(\Omega^{\infty-n}E_{\alpha}), K) \simeq \Psi(K)\mathcal{D}(\Omega^{\infty-n}E_{\alpha})$$
$$\operatorname{Map}_{\mathcal{B}}(\mathcal{D}(\Omega^{\infty-n}E_{\alpha}), K') \simeq \Psi(K')\mathcal{D}(\Omega^{\infty-n}E_{\alpha})$$

It follows that  $e_{\alpha}(K) \simeq e_{\alpha}(K')$ . Since the functors are jointly conservative, we conclude that f is an equivalence.



## Proof of Equivalence

To prove that  $X \to \Psi \circ \Phi(X)$  is an equivalence, by the proposition of tangent complex. it suffice to show that for each  $\alpha \in T$ , the induced map

$$\theta: X(E_{\alpha}) \to (\Psi \circ \Phi)(X)(E_{\alpha}) \simeq e_{\alpha}(\Phi X)$$

is equivalence of spectra.

Every formal moduli problems admits a smooth hypercovering by "affine" objects.

### Proposition

Let  $(\mathcal{A}, \{E_{\alpha}\}_{\alpha \in T})$  be a deformation context and  $X : \mathcal{A}^{\operatorname{art}} \to \mathcal{S}$  be a formal moduli problem. Then there exists a simplicial objects  $X_{\bullet}$  in  $\operatorname{Moduli}_{/X}^{A}$  with the following properties:

- 1. Each  $X_n$  is prorepresentable.
- 2. For each  $n \geq 0$ , let  $M_n(X_{\bullet})$  denote the matching object of the simplicial object  $X_{\bullet}$ . Then the canonical map  $X_n \to M_n(X_{\bullet})$  is smooth.

In particular, X is equivalent to the geometric realization  $|X_{\bullet}|$  in  $\operatorname{Fun}(A^{\operatorname{art}},\mathcal{S})$ .

$$\theta: X(E_{\alpha}) \to (\Psi \circ \Phi)(X)(E_{\alpha}) \simeq e_{\alpha}(\Phi X)$$

Choose a simplicial object  $X_{\bullet}$  of  $\operatorname{Moduli}_{/X}^{A}$  satisfying the above proposition. For each  $a \in A^{\operatorname{art}}$ .

- 1.  $X_{\bullet}(a)$  is an hypercovering of X(A),  $|X_{\bullet}(A)| \to X(A)$  is a homotopy equivalence.
- 2. X is a colimit of the diagram  $X_{\bullet}$  in the  $\infty$ -category  $\operatorname{Fun}(A^{\operatorname{art}}, \mathcal{S})$ .
- 3. Similarly,  $X(E_{\alpha})$  is equivalent to the geometric realization  $|X_{\bullet}(E_{\alpha})|$ .
- 4. Since  $\Phi$  preserves small colimits and  $e_{\alpha}$  preserves sifted colimits.

$$e_{\alpha}(\Phi(X)) \simeq e_{\alpha}(\Phi|X_{\bullet}|) \simeq |e_{\alpha}(\Phi X_{\bullet})|.$$

- 5. It follows that  $\theta$  is a geometric realization of a simplicial morphism  $\theta_{\bullet}: X_{\bullet}(E_{\alpha}) \to e_{\alpha}(\Phi X_{\bullet}).$
- 6. It suffices to prove that each  $\theta_n$  is an equivalence.
- 7. Equivalent to prove that  $X_n \to (\Psi \circ \Phi)(X_n)$  is an equivalence.



When X is prorepresentable, since  $\Phi$  and  $\Psi$  both commutes with filtered colimits. We may reduce to the case  $X = \operatorname{Spf}(A)$  for some  $A \in \mathcal{A}^{art}$ . But  $\Phi(\operatorname{Spf}(A)) = \mathcal{D}(A)$ , it is equivalent to prove that for each  $B \in \mathcal{A}^{art}$ , the map

$$\operatorname{Map}_{\mathcal{A}(A,B)} \to \operatorname{Map}_{\mathcal{B}}(\mathcal{D}(B),\mathcal{D}(A)) \simeq \operatorname{Map}_{\mathcal{A}}(A,\mathcal{D}'\mathcal{D}(B)).$$

This a consequence form above proposition.





# Formal Moduli Problems in Different Graded Algebras

- 1.  $\mathrm{Cdga}_k^{aug}$  is the  $\infty$ -category of augmented commutative differential graded algebras.
- 2. A morphism in  $\operatorname{Cdga}_k^{aug}$  is called elementary if it is a pullback of  $k \to k \oplus k[n]$  for some  $n \ge 1$ , where  $k \to k \oplus k[n]$  is the square zero extension of k by k[n].
- 3. A morphism in  $\mathrm{Cdga}_k^{aug}$  is called small if it is a finite composition of elementary morphisms.
- 4. An object in  $\operatorname{Cdga}_k^{aug}$  is called small if the augmentation morphism  $\epsilon:A\to k$  is small.

## Proposition

An object  $\operatorname{Cdga}_k^{aug}$  is small if and only if the following conditions hold:

- 1.  $H^n(A) = \{0\}$  for n positive and for n sufficiently negative.
- 2. All cohomology groups  $H^n(A)$  are finite dimensional over k.
- 3.  $H^0(A)$  is a local ring with maximal ideal m, and the morphism  $H^0(A)/m \to k$  is an isomorphism.



### Definition

A formal moduli problem is an  $\infty$ -functor  $X:(Cdga)_k^{sm}\to \mathcal{S}$  satisfying the following two conditions:

- 1. X(k) is contractible.
- 2. X perserves pull-back along small morphisms.

The second condition means that given a Cartesian diagram

$$\begin{array}{ccc}
N \longrightarrow A \\
\downarrow & & \downarrow \\
M \longrightarrow B
\end{array}$$

in Cdga where  $A \rightarrow B$  is small, then

$$\begin{array}{ccc} X(N) & \longrightarrow X(A) \\ & & \downarrow \\ X(M) & \longrightarrow X(B) \end{array}$$



is Cartesian.

The second condition is stable under composition and pullback. We can replace small morphism in the condition by  $k \to k \oplus k[n]$ .

### Theorem

There is a equivalence of  $\infty$ -categories  $\operatorname{dgLie}_k \to \operatorname{Moduli}_k$ .



## **Chevalley-Eilenberg Complex**

For any differential graded Lie algebra  $\mathfrak{g}$ , we can construct the homological and cohomological Chevalley-Eilenberg complex  $CE_{\bullet}$ 

- 1. As vector space  $CE_{\bullet} = S(\mathfrak{g}[1])$  is the graded symmetric algebra of  $\mathfrak{g}[1]$ . The differential is obtained by extending, as a degree graded coderivation. The complex  $CE_{\bullet}$  is actually counital, conilpotent cocommutative coalgebra object in the category of complexes.
- 2.  $CE^{\bullet}$  is the linear dual of  $CE_{\bullet}(\mathfrak{g})$ , it is an augmented cdga.

$$CE_{ullet}(\mathfrak{g}) \simeq k \overset{L}{\otimes}_{U(\mathfrak{g})} k \simeq \operatorname{Tor}_{ullet}^{U(\mathfrak{g})}(k,k)$$

$$CE^{ullet}(\mathfrak{g}) \simeq R\operatorname{Hom}_{U(\mathfrak{g})}(k,k) \simeq \operatorname{Ext}_{U(\mathfrak{g})}^{ullet}(k,k)$$



The Chevalley-Eilenberg construction preserves weak equivalence, thus defining an functor

$$CE^{\bullet}: \operatorname{Lie}_{k}^{op} \to \operatorname{CAlg}_{k}^{aug}$$

This functor commutes with small colimits.

The  $\infty$ -category  $\text{Lie}_k$  is presentable, so  $\text{\it CE}^{\bullet}$  admits a left adjoint. We denote this adjoint  $\mathcal{D}$ .

$$\mathcal{D}: \mathrm{CAlg}_k^{aug} \leftrightarrows \mathrm{Lie}_k^{op}: \mathit{CE}^{\bullet}$$

We define an  $\infty$ -functor form  $\mathrm{Lie}_k$  to  $\mathrm{Fun}(\mathrm{CAlg}_k^{aug},\mathcal{S})$ 

$$\Delta(\mathfrak{g}) = \operatorname{Hom}_{\operatorname{Lie}_{\iota}^{op}}(\mathfrak{g}, \mathcal{D}(-)) = \operatorname{Hom}_{\operatorname{Lie}_{\iota}}(\mathcal{D}(-), \mathfrak{g})$$



A differential graded Lie algebra L is good if there exists a finite chain  $0 = L_0 \to L_1 \to \cdots \to L_n = L$  such that each of these morphism appears in the pushout diagram

$$\operatorname{free} k[-n_i - 1] \longrightarrow L_i \\
\downarrow \\
\{0\} \longrightarrow L_{i+1}$$

is

#### Lemma

If  $\mathfrak g$  is good, the counit morphism  $\mathcal DCE^{\bullet}(\mathfrak g)$  in  $\mathrm{Lie}_k^{op}$  is an equivalence.



If we have a cartesian diagram

$$\begin{array}{ccc} N & \longrightarrow & k \\ \downarrow & & \downarrow \\ M & \longrightarrow & k \oplus k[n] \end{array}$$

where N and M are small, then

$$\mathcal{D}(N) \longrightarrow \{0\}$$

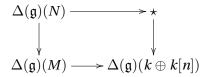
$$\downarrow \qquad \qquad \downarrow$$

$$\mathcal{D}(M) \longrightarrow \mathcal{D}(k \oplus k[n])$$

is also cartesian in  $\operatorname{Lie}_{k}^{op}$  and therefore







is also cartesian in sSet. So  $\Delta$  is an object of FMP  $_l$ . Hence  $\Delta$  factor through the category FMP  $_k$ .



# Formal Moduli Problem for Associative Algebras

Assume that k is a field,  $X : Alg_k^{art} \to \mathcal{S}$  be a functor. We will say that X is a formal  $E_1$ -moduli problem if it satisfies the following conditions:

- 1. X(k) is contractible.
- 2. For every pullback diagram  $\sigma$



in  $\mathrm{Alg}_k^{\mathrm{art}}$  where the underlying maps  $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$  are surjective. Then  $X(\sigma)$  is a pull back square.

### Theorem

Let k be a field. Then there is an equivalence of  $\infty$ -categories

$$Alg_k^{aug} \to Moduli_k^{(1)}$$
.



## Moduli Problem for $\mathbb{E}_n$ algebras

There is a diagram

$$\cdots \to \operatorname{Alg}_k^{(3)} \to \operatorname{Alg}_k^{(2)} \to \operatorname{Alg}_k^{(1)} \simeq \operatorname{Alg}_k,$$

where  $Alg^{(n)}$  denote the  $\infty$ -category of  $E_n$  algebras over k.

We say that  $A \in Alg_k^{(n)}$  is Artinian if its image in  $Alg_k$  is Artinian.

 $X: \mathrm{Alg}_k^{(n),art} \to \mathcal{S}$  be a functor. We will say that X is a formal  $\mathrm{E}_n$ -moduli problem if it satisfies the following conditions:

- 1. X(k) is contractible.
- 2. For every pullback diagram  $\sigma$



in  $\mathrm{Alg}_k^{(n),\mathrm{art}}$  where the underlying maps  $\pi_0R_0\to\pi_0R_{01}\leftarrow\pi_0R_1$  are surjective.

Then  $X(\sigma)$  is a pull back square.

#### Theorem

Let k be a field. Then there is an equivalence of  $\infty$ -categories

$$\operatorname{Alg}_k^{(n),aug} \to \operatorname{Moduli}_k^{(n)}.$$

Morever, the diagram

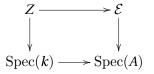
$$\begin{array}{ccc} \operatorname{Alg}_k^{(n),aug} & \xrightarrow{\Psi} \operatorname{Moduli}_k^n \\ & & \downarrow^{m_A} & & \downarrow^{\Sigma^{-n}T} \\ \operatorname{Mod}_k & \longrightarrow \operatorname{Sp} \end{array}$$

commutes up to homotopy.



## Deformation as Formal Moduli Problems

Given a smooth scheme Z over k, then formal deformation theory of Z deal with the equivalence classes of Cartesian diagrams



where A is a local artinian algebra with residue field k. This define a deformation functor  $\mathrm{Def}_Z$  from the category of local artinian algebra to sets.

### Theorem

When  $A = k[t]/t^2$ . There is a bijection between the isomorphism class of X over  $\operatorname{Spec}(k[t]/t^2)$  and the cohomology  $H^1(Z, T_Z)$ .

Let  $f: X \to S$  be a scheme, and  $t: S \to S'$  be a square zero infinitesimal thickening, which is morphism of scheme with the kernel

$$\mathcal{I} = \ker(\mathcal{O}_{S'} \to \mathcal{O}_S)$$

satisfying  $\mathcal{I}^2 = 0$ . Given a  $\mathcal{O}_X$ -module  $\mathcal{G}$ , and a morphism  $\mathcal{I} \to \mathcal{G}$  of  $\mathcal{O}_X$  module. We ask whether we can find a  $\mathcal{M}$  fitting into the following diagram

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{M}(?) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{S'} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

$$(1)$$

and what situation the solution is unique?



#### Theorem

In the situation above we have

- 1. There is a canonical element  $\zeta \in \operatorname{Ext}^2_{\mathcal{O}_X}(L_{X/S},\mathcal{G})$  whose vanishing is a sufficient and necessary condition for the existence of a solution to the above diagram.
- 2. If there exists a solution, then the set of isomorphism classes of solution is principal homogeneous under  $\operatorname{Ext}^1_{\mathcal{O}_Y}(L_{X/S},\mathcal{G})$ .
- 3. Given a solution X', the set of automorphisms of X' fitting into the diagram is canonically isomorphic to  $\operatorname{Ext}^0_{\mathcal{O}_X}(L_{X/S},\mathcal{G})$



# Deformation Theory in the Higher Categorical Case

Let k be field, C be a stable k-linear  $\infty$ -category, and  $E \in C$ 

$$Def : Alg^{art} \to S$$

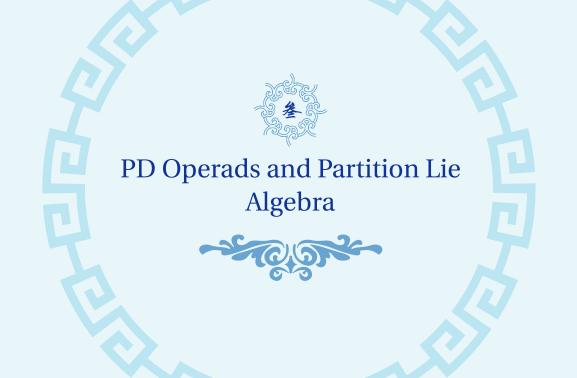
$$B \mapsto (R \operatorname{Mod}_B(\mathcal{C}) \times_{\mathcal{C}} E)^{\simeq}$$

### Theorem

Let k be field,  $\mathcal{C}$  be a stable k-linear  $\infty$ -category, and  $E \in \mathcal{C}$ . Let  $\Psi : \mathrm{Alg}_k^{aug} \to \mathrm{Moduli}_k^{(1)}$  be the equivalence of  $\infty$ -category of formal moduli problem. Then there is an equivalence of formal  $\mathbb{E}_1$ -moduli problems

$$\mathrm{Def}_E \simeq \Psi(k \oplus \mathrm{End}(E)).$$





## Partition Lie Algebra

### Definition

The Monad  $\operatorname{Lie}_{k,\Delta}^{\pi}$  is defined by the following properties

- 1. If V is a finite dimensional k-vector space, then  $\operatorname{Lie}_{k,\Delta}^\pi(V)$  is the linear dual of the algebraic cotangent fiber of  $k \oplus V^\vee$ , the trivial square-zero extension of k by  $V^\vee$ .
- 2. If  $V \simeq \operatorname{Tot}(V^{\bullet})$  is represented by a cosimplicial k-vector space  $V^{\bullet}$ , then

$$\mathrm{Lie}_{k,\Delta}^{\pi}(V) = \underset{n}{\oplus} \mathrm{Tot}(\widetilde{C}^{\bullet}(\Sigma|\Pi_{n}|^{\diamond},k) \otimes (V^{\bullet})^{\otimes n})^{\Sigma_{n}}.$$

Here  $\widetilde{C}^{\bullet}(\Sigma|\Pi_n|^{\diamond},k)$  denote the k-valued cosimplices on the space  $\Sigma|\Pi_n|^{\diamond}$ , the functor  $(-)^{\Sigma}$  takes the strict fixed points, and the tensor product is computed in cosimplicial k-modules.

- 3. The functor  $\mathrm{Lie}_{k,\Delta}^\pi$  commuted with filtered colimits and geometric realisations.
- 4. The tangent fiber  $T_X$  of any  $X \in \text{Moduli}_{k,\Delta}$  has the structure of a  $\text{Lie}_{k,\Delta}^{\pi}$ -algebra.



### Theorem (Brantner-Mathew, 2019)

If k is a field, there is an equivalence of  $\infty$ -categories

$$\mathrm{Moduli}_{k,\Delta} \simeq \mathrm{Alg}_{\mathrm{Lie}_{k,\Delta}^{\pi}}$$

between formal moduli problems and partition Lie algebra k. k. It sends a formal moduli problem  $X \in \mathrm{Moduli}_{k,\Delta}$  to its tangent fibre  $T_X$  equipped with a suitable partition Lie algebra structure.

