

# Chromatic Homotopy Theory

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# Tensor-Triangulated Geometry



# Stable homotopy category

Brown representability theorem :

Generalized cohomology theories of  $\text{Top} \longleftrightarrow \text{Spectra}$

**Stable homotopy category** (closed symmetric monoidal category)

Models of Spectra: S-Modules, symmetric spectra, orthogonal spectra

Modern approach:  $\infty$ -category of spectra,  $\text{Sp}$

◼ ring spectra:  $\text{Alg}(\text{Sp})$

◼  $E_\infty$ -ring spectra :  $\text{CAlg}(\text{Sp})$

◼  $H_\infty$ -ring spectra :  $\text{CAlg}(\text{ho}(\text{Sp}))$

Waldhausen's version of *braver new algebra* of abelian groups: The category  $\text{Sp}$  of spectra should be thought of as a homotopical enrichment of the derived category  $\mathcal{D}_{\mathbb{Z}}$



# Local-to-global principle

The Hasse square is a pullback square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_p \prod_p \mathbb{Z}_p \end{array}$$

This is the special case of a local-to-global principle for any chain complex  $M \in \mathcal{D}_{\mathbb{Z}}$ .

$$\begin{array}{ccc} M & \longrightarrow & \prod_p M_p^\wedge \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes M & \longrightarrow & \mathbb{Q} \otimes_p \prod_p M_p^\wedge \end{array}$$

which is a homotopy pullback square, where  $M_p^\wedge$  denote the derived p-completion (p-local and  $\text{Ext}^i(\mathbb{Q}, M_p^\wedge) = 0$ , for  $i = 0, 1$ .)



# The Category $\mathcal{D}_{\mathbb{Z}}$

$\mathcal{D}_{\mathbb{Q}}$ : The derived category of  $\mathbb{Q}$ -vector spaces.

$(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$ : The category of derived  $p$ -complete complexes of abelian groups.

▣  $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$  is compactly generated by  $\mathbb{Z}/p$ , any object  $X \in (\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$  is trivial if and only if  $X \otimes \mathbb{Z}/p$  is trivial.

▣ The only proper localizing subcategory (triangulated subcategory closed under shifts and colimits) of  $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$  is  $(0)$ .

▣ Any object  $M \in \mathcal{D}_{\mathbb{Z}}$  can be reassembled from its derived  $p$ -completions  $M_p^{\wedge} \in (\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$ , its rationalization  $Q \times M \in \mathcal{D}_{\mathbb{Q}}$ , together with the gluing information specified in the pullback square on last page.

$$\{\mathbb{Q} \text{ and } F_p \text{ for } p \text{ prime}\} \leftrightarrow \{\mathcal{D}_{\mathbb{Q}} \text{ and } (\mathcal{D}_{\mathbb{Z}})_p^{\wedge} \text{ for } p \text{ prime}\}$$



# Examples of tensor-triangulated categories

1. The category of spectra.
2. The derived category  $D(R)$  of a commutative ring  $R$ .
3. The  $\infty$ -category  $\mathrm{Mod}_R$  of modules over an  $E_\infty$ -ring spectrum  $R$ .
4. The quasi-coherent sheaves complexes over a scheme (algebraic stack).
5.  $\mathrm{Fun}(K, \mathcal{C})$  when  $K$  is a  $\infty$ -category and  $\mathcal{C}$  is a tensor-triangulated category. If  $K = BG$ , then this functor category are those objects in  $\mathcal{C}$  with a  $G$ -action.
6. Derived category of geometric motives  $DM_{gm}(S) \subset DM(S)$  constructed by Voevodsky.
7.  $SH_{gm}^{\mathbb{A}^1}(S) \subset SH^{\mathbb{A}^1}(S)$  of the stable  $\mathbb{A}^1$  homotopy theory.
8. Homotopy category of Fukaya category  $\mathrm{Fuk}(X)$  of a Calabi-Yau manifold  $X$  (symmetric tensor is induced by its mirror).
9.  $kG - \mathrm{stmod} = \frac{kG - \mathrm{mod}}{kG - \mathrm{proj}} \cong \frac{D^b(kG - \mathrm{mod})}{D^{\mathrm{perf}}(kG)}$  in modular representation theory, for  $G$  a finite group.
10. Tensor-triangulated category of non-commutative motives by Kontsevich.
11.  $G$ -equivariant KK-theory (or its stabilization E-theory) of  $C^*$ -algebras in Alain Connes's non-commutative geometry.



# Tensor-triangulated category

## Definition

A tensor-triangulated category, is a triangulated category  $\mathcal{K}$  together with a symmetric monoidal category structure

$$\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$$

which is exact in each variable.

- ▣ A thick subcategory  $\mathcal{J} \subset \mathcal{K}$  is a triangular subcategory closed under direct summands: if  $X \oplus Y \in \mathcal{J}$ , then  $X, Y \in \mathcal{J}$ .
- ▣  $\mathcal{J} \subset \mathcal{K}$  is a tensor-triangular ideal if  $\mathcal{K} \otimes \mathcal{J} \subset \mathcal{J}$ .

## Definition

A prime  $\mathcal{P} \subset \mathcal{K}$  is a proper tensor-triangular ideal such that  $X \otimes Y \in \mathcal{P}$  implies  $X \in \mathcal{P}$  or  $Y \in \mathcal{P}$ .

# Balmer's Spectrum

## Definition

For  $\mathcal{K}$  a tensor-triangular category, we define

$$\mathrm{Spc}(\mathcal{K}) = \{\mathcal{P} \subset \mathcal{K} \mid \mathcal{P} \text{ is prime}\},$$

$$\mathrm{Supp}(X) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid X \notin \mathcal{P}\}.$$

The Supp has the following properties:

1.  $\mathrm{Supp}(0) = \emptyset$  and  $\mathrm{Supp}(\mathbb{I}) = \mathrm{Spc}(\mathcal{K})$ .
2.  $\mathrm{Supp}(a \oplus b) = \mathrm{Supp}(a) \cup \mathrm{Supp}(b)$ , for every  $a, b \in \mathcal{K}$
3.  $\mathrm{Supp}(\Sigma a) = \mathrm{Supp}(a)$  for every  $a \in \mathcal{K}$ .
4.  $\mathrm{Supp}(c) \subset \mathrm{Supp}(a) \cup \mathrm{Supp}(b)$  for every distinguished triangle  $a \rightarrow b \rightarrow c \rightarrow \Sigma a$ .
5.  $\mathrm{Supp}(a \otimes b) = \mathrm{Supp}(a) \cap \mathrm{Supp}(b)$  for every  $a, b \in \mathcal{K}$ .

We define a topology on  $\mathrm{Spc}(\mathcal{K}) : \{\mathrm{Supp}(X)\}_{X \in \mathcal{K}}$  as a basis of closed subsets.



# Ideal-Thomason Subset

## Definition

For every subset  $V \subseteq \mathrm{Spc}(\mathcal{K})$ , we can associate a tensor-triangular ideal

$$\mathcal{K}_V = \{X \in \mathcal{K} \mid \mathrm{Supp}(X) \subseteq V\}.$$

A subset  $V \subseteq \mathrm{Spc}(\mathcal{K})$  is called a Thomason subset if it is the union of the complements of a collection of quasi-compact open subsets  $V = \bigcup_{\alpha} V_{\alpha}$  where each  $V_{\alpha}$  is closed with quasi-compact complement.

## Theorem

The assignment  $V \rightarrow \mathcal{K}_V$  defines a order-preserving bijection between the Thomason subsets  $V \subset \mathrm{Spc}(\mathcal{K})$  and the tensor-triangular ideal.

# Examples: stable homotopy category

There is a map  $\phi : S^0 \rightarrow \tau_{\leq 0} S^0 \simeq H\mathbb{Z}$ ,

$$\mathrm{Sp} \simeq \mathrm{Mod}_{S^0}(\mathrm{Sp}) \xrightarrow{\phi^*} \mathrm{Mod}_{H\mathbb{Z}}(\mathrm{Sp}) \simeq \mathcal{D}_{\mathbb{Z}}$$


$$\mathrm{Spc}(\mathcal{D}_{\mathbb{Z}}) \xrightarrow{\mathrm{Spc}(\phi^*)} \mathrm{Spc}(\mathrm{Sp}) \xrightarrow{\rho} \mathrm{Spec}(\mathbb{Z})$$


Question: What is the inverse image of the irreducible building block  $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$  ? Answer:  
There are infinitely many blocks in  $\mathrm{Sp}$  between  $(0)$  and  $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$




The Balmer's Spectrum of classical stable homotopy category (Hopkins-Smith, 1988-1996) is the following topological space.

$$\begin{array}{cccc}
 \mathcal{P}_{2,\infty} & \mathcal{P}_{3,\infty} & \cdots & \mathcal{P}_{3,\infty} \cdots \\
 \vdots & \vdots & & \vdots \\
 \mathcal{P}_{2,n+1} & \mathcal{P}_{3,n+1} & \cdots & \mathcal{P}_{p,n+1} \cdots \\
 \mathcal{P}_{2,n} & \mathcal{P}_{3,n} & \cdots & \mathcal{P}_{p,n} \cdots \\
 \vdots & \vdots & & \vdots \\
 \mathcal{P}_{2,2} & \mathcal{P}_{3,2} & \cdots & \mathcal{P}_{p,2} \cdots \\
 & & \mathcal{P}_{0,1} & 
 \end{array}$$

  $\mathcal{P}_{0,1} = \ker(SH^c \rightarrow SH^c \cong D^b(\mathbb{Q})), \mathcal{P}_{n,\infty} = \ker(SH^c \rightarrow SH^c_{(p)}).$

  $\mathcal{P}_{p,n} = \ker(SH^c \rightarrow SH^c_{(p)} \rightarrow \mathbb{F}_p[v_{n-1}^{\pm 1}] - grmod)$  of localization at  $p$  and  $(n-1)$  Morava K-theory  $K_{p,n-1}$ .

- 
- ▣ The higher point belongs to the closure of the lower one.
  - ▣ A closed subset is either empty, or the whole  $\mathrm{Spc}(SH^c)$ , or a finite union of closed points  $\{\mathcal{P}_{p,\infty}\}$  and of columns

$$\overline{\{\mathcal{P}_{p,m_p}\}} = \{\mathcal{P}_{p,n} \mid m_p \leq n \leq \infty\}$$



# Examples

## Theorem(Thomason, 1997)

Let  $X$  be a quasi-compact and quasi-separated scheme. Then there is a homeomorphism of topological space

$$|X| \xrightarrow{\cong} \mathrm{Spc}(D^{perf}(X))$$

$$x \longmapsto \mathcal{P}(X)$$

where  $\mathcal{P}(x) = \{Y \in D^{perf}(X) \mid Y_x \cong 0\}$

## Corollary

Let  $A$  be a commutative ring,  $K^b(A - proj) \cong D^{perf}(A)$ . Then we have

$$\mathrm{Spec}(K^b(A - proj)) \cong \mathrm{Spec}(A).$$

# Examples

## Theorem (Benson-Carlson-Richard , 1997)

Let  $G$  be a finite group, then there is a homeomorphism

$$\mathrm{Spc}(kG - \mathrm{stmod}) \cong \mathrm{Proj}(H^\bullet(G, k)).$$

## Theorem (Balmer-Sanders , 2017)

Let  $G$  be a finite group. Then every tensor triangular prime in  $SH(G)^c$  is of the form  $\mathcal{P}(H, p, n)$  for a unique subgroup  $H \subset G$  up to conjugation, where

$$\mathcal{P}(H, p, n) \cong (\Phi^H)^{-1}(\mathcal{P}_{p,n})$$

is the preimage under geometric  $H$ -fixed points  $\Phi^H : SH(G)^c \rightarrow SH^c$ .

If  $K \triangleleft H$  is a normal subgroup of index  $p > 0$ , then  $\mathcal{P}(K, p, n+1) \subset \mathcal{P}(H, p, n)$ .



# Chromatic Homotopy Theory



# Formal Groups

Let  $R$  be a complete local ring with residue field characteristic  $p > 0$ ,  $C_R$  denote the category of local Noetherian  $R$ -algebras. We define

$$\hat{\mathbb{A}}^1(A) := C_R(R[[t]], A)$$

A commutative one-dimensional formal group over  $R$  is a functor

$$G : C_R \rightarrow \mathbf{Ab}$$

which is isomorphic to  $\hat{\mathbb{A}}^1$ .

$$\mathcal{O}_G \rightarrow \mathcal{O}_{G \times G} \cong \mathcal{O}_G \otimes \mathcal{O}_G$$

$\mathcal{O}_G$  is just  $R[[X]]$  and  $\mathcal{O}_G \otimes \mathcal{O}_G$  is  $R[[X]] \otimes_R R[[Y]] = R[[X, Y]]$ .

$$\begin{array}{ccc} \phi : & R[[X]] & \rightarrow & R[[X, Y]] \\ & X & \rightarrow & f(X, Y) \end{array}$$





# Formal Group Laws

## Definition

Formal group law :  $F \in R[[x_1, x_2]]$

☐  $F(x, 0) = F(0, x) = x$  (Identity)

☐  $F(x_1, x_2) = F(x_2, x_1)$  (Commutativity)

☐  $F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$  (Associativity)

There exists a ring  $L$  and  $F_{univ}(x, y) \in L[[x, y]]$

$$\{\text{Formal Group Law over } R\} \longleftrightarrow \{L \rightarrow R\}$$

such that  $F(x, y) \in R[[x, y]]$  over  $R$ ,

$$f^*(F_{univ}(x, y)) = F(x, y).$$

## Lazard's Theorem

$$L \cong \mathbb{Z}[t_1, t_2, \dots]$$

# Heights of Formal Groups

Let  $f(x, y) \in R[[x, y]]$

1. If  $n = 0$ , we set  $[n](t) = 0$ .
2. If  $n > 0$ , we set  $[n](t) = f([n-1](t), t)$ .

P-series  $p[t]$  is either 0 or equals  $\lambda t^{p^n} + O(t^{p^{n+1}})$  for some  $n > 0$ .

## Definition

Let  $v_n$  denote the coefficient of  $t^{p^n}$  in the p-series,  $f$  has height  $\leq n$  if  $v_i = 0$  for  $i < n$ ,  $f$  has height exactly  $n$  if it has height  $\leq n$  and  $v_n$  is invertible.

## Examples

- Formal multiplicative group  $f(x, y) = x + y + xy$ ,  $[n](t) = (1 + t)^n - 1$ . If  $p = 0$  in  $R$ , then  $[p](t) = (1 + t)^p - 1 = t^p$ , so  $f$  has height 1.
- Formal additive group  $f(x, y) = x + y$ , if  $p = 0$  in  $R$ . Then  $[p](t) = 0$ , so  $f$  has infinite height.

# Complex Oriented Cohomology Theories

## Definition (Complex Orientation)

Let  $E$  be cohomology theory. Then a complex orientation of  $E$  is a choice  $x \in E^2(\mathbb{C}P^\infty)$  which restricts to 1 under the composite

$$E^2(\mathbb{C}P^\infty) \rightarrow E^2(\mathbb{C}P^1) = E^2(S^2) \cong E^0(*)$$

$$E^*(\mathbb{C}P^\infty) \cong E^*(*)[[t]] = (\pi_* E)[[t]]$$

$$(\pi_* E)[[t]] \cong E^*(\mathbb{C}P^\infty) \rightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong (\pi_* E)[[x, y]]$$

$$\{\text{complex oriented cohomology theory } E\} \rightarrow \text{Formal Groups } G_E = \text{Spf } E^0(\mathbb{C}P^\infty).$$

$$E \longrightarrow G_E = \text{Spf } E^0(\mathbb{C}P^\infty).$$

## Theorem (Quillen, 1969)

$MU$  is the universal complex oriented cohomology theory,  $L \cong \pi_* MU$ . For  $E$  complex oriented,  $MU \rightarrow E$ , induce  $L = \pi_* MU \rightarrow \pi_* E$ .

# The Landweber Exact Functor Theorem

If we already have a ring map  $L \rightarrow R$ , can we construct a complex oriented cohomology theory  $E$  such that  $R = \pi_* E$ ?

$$E_*(X) = MU_*(X) \otimes_{\pi_* MU} R = MU_*(X) \otimes_L R$$

## Landweber's Exact Functor Theorem, 1976

Let  $M$  be a module over the Lazard ring  $L$ . Then  $M$  is flat over  $\mathcal{M}_{FG}$  if and only if for every prime number  $p$ , the elements  $v_0 = p, v_1, v_2, \dots \in L$  form a regular sequence for  $M$

# Lubin-Tate Theory

**Deformation of formal groups:** Let  $G_0$  be a formal group over a perfect field  $k$  with characteristic  $p$ , then a deformation of  $G_0$  to  $R$  is a triple  $(G, i, \Psi)$  satisfying

1.  $G$  is a formal group over  $R$ ,
2. There is a map  $i : k \rightarrow R/m$ ,
3. There is an isomorphism  $\Psi : \pi^* G \cong i^* G_0$  of formal groups over  $R/m$ .

## Lubin-Tate's Theorem, 1966

There is a universal formal group  $G$  over  $R_{LT} = W(k)[[v_1, \dots, v_n - 1]]$  in the following sense: for every infinitesimal thickening  $A$  of  $k$ , there is a bijection

$$\mathrm{Hom}_{/k}(R_{LT}, A) \rightarrow \mathrm{Def}(A).$$

# Morava E-theories and Morava K-theories

Using Landweber exact functor theorem, there is a even periodic spectrum  $E(n)$

$$\pi_* E(n) = W(k)[[v_1, \dots, v_{n-1}]][\beta^{\pm 1}]$$

## Theorem (Goerss-Hopkins-Miller)

The spectrum  $E(n)$  admits a unique  $E_\infty$ -ring structure.

$M(k)$  denote the cofiber of the map  $\sum^{2k} MU_{(p)} \rightarrow MU_{(p)}$  given by the multiplication by  $t_k$ .

Let  $K(n)$  denote the smash product

$$MU_{(p)}[v_n^{-1}] \otimes_{MU_{(p)}} \bigotimes_{k \neq p^n - 1} M(k).$$

This spectrum  $K(n)$  is called **Morava K-theory**. The homotopy groups of  $K(n)$  is

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}] / (t_0, t_1, \dots, t_{p^n-2}, t_{p^n}, \dots) \cong \mathbb{F}_p[v_n^{\pm 1}]$$

# Properties of Morava K-theories

- ▣ A commutative evenly graded ring is a graded field every nonzero homogeneous element is invertible. Equivalently,  $R$  is a field or  $R \simeq k[\beta^\pm]$ .
- ▣ We say a homotopy associative ring spectrum is a field if  $\pi_*E$  is a graded field.

## Example

For every prime  $p$  and every integer  $n$ ,  $K(n)$  is a field.

## Proposition

- ▣ If  $E$  is an field such that  $E \otimes K(n)$  is nonzero, then  $E$  admits a structure of  $K(n)$ -module.
- ▣ Let  $E$  be complex-oriented ring spectrum of height  $n$  and  $\pi_*E \simeq \mathbb{F}_p[\nu_n^{\pm 1}]$ . Then  $E \simeq K(n)$ .

# Localization

Let  $S$  be a set of prime numbers, for example  $S = (p)$ .

- ▣ A ring  $R$  is  $S$ -local, if all prime numbers not in  $S$  is invertible in  $R$ .
- ▣ A group  $G$  is said to be  $S$ -local if the  $p^{\text{th}}$  power map  $G \rightarrow G$  is a bijection for  $p \notin S$ .
- ▣ If  $G$  is abelian,
  1.  $G$  is  $S$ -local;
  2.  $G$  admits a structure of  $\mathbb{Z}_S$ -module (necessarily unique);

## Definition

A spectrum  $X$  is called  $S$ -local if its homotopy groups are  $S$ -local abelian groups.

The  $S$ -localization can be constructed as the Bousfield localization of spectra with respect to the Moore spectrum  $M(\mathbb{Z}_S)$



# Localization

The general idea of localization at a spectrum  $E$  is to associate to any spectrum  $X$  the “part of  $X$  that  $E$  can see”, denoted by  $L_EX$ .  $L_E$  is a functor with the following equivalent properties:

□□  $E \wedge X \simeq * \Rightarrow L_EX \simeq *.$

□□ If  $X \rightarrow Y$  induces an equivalence  $E \wedge X \rightarrow E \wedge Y$  then  $L_EX \rightarrow L_EY$ .



# Bousfield Localization

Let  $\mathcal{C}$  be a full subcategory of  $\mathrm{Sp}$ , which is closed under shifts and homotopy colimits, and can be generated by small subcategory under homotopy colimits.

If  $X$  is a spectrum, define  $G(X)$  to be the homotopy colimit of all  $Y \in \mathcal{C}$  with a map to  $X$ .

We have a counit map  $\nu : G(X) \rightarrow X$ , and we let  $L(X)$  denote the cofiber of  $\nu$ , then we have a cofiber sequence

$$G(X) \rightarrow X \rightarrow L(X).$$

A spectrum is  $\mathcal{C}$ -local if every map  $Y \rightarrow X$  is nullhomotopic when  $Y \in \mathcal{C}$ . We denote the category of  $\mathcal{C}$ -local spectra as  $\mathcal{C}^\perp$ .



## Bousfield localization

Let  $G_E$  the collection of E-acyclic spectra. We say that a spectrum is E-local if every map for every  $Y \in G_E$ , the map  $Y \rightarrow X$  is nullhomotopic.

We have a cofiber sequence

$$G_E(X) \rightarrow X \rightarrow L_E(X).$$

where  $G_E(X)$  is E acyclic and  $L_E(X)$  is E-local. This functor is called Bousfield localization with respect to E.

The map  $X \rightarrow L_E(X)$  is characterized up to equivalence by two properties.

1. The spectrum  $L_E(X)$  is E-local.
2. The map  $X \rightarrow L_E(X)$  is an E-equivalence.

## Theorem

A spectrum  $X$  is E-local if and only if for each E-equivalence  $S \rightarrow T$ , the induced map  $[T, X] \rightarrow [S, X]$  is an isomorphism.

# Moore Spectrum

For  $G$  an abelian group, then the Moore spectrum  $MG$  of  $G$  is the spectrum characterized by having the following homotopy groups:

1.  $\pi_{<0}MG = 0$ ;
2.  $\pi_0(MG) = G$ ;
3.  $H_{>0}(MG, Z) = \pi_{>0}(MG \wedge HZ) = 0$ .

A basic special case of E-Bousfield localization of spectra is given by  $E = MA$  the Moore spectrum of an abelian group  $A$ .

1. For  $A = Z_{(p)}$ , this is  $p$ -localization.
2. For  $A = F_p$ , this is  $p$ -completion
3. For  $A = \mathbb{Q}$ , this is the rationalization .



# Examples of Localization

## Theorem

p-Localization is a smashing localization:

$$L_{MZ_{(p)}}X \simeq MZ_{(p)} \wedge X$$

We denote this as  $L_{MZ_{(p)}}X \simeq X_{(p)}$ , which is called the Bousfield p-localization

A spectrum E is p-complete, if  $\pi_*E$  is a (p)-adic complete ring. Bousfield localization at the Moore spectrum  $MF_p$  is p-completion to p-adic homotopy theory.

## Theorem

The localization of spectra at the Moore spectrum  $MF_p$  is given by the mapping spectrum out of  $\Omega M\mathbb{Z}/p^\infty$ :

$$L_p = L_{MF_p}X \simeq [\Omega M\mathbb{Z}/p^\infty, X]$$

where  $\mathbb{Z}/p^\infty = \mathbb{Z}[1/p]/\mathbb{Z}$ . We denote this spectrum  $L_p = L_{MF_p}X$  as  $X_p^\wedge$

# Examples of Localization

## Theorem

$L_{M\mathbb{Q}}X = X \wedge L_{\mathbb{Q}}S^0 = X \wedge M\mathbb{Q} = X \wedge H\mathbb{Q}$  is smashing, we call this as the rationalization of  $X$ , denote it as  $L_{\mathbb{Q}}X$ .

## Examples

Localization with respect to  $E(n)$  and  $K(n)$ .

☐  $L_{E(n)}$ , behaves like restriction to the open substack

$$\mathcal{M}_{FG}^{\leq n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbb{Z}_{(p)}.$$

☐  $L_{K(n)}$ , behaves like completion along the locally closed substack

$$\mathcal{M}_{FG}^n \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbb{Z}_{(p)}.$$

# Localization with respect to $E(n)$ and $K(n)$

## Lemma

The Spectrum  $E(n)$  is Bousfield equivalent to  $E(n) \times K(n)$ . Here  $E(0) = H\mathbb{Q}[\beta^{\pm}]$  which is Bousfield equivalent to  $H\mathbb{Q}$ .

So a spectrum is  $E(n)$ -acyclic if and only if it is both  $E(n)$ -acyclic and  $K(n)$ -acyclic.

$$L_{E(n)}(X) \cong L_{K(0) \vee K(1) \dots K(n)}(X).$$

There is pullback square

$$\begin{array}{ccc} L_{E(n)}X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{E(n-1)}X & \longrightarrow & L_{E(n-1)}(L_{K(n)}X) \end{array}$$

This come from  $L_{E(n-1)}X$  is  $K(n)$ -acyclic and the following Lemma



### Lemma

Let  $E, F, X$  be spectra with  $E_*L_FX = 0$ . Then there is a homotopy pullback square.

$$\begin{array}{ccc} L_{E \vee F}X & \longrightarrow & L_EX \\ \downarrow & & \downarrow \\ L_FX & \longrightarrow & L_F(L_EX) \end{array}$$

So we have the following **Suillivan arithmetic square** for  $E = \bigvee_p M(Z/p), F = H\mathbb{Q}$

$$\begin{array}{ccc} X & \longrightarrow & \prod_p L_pX \\ \downarrow & & \downarrow \\ L_{\mathbb{Q}}X & \longrightarrow & L_{\mathbb{Q}}(\prod_p L_pX) \end{array}$$

In chromatic homotopy, we often care the Bousfield localization with respect to the Morava E-theories and Morava K-theories.



# Nilpotence

We say that a collection of ring spectra  $\{E^\alpha\}$  detect nilpotence if for any p-local ring spectra  $R$ ,  $x \in \pi_m R$  is sent to zero in  $E_0^\alpha R$  for all  $\alpha$ , then  $x$  is nilpotent in  $\pi_* R$ .

## Nilpotence Theorem (Devnatz-Hopkins-Smith, 1988)

For any ring spectrum  $R$ , the kernel of the map  $\pi_* R \rightarrow MU_* R$  consists of nilpotent elements. In particular, the single  $MU$  detects nilpotence.

## Theorem

The spectra  $\{K(n)\}_{0 \leq n \leq \infty}$  detect nilpotence.

Let  $E$  be a nonzero p-local ring spectrum, then  $E \otimes K(n)$  is nonzero for some  $0 \leq n \leq \infty$ . If not, every element of  $\pi_0 E$  is nilpotent, so  $\mathbb{I} \in \pi_0 E$  is nilpotent, so that  $E \simeq 0$ .



# Thick Subcategories

Let  $\mathcal{C}$  be a full subcategory of finite p-local spectra. We say that  $\mathcal{C}$  is **thick** if it contains 0, closed under fiber and cofibers, and every retract of a spectrum belong to  $\mathcal{C}$  also belongs to  $\mathcal{C}$ .

## Lemma

Let  $X$  be a finite p-local spectrum, if  $K(n)_*(X) \simeq 0$  for some  $n > 0$ . Then  $K(n-1)_*(X) = 0$ .

We say that a p-local finite spectrum has type  $n$  if  $K(n)_*(X) \neq 0$  and  $K(m)_*(X) = 0$  for  $m < n$ .  $X$  has type 0 if  $H_*(X, \mathbb{Q}) \simeq 0$ .

We let  $\mathcal{C}_{\geq n}$  be the category of p-local spectra which has type  $\geq n$ .

## Thick Subcategory Theorem

Let  $\mathcal{T}$  be a thick subcategory of finite p-local spectra. Then  $\mathcal{T} = \mathcal{C}_{\geq n}$  for some  $0 \leq n \leq \infty$ .

# Different Localizations

We have an adjunction

$$\text{inclusion} : G_E = \{E - \text{acyclic}\} \rightleftarrows \text{Sp} : G_E$$

Localization with respect to  $E$  means localization with respect to  $G_E$ .

$$G_E \hookrightarrow \text{Sp} \xrightarrow{L_E} E - \text{local} = (G_E)^\perp$$

$$G_E(X) \longrightarrow X \longrightarrow L_E(X)$$

We know  $E(n)$  acyclic means  $E(n-1)$  acyclic and  $K(n)$ -acyclic, but  $\ker L_E = G_E = \{E(n) - \text{acyclic}\}$ , so we get inclusions

$$0 = \ker(id) \subset \ker(L_{E(\infty)}) \cdots \subset \ker(L_{E(n)}) \subset \ker(L_{E(n-1)}) \cdots \ker(L_{E(0)}) \subset \text{Sp}$$

by taking orthocomplement, we get

$$0 \subset E(1)\text{-local Sp} \subset \cdots \subset E(n-1)\text{-local Sp} \subset E(n)\text{-local Sp} \subset \cdots$$



# Different Localization

We have  $K(n)_*(X) = 0 \Rightarrow K(n-1)_*(X) = 0$ .

$$\mathcal{C}_{\geq n} = \{X \in \mathrm{Sp}_{(p)} \mid X \text{ has type } \geq n, \text{ i.e., } K(m)_*X = 0, m < n\}$$

So we have sequence

$$(0) \subset \cdots \subset \mathcal{C}_{\geq n+1} \subset \mathcal{C}_{\geq n} \subset \cdots \subset \mathcal{C}_{\geq 0} = \mathrm{Sp}$$

by taking orthocomplement, we get

$$\mathcal{C}_{\geq 0} \text{ local spectra} \subset \cdots \subset \mathcal{C}_{\geq n} \text{ local spectra} \subset \mathcal{C}_{\geq n+1} \text{ local spectra} \subset \cdots$$

## Telescope Localization

The telescope localization  $L_n^t$ : Localization with respect to  $\mathcal{C}_{\geq n+1}$ .

$$C(X) \rightarrow X \rightarrow L_n^t(X).$$

where  $C(X)$  is a filtered colimit of object in  $\mathcal{C}_{\geq n+1}$

# Different Localizations

## Definition

We say a localization functor  $L$  is a smash localization if  $L(X) = K \wedge X$  for a  $K$ .

The following conditions are equivalent

1.  $L$  preserves homotopy colimits.
2.  $C^\perp \subset \mathcal{S}p$  is stable under homotopy colimits
3.  $G$  preserves homotopy colimits.
4.  $L(X) = K \wedge X$ .

## Examples

- $L_{E(n)}$  is a smash localization.
- $L_n^t$  is a smash localization.
- Rationalization and  $p$ -localization is a smash localization.

For any smashing localization  $L$

$$\ker(L_n^t) \subset \ker(L) \subset \ker(L_{E(n)})$$

So there is a comparison

$$L_n^t \rightarrow L \rightarrow L_{E(n)}$$

**Telescope Conjecture**

$$L_n^t \simeq L_{E(n)}$$

# The periodicity theorem: find a type $n$ spectrum

Consider the cofiber sequence

$$\Sigma^k X \xrightarrow{f} X \rightarrow X/f$$

If we have  $X$  has type  $\leq n$ , we hope  $X/f$  has type  $\leq n + 1$

## Definition

Let  $X$  be finite  $p$ -local spectrum, a  $\nu_n$  self map is a map  $f : \Sigma^q X \rightarrow X$  and satisfying the following,

1.  $f$  induces an isomorphism  $K(n)_*(X) \rightarrow K(n)_*X$ .
2. The induced map  $K(m)_*(X) \rightarrow K(m)_*(X)$  is nilpotent, for  $m \neq n$ .

## Theorem

Let  $X$  be a finite  $p$ -local spectrum of type  $\geq n$ , then  $X$  admits a  $\nu_n$ -self map.

# Telescopic Localization

$$X \xrightarrow{f} \Sigma^{-k}(X) \xrightarrow{f} \Sigma^{-2k}(X) \xrightarrow{f} \dots$$

Let  $X[f^{-1}]$  denote the colimit of this sequence.

## Proposition

1. If  $X \in \mathcal{C}_{\geq n}$ , then  $L_n^t(X) \simeq X[f^{-1}]$ .
2. There is a fiber sequence

$$\lim_{\substack{\rightarrow \\ k_0, \dots, k_n}} \Sigma^{-n} X / (v_0^{k_0}, \dots, v_n^{k_n}) \rightarrow X \rightarrow L_n^t(X).$$



# Monochromatic

Let  $L_n(X) = L_{E(n)}(X)$ , then we have the following chromatic tower.

$$\begin{array}{ccccccc} & M_n(X) & & M_2(X) & & M_1(X) & & M_0(X) = H\mathbb{Q} \wedge X \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & L_n(X) & \longrightarrow & \cdots \longrightarrow & L_2(X) & \longrightarrow & L_1(X) & \longrightarrow & L_0(X) = H\mathbb{Q} \wedge X \end{array}$$

where the monochromatic layers  $M_n(X)$  are defined by the fiber sequence.

$$M_n(X) \rightarrow L_n(X) \rightarrow L_{n-1}(X)$$

The following is the chromatic convergence theorem proved by Hopkins- Ravenel.

## Chromatic Convergence Theorem

Then Canonical Map  $X \rightarrow \lim_n L_n X$  is an equivalence for a  $p$  local finite spectrum  $X$ .

### Definition

Monochromatic A spectrum  $X$  is monochromatic of height  $n$  if it is  $E(n)$ -local and  $E(n-1)$ -acyclic.

We let  $\mathcal{M}_n$  denote the category of all spectra which are monochromatic of height  $n$ .

### Theorem

There is a equivalence of category between the homotopy category of monochromatic spectra of height  $n$  and the homotopy category of  $K(n)$ -local spectra, which is given by the functor

$$L_{K(n)} : \mathcal{M}_n \rightleftarrows K(n) \text{ local spectra} : M_n$$

# K(n)-Local Spectra

1.  $\mathrm{Sp}_{K(n)}$  is compactly generated by  $L_E(n)F$ , for any type  $n$  spectrum  $F$ , an object  $X \in \mathrm{Sp}_{K(n)}$  is trivial if and only if  $X \wedge K(n)$  is trivial.
2. The only proper localizing subcategory of  $\mathrm{Sp}_{K(n)}$  is  $(0)$ .
3. A spectrum  $X \in \mathrm{Sp}_{E(n)}$  can be reassembled from  $L_{K(n)}X$ ,  $L_{E(n-1)}X$ , together with the gluing information.

$$\begin{array}{ccc} L_{E(n)}X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{E(n-1)}X & \longrightarrow & L_{E(n-1)}(L_{K(n)}X) \end{array}$$


The chromatic approach to  $\pi_* S_{(p)}^0$ :

1. Compute  $\pi_* L_{K(n)} S^0$  for each  $n$ .
2. Understanding the gluing of above square.
3. Using chromatic convergence  $\lim_n \pi_* L_{E(n)} S^0 \cong \pi_* S_{(p)}^0$





# From Algebra to Algebraic Topology



How do we detect topological structure from algebraic information?

■  $E_*$  module structure with symmetry  $\implies$  Fixed point spectral sequence.

■  $(E_*, E_*E)$  module structure  $\implies$  Adams spectral sequence



# Morava Stabilizer Groups

We let  $G_0$  denote a formal group of height  $n$  over a perfect field  $k/\mathbb{F}_p$

The small Morava stabilizer group  $\text{Aut}_k(G_0)$  is the group of automorphism of  $G_0$  with coefficients in  $k$ ,

$$\text{Aut}(G_0) = \{f(x) \in k[[x]] : f(G_0(X, Y)) = G_0(f(x), f(y)), f'(0) \neq 0\}$$

Since  $G_0$  is defined over  $k$ , the Galois group  $\text{Gal} = \text{Gal}(k/\mathbb{F}_p)$  act on  $G_0$  by acting on the coefficients.

The Morava stabilizer group  $\mathbb{G}_n$  is defined by

$$\mathbb{G}_n = \text{Gal}(k/\mathbb{F}_p) \ltimes \text{Aut}(G_0)$$

# Morava Stabilizer Groups

$$(G_0, k) \longrightarrow \text{Morava E-theory } E(G_0, k)$$

Does the action  $\mathbb{G}_n$  lift to  $E(G_0, k)$ ?

**Theorem (Devnatz-Hopkins, Goerss-Hopkins-Miller)**

The Morava stabilizer group acts on  $E_n$ , and it gives essential all automorphisms of  $E(n)$

$$E(n)^{h\mathbb{G}_n} \simeq L_{K(n)} S^0$$

**Example**

When  $p$  is odd and  $n=1$ ,  $L_{K(1)}(S)$  is the spectrum  $\widehat{KU}^{\psi^g=1}$

# Homotopy fixed point spectral sequence

If we  $E_*$  module structure with an action of Morava stabilizer group  $\mathbb{G}_n$ , how can we get  $L_{K(n)}S^0$ ?

$$\mathrm{Sp}_{K(n)} \longrightarrow \{ \text{Morava Modules} : E_* \text{ module structure with action of } \mathbb{G}_n \}$$

## Proposition

There is a homotopy fixed point spectral sequence (descent spectral sequence)

$$E_2^{s,t} = H_{gp}^s(G; \pi_t(X)) \implies \pi_{t-s}(X^{hG})$$

similarly for  $X_{hG}$ ,  $X^{tG}$ .

We have  $E(n)^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$ , so

$$E_2^{s,t} \cong H_{gp}^s(\mathbb{G}, E(n)_t) \implies \pi_{t-s}L_{K(n)}S^0$$



# The structure of Morava stabilizer group

For  $f$  a formal group law over  $\mathbb{F}_p$ .

$$\text{End}f = \{g(t) \in tR[[t]] \mid f(g(x), g(y)) = gf(x, y)\}$$

## Proposition

$\text{End}(f)$  is a noncommutative local ring: The collection non-invertible elements is the left ideal generated by  $\pi(t) = \nu(t^p)$ , where  $\nu f^p(x, y) = f(\nu(x), \nu(y))$ .

Let  $D = \mathbb{Q} \otimes \text{End}(f)$ .

## Lemma

$D$  is a central division algebra over  $\mathbb{Q}_p$ . And  $\text{End}(f) = \{x \in D : \nu(x) \geq 0\}$ .

# Morava Stabilizer Group

$$\det : \mathbb{G}_n \rightarrow \mathbb{Z}_p^\times \quad \det : \mathbb{S}_n \rightarrow \mathbb{Z}_p^\times$$

Composition with  $\mathbb{Z}_p^\times / \mu \cong \mathbb{Z}_p$ .

$$\zeta_n : \mathbb{G}_n \rightarrow \mathbb{Z}_p.$$

Let  $\mathbb{G}_n^1 = \ker \zeta_n$ , we have

$$\mathbb{G}_n \cong \mathbb{G}_n^1 \rtimes \mathbb{Z}_p, \quad \mathbb{S}_n \cong \mathbb{S}_n^1 \rtimes \mathbb{Z}_p.$$

As a consequence of  $\mathbb{G}_n / \mathbb{G}_n^1 \rtimes \mathbb{Z}_p$ , there is a equivalence  $L_{K(n)} S^0 \simeq (E_n^{h\mathbb{G}_n^1})^{h\mathbb{Z}_p}$ .

$$L_{K(n)} S^0 \longrightarrow E_n^{h\mathbb{G}_n^1} \xrightarrow{\psi-1} E_n^{h\mathbb{G}_n^1} \xrightarrow{\delta} \Sigma L_{K(n)} S^0.$$



# The action of Morava stabilizer group

Let  $F_n$  be the universal deformation over  $(E_n)_0$  of  $G_0$ . If we have  $\alpha = (f, \sigma) \in \mathbb{G}_n$ . The universal property of  $F_n$  implies that there is ring isomorphism  $\alpha_* : (E_n)_0 \rightarrow (E_n)_0$  and an isomorphism of formal group laws  $f_\alpha : \alpha_* F_n \rightarrow F_n$ .

And the action can extend to  $(E_n)_* \cong \mathbb{W}_n[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$

1.  $\alpha = (id, \sigma)$  for  $\sigma \in \text{Gal}(k/\mathbb{F}_p)$ . Then the action is action of Galois group on  $\mathbb{W}_n$ .
2. If  $\omega \in \mathbb{S}_n$  is a primitive  $(p^n - 1)$ -th root of the unity, then  $\omega_*(u_i) = \omega^{p^i - 1} u_i$  and  $\omega_*(u) = \omega u$ .
3.  $\Psi \in \mathbb{Z}_p^\times \subset \mathbb{S}_n$  is the center, then  $\psi_*(u_i) = u_i$  and  $\Psi_* u = \Psi u$ .

## Theorem (Devnatz-Hopkins)

Let  $1 \leq i \leq n-1$  and  $f = \sum_{j=0}^{n-1} f_j \xi_j \in \mathbb{S}_n$ , where  $f_j \in \mathbb{W}_n$ . Then modulo  $(p, u_1, \dots, u_{n-1})^2$ ,

$$f_*(u) \equiv f_0 u + \sum_{j=1}^{n-1} f_{n-j}^{\sigma^j} u u_j \quad f_*(u u_i) \equiv \sum_{j=1}^i f_{i-j}^{\sigma^j} u u_j + \sum_{j=i+1}^n p f_{n+i-j}^{\sigma^j} u u_j$$

# Stable Homotopy Groups of Sphere

## Lemma

The  $K(1)$ -local sphere  $L_{K(1)}S$  is given by the homotopy fiber of the map  $\Psi^g - 1 : \widehat{KU} \rightarrow \widehat{KU}$ .

$$\begin{aligned}\pi_{2n}(\widehat{KU}^{\Psi^g-1}) &\simeq 0 \\ \pi_{2n-1}(\widehat{KU}^{\Psi^g-1}) &\simeq \mathbb{Z}^p / (g^n - 1).\end{aligned}$$

By this theorem, we can compute the homotopy group of  $L_{K(1)}S$

$$\pi_n L_{K(1)}S = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Q}_p / \mathbb{Z}_p & n = -2 \\ \mathbb{Z} / p^{k+1} \mathbb{Z} & n+1 = (p-1)p^k m, p \nmid m \\ 0 & \text{otherwise} \end{cases}$$



Let  $im(J)_n$  denote the image of the composition map

$$\pi_n(O) \rightarrow \pi_n(S) \rightarrow \pi_n(S_{(p)})$$

The relation of image of  $J$  and the  $L_{(K(1))}S$  is described as

### Theorem

For  $n > 0$ , the Bousfield Localization at  $E(1)$ ,  $S_{(p)} \rightarrow L_{E(1)}S$  induces an isomorphism

$$im(J)_n = \pi_n(L_{E(1)}S)$$

In particular,  $\pi_n S_{(p)} \rightarrow \pi_n L_{E(1)}S$  is surjective.

By this theorem and the computation of  $L_{(E(1))}S$ , we can get

$$\pi_{2n}(KU) \rightarrow \pi_{2n-1}(U) \xrightarrow{J} \pi_{2n-1}(S) \rightarrow \pi_{2n-1}(\widehat{KU}^{\Psi^g-1})$$

is surjective, and for  $n > 0$ ,

$$im(\pi_* J)_{(p)} = \begin{cases} \mathbb{Z}/p^{k+1} & n = (p-1)p^k m, p \nmid m \\ 0 & (p-1) \nmid n. \end{cases}$$

# Adams Spectral Sequence

There is an equivalence

$$D(R) \cong \text{Mod}_{HR}(\text{Sp})$$

Homology forget the  $\mathcal{A}_p$ -module structure.

$$\begin{array}{ccc} & & \text{Mod}_{\mathcal{A}_p}^{\text{graded}} \\ & \nearrow^{H^*(-, \mathbb{F}_p)} & \downarrow \text{forget} \\ \text{Sp}^{op} & \xrightarrow{H^*(-, \mathbb{F}_p)} & \text{Mod}_{\mathbb{F}_p}^{\text{graded}} \end{array}$$

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}_p}^{s,t}(H^*Y, H^*X) \Longrightarrow [X, Y_p^\wedge]_{t-s}$$

1.  $E_2^{s,t} = \text{Ext}_{\mathcal{A}_p}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*(\mathbb{S})_p$



### E based Adams spectral sequence

There exists a cohomological spectral sequence  $E_*^{*,*}$  such that

$$E_2^{s,t} = \text{Ext}_{E^*E}^{s,t}(E^*Y, E^*X) \Longrightarrow [X, \Sigma^{t-s}Y]_E$$

where  $[X, \Sigma^{t-s}Y]_E$  is the set of stable homotopy class from  $X$  to  $Y$  in an  $E$ -localization.

# Power Operations

Suppose  $\mathcal{C}$  is a tensor triangulated category (presentable stable symmetric monoidal  $\infty$  category), then the functor

$$\pi_0 : \mathrm{CAlg}(\mathcal{C}) \longrightarrow \mathrm{Set}, R \mapsto \pi_0 \mathrm{Map}_{\mathcal{C}}(\mathbb{I}, R)$$

is represented by the free commutative algebra on a copy of the unit,  $\mathbb{I}\{t\}$ .

We can define the power operation on  $\pi_0 R$  which is given by the elements of

$$\pi_0 \mathbb{I}\{t\} = \pi_0 \bigoplus \mathbb{I}_{h\Sigma_s}^{\otimes} \cong \pi_0 \bigoplus \mathbb{I}_{h\Sigma_s}.$$

## Definition

To each object  $P \in \pi_0 \mathbb{I}_{h\Sigma_r}$ , we define the power operation of weight  $r$  by sending a class  $x \in \pi_0 R = [\mathbb{I}, R]$  to be the composite

$$\mathbb{I} \xrightarrow{P} \mathbb{I}_{h\Sigma_r} \hookrightarrow \bigoplus_s \mathbb{I}_{h\Sigma_s} \cong \mathbb{I}\{t\} \xrightarrow{t \mapsto x} R.$$



# Power Operations

If  $E$  is a structured commutative ring spectra (ie, a commutative S-algebra), we have a map  $E^*(X) \rightarrow E^*(X^m)$  given by  $x \rightarrow x^{\times m}$ , then there is **total m-th power operation**

$$P_m : E^0(X) \rightarrow E^0(X \times B\Sigma_m)$$

If  $h^*$  is a multiplicative cohomology theory, that is, we have map:  $h^p(X) \otimes h^q(X) \rightarrow h^{p+q}(X)$ . Then we have the m-th power map

$$h^q(X) \rightarrow h^{mq}(X) : \quad x \mapsto x^m.$$

Let  $R$  be a commutative S-algebra in the context of EKMM category, and  $M$  is an  $R$ -module, then we can define a free commutative  $R$ -algebra on  $M$ :

$$\mathbb{P}_R M = \bigvee_{m \geq 0} \mathbb{P}_R^m(M) \cong \bigvee_{m \geq 0} (M \wedge_R \cdots \wedge_R M)_{h\Sigma_m}$$

And if  $A$  is commutative  $R$ -algebra  $A$ , then we have a map

$$\mu : \mathbb{P}_R A \rightarrow A.$$



If  $A$  is a commutative  $R$ -algebra.

1. We can choose a  $\alpha : R \rightarrow \mathbb{P}_R^m(R) \cong R \wedge B\Sigma^+$
2. For any element  $x \in \pi_0 A$  which is represented by  $f_x : R \rightarrow A$ .
3. We define an element  $Q_\alpha(x) \in \pi_0 A$  which is represented by the following composite

$$R \xrightarrow{\alpha} \mathbb{P}_R^m(R) \xrightarrow{\mathbb{P}_R^m(f_x)} \mathbb{P}_R^m(A) \subset \mathbb{P}_R(A) \xrightarrow{\mu} A$$

So we have defined a map  $Q_\alpha : \pi_0 A \rightarrow \pi_0 A$ . And we can also define  $Q_\alpha : \pi_q A \rightarrow \pi_{q+r} A$  if

$$\alpha : \Sigma^{q+r} R \rightarrow \mathbb{P}_R^m(\Sigma^q R) \cong R \wedge B\Sigma_m^{qV_m}.$$



# Example of Power Operations

Let  $H = H\mathbb{F}_2$  is the mod 2 Maclane spectrum, if  $A$  is a commutative  $H$ -algebra spectrum, then  $\pi_* A$  is a graded commutative  $\mathbb{F}_2$ -algebra.  $Q^r : \pi_q A \rightarrow \pi_{q+r} A$

$$\square\square Q^r(x+y) = Q^r(x) + Q^r(y).$$

$$\square\square Q^r(xy) = \sum Q^i(x) Q^{r-i}(y).$$

$$\square\square Q^r Q^s(x) = \epsilon_{r,s}^{i,j} Q^i Q^j(x) \text{ if } r > 2s, \text{ where } i \leq 2j.$$

if  $A = \text{Fun}(\Sigma^\infty X, H\mathbb{F}_2)$ , then the power operations are Steenrod operations on  $H^*(X, \mathbb{F}_2)$ .

## Power Operations in K-theory

If  $K$  is the complex K-theory spectrum, and  $A$  is a  $p$ -complete  $K$ -algebra.  $\psi^p : \pi_0 A \rightarrow \pi_0 A$ .

$$\square\square \psi^p(x+y) = \psi^p(x) + \psi^p(y).$$

$$\square\square \psi^p(x) \equiv x^p \pmod{p}.$$

$$\square\square \psi(xy) = \psi(x)\psi(y).$$

# Power Operation in Morava E-theories

## Theorem (Rezk)

There exists a monad  $\mathbb{T}$  on the category of discrete  $E_0$ -modules whose category of algebras  $\text{Alg}_{\mathbb{T}}$  is the image of the functor  $\pi_0(-)$  on commutative E-algebras.

$$\begin{array}{ccc} & & \text{Alg}_{\mathbb{T}} \\ & \nearrow \pi_0 & \downarrow U_{\mathbb{T}} \\ \text{CAlg}_E^{\wedge} & \xrightarrow{\pi_0} & \text{CRing}_{E_0} \end{array}$$

In the case  $n = 1$  and  $E = E(\mathbb{F}_p, \mathbb{G}_m) = KU_p$ .  $\text{Alg}_{\mathbb{T}}$  can be identified with the category  $\text{CRing}_{\delta}$ -rings. If  $R$  is a  $T(1)$ -local commutative  $KU_p$  algebra, then there is a operation  $\delta : \pi_0(R) \rightarrow \pi_0(R)$  which act as a  $p$ -derivation

$$\psi(x) = x^p + p\delta(x)$$

For formal reasons, the forgetful functor  $U_{\mathbb{T}} : \text{Alg}_{\mathbb{T}} \rightarrow \text{CRing}_{E_0}$  admits both left and right adjoint

$$U_{\mathbb{T}} : \text{Alg}_{\mathbb{T}} \rightleftarrows \text{CRing}_{E_0} : W_{\mathbb{T}}$$

$$F_{\mathbb{T}} : \text{CRing}_{E_0} \rightleftarrows \text{Alg}_{\mathbb{T}} : U_{\mathbb{T}}$$

In the case of  $\text{Alg}_{\mathbb{T}} = \text{CRing}_{\delta}$  at height 1, we have  $W_{\mathbb{T}}(A) = W(A) = \pi_0 E(A)$ . By composing with the adjunction

$$(-/p)^{\sharp} : \text{CRing} \rightleftarrows \text{Perf}_{\mathbb{F}_p} : \text{Incl}$$

We obtain an adjunction

$$(U(-)/p)^{\sharp} : \text{CRing}_{\delta} \rightleftarrows \text{Perf}_{\mathbb{F}_p} : \pi_0 E(-)$$

This adjunction can be generalize to any height.

**Theorem (Burklund-Schlank-Yuan, 2022)**

There is an adjunction

$$(U(-)/m)^{\sharp} : \text{Alg}_{\mathbb{T}} \rightleftarrows \text{Perf}_k : \pi_0 E(-)$$

where the right adjoint  $\pi_0 E(-)$  is fully faithful.

### Theorem (Rezk)

Let  $A$  be a  $K(n)$ -local  $E$ -Algebra, then the power operation of the homotopy group of  $A$  has the structure of an amplified  $\Gamma$ -ring.

We say that a graded  $\Gamma$ -algebra  $B$  satisfies the congruence condition if for all  $x \in B_0$ ,

$$x\sigma \equiv x^p \bmod pB.$$

### Theorem

An object  $B \in \text{Alg}_\Gamma^*$  which is  $p$ -torsion free, then  $B$  admits the structure of a  $\mathbb{T}$ -algebra if and only if  $B$  satisfies the congruence condition.

# Sheaves on the Categories of Deformations

Let  $R$  be complete local ring whose residue has characteristic  $p$ . Let  $\phi : R \rightarrow R, x \mapsto x^p$  be the Frobenius map.

The **Frobenius isogeny**  $\text{Frob} : G \rightarrow \phi^* G$  is induced by the relative Frobenius map on rings.

We write  $\text{Frob}^r : G \rightarrow (\phi^r)^* G$  which is the composition  $\phi^*(\text{Frob}^{r-1}) \circ \text{Frob}$



Let  $(G, i, \alpha)$  and  $(G', i', \alpha')$  be two deformation of  $G_0$  to  $R$ . A deformation of  $\text{Frob}^r$  is a homomorphism  $f : G \rightarrow G'$  of formal groups over  $R$  which satisfying

1.  $i \circ \phi^r = i'$  and  $i^*(\phi^r)^* G_0 = (i')^* G_0$ .

$$\begin{array}{ccc} k & \xrightarrow{i'} & R/m \\ \phi^r \downarrow & \nearrow i & \\ K & & \end{array}$$

2. the square

$$\begin{array}{ccc} i^* G_0 & \xrightarrow{i^*(\text{Frob}^r)} & i^*(\phi^r)^* G_0 \\ \alpha \downarrow & & \downarrow \alpha' \\ \pi^* G & \xrightarrow{\pi^*(f)} & \pi^* G' \end{array}$$

of homomorphisms of formal groups over  $R/m$  commutes.



We let  $\text{Def}_R$  denote the category whose objects are deformations of  $G_0$  to  $R$ , and whose morphisms are homomorphisms which are deformation of  $\text{Frob}^r$  for some  $r \geq 0$ . Say that a morphism in  $\text{Def}_R$  has **height**  $r$ , if it is a deformation of  $\text{Frob}^r$ .

### Proposition

Let  $G$  be deformation of  $G_0$  to  $R$ , then the assignment  $f \rightarrow \text{Ker} f$  is a one-to-one correspondence between the morphisms in  $\text{Sub}_R^r$  with source  $G$  and the finite subgroup of  $G$  which have rank  $p^r$ .

For the following, Let  $G_E = G_{\text{univ}}/E_0$  be the universal deformation of  $G_0$ .

# Deformation of Frobenius

## Theorem (Strickland, 97)

Let  $G_0/k$  be a formal group of height  $h$  over a perfect field  $k$ . For each  $r > 0$ , there exists a complete local ring  $A_r$  which carries a universal height  $r$  morphism  $f_{univ}^r : (G_s, i_s, \alpha_s) \rightarrow (G_t, i_t, \alpha_t) \in Sub^r(A_r)$ . That is the operation  $f_{univ}^r \rightarrow g^*(f_{univ}^r)$  define a bijective relation from the set of local homomorphism  $g : A_r \rightarrow R$  to the set  $Sub_R^r$ . Furthermore, we have:

1.  $A_0 \approx W(k)[[v_1, \dots, v_{h-1}]]$ .
2. Under the map  $s : A_0 \rightarrow A_r$  which classifies the source of the universal height  $r$  map, i.e.  $G_s = i^* G_E$ , and  $A_r$  is finite and free as an  $A_0$  module.
3. Under the map  $t : A_0 \rightarrow A_r$  which classifies the target of the universal height  $r$  map, i.e.  $G_t = t^* G_E$

So there is a bijection

$$\{g : A_r \rightarrow R\} \rightarrow Sub^r(R)$$

$$g \mapsto g^*(f_{univ}^r)(g^* G_s \rightarrow g^* G_t)$$



Thus,  $Sub = \coprod Sub^r$  is a affine graded-category scheme. In particular, there are ring maps:

$$s = s_k, t = t_k : A_0 \rightarrow A_k,$$

which is induced by  $E^0$  cohomology on  $B\Sigma \rightarrow *$

$$\mu = mu_{k,l} : A_{k+l} : A_{k+l} \rightarrow A_k^s \otimes_{A_0} {}^t A_l$$

which classifying the source, target, and composite of morphisms.

### Theorem (Strickland, 1998)

The ring  $A[r]$  in the universal deformation of Frobenius is isomorphic to  $E^0(B\Sigma_{p^r})/I$ , i.e,

$$A[r] \cong E^0(B\Sigma_{p^r})/I$$

where  $I$  is transfer ideal.

So for the power operation

$$R^k(X) \rightarrow R^k(X \times B\Sigma_m)$$

For  $x = *$ , we have  $\pi_0 R \rightarrow E^0(B\Sigma_{p^r})/I \otimes \pi_0 R = A[r] \otimes \pi_0 R$ . This make  $\pi_0 R$  becomes a  $\Gamma$ -module.

# Andre-Quillen Cohomology Groups

Let  $A$  be a commutative ring,  $B$  be an  $A$ -algebra, and  $M$  be a  $B$ -module. The André-Quillen cohomology groups are the derived functors of the derivation functor  $\text{Der}_A(B, M)$ .

Morphisms of commutative rings  $A \rightarrow B \rightarrow C$  and a  $C$ -module  $M$ , there is a three-term exact sequence of derivation modules:

$$0 \rightarrow \text{Der}_B(C, M) \rightarrow \text{Der}_A(C, M) \rightarrow \text{Der}_A(B, M)$$

Let  $P$  be a simplicial cofibrant  $A$ -algebra resolution of  $B$ . André notates the  $q$ th cohomology group of  $B$  over  $A$  with coefficients in  $M$  by  $H^q(A, B, M)$ , while Quillen notates the same group as  $D^q(B/A, M)$ . The  $q$ -th André-Quillen cohomology group is:

$$D^q(B/A, M) = H^q(A, B, M) \stackrel{\text{def}}{=} H^q(\text{Der}_A(P, M))$$

Let  $L_{B/A}$  denote the relative cotangent complex of  $B$  over  $A$ . Then we have the formulas:

$$D^q(B/A, M) = H^q(\text{Hom}_B(L_{B/A}, M))$$

$$D_q(B/A, M) = H_q(L_{B/A} \otimes_B M)$$



In general , let  $C$  be an operad,  $A$  is an  $C$  -algebra,  $M$  is an Module. The square zero extension  $M \rtimes A$  is a new  $A$  -algebra

We have definitions of derivation

$$\mathcal{D}|\nabla_C(X, M) := \text{Alg}_{C/A}(X, M \rtimes A)$$

We can form the simplicial module  $K(M, n)$  over  $A$  whose normalization  $NK(M, n) \cong M$ . And define  $K_A(M, n) = K(M, n) \rtimes A$ .

We define the Andre-Quillen Cohomology of  $X$  with coefficients in  $M$  by the formula

$$D_C^n(X, M) = [X, K_A(M, m)]_{s\text{Alg}/A} \cong \pi_0 \text{Map}_{s\text{Alg}/A}(X, K_A(M, n))$$

$$D_C^n(X, M) \cong \pi_{-n} \text{hom}_{s\text{Alg}/A}(X, K_A M)$$

**Lemma**

$$D_C^n(X, M) = H^n N(\mathcal{D}|\nabla_C(Y, M))$$

where  $Y$  is some cofibrant model for  $X$  and  $N$  is some normalization functor from comsimplicial  $k$ -module to cochain complex.

Let  $X \rightarrow Y$  be a morphism of  $\mathcal{F}$ -algebra in spectra. There is second quadrant spectral sequence with  $E_2$  term

$$E_2^{0,0} = \text{Hom}_{E_*\mathcal{F}}(E_*X, Y_*)$$

and

$$E_2^{s,t} = D_{E_*T}^s(E_*X, \Omega^t Y_*)$$

converge to

$$\pi_{t-s}(\text{Map}_{\text{Alg}_F}(X, Y), \phi)$$



# Goerss-Hopkins Obstruction Theory

## Goerss-Hopkins Obstruction Theory

Let  $R$  and  $S$  be  $E$ -local  $E_\infty$ -rings, and let  $A = E_*R$  and  $B = E_*S$ . Given a map  $\phi : A \rightarrow B$  of commutative algebras in  $E_*E$ -comodules, there exists an inductively defined sequence of obstructions valued in

$$\mathrm{Ext}_{\mathrm{Mod}_A(\mathrm{Comod}_{E_*E})}^{n+1,n}(L_{A/E_*}, B)$$

which vanishes iff there is an  $E_\infty$ -ring map  $\tilde{\phi} : R \rightarrow S$  such that  $E_*(\tilde{\phi}) = \phi$ .

# Elliptic Cohomology

An elliptic cohomology consists of

- ▣ An even periodic spectrum  $E$ .
- ▣ An elliptic curve  $C$  over  $\pi_0 E$ .
- ▣  $\phi : G_E \cong \hat{C}$

We denote this data as  $(E, C, \phi)$

## Theorem(Goerss-Hopkins-Miller-Lurie)

There is a sheaf  $\mathcal{O}_{tmf}$  of  $E_\infty$ -ring spectra over the stack  $\overline{\mathcal{M}}_{ell}$  for the *étale* topology. For any *étale* morphism  $f : \mathrm{Spec}(R) \rightarrow \overline{\mathcal{M}}_{ell}$ , there is a natural structure of elliptic spectrum  $(\mathcal{O}_{tmf}(f), C_f, \phi)$ , satisfying  $\pi_0 \mathcal{O}_{tmf}(f) = R$ , and  $C_f$  is a generalized elliptic curve over  $R$  classified by  $f$ .

$Tmf = \mathcal{O}_{tmf}(\overline{\mathcal{M}}_{ell} \rightarrow \overline{\mathcal{M}}_{ell})$ , topological modular forms.



# Topological Automorphic Forms

## Theorem

let  $M_{pd}^n$  denote the moduli stack of one dimensional height  $n$   $p$ -divisible group, then there is a sheaf of  $E_\infty$ -ring space,  $\mathcal{O}^{top}$  on the etale site. such that for any

$$E := \mathcal{O}^{top}(\mathrm{Spec} R \xrightarrow{G} M_{pd}^n)$$

we have

$$F_E = G^0$$

where  $G_0$  is the formal part of the  $p$ -divisible group  $G$ .

The main issue of this construction is that fro a general  $n$ -dimensional abelian variety, their associated  $p$ -divisible group are not 1-dimensional.

PEL Shimura stacks are moduli stacks of abelian varieties with the extra structure of Polarization, Endomorphisms, and Level structure . A class of PEL Shimura stacks (associated to a rational form of the unitary group  $U(1, n_1)$ ) whose PEL data allow for the extraction of a 1-dimensional  $p$ -divisible group satisfying the hypotheses of above theorem.



# Orientations



# Obstructions to $H_\infty$ -maps

$$\begin{array}{ccc}
 H_\infty \mathcal{C} & \longrightarrow & (\text{formal groups with descent data}) \\
 \downarrow & & \downarrow \\
 \text{homogeneous spectra } \mathcal{C} & \longrightarrow & (\text{formal groups})
 \end{array}$$

**Theorem (Ando-Hopkins-Strickland, 2004)**

The rule which associates a level structure

$$l : A \rightarrow i^* G(R)$$

to a map  $\psi_l^E : \mathrm{Spf} R \rightarrow S_E$  given by the ring map  $\pi_0 E \xrightarrow{D_A} \pi_0 E^{BA^*} \rightarrow R$  and the isogeny

$$\psi_l^{G/E} : i^* G \rightarrow \psi_l^* G$$

is descent data for level structure on the formal group  $G$  over  $S_E$ .

$\mathcal{L}$  is a line bundle over  $G$ . Given a subset  $I \subset \{1, \dots, k\}$ ,  $\sigma_I : G_S^k \rightarrow G$  defined by  $\sigma_I(a_1, \dots, a_k) = \sum_{i \in I} a_i$ .

We define a line bundle over  $G_S^k$  by

$$\Theta^k(\mathcal{L}) = \bigotimes_{I \subset \{1, \dots, k\}} (\mathcal{L}_I)^{(-1)^{|I|}}$$

And set  $\Theta^0(\mathcal{L}) = \mathcal{L}$ .

$$\begin{aligned}\Theta^0(\mathcal{L})_a &= \mathcal{L}_a \\ \Theta^1(\mathcal{L})_a &= \frac{\mathcal{L}_0}{\mathcal{L}_a} \\ \Theta^2(\mathcal{L})_{a,b} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b}}{\mathcal{L}_a \otimes \mathcal{L}_b} \\ \Theta^3(\mathcal{L})_{a,b,c} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b} \otimes \mathcal{L}_{a+c} \otimes \mathcal{L}_{b+c}}{\mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c \otimes \mathcal{L}_{a+b+c}}\end{aligned}$$



### Definition

A  $\Theta^k$  structure on a line bundle  $\mathcal{L}$  over a group  $G$  is a trivialization  $s$  of the line bundle  $\Theta^k(\mathcal{L})$  such that

1. For  $k > 0$ ,  $s$  is a rigid section.
2.  $s$  is symmetric, i.e., for each  $\sigma \in \Sigma_k$ , we have  $\xi_\sigma \pi_\sigma^* s = s$ .
3. The section

$s(a_1, a_2, \dots) \otimes s(a_0 + a_1, a_2, \dots)^{-1} \otimes s(a_0, a_1 + a_2, \dots) \otimes s(a_0, a_1, \dots)^{-1} \otimes$   
corresponds to 1.

If  $g : MU\langle 2k \rangle \rightarrow E$  is an orientation, then the composition

$$((\mathbb{C}P^\infty)^k)^V \rightarrow MU\langle 2k \rangle \rightarrow E$$

represents a rigid section  $s$  of  $\Theta^k(I_G(0))$

### Theorem

For  $0 \leq k \leq 3$ , the maps of ring spectra  $MU\langle 2k \rangle \rightarrow E$  are in one to one correspondence with  $\Theta^k$ -structures on  $\mathcal{I}(0)$  over  $G_E$ .

### Theorem (Ando-Hopkins-Strickland, 2004)

Let  $g : MU\langle 2k \rangle \rightarrow E$  be a homotopy multiplicative map,  $s = s_g$  be the section of  $\Theta^k(I_G(0))$  as before. If the map  $g$  is  $H_\infty$ , then for each level structure

$$A \xrightarrow{l} i^*G,$$

the section  $s$  satisfy the identity

$$\tilde{N}_{\psi_l^{G/E}} s = (\psi_l^E)^* i^* s$$

And if  $k \leq 3$ , the converse is true.

Using this theorem, they proved the  $\sigma$  orientation of an elliptic spectrum is an  $H_\infty$  map. Zhu (2020) proved that the map  $MU\langle 0 \rangle \rightarrow E$  coming from a coordinate of  $\mathrm{Spf} E^0(\mathbb{C}^\infty)$  is a  $H_\infty$  map, since the map satisfying the condition above, which is called norm coherence.

# Obstructions to $E_\infty$ -maps

Hopkins-Lawson obstruction theory (2018): There exists a diagram of  $E_\infty$ -ring spectra

$$\mathbb{S} \rightarrow MX_1 \rightarrow MX_2 \rightarrow MX_3 \rightarrow \cdots$$

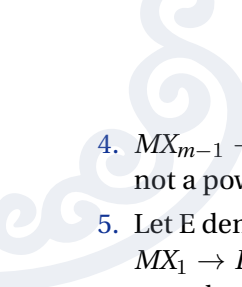
such that the following hold:

1.  $\lim MX_n \rightarrow MU$  is an equivalence.
2.  $\mathrm{Map}_{E_\infty}(MX_1, E) \simeq \mathrm{Or}(E)$  for each  $E_\infty$ -ring  $E$ .
3. Given  $m > 0$  and an  $E_\infty$ -ring  $E$ , there is a pull back square

$$\begin{array}{ccc} \mathrm{Map}_{E_\infty}(MX_m, E) & \longrightarrow & \mathrm{Map}_{E_\infty}(MX_{m-1}, E) \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \mathrm{Map}_*(F_m, \mathrm{Pic}(E)) \end{array}$$


where  $F_m$  is a certain pointed space.



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4.  $MX_{m-1} \rightarrow MX_m$  is a rational equivalence if  $m > 1$ , a  $p$ -local equivalence if  $m$  is not a power of  $p$ , and a  $K(n)$ -local equivalence if  $m > p^n$ .
  5. Let  $E$  denote an  $E_\infty$  such that  $\pi_* E$  is  $p$ -local and  $p$ -torsion free. Then an  $E_\infty$ -map  $MX_1 \rightarrow E$  extends to an  $E_\infty$  map  $MX_p \rightarrow E$  if and only if the corresponding complex orientation of  $E$  satisfies the Ando criterion.

**Theorem (Senger, 2022)**

Let  $E$  denote a height  $\leq 2$  Landweber exact  $E_\infty$ -ring whose homotopy groups is concentrated in even degrees. Then any complex orientation  $MU \rightarrow E$  which satisfies the Ando criterion lifts uniquely up to homotopy to an  $E_\infty$ -ring map  $MU \rightarrow E$ .





The proof of Senger's theorem was based on E-cohomology of some certain spaces. We have the following pullback square.

$$\begin{array}{ccc} E & \longrightarrow & \prod_p E_p^\wedge \\ \downarrow & & \downarrow \\ E_{\mathbb{Q}} & \longrightarrow & (\prod_p E_p^\wedge)_{\mathbb{Q}} \end{array}$$

$\mathrm{Map}_{E_\infty}(MU, R) \simeq \mathrm{Or}(R)$  for a rational  $E_\infty$ -ring  $R$ , and  $\pi_1 \mathrm{Map}_{E_\infty}(MU, R) \cong \pi_1 \mathrm{Or}(R) \cong 0$ , if  $R$  is concentrated in even degrees.

$$\begin{array}{ccc} \pi_0 \mathrm{Map}_{E_\infty}(MU, R) & \longrightarrow & \pi_0 \mathrm{Map}_{E_\infty}(MU, \prod_p E_p^\wedge) \\ \downarrow & & \downarrow \\ \pi_0 \mathrm{Or}(E_{\mathbb{Q}}) & \longrightarrow & \pi_0 \mathrm{Or}((\prod_p E_p^\wedge)_{\mathbb{Q}}) \end{array}$$

$$\begin{array}{ccc} \pi_0 \mathrm{Or}(E) & \longrightarrow & \pi_0 \mathrm{Or}(\prod_p E_p^\wedge) \\ \downarrow & & \downarrow \\ \pi_0 \mathrm{Or}(E_{\mathbb{Q}}) & \longrightarrow & \pi_0 \mathrm{Or}((\prod_p E_p^\wedge)_{\mathbb{Q}}) \end{array}$$



It suffices to lift the induced complex orientation of  $E_p^\wedge$ .

Assume that  $E$  is  $p$ -complete. So we only need to prove

$$\pi_0 \mathrm{Map}_{E_\infty}(MX_{p^2}, E) \rightarrow \pi_0 \mathrm{Map}_{E_\infty}(MX_p, E)$$

is surjective.

There is a cofiber sequence.

$$\mathrm{Map}_{E_\infty}(MX_{p^2}, E) \rightarrow \mathrm{Map}_{E_\infty}(MX_p, E) \rightarrow \mathrm{Map}_*(F_{p^2}, \mathrm{Pic}(E))$$

and a equivalence

$$\mathrm{Map}_{E_\infty}(F_m, \mathrm{Pic}(E)) \simeq \mathrm{Hom}(\Sigma^\infty F_m, \mathrm{pic}(E)) \simeq \mathrm{Hom}(\Sigma^\infty F_m, \Sigma E).$$

It suffices to show that

$$E^1(\Sigma^\infty F_{p^2}) \simeq 0$$



Lemma (Senger , 2022)

$$E^{2n}(F_p) \cong E^{2n+1}(F_{p^2}) \cong 0.$$

Let  $L_m$  denote the nerve of the poset of proper direct sum decomposition of  $\mathbb{C}^m$ , and  $(L_m)^\diamond$  is its unreduced suspension.

$$F_m \simeq ((L_m)^\diamond \wedge S^{2m})_{hU(m)}.$$