Lecture Notes on Chromatic Homotopy Theory

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1 Introduction

By Brown's representatilty theorem, a cohomology theory of topological spaces corresponds to a spectrum. All spectra form a closed symmetric monoidal category, called the stable homotopy category. Studying the stable homotopy category is a central topic in algebraic topology. There are many models of spectra making it become a closed symmetric monoidal category. See [Ada74] for an early discussion of stable homotopy category, [EKMM97] for S-module approach, and [Lur17] for ∞-category approach.

Chromatic homotopy theory uses chromatic localizations and the chromatic filtration to study stable homotopy category. The heart of chromatic homotopy theory is complex oriented cohomology theories. Each complex oriented cohomology theory can associate an one dimensional formal group. Studying the associated formal groups can help us understand complex oriedted cohomology theories. Like heights of formal groups can distinguish different complex oriented cohomology theories. Choosing a coordinate of a formal can produce a formal group law. Quillen [Qui69] proved that the complex cobordism MU is the universal complex oriented cohomology theory and its associated formal group law is the universal formal group law over the Lazard ring. Using the Landweber exact functor theorem [Lan76], one can construct many complex oriented cohomology theories. Morava E-theories are just constructed by using this theorem. Morava K-theories are another important complex oriented cohomology theories in chromatic homotopy theory, which is constructed by tensoring some certain spectra together. Localizing with respect to Morava E-theories and Morava K-theories is the most common method in chromatic homotopy theory when dealing with spectra. Another very important examples in chromatic homotopy theory are elliptic cohomology theories, and their global section, the topological modular forms, which is useful in quantum field theory.

We review definitions and results in chromatic homotopy theory in this chapter. More details can be found in [Hop99], [Mil94], [And95], [Lur10] and [Pet18].

2 Complex Oriented Cohomology theory

2.1 Moduli Stack of Formal Groups

We first review some classical results in formal groups and formal group laws more details can be found in the chapter 'Formal Groups and P-divisible groups'.

Definition 2.1 *A commutative one-dimensional formal group over R is a functor*

$$F : C_R \to \mathbf{Ab}$$

which is isomorphic to $\hat{\mathbb{A}}^1$ as a functor.

A coordinate X on F is a natural isomorphism $x : F \to \hat{\mathbb{A}}^1 = \hat{\mathbb{A}}^1_R$ of functors to pointed set. It gives an isomorphism $\Gamma(F, \mathcal{O}_F) \cong R[[X]]$.

Definition 2.2 *A formal group law over a ring R is a formal power series $F \in R[\![x_1, x_2]\!]$ satisfying the following conditions*

- $F(x, 0) = F(0, x) = x$ *(Identity)*
- $F(x_1, x_2) = F(x_2, x_1)$ *(Commutativity)*
- $F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$ *(Associativity) If R is a graded ring, we require F to be homogeneous of degree 2 where $|x_1| = |x_2| = 2$.*

Theorem 2.3 *1. There is a universal formal group law $F_{univ}(x, y) \in L[\![x, y]\!]$ over a ring L, such that for any other formal group law $F(x, y) \in R[\![x, y]\!]$ over a ring R, there is a ring morphism $f : L \to R$ such that $f^*(F_{univ}(x, y)) = F(x, y)$.*

2. $L \cong \mathbb{Z}[t_1, t_2, \cdots]$.

Let $f(x, y) \in R[\![x, y]\!]$ be a formal group law over a commutative ring R. For every non-negative integer n, we define the n-series $[n](t) \in R[\![t]\!]$ as

1. If $n = 0$,we set $[n](t) = 0$.
2. If $n > 0$, we set $[n](t) = f([n-1](t), t)$.

For every integer n, the n-series $[n](t)$ determine a homomorphism from the formal group law f to itself, i.e., we have $f([n](x), [n](y)) = [n]f(x, y)$.

Proposition 2.4 *Let R be commutative ring, $p = 0$ in R. If f is formal group law over R, then the p-series $p[t]$ is either 0 or equals $\lambda t^{p^n} + O(t^{p^n+1})$ for some $n > 0$.*

Definition 2.5 *Fro a formal group law over a commutative ring R . Let v_n denote th coefficient of t^Pn in the p-series. We say that f has height $\leq n$ if $v_i = 0$ fro $i < n$, we say f has height exactly n if it has height $\leq n$ and v_n is invertible.*

Example 2.6 *The formal multiplicative group $f(x, y) = x + y + xy$, which the n-series is $[n](t) = (1 + t)^n - 1$. If $p = 0$ in R, then $[p](t) = (1 + t)^p - 1 = t^p$, so f has exactly height 1.*

Example 2.7 *The formal additive group $f(x, y) = x + y$, if $p = 0$ in R. Then $[p](t) = 0$, so f has infinite height.*

We know that a coordinate of a formal group determine a formal group law. So a natural question is if we already have a formal group law, how do we get a formal group. We define a new functor

$$G_f : \mathrm{Alg}_R \to \mathbf{Ab} \quad A \mapsto \{a \in A : \exists n, a^n = 0\}$$

This definition make sense, because if a and b are nilpotent, then the sum only have finte nonzero terms. We call this G_f the formal associated to f.

Definition 2.8 *Let G be a formal group over R. The Lie algebra of G is the abelian group ,*

$$\mathfrak{g} = \ker(G(R[t]/(t^2)) \to G(R).$$

We notice that if $G = G_f$ for some formal group law, then we have an isomorphism $\mathfrak{g} \simeq tR[t]/(t^2) \simeq R$. Actually if a formal group comes form a formal group law if and only its lie algebra is isomorphic to R.

There is a geometric interpretation of the height of a formal group. Let $\mathcal{F} : \mathrm{Alg}_R \to \mathbf{Ab}$ be a formal group of height exactly n. Then $\mathcal{F}[p] = \ker(\mathcal{F} \xrightarrow{p})$ is representable by a finite flat group scheme of rank p^n. If we assume $\mathcal{F}$ is defined by a formal group law $f(x, y)$ whose p-series $[p](t) = \lambda p^{t^n} + \cdots$ where λ is invertible. Then we have $\mathcal{F}[p] = \mathrm{Spec}R[[t]]/(\lambda t^{p^n} + \cdots)$

We know that a formal group over R corresponds to a morphism $L \to R$, where the ring L is the Lazard ring, there fore can be understood as a quasicoherent sheaf on the affine scheme $\mathrm{Spec}L$. This sheaf admits an action of the affine group scheme $G = \mathrm{Spec}Z[b_1, b_2, ..]$ which assigns to each R to the group

$$\{g \in R[[t]] : g(t) = t + b_1t^2 + b_2t^3 + \cdots\}$$

and the action is given by

$$G \times FGL(R) \to FGL(R), \quad (g, f(x, y)) \mapsto gf(g^{-1}(x), g^{-1}(y)) \in FGL(R).$$

Definition 2.9 *We define the moduli stack of formal groups to be the quotient stack $\mathrm{Spec}L/G^+$, and denote it by $\mathcal{M}_{FG}$*

The stack $\mathcal{M}_{FG}$ assign each completer local ring R to the groupoid of formal groups over R. One can prove that it is isomorphic to the quotient stack $\mathrm{Spec}L/G^+$, which is our definition above. Where L is the Lazard ring and G^+ represented the group of isomorphism of formal group law.

By the concept of height, we can define some sub stack which satisfying some height condition. For every Z_p-algebra R, we define $\mathcal{M}_{FG}^{\leq n}$ to be the groupoid of formal groups of height $\leq n$ over R. Then $\mathcal{M}_{FG}^{\leq}$ can be viewed as a closed substack of $\mathcal{M}_{FG} \times \mathrm{Spec}(Z_p)$. Actually we have

$$\mathcal{M}_{FG}^{\leq n} \cong \mathrm{Spec}(L_p/(v_0, \cdots, v_{n-1}))/G^+$$

An important problem is to classify all formal group laws, we have the following result.

Theorem 2.10 *Two formal group laws $f(x, y), f'(X, y) \in k[[x, y]]$ over a algebraically closed field k are isomorphic if and only they have the same height.*

2.2 Complex Oriented Cohomology Theory

Complex oriented cohomology theory is the center of chromatic approach to stable homotopy theory. So we first review some basic conception of complex oriented cohomology theory.

We say that a cohomology theory E is multiplicative if there is a map $E^p(X) \otimes E^q(Y) \to E^{p+q}(X)$ for every topological space and every integer p, q.

Definition 2.11 *A multiplicative cohomology theory E is even periodic if $E^i(pt) = 0$ whenever i is odd and there exists $\beta \in E^{-2}(pt)$ such that multiplication with β induces an isomorphism $E^n(-) \cong E^{n-2}(-)$ for all n.*

Definition 2.12 *A complex orientation of E is a natural, multiplicative, collection of Thom classes $\mathcal{U}_V \in \widetilde{E}^{2n}(Th(V))$ for all complex vector bundles $V \to X$, where $\dim_{\mathbb{C}} V = n$, and satisfying the following condition*

- $f^*(\mathcal{U}_V) = \mathcal{U}_{f^*V}$ *for* $f : Y \to X$
- $\mathcal{U}_{V_1 \otimes V_2} = \mathcal{U}_{V_1} \cdot \mathcal{U}_{V_2}$.
- *For any $x \in X$, the class $\mathcal{U}_V$ maps to 1 under the composition*

$$\widetilde{E}^{2n}(Th(V)) \to \widetilde{E}^{2n}(Th(V_x)) \xrightarrow{\approx} \widetilde{E}^{2n}(S^{2n}) \xrightarrow{\approx} E^0(pt).$$

Proposition 2.13 *Let E be cohomology theory. Then any class $x \in E^2(\mathbb{C}P^\infty)$ restricting to 1 under the composite*

$$E^2(\mathbb{C}P^\infty) \to E^2(\mathbb{C}P^1) = E^2(S^1) \cong E^0(*)$$

extends in a unique way to a complex orientation of E.

We know that $E^2(\mathbb{C}P^\infty)$ is set of morphisms of spectrum $e : \Sigma^{\infty-2}(\mathbb{C}P^\infty) \to E$.

If there is a unit map $e : \mathbb{S} \to E$, then E is complex orientable if the map e factor as a composition

$$\mathbb{S} \simeq \Sigma^{\infty-2}\mathbb{C}P^1 \to \Sigma^{\infty-2}\mathbb{C}P^\infty \to E$$

By using the Atiyah-Hirzebruch spectral sequence

$$H^p(X, E^q(*)) \Rightarrow E^{p+q}(X)$$

The complex orientation of E determines an isomorphism

$$E^*(\mathbb{CP}^\infty) \cong E^*(*)[\![t]\!] = (\pi_* E)[[t]]$$

for some generator $t \in E^2(*)$. Furthermore given such an isomorphism, is equivalent to a complex orientation. In particularly, any even periodic theory is complex orientable.

We know that there is a multiplication map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

(We can view $\mathbb{C}^\infty$ as function space $\mathbb{C}[x]$, then we get a commutative and associative multiplication on $\mathbb{C}P^\infty$).

And still using the Atiyah-Hirzebruch spectrtal sequence, we can get $E^*(\mathbb{C}P^\infty \times bcP^\infty) \cong (\pi_* E)[[x, y]]$. We then get a map

$$(\pi_* E)[[t]] \cong E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times bcP^\infty) \cong (\pi_* E)[[x, y]]$$

We let $f(x, y) \in (\pi_* E)[[x, y]]$ denote the image of t under this map. It is easy to prove that $f(x, y)$ is a formal group law.

2.3 Complex Cobordism Spectrum MU

Let $EU(n) \to BU(n)$ be the universal bundle over the classifying space $BU(n)$, then we define spectrum $MU(n)$ to be $\Sigma^{\infty-2n}BU(n)/BU(n-1)$ which $BU(n)/BU(n-1)$actually is $Thv(EU(n))$, the Thom space of $EU(n)$.

And we define a new spectrum $MU = \lim MU(n)$. This spectrum MU is called the complex cobordism. The n-th homotopy group is just the bordsim group of n-dimensional complex manifold. MU admits a E_∞ structure since there is a diagram commutes up to homotopy for any complex oriented spectrum.

There is a diagram commutes up to homotopy.

$$\begin{tikzcd} MU(m) \times MU(n) \arrow[r] \arrow[d] & MU(m+n) \arrow[d] \\ E \otimes E \arrow[r] & E \end{tikzcd}$$

Theorem 2.14 () *[Quillen's theorem]* MU *is the universal complex oriented cohomology theory, ie* $L \cong \pi_* \mathrm{MU}$

2.4 The Landweber Exact Functor Theorem

We have seen that if E is a complex oriented cohomology theory. Then $\pi_* E$ is an algebra over the Lazard ring $L = \pi_* MU$. So it nature to ask a question: if we have a ring map $L \to R$, can we construct a complex oriented cohomology theory such that $R = \pi_* E$. There is a natural way to construct such cohomology theory by defining

$$E_*(X) = MU_*(X) \otimes_{\pi_* MU} R = MU_*(X) \otimes_L R$$

However the axiom of cohomology theory require some exactness of a certain sequence, the functor $- \otimes_L R$ general preserve exact sequence. If R is flat over L, then there is no problem. But this condition is too restrictive because the Lazard ring is too big. there is a weaker condition proved by Landweber.

We recall that if R is a commutative ring, and M is an R-module, then a sequence of elements $x_0, x_1, x_2, \cdots \in R$ is said to be regular for M if x_0 is not a zero divisor on M, x_1 is not a zero divisor on M/x_0M, x_2 is not a zero divisor on $M/(x_0M + x_1M)$, and so on.

Theorem 2.15 () *[The Landweber Exact Functor Theorem] Let M be a module over the Lazard ring. Then M is flat over $\mathcal{M}_{FG}$ if and only if for every prime number p, the elements $v_0 = p, v_1, v_2, \cdots \in L$ form a regular sequence for M.*

Let A be an even, periodic cohomology theory with $A(*) = k$. If k is a field with characteristic zero, then any 1-dimensional formal group over k is isomorphic to the formal additive group, correspondingly, A is equivalent to periodic ordinary cohomology theory with coefficients in k. If k is a field with characteristic p, then the formal group $\widehat{G} = \mathrm{Spf} A(CP^\infty)$ is classified up to isomorphism(over $\bar{k}$) by the height of the formal group.

2.5 Construction of Even Periodic Cohomology Theories

Definition 2.16 *Let R be a commutative ring and $\mathcal{L}$ be an invertible R-module. An $\mathcal{L}$-twisted formal group law is a formal power series*

$$f(x,y) = \sum a_{i,j}x^i y^j$$

where $a_{i,j} \in \mathcal{L}^{\otimes i+j-1}$ which satisfies the identities

$$f(x,y) = f(y,x), f(x,0) = x, f(x, f(y,z)) = f(f(x,y),z).$$

When $\mathcal{L} = R$, we find that the $\mathcal{L}$-twisted formal group law js just the formal group law over R.

Like a formal group law can determine a formal group, every $\mathcal{L}$-twisted formal group law determine a formal group $\mathrm{Spf}R[[\mathcal{L}]] = \mathrm{Spf}(\prod_n \mathcal{L}^{\otimes n})$, which is a functor

$$A \mapsto \mathrm{Hom}_R(\mathcal{L}, \sqrt{A}),$$

where $\sqrt{A}$ is the ideal of nilpotent elements of A, We still denote this formal group as G_f. And it is not hard to find if f is a $\mathcal{L}$-twisted formal group law, then there is a canonical isomorphism $\mathfrak{g}_{G_f} \simeq \mathcal{L}^{-1}$, where $\mathfrak{g}_{G_f}$ is the Lie algebra over G_f.

Actually, the converse is also true. That is to say, if we have a formal group G over R, whose lie algebra is $\mathfrak{g}$. Then there exists a $\mathfrak{g}^{-1}$ twisted formal group law f' and an isomorphism $G_{f'} \cong G$ lifts the lie algebra isomorphism $\mathfrak{g}_{G_{f'}} \simeq \mathfrak{g}$.

One can prove that an $\mathcal{L}$-twisted formal group law over R, is equivalent to a formal group law over $\bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$ where $\mathcal{L}^{\otimes n}$ has degree 2n. So this is equivalent a map of graded rings $L \to \bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$.

Lemma 2.17 *let f be an $\mathcal{L}$-twisted formal group law over R. Then we have the following equivalent conditions*

1. *G_f is classified by a flat map $q : \mathrm{Spec}R \to \mathcal{M}_{FG}$.*
2. *The graded L-module $\bigoplus \mathcal{L}^{\otimes n}$ is Landweber-exact.*

By using this lemma, we can construct many even periodic cohomology theory.

$$(E_R)_*(X) = MU_*(X) \otimes_L (\bigoplus_n \mathcal{L}^{\otimes n})$$

Example 2.18 *If $R = L$ is the Lazard ring, and $\mathcal{L} = R$ is trival, so $\bigoplus_n \mathcal{L}^{\otimes n} \cong L[\beta^{\pm}]$. So*

$$(E_R)_*(X) = MU_*(X) \otimes_L L[\beta^{\pm}] \cong MU_*(X)[\beta^{\pm}].$$

This spectrum is called the periodic complex bordism spectra *and is denoted by MP.*

Example 2.19 *Let f be a $\mathcal{L}$ twisted formal group law over a commutative ring R satisfying the theorem's condition, if we choose an isomorphism $\mathcal{L} \simeq R$. Then we get a homology theory*

$$(E_R)_*(X) = MU_*(X) \otimes_L R[\beta^{\pm}] \simeq MP_*(X) \otimes_L R.$$

In particular, we have $(E_R)_0(X) \otimes_L R = MU_{even}(X) \otimes_L R$.

3 Morava E- theories and Morava K-theories

We first review the Lubin-Tate theory. Let R be a local Artin ring having residue field k.

Deformation of formal groups: Let G_0 be a formal group over a perfect field k with characteristic p, then a deformation of G_0 to R is a triple (G, i, Ψ) satisfying

- G is a formal group over R,
- There is a map $i : k \to R/m$
- There is an isomorphism $\Psi : \pi^* G \cong i^* G_0$ of formal groups over R/m.

The following result due to [LT66] gives a universal deformation of the formal groups.

Theorem 3.1 *There is a universal formal group G over $R = W(k)[[v_1, \cdots, v_n - 1]]$ in the following sense: for every local Artin ring with residue filed k, we have a bijection*

$$\mathrm{Hom}_{/k}(R, A) \to \mathrm{Def}(A).$$

Morava E-Theories

Let k be a perfect field of characteristic p, f is a formal group law of height n over k. By Lubin -Tate's theorem, the deformation of by is classified by the ring $R = W(k)[\![v_1, \cdots, v_{n-1}]\!]$. Notice that this universial deformation over R is Landweber-exact: the sequence $v_0 = p, v_1, \cdots, v_{n-1}$ is regular,and v_n has invertible image in $R/v_1, \cdots, v_n$.

So using the construction in the last section, there is a even periodic spectrum $E(n)$ with

$$\pi_* E(n) = W(k)[\![v_1, \cdots, v_{n-1}]\!][\beta^{\pm 1}]$$

where β has degree 2. It's called **Morava E-theory** (also sometimes called Lubin-Tate theory or completed Johnson-Wilsdon theory). The cohomology theory $E(n)$ not only depends on n, but also a choice of field k and a formal group of height n over k.

The following theorem is very important, which implies that $E(n)$ is a commutative object in the category of spectra.

Theorem 3.2 *The Spectrum E is a commutative S-algebra [GH04].*

Morava K-Theories

For a prime p, we can consider the p-local complex cobordism spectrum $MU_{(p)}$ whose homotopy groups are $\pi_* MU_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, \cdots,]$, and we may assume that $v_i = t^{p^i - 1}$ for each $i > 0$.

For each integer k, let $M(k)$ denote the cofiber of the map $\sum^{2k} MU_{(p)} \to MU_{(p)}$ given by the multiplication by t_k. One can prove that each $M(k)$ admits a unital and homotopy associative multiplication .

Let $K(n)$ denote the smash product

$$MU_{(p)}[v_n^{-1}] \otimes_{MU_{(p)}} \bigotimes_{k \neq p^n - 1} M(k).$$

This spectrum $K(n)$ is called **Morava K-theory**. It is easy to calculate the homotopy groups of $K(n)$.

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}]/(t_0, t_1, \cdots t_{p^n-2}, t_{p^n}, \cdots) \cong \mathcal{F}_p[v_n^{\pm 1}]$$

where v_n has degree $2(p^n - 1)$ The following discussion implies that Morav K-theory plays a important role in the study of Sp..

- A commutative evenly graded ring is a graded field every nonzero homogeneous element is invertible. Equivalently, R is a field or $R \simeq k[\beta^{\pm}]$.
- We say a homotopy associative ring spectrum is a field if $\pi_* E$ is a graded filed.

Example 3.3 *For every prime p and every integer n, $K(n)$ is a field.*

Definition 3.4 *A ring spectrum $K \in \mathrm{Sp}$ is said to be a field if every K-module splits into a wedge of shifted copies of K.*

If K is field object, we have

$$K_*(X \wedge Y) \cong K_*(X) \otimes_{K_*} K_*(Y)$$

Theorem 3.5 () *[Hopkins-Smith] Any field object in Sp splits (additively) into a wedge of shifted copies of Morava K-theories. Morever, if R is a ring spectrum such that $K(n)_*(R) = 0$ for all $0 \leq n \leq \infty$, then $R \simeq 0$.*

4 The Chromatic Localization

Let S be a set of prime numbers, We say that a ring R is S-local, if all prime numbers not in S is invertible in R. And for the ring of integers $\mathbf{Z}$, let $\mathbf{Z}_S$ be the localization of $\mathbf{Z}$ at S, i.e . $\mathbf{Z}_S = \{a/b, a \in \mathbf{Z}, b \notin S\}$

A group G is said to be S-local if the p^{th} power map $G \to G$ is a bijection for all p not in S.

If G is abelian, then this map is a group homomorphism and is generally written as multiplication by p. In this case we have.

1. G is S-local;

2. G admits a structure of Z_S-module (necessarily unique);

Definition 4.1 *A spectrum X is called S-local if its homotopy groups are S-local abelian groups.*

The S-localization of a spectrum is its reflection into S-local spectra. The S-localization, may be constructed as the Bousfield localization of spectra with respect to the Moore spectrum $S(Z_S)$

4.1 Bousfield Localization

We recall that a map $f : X \to Y$ of spectrum is an E-equivalence if it induces isomorphism on E-homology groups.

Let $\mathcal{C}$ be a full subcategory of *Sp* of spectra, which is closed under shifts and homotopy colimits, and can be generated by small subcategory under homotopy colimits.

If X is a spectrum, define $G(X)$ to be the homotopy colimit of all $Y \in \mathcal{C}$ with a map to X.

We have a counit map $v : G(X) \to X$, and we let $L(X)$ denote the cofiber of v, then we have a cofiber sequence

$$G(X) \to X \to L(X).$$

A spectrum is $\mathcal{C}$-local if every may $Y \to X$ is nullhomotopic when $Y \in \mathcal{C}$. We denote the category of $\mathcal{C}$-local spectra as $\mathcal{C}^{\perp}$.

The general idea of localization at a spectrum E is to associate to any spectrum X the "part of X that E can see", denoted by $L_E X$. In particular, it is desirable that LE is a functor with the following equivalent properties:

$$E \wedge X \simeq * \Rightarrow L_E X \simeq *$$

If $X \to Y$ induces an equivalence $E \wedge X \to E \wedge Y$ then $L_E X \to L_E Y$.

Definition 4.2 *Fix a spectrum E. We say that a spectrum is E-acyclic if the smash product is $X \otimes E$ is 0. We denote G_E the collection of E-acyclic spectra. We say that a spectrum is E-local if every map for every $Y \in G_E$, the map $Y \to X$ is nullhomotopic.*

Then by the construction above, we have a cofiber sequence

$$G_E(X) \to X \to L_E(X).$$

where $L_E(X)$ is E-local. this functor is called Bousfield localization with respect to E.

And the map $X \to L_E(X)$ is characterized up to equivalence by two properties.

1. *The spectrum $L_E(X)$ is E-local.*

2. *The map $X \to L_E(X)$ is an E-equivalence.*

Lemma 4.3 *A spectrum X is E-local if and only if for each E-equivalence $S \to T$, the induced map $[T, X] \to [S, X]$ is an isomorphism.*

For G an abelian group, then the Moore spectrum MG of G is the spectrum characterized by having the following homotopy groups:

1. $\pi_{<0} MG = 0$;

2. $\pi_0(MG) = G$;

3. $H_{>0}(MG, Z) = \pi_{>0}(MG \wedge HZ) = 0$.

A basic special case of E-Bousfield localization of spectra is given by $E = MA$ the Moore spectrum of an abelian group A. For $A = Z_{(p)}$ this is p-localization, for $A = F_p$ this is p-completion, for $A = \mathbf{Q}$ is the rationalization of X.

Theorem 4.4 *p-localization is a smashing localization:*

$$L_{MZ_{(p)}} X \simeq MZ_{(p)} \wedge X$$

We denote this as $L_{MZ_{(p)}} X \simeq X_{(p)}$, which is called the Bousfield p-localization

A spectrum E is p-complete, if $\pi_* E$ is a (p)-adic complete ring. Bousfield localization at the Moore spectrum SF_p is p-completion to p-adic homotopy theory.

Theorem 4.5 *The localization of spectra at the Moore spectrum $MF_p$is given by the mapping spectrum out of $\Omega M\mathbf{Z}/p^\infty$:*

$$L_p = L_{MF_p} X \simeq [\Omega M\mathbf{Z}/p^\infty, X]$$

where $\mathbf{Z}/p^\infty = \mathbf{Z}[1/p]/\mathbf{Z}$. *We denote this spectrum* $L_p = L_{MF_p}X$ *as* $X_p^\wedge$

Theorem 4.6 $L_{M\mathbf{Q}}X = X \wedge L_{\mathbf{Q}}S^0 = X \wedge M\mathbf{Q} = X \wedge H\mathbf{Q}$ *is smashing, we call this as the rationalization of* X*, denote it as* $L_{\mathbf{Q}}X$.

Proposition 4.7 *Let* E*,* F*,*X *be spectra with* $E_*L_FX = 0$. *Then there is a homotopy pullback square*

$$\begin{tikzcd} L_{E\vee F}X \arrow[r] \arrow[d] & L_EX \arrow[d] \\ L_FX \arrow[r] & L_F(L_EX) \end{tikzcd}$$

So we have the following **Suillivan arithmetic square** for $E = \bigvee_p M(Z/p), F = H\mathbf{Q}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p L_pX \arrow[d] \\ L_{\mathbf{Q}}X \arrow[r] & L_{\mathbf{Q}}(\prod_p L_pX) \end{tikzcd}$$

In chromatic homotopy, we often cares the Bousfield localization with respect to the Morava E-theories and Morava K-theories.

Example 4.8 *Bousfield Localization with respect to Morava E-theory* $E(n)$*,* $L_{E(n)}$*.And one can prove that* $L_{E(n)}$ *behaves like restriction to the open substack* $\mathcal{M}_{FG}^{\leq n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbf{Z}_{(p)}$.

Theorem 4.9 *The localization functor* $L_{E(n)}$ *is smash, i.e., it commutes with finite products.*

Example 4.10 *Bousfield Localization with respect to Morava K-theory* $K(n)$*,* $L_{K(n)}$*.And one can prove that* $L_{K(n)}$ *behaves like completetion the locally closed substack* $\mathcal{M}_{FG}^{\leq n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbf{Z}_{(p)}$.

We define

$$L_n(X) := L_{K(0)\vee K(1)\vee\cdots\vee K(n)}X$$

In fact, for every n there exists a spectrum $E(n)$ with coefficents $E(n)_* = Z_{(p)}[v_1, \cdots, v_n][v_n^{-1}]$ called the Johnson-Wilson spectrum. that has property, E_n-acyclic spectra is equal to $K(0) \vee K(1) \vee \cdots \vee K(n)$-acyclic spectra. So we have $L_n = L_{E(n)}$. And it has been proved that $L_{E(n)} = L_{E_n}$, where E_n is the Morava E-spectrum.

4.2 Nilpotence

We say that a collection of ring spectra $\{E^\alpha\}$ detect nilpotence if for any p-local ring spectra R, $x \in \pi_m R$ is send to zero in $E_0^\alpha R$ for all α, then x is nilpotent in $\pi_* R$.

Theorem 4.11 () *[(Devinatz-Hopkins-Smith, 1988)] For any ring spectrum R, the kernel of the map $\pi_* R \to MU_* R$ consists of nilpotent elements. In particular, the single MU detects nilpotence.*

Theorem 4.12 *The spectra $\{K(n)\}_{0\leq n\leq\infty}$ detect nilpotence.*

Let E be a nonzero p-local ring spectrum, then $E \otimes K(n)$ is nonzero for some $0 \leq n \leq \infty$. If not, every element of $\pi_0 E$ is nilpotent, so $\mathbb{I} \in \pi_0 E$ is nilpotent, so that $E \simeq 0$.

4.3 Thick Subcategories

We first recall the definition of think subcategory. Let $\mathcal{C}$ be a full subcategory of finite p-local spectra. We say that $\mathcal{C}$ is **thick** if it contains 0 , closed under fiber and cofibers, and every retract of a spectrum belong to $\mathcal{C}$ also belongs to $\mathcal{C}$.

Lemma 4.13 *Let X be a finite p-local spectrum, if $K(n)_*(X) \simeq 0$ for some $n > 0$. Then $K(n-1)_*(X) = 0$.*

Definition 4.14 *We say that a p-local finite spectrum has type n if $K(n)_*(X) \neq 0$ and $K(m)_*(X) = 0$ for $m < n$. And we let $\mathcal{C}_{\geq n}$ be the category of p-local spectra which has type $\geq n$.*

Theorem 4.15 () *[Thick Subcategory Theorem] Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. Then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $0 \leq n \leq \infty$.*

Consider the cofiber sequence

$$\Sigma^k X \xrightarrow{f} X \to X/f$$

If we have X has type $\leq n$, we hope X/f has type $\leq n+1$

Definition 4.16 *Let X be finite p-local spectrum, a v_n self map is a map $f : \Sigma^q X \to X$ and satisfying the following,*

1. *f induces an isomorphism $K(n)_*(X) \to K(n)_* X$.*
2. *The induced map $K(m)_*(X) \to K(m)_*(X)$ is nilpotent, for $m \neq n$.*

Theorem 4.17 () *[Periodicity Theorem] Let X be a finite p-local spectrum of type $\geq n$, then X admits a v_n-self map.*

4.4 Telescopic Localization

We have an adjunction

$$\text{inclusion} : G_E = \{E - \text{acyclic}\} \leftrightarrow \text{Sp} : G_E$$

Localization with respect to E means localization with respect to G_E.

$$G_E \hookrightarrow \text{Sp} \xrightarrow{L_E} E - \text{local} = (G_E)^\perp$$

$$G_E(X) \longrightarrow X \longrightarrow L_E(X)$$

We know $E(n)$ acyclic means $E(n-1)$ acyclic and $K(n)$-acyclic, but $\ker L_E = G_E = \{E(n) - \text{acyclic}\}$, so we get inclusions

$$0 = \ker(id) \subset \ker(L_{E(\infty)}) \cdots \subset \ker(L_{E(n)}) \subset \ker(L_{E(n-1)}) \cdots \ker(L_{E(0)}) \subset \text{Sp}$$

by taking orthocomplement, we get

$$0 \subset E(1)\text{-local Sp} \subset \cdots \subset E(n-1)\text{-local Sp} \subset E(n)\text{-local Sp} \subset \cdots$$

We have $K(n)_*(X) = 0 \Rightarrow K(n-1)_*(X) = 0$.

$$\mathcal{C}_{\geq n} = \{X \in \text{Sp}_{(p)} | X \text{ has type } \geq n, i.e., K(m)_* X = 0, m < n\}$$

So we have sequence

$$(0) \subset \cdots \subset \mathcal{C}_{n+1} \subset \mathcal{C}_{\geq n} \subset \cdots \subset \mathcal{C}_{\geq 0} = \text{Sp}$$

by taking orthocomplement, we get

$$\mathcal{C}_{\geq 0} \text{ local spectra} \subset \cdots \subset \mathcal{C}_{\geq n} \text{ local spectra} \subset \mathcal{C}_{\geq n+1} \text{ local spectra} \subset \cdots$$

Definition 4.18 *Telescope Localization The telescope localization L_n^t: Localization with respect to $\mathcal{C}_{\geq n+1}$.*

$$C(X) \to X \to L_n^t(X).$$

where $C(X)$ is a filtered colimit of object in $\mathcal{C}_{\geq n+1}$

Definition 4.19 *We say a localization functor L is a smash localization if $L(X) = K \wedge X$ for a K.*

The following conditions are equivalent

1. L preserves homotopy colimits.
2. $C^{\perp} \subset \mathrm{Sp}$ is stable under homotopy colimits
3. G preserves homotopy colimits.
4. $L(X) = K \wedge X$.

Example 4.20

- *$L_{E(n)}$ is a smash localization.*
- *L_n^t is a smash localization.*
- *Realization and p-localization is a smash localization.*

For any smashing localization L

$$\ker(L_n^t) \subset \ker(L) \subset \ker(L_{E(n)})$$

So there is a comparison

$$L_n^t \to L \to L_{E(n)}$$

The famous **Telescope Conjecture** says that

$$L_n^t \simeq L_{E(n)}$$

$$X \xrightarrow{f} \Sigma^{-k}(X) \xrightarrow{f} \Sigma^{-2k}(X) \xrightarrow{f} \cdots$$

Let $X[f^{-1}]$ denote the colimit of this sequence.

Proposition 4.21

1. *If $X \in \mathcal{C}_{\geq n}$, then $L_n^t(X) \simeq X[f^{-1}]$.*
2. *There is a fiber sequence*

$$\lim_{\substack{\rightarrow \\ k_0, \cdots, k_n}} \Sigma^{-n} X/(v_0^{k_0}, \cdots, v_n^{k_n}) \to X \to L_n^t(X).$$

5 The Chromatic Filtration

5.1 Monochromatic

Let $L_n(X) = L_{E(n)(X)}$, then we have the following chromatic tower.

$$\begin{tikzcd}
 & M_n(X) \arrow[d] & & M_2(X) \arrow[d] & M_1(X) \arrow[d] & M_0(X) = H\mathbb{Q} \wedge X \arrow[d] \\
\cdots \arrow[r] & L_n(X) \arrow[r] & \cdots \arrow[r] & L_2(X) \arrow[r] & L_1(X) \arrow[r] & L_0(X) = H\mathbb{Q} \wedge X
\end{tikzcd}$$

where the monochromatic layers $M_n(X)$ are defined by the fiber sequence.

$$M_n(X) \to L_n(X) \to L_{n-1}(X)$$

The following is the chromatic covergence theorem proved by Hopkins- Ravenel.

Theorem 5.1 () *[Chromatic Covergence Theorem] Then Canonical Map $X \to \lim_n L_n X$ is an equivalence for all finite spectra X.*

Definition 5.2 *A spectrum X is monochromatic of height n if it is $E(n)$-local and $E(n-1)$- acyclic.*

We let $\mathcal{M}_n$ denote the category of all spectra which are monochromatic of height n. One can prove that this category is compactly generated, that is every object $\mathcal{M}_n$ can be write as a filtered colimit of compact objects of $\mathcal{M}_n$.

Proposition 5.3 *There is a equivalence of category between the homotopy category of monochromatic spectra of height n and the homotopy category of $K(n)$-local spectra, which is given by the functor*

$$L_{K(n)} : \mathcal{M}_n \rightleftharpoons \textit{K(n) local spectra} : M_n$$

6 Morava Stabilizer Group

We let the Γ_n denote the Honda Formal Group Law of height n, It is formal group calssified by the map

$$BP_* \cong Z_{(p)}[v_1, v_2, \cdots] \to \mathbb{F}_{p^n}$$

that sends v_n to 1, and $(p, v_1, \cdots, v_{n-1}, v_{n+1}, \cdots)$ to 0.

The small Morava stabilizer group $Aut_{\mathbb{F}_{p^n}}(\Gamma_n)$ is the group of automorphism of Γ_n with coefficients in $\mathbb{F}_{p^n}$,

$$Aut(\Gamma_n) = \{f(x) \in \mathbb{F}_{p^n}[[x]] : f(\Gamma_n(X,Y)) = \Gamma_n(f(x), f(y)), f'(0) \neq 0\}$$

Since Γ_n is defined over $\mathbb{F}_{p^n}$, the Galois group $Gal = Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ act on G_n by acting on the coefficients.

Definition 6.1 *The Morava stabilizer group $\mathbb{G}_n$ is defined by*

$$\mathbb{G}_n = \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p) \ltimes Aut(\Gamma_n)$$

Theorem 6.2 () *[Devinatz-Hopkins, Goerss-Hopkins-Miller] The Morava Stablizer group acts on E_n,and*

$$E(n)^{\mathbb{G}_n} \simeq L_{K(n)}S^0$$

Example 6.3 *When p is odd and n=1, $L_{K(1)}(S)$ is the spectrum $\widehat{KU}^{\psi^g=1}$*

Hence: the image of J is essentially the first chromatic layer of the sphere spectrum.

If we E_* module structure with an action of Morava stabilizer group $\mathbb{G}_n$, how can we get $L_{K(n)}S^0$?

$$\mathrm{Sp}_{K(n)} \longrightarrow \{ \text{ Morava Modules } : E_* \text{ module structure with action of } \mathbb{G}_n\}$$

Proposition 6.4 *There is a homotopy fixed point spectral sequence (descent spectral sequence)*

$$E_2^{s,t} = H^s_{gp}(G; \pi_t(X)) \Longrightarrow \pi_{t-s}(X^{hG})$$

similarly for X_{hG}, X^{tG}.

We have $E(n)^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$, so

$$E_2^{s,t} \cong H^s_{gp}(\mathbb{G}, E(n)_t) \Longrightarrow \pi_{t-s}L_{K(n)}S^0$$

For f a formal group law over $\bar{\mathbf{F}}_p$.

$$\mathrm{End}f = \{g(t) \in tR[\![t]\!] \quad | \quad f(g(x), g(y)) = gf(x,y)\}$$

Proposition 6.5 *End(f) is a noncommutative local ring: The collection non-invertible elements is the left ideal generated by $\pi(t) = \nu(t^p)$, where $\nu f^p(x,y) = f(\nu(x), \nu(y))$.*

Let $D = \mathbf{Q} \otimes \mathrm{End}(f)$.

Lemma 6.6 *D is a central division algebra over $\mathbf{Q}_p$. And* $\mathrm{End}(f) = \{x \in D : v(x) \geq 0\}$.

Let F_n be the universal deformation over $(E_n)_0$ of G_0. If we have $\alpha = (f, \sigma) \in \mathbb{G}_n$. The universal property of F_n implies that there is ring isomorphism $\alpha_* : (E_n)_0 \to (E_n)_0$ and an isomorphism of formal group laws $f_\alpha : \alpha_* F_n \to F_n$.

And the action can extends to $(E_n)_* \cong \mathbb{W}_n[\![u_1, \cdots, u_{n-1}]\!][u^{\pm 1}]$

1. $\alpha = (id, \sigma)$ for $\sigma \in \mathrm{Gal}(k/\mathbf{F}_p)$. Then the action is action of Galois group on $\mathbb{W}_n$.

2. If $\omega \in \mathbb{S}_n$ is a primitive $(p^n - 1)$-th root of the unity, then $\omega_*(u_i) = \omega^{p^i - 1} u_i$ and $\omega_*(u) = \omega u$.

3. $\Psi \in \mathbb{Z}_p^\times \subset \mathbb{S}_n$ is thee center, then $\psi_*(u_i) = u_i$ and $\psi_* u = \psi u$.

Theorem 6.7 *Devinatz-Hopkins Let $1 \leq i \leq n-1$ and $f = \sum_{j=0}^{n-1} \in \mathbb{S}_n$„ where $f_j \in \mathbb{W}_n$. Then modulo $(p, u_1, \cdots u_{n-1})^2$,*

$$f_*(u) \equiv f_0 u + \sum_{j=1}^{n-1} f_{n-j}^{\sigma^j} u u_j \qquad f_*(u u_i) \equiv \sum_{j=1}^{i} f_{i-j}^{\sigma^j} u u_j + \sum_{j=i+1}^{n} p f_{n+i-j}^{\sigma^j} u u_j$$

7 Elliptic Cohomology

Elliptic curves are very important objects in arithmetic geometry, it is the most simple nontrival example in algebraic geometry. But it still can gives us some interesting examples.One can see [Sil09] for information of elliptic curves and [KM85] for the moudli stack and level structures of the elliptic curves. A natural question is if we do completetion for a elliptic curve,then we got a one dimensional formal group, so we can get a formal group, can this formal group can give us a good cohomology theory. The answer is yes.

Definition 7.1 *An elliptic cohomology theory is a generalized cohomology theory E, which can be representated by a spectrum E, satisfying.*

1. *E is an even periodic spectrum.*

2. *There exists a elliptic curve C over $\pi_0 E$.*

3. *The formal group law assocaited E, $\phi : G_E \cong \hat{C}$*

We denote this data as (E, C, ϕ)

Theorem 7.2 *There is a sheaf $\mathcal{O}_{tmf}$ of E_∞-ring spectra over the stack $\overline{\mathcal{M}}$ for the etale topology. For any etale morphism $f : \mathrm{Spec}(R) \to \overline{\mathcal{M}}$ there is a natural structure of elliptic spectrum $(\mathcal{O}_{tmf}(f), C_f, \phi)$, satisfying $\pi_0\mathcal{O}_{tmf}(f) = R$, and C_f is the generalized elliptic curve over R classified by f.*

Let $Tmf = \mathcal{O}_{tmf}(\overline{\mathcal{M}} \to \overline{\mathcal{M}})$, the specturm topological modular forms.

Let $TMF = \mathcal{O}_{tmf}(\mathcal{M} \to \overline{\mathcal{M}})$, the periodic specturm of topological modular forms

Let $tmf = \tau_{\geq 0}\mathcal{O}_{tmf}(\bar{\mathcal{M}}_{ell}$ be the connect cover of Tmf.

We know that the modular form can be viewed as global sections of the mouli stack of elliptic curve over complex plane $\mathbb{C}$. And it is easy to see that if we take homotpy group of the toplogical modular forms , then we can get the ordinary moudlar forms.

The construction of the topological modular forms is complicated, one can see [DFHH14] for more details.

8 Power Operation in Morava E-theory

For power operations in Morava E-theory, one can see [Rez06], [Rez09] and [Rez13]. Direct computation was in [Rez08] for height 2 at the prime 2, and [Zhu14] for height 2 at prime 3. The case of height > 2 is still lack of computation. The final results are

Theorem 8.1 *Let A be a $K(n)$-local E-Algebra, then the power operation of the homotopy group of A has the structure of an amplified Γ-ring.*

Now we that the spectrum of Morava E-theory is a commutative S-algebra. It has the form

$$P_m : E^0(X) \to E^0(X \times B\Sigma_m)$$

So for a map $f : \Sigma^\infty X \to E$, P_m map it to the following map

We say that a graded Γ-algebra B satisfies thee congruence condition if for all $x \in B_0$,

$$x\sigma \equiv x^p modpB.$$

Theorem 8.2 *An object $B \in \mathrm{AlgAlg}^*_\Gamma$ which is p-torsion free, then B admits the structure of a $\mathbb{T}$-algebra if and only B satisfies the congruence condition.*

8.1 Sheaves on the Categories of Deformations

We recall the deformation of formal groups. Let G_0 be a formal group over a perfect field k with characteristic p, then a deformation of G_0 to R is a triple (G, i, Φ) satisfying

- G is a formal group over R,
- There is a map $i : k \to R/m$
- There is an isomorphism $\Phi : \pi^* G \cong i^* G_0$ of formal groups over R/m.

For any G_0/k, there is an even periodic ring spectrum $E = E_{G_0}$ associated to the universal deformation of G_0, with $\pi_0 E = A_{G_0}$, and $G_E = G_{univ}$. It can be constructed as the spectrum representing a Landweber exact cohomology theory.

Let R be complete local ring whose residue has characteristic p. Let $\phi : R \to R, x \mapsto x^p$ be the Frobenius map.

Definition 8.3 *For each formal group G over R, the **Frobenius isogeny*** $\mathrm{Frob} : G \to \phi^* G$ *is the homomorphism of formal group over R induced by the relative Frobenius map on rings. We write* $\mathrm{Frob}^r : G \to (\phi^r)^* G$ *which is the composition* $\phi^*(\mathrm{Frob}^{r-1}) \circ \mathrm{Frob}$

Let (G, i, α) and $(G', i'\alpha')$ be two deformation of G_0 to R. A deformation of Frob^r is a homomorphism $f : G \to G'$ of fromal groups over R which satisfying

1. $i \circ \phi^r = i'$ and $i^*(\phi^r)^* G_0 = (i')^* G_0$.

$$\begin{tikzcd} k \arrow[r, "i'"] \arrow[d, "\phi^r"'] & R/m \\ K \arrow[ur, "i"'] & \end{tikzcd}$$

2. the square

$$\begin{tikzcd} i^* G_0 \arrow[r, "i^*(\mathrm{Frob}^r)"] \arrow[d, "\alpha"'] & i^*(\phi^r)^* G_0 \arrow[d, "\alpha'"] \\ \pi^* G \arrow[r, "\pi^*(f)"] & \pi^* G' \end{tikzcd}$$

   of homomorphisms of formal groups over R/m commutes.

We let Def_R denote the category whose objects are deformations fo G_0 to R, and whose morphisms are homomorphsim which are deformation of Frob^r for some $r \geq 0$. Say that a morphism in Def_R has **height** r, if it is a deformation of Frob^r.

Proposition 8.4 *Let G be deformation of* G_0 *to R, then the assignment* $f \to \mathrm{Ker} f$ *is a one-to-one correspondence between the morphisms in* Sub_R^r *with source G and the finite subgroup of G which have rank* p^r.

For the following, Let $G_E = G_{univ}/E_0$ be the universal deformation of G_0.

Theorem 8.5 *Let G_0/k be a formal group of height h over a perfect field k. For each $r > 0$, there exists a complete local ring A_R which carries a universal height r morphism $f^r_{univ} : (G_s, i_s, \alpha_s) \to (G_t, i_t, \alpha_t) \in Sub^r(A_r)$. That is the operation $f^r_{univ} \to g^*(f^r_{univ})$ define a bijective relation from the set of local homomorphism $g : A_r \to R$ to the set Sub^r_R. Furthermore, we have:*

1. *$A_0 \approx W(k)[[v_1, \cdots, v_{h-1}]]$.*
2. *Under the map $s : A_0 \to A_r$ which classifiers the source of the universal height r map, i.e. $G_s = i^* G_E$, and A_r is finite and free as an A_0 module.*
3. *Under the map $t : A_0 \to A_r$ which classifies the target of the universal height r map, i.e. $G_t = t^* G_E$*

So there is a bijection

$$\{g : A_r \to R\} \to Sub^r(R)$$

given by

$$g \to g^*(f^r_{univ})(g^* G_s \to g^* G_t)$$

Proof. The case $r = 0$ is just the case of Lubin-Tate theory. For $r > 0$ this a theorem of Strickland [Str97]. ■

Thus, $Sub = \coprod Sub^r$ is a affine graded-category scheme. In particular, there are ring maps:

$$s = s_k, t = t_k : A_0 \to A_k,$$

which is induced by E^0 cohomology on $B\Sigma \to *$

$$\mu = \ mu_{k,l} : A_{k+l} : A_{k+l} \to A_k{}^s \otimes_{A_0} {}^t A_l$$

which classifying the source,target, and composite of morphisms.

Theorem 8.6 *The ring $A[r]$ in the universal deformation of Frobenuis is isomorphic to $E^0(B\Sigma_{p^r})/I$,i.e,*

$$A[r] \cong E^0(B\Sigma_{p^r})/I$$

where I is transfer ideal.

So for the power operation

$$R^k(X) \to R^k(X \times B\Sigma_m)$$

For $x = *$, we have

$$\pi_0 R \to E^0(B\Sigma_{p^r})/I \otimes \pi_0 R = A[r] \otimes \pi_0 R$$

This make $\pi_0 R$ becomes a Γ-module.

8.2 Character Theory

Hopkins-Kunh-Ranvel defined a generalized character theory

$$\chi : E^0(BG) \to Cl_n(G, C_0)$$

Theorem 8.7 *There is a commutative diagram*

$$\begin{tikzcd} E^0(BG) \arrow[r, "P_m"] \arrow[d, "\chi"] & E^0(BG \times B\Sigma_M) \arrow[d, "\chi"] \\ Cl_n(G, C_0) \arrow[r] & Cl_n(G \times B\Sigma_m, C_0) \end{tikzcd}$$

9 Chromatic Algebraic Geometry

The aim of this chapter is using idea from Burklund-Schlank-Yuan's [BSY22] result to establish an algebraic geometry which can describe the information of chromatic homotopy theory.

9.1 Spherical Witt Vectors

For a perfect $\mathbf{F}_p$-algebra A, in [Lur18], Lurie constructed constructed an spectrum $\mathbb{W}(A)$ called the spherical Witt vectors, which is a p-complete flat $\mathbb{S}_p$ algebra such that

$$\pi_* \mathbb{W}(A) \cong ((\pi_* \mathbb{S}_p) \otimes W(A))_p.$$

Proposition 9.1 *There is an Witt-tilt adjunction*

$$\mathbb{W}(-) : Perf \rightleftarrows \mathrm{CAlg}(\mathrm{Sp}_p) : (-)^\flat$$

where the right adjoint $(-)^b$ is computed by the inverse limit along Frobenius on $\pi_0(-)/p$.

If $\mathcal{C}$ be p-complete stable presentable symmetric monoidal category, then the unit $\mathrm{Sp}_p \to \mathcal{C}$ in $\mathrm{CAlg}(Pr^L)$ provide with an adjunction

$$l^* : \mathrm{CAlg}(\mathrm{Sp}_p) \leftrightarrows \mathrm{CAlg}(\mathcal{C}) : l_*$$

We can compose with the Witt-tilt adjunction , then we get an adjunction

$$\mathrm{Perf} \rightleftarrows \mathrm{CAlg}(\mathcal{C})$$

Let $B = (l_* Id_{\mathcal{C}})^\flat$, we can rewrite this as

$$W_{\mathcal{C}}(-) : \mathrm{Perf}_B \leftrightarrows \mathrm{CAlg}(\mathcal{C}) : (-)^\flat_{\mathcal{C}}$$

where the adjunction is given by $W_{\mathcal{C}}(A) = Id_{\mathcal{C}} \otimes_{l^*\mathbb{W}(B)} \mathbb{W}(A)$ and $(R)^\flat_{\mathcal{C}} = l_*(R)^\flat$.

For the case $\mathcal{C} = \mathrm{Mod}^\wedge_{E(k)}$, we have

$$(R)^\flat \cong (\pi_0 R)^\flat \cong (\pi_0(R)/m)^\flat$$

So we have an adjunction

$$\mathbb{W}_{E(k)}(-) : \mathrm{Perf}_k \leftrightarrows \mathrm{CAlg}^\wedge_{E(k)} : (-)^b$$

But we know that there is an adjunction for the Lunbin-Tate functor

$$E(-) : \mathrm{Perf}_k \to \mathrm{CAlg}^\wedge_{E(L)} : (\pi_0(-)/m)^\flat$$

where m is the Landweber ideal $(p, v_1, \cdots, v_{n-1})$ denote the inverse limits along Frobenius. And they satisfying the conditions

Lemma 9.2 $\mathbb{W}_{E(k)}(-) : \mathrm{Perf}_k \leftrightarrows \mathrm{CAlg}^\wedge_{E(k)}$ *is equivalent to the functor* $E(-)$

1. E(-) is fully-faithful and preserve arbitrary products.
2. The tilt can be computed via any of

$$(-)^\flat, \pi_0()^\flat, (\pi_0(-)/m)^\flat$$

3. ....
4. $R \in \mathrm{CAlg}^\wedge_{E(k)}$ belongs to the essential image of E(-) i f and only if $R//m$ has vanishing odd homotopy groups and $\pi_0 R/m = \pi_0(R//m)$ is perfect.

We knot that if R is a commutative ring spectrum. Then the commutative ring $\pi_0 R$ comes equipped with additional structures from power operation. This additional algebra structure was studied bu Rezk in the case when R is a T(n)-local commutative E-algebra. He constructe a monoad $\mathbb{T}$ on the category of discrete E_0-modules whose categroy of algebras $\mathrm{Alg}_{\mathbb{T}}$ is the image of the functor $\pi_0()$ on commutative E-algebras.

$$\begin{tikzcd} & \mathrm{Alg}_{\mathbb{T}} \arrow[d, "U_{\mathbb{T}}"] \\ \mathrm{CAlg}_E^{\wedge} \arrow[ur, dotted, "\pi_0"] \arrow[r, "\pi_0"] & \mathrm{CRing}_{E_0} \end{tikzcd}$$

In the case $n = 1$ and $E = E(\mathbf{F}_p, \mathbb{G}_m) = KU_p$. $\mathrm{Alg}_{\mathbb{T}}$ can be identified with the category CRing_δ-rings. If R is a $T(1)$-local commutative KU_p algebra, then there is a operation $\delta : \pi_0(R) \to \pi_0(R)$ which act as a p-derivation

$$\psi(x) = x^p + p\delta(x)$$

For formal reasons, the forgetful functor $U_{\mathbb{T}} : \mathrm{Alg}_{\mathbb{T}} \to \mathrm{CRing}_{E_0}$ admits both left and right adjoint

$$U_{\mathbb{T}} : \mathrm{Alg}_{\mathbb{T}} \rightleftarrows \mathrm{CRing}_{E_0} : W_{\mathbb{T}}$$

$$F_{\mathbb{T}} : \mathrm{CRing}_{E_0} \rightleftarrows \mathrm{Alg}_{\mathbb{T}} : U_{\mathbb{T}}$$

In the case of $\mathrm{Alg}_{\mathbb{T}} = \mathrm{CRing}_\delta$ at height 1, we have $W_{\mathbb{T}}(A) = W(A) = \pi_0 E(A)$. By composing with the adjunction

$$(-/p)^\sharp : \mathrm{CRing} \rightleftarrows \mathrm{Perf}_{\mathbf{F}_p} : Incl$$

We obtain an adjunction

$$(U(-)/p)^\sharp : \mathrm{CRing}_\delta \rightleftarrows \mathrm{Perf}_{\mathbf{F}_p} : \pi_0 E(-)$$

This adjunction can be generalize to any height.

Theorem 9.3 *There is an adjunction*

$$(U(-)/m)^\sharp : \mathrm{Alg}_{\mathbb{T}} \rightleftarrows \mathrm{Perf}_k : \pi_0 E(-)$$

where the right adjoint $\pi_0 E(-)$ is fully faithful.

9.2 Nilpotence Detecting

Theorem 9.4 *Let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, then there exists a perfect algebra A of Kull dimension 0 and a nilpontence detecting map*

$$R \to E(A).$$

Proof. See Section 5 of [BSY22, Theorem 5.1]. ∎

Corollary 9.5 *Let $0 \neq R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, then there exists an algebraically closed field and a map*

$$R \to E(L)$$

in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$.

Definition 9.6 *We say that a presentable ∞-category $\mathcal{C}$ is α-Nullstellensation for a regular cardinal α if every α-compact non-terminal object admit some map to the initial object. We say that an object $R \in \mathcal{C}$ is α-Nullstellensation if R is non-terminal and $\mathcal{C}_{R/}$ is α-Nullstellensation. We say R is Nullstellensation if it is ω-Nullstellensation.*

Theorem 9.7 *For any $0 \neq R \in \mathrm{CAlg}(Sp_{T(n)})$, R is α-Nullstellensation if and only if there exists some algebraic closed field L, cardinality $|L| \leq \alpha$ such that*

$$R = E(L).$$

Let $\mathcal{C}$ denote the category of products of algebraically fields equipped witha formal group of height n.

Definition 9.8 *The constructible topology of a $T(n)$-local spectra is defined by*

$$\mathrm{Spec}^{cons}_{T(n)} : \mathrm{CAlg}(\mathrm{Sp}_{T(n)})^{op} \to \mathrm{CHaus}$$

which is the Kan extension given by

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[rr, "(A{,}\mathbb{H}_0)\to \mathrm{Spec}_{Zar}(A)"] \arrow[dr] & {} \arrow[d, phantom, "\Downarrow"] & \mathrm{CHaus} \\
& \mathrm{CAlg}(\mathrm{Sp}_{T(n)})^{op} \arrow[ur, dotted, "\mathrm{Spec}^{cons}_{T(n)}"'] &
\end{tikzcd}$$

The constructible spectrum functor have the following properties

1. $\mathrm{Spec}^{cons}_{T(n)}$ is empty if and only if $R = 0$.
2. If L is algebraically closed filed L, $\mathrm{Spec}^{cons}_{T(n)}(E(L))$ is a point.
3. For every $q \in \mathrm{Spec}^{cons}_{T(n)}(R)$, there exists an algebraically closed filed L and a map $R \to E(L)$ such that q is the image of

$$\{*\} = \mathrm{Spec}^{cons}_{T(n)} E(L) \to \mathrm{Spec}^{cons}_{T(n)}(R).$$

4. A subset $U \subset \mathrm{Spec}^{cons}_{T(n)}(R)$ is closed if and only if there exists a map of $T(n)$-local commutative spectra $R \to S$ such that U is the image

$$\mathrm{Spec}^{cons}_{T(n)}(S) \to \mathrm{Spec}^{cons}_{T(n)}(R).$$

5. Given a map of spectra $S \leftarrow R \to T$, the natural comparison map

$$\mathrm{Spec}^{cons}_{T(n)}(S \otimes_R T) \to \mathrm{Spec}^{cons}_{T(n)}(S) \otimes_{\mathrm{Spec}^{cons}_{T(n)}(R)} \mathrm{Spec}^{cons}_{T(n)}(T)$$

is surjective.

Lemma 9.9 *For a perfect k-algebra, there is a natural isomorphism*

$$\mathrm{Spec}^{cons}_{T}(n)(E(A)) \cong \mathrm{Spec}^{cons}_{CRing}(A)$$

Theorem 9.10 *A map $R \to A$ in* $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ *detects nilpotence if and only if the associated map on constructible spectrum is surjective,*

References

[Ada74] John Frank Adams. *Stable homotopy and generalised homology.* University of Chicago press, 1974.

[And95] Matthew Ando. Isogenies of formal group laws and power operations in the cohomology theories en. *Duke Mathematical Journal*, 79(2):423–485, 1995.

[BSY22] Robert Burklund, Tomer M Schlank, and Allen Yuan. The chromatic nullstellensatz. *arXiv preprint arXiv:2207.09929*, 2022.

[DFHH14] Christopher L Douglas, John Francis, André G Henriques, and Michael A Hill. *Topological modular forms*, volume 201. American Mathematical Soc., 2014.

[EKMM97] AD Elmendorf, I Kriz, MA Mandell, and JP May. *Rings, modules, and algebras in stable homotopy theory*, volume 47. 1997.

[GH04] Paul G Goerss and Michael J Hopkins. Moduli spaces of commutative ring spectra. *Structured ring spectra*, 315(151-200):22, 2004.

[Hop99] Michael Hopkins. Complex oriented cohomology theories and the language of stacks. *Course notes*, 1999.

[KM85] Nicholas M Katz and Barry Mazur. *Arithmetic Moduli of Elliptic Curves*. Number 108. Princeton University Press, 1985.

[Lan76] Peter S Landweber. Homological properties of comodules over mu*(mu) and bp*(bp). *American Journal of Mathematics*, pages 591–610, 1976.

[LT66] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal lie groups. *Bulletin de la Société Mathématique de France*, 94:49–59, 1966.

[Lur10] Jacob Lurie. Lecture on chromatic homotopy theory. *Lecture notes online*, 2010.

[Lur17] Jacob Lurie. *Higher Algebra*. 2017.

[Lur18] Jacob Lurie. Elliptic cohomology ii: Orientations. 2018.

[Mil94] Haynes Miller. Notes on cobordism. *Notes typed by Dan Christensen and Gerd Laures based on lectures of Haynes Miller*, 1994.

[Pet18] Eric Peterson. *Formal geometry and bordism operations*, volume 177. Cambridge University Press, 2018.

[Qui69] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bulletin of the American Mathematical Society*, 75(6):1293–1298, 1969.

[Rez06] Charles Rezk. Lectures on power operations. *Lecture notes online*, 2006.

[Rez08] Charles Rezk. Power operations for morava e-theory of height 2 at the prime 2. *arXiv preprint arXiv:0812.1320*, 2008.

[Rez09] Charles Rezk. The congruence criterion for power operations in morava e-theory. *Homology, Homotopy and Applications*, 11(2):327–379, 2009.

[Rez13] Charles Rezk. Power operations in morava e-theory: structure and calculations (draft). *preprint*, 2013.

[Sil09] Joseph H Silverman. *The arithmetic of elliptic curves*, volume 106. Springer Science & Business Media, 2009.

[Str97] Neil P Strickland. Finite subgroups of formal groups. *Journal of Pure and Applied Algebra*, 121(2):161–208, 1997.

[Zhu14] Yifei Zhu. The power operation structure on morava e–theory of height 2 at the prime 3. *Algebraic & Geometric Topology*, 14(2):953–977, 2014.