Research Statement

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My research interests are Algebraic Topology, Algebraic Geometry, Number Theory and Theoretical Physics. I focus on derived methods on these fields, specifically, I am working on following problems.

1 Derived Level Structures

In [Lur09] and [Lur18a], Lurie uses spectral algebraic geometry methods give a proof of Goerss-Hopkins-Miller theorem for topological modular forms. Except the application of elliptic cohomology, Lurie also proved the E_∞ structures of Morava E-theories [Lur18a], which use the spectral version of deformations of formal groups and p-divisible groups. There are more and more its applications in algebraic topology. Like topological automorphic forms [BL10], the construction of equivariant topological modular forms [GM20], elliptic Hochschild homology [ST23] and so on.

The moduli problem of deformations of formal groups with level structures is also representable and moduli spaces of different levels form a Lubin-Tate tower [RZ96], [FGL08]. We know that the universal objects of deformation of formal groups have higher algebra analogues which are Morava E-theories. A natural question is what the higher categorical analogue of the moduli problem of deformations with level structures is? And can we find a higher categorical analogue of the Lubin-Tate tower. Unfortunately, the representable object of deformations with level structures is not *é*tale over the universal deformations, so we can't use the Goerss-Hopkins-Miller theorem directly. Except this, in the computation of unstable homotopy groups of spheres, after applying the EHP-spectral sequences and Bousfield-Kuhn functor, we find some terms in E_2-page also comes form the universal deformation of isogenies of form groups. They are computed by the Morava E-theories on the classify spaces of symmetric groups [Str97], [Str98]. They can be viewed as sheaves on the Lubin-Tate tower. We hope a more conceptual view about this fact in the higher categorical Lubin-Tate tower.

Question 1.1 *What is the higher categorical analogues of level structures? Can they form good derived moduli spaces?*

In the upcoming paper [Ma23], we give some attempts on this problem, we define the derived level structures in the context of spectral algebraic geometry and give some representable results about moduli problems associated with derived level structures, at least in the case of spectral elliptic curves.

Definition 1.2 *Let X/S be a spectral elliptic curves over a non-connective spectral Deligne-Mumford stack S. In the level of objects, a derived level structure is a morphism of spectral Deligne-Mumford stacks $\phi : D \to X$, such that $D \to X$ is a closed immersion and the associated sheaf is a line bundle, $D \to S$ is flat, proper, and locally almost of finite presentation, and $D^\heartsuit \to X^\heartsuit$ is a classical level structure.*

Proposition 1.3 *Let E/R be a spectral elliptic curve, then the functor*

$$\begin{aligned} \underline{\mathrm{Level}}_{E/R} \quad : \quad & \mathrm{CAlg}_R^{cn} \to \mathcal{S} \\ & R' \mapsto \underline{\mathrm{Level}}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is represented by a closed substack $S(A)$ of $\mathrm{CDiv}_{X/R}$. Moreover, $S(A)$ is affine and locally almost of finite presentation over R.

Proposition 1.4 *[Ma23] The moduli problem*

$$\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}, \quad R \mapsto \mathrm{Ell}(\mathcal{A})(R)^{\simeq}$$

is represented by a spectral Deligne-Mumford stack in the sense of [Lur18b], where $\mathrm{Ell}(\mathcal{A})(R)$ is the ∞-groupoid consists of pairs (E, ϕ), where E is a spectral elliptic curve over R, and $\phi : D \to E$ is a derived level structure.

As we said, what we want is the higher categoric analogue of Lubin-Tate tower, so we need to consider the moduli problem of spectral derived deformations with derived level structures.

Let G_0 be a p-divisible group over a commutative ring R_{G_0}. We consider the following functor:

$$\begin{aligned} \mathcal{M}_{\mathcal{A}}^{or,level} \quad : \quad & \mathrm{CAlg}_k^{cn} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, \mathcal{A}) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, \mathcal{A})$ is the ∞-category whose objects are four-tupes (G, ρ, e, η)

1. G is a spectral p-divisible group over R.
2. ρ is an equivalence class of G_0 taggings of R (which we can view as $\rho : G \times_R k$).
3. e is an orientation of the identity component of G.
4. $\eta : D \to G$ is a derived level structure.

Theorem 1.5 *[Ma23] The functor $\mathcal{M}_{\mathcal{A}}^{or,level}$ is representable by a formal spectral Deligne-Mumford Stack $\mathrm{Spf}\mathcal{JL}$, where $\mathcal{JL}$ is a E_∞-ring which is finitely generated over $R_{G_0}^{or}$, here $R_{G_0}^{or}$ is the orientated deformation ring defined in [Lur18a].*

We call this spectrum Jacquet-Langlands spectrum.

Question 1.6 *What properties does these spectra $\mathcal{JL}$ have?*

It is easy to see that these $\mathcal{JL}$ admit a action of $GL_n(Z/p^mZ) \times \mathrm{Aut}(G_0)$. In the classical algebraic geometry, the Lubin-Tate tower can be used to realize the Jacquet-Langlands correspondence [HT01]. So a natural question is that whether there is a topological realization of the Jacquet-Langlands correspondence. Actually, in a recent paper [SS23], they already realized the topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They consider the degenerate level structures such that representing objects are *é*tale over representing objects of universal deformations. We hope that our derived level structure can also realize the topological Lubin-Tate tower, and we hope to find a relation with the construction of degenerating level structures.

2 Derived Rings

It follows that [BMS19], some topological realizations of classical cohomology rings may have a good structures, like the topological Hochschild homology of quasiregualr semiperfectoid rings. These leads to the establishment of some special p-adic cohomology theories, Breuil-Kisin cohomology theory and its refinement, prismatic cohomology. The heart of this topic is δ-rings and its topological realization derived δ-rings [Hol23].

Question 2.1 *Does these derived rings follows form derived moduli problems?*

Actually, in [Lur18a], Lurie constructed the spherical Witt-vectors, which follows form derived moduli problems, thickenings of relatively perfect morphisms . And it has many application in chromatic homotopy theory, like [BSY22] and [Ant23]. We hope to establish more derived moduli problems, to give us more understand of these derived rings. And we hope the spectrum $\mathcal{JL}$ is a good p-adic cohomology theory. It will give us more arithmetic information.

Question 2.2 *Computation of $\mathcal{JL}$ theory of some p-adic rings, especially for perfectoid rings.*

We consider the spherical Witt-vector functor defined in [Lur18a] and [BSY22].

$$\mathbb{SW} : \mathrm{Perf} \to \mathrm{CAlg}(\mathrm{Sp}_p).$$

By this functor and the classical algebraic methods, like power operations, periodictiy, thick subcategories, it may give us more information about the ∞-category of derived δ-rings. The reason we consider chromatic methods is the appearance of v_1 periodic elements in the computation of topological Hochschild homology. We hope through the study of global property of derived δ-rings, we can find more computation methods of K-theory and its local variant of perfectoid rings.

Question 2.3 *The structures and classification of derived δ-rings.*

3 Topological Langlands Correspondence

Let E be a local field, G be a reductive group over E. The classical local Langlands correspondence predict that for any irreducible smooth representation π of $G(E)$, we can naturally associate an L-parameter

$$\phi_E : W_E \to \widehat{G}(\mathbb{C}).$$

And the geometric Langlands correspondence actually aim to construct an equivalence of categories

$$D(\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}}(X)) \simeq D(\mathcal{D}(\mathrm{Bun}_G))$$

From the derived category of quasi-coherent sheaf on $\widehat{G}$ local systems on X and the derived categories of D-modules on the moduli stack of G-bundles over X [BD91]. Due to the work of Fargues-Scholze [FS21], the arithmetic local Langlands correspondence can also be some kinds of geometric Langlands correspondence, but in the perfectoid world.

We hope to establish an topological version of the classical Langlands correspondence, which means the we construct representations on the category of spectra. By the construction of Jacquet-Langlands spectra in the conjecture above, Let $\mathbb{G}$ be a formal group over a field of characteristic p, $\mathcal{JL}$ be its ℓ-adic complete Jacquet-Langlands spectrum. Let X be a spectrum with an action of $\mathrm{Aut}(\mathbb{G}_n)$. We have the following conjecture.

Conjecture 3.1 *The function spectrum $F(X, \mathcal{JL})$ admits an action of $GL_n(\mathbb{Z}_p)$ and all its homotopy groups are $\mathbb{Z}_l$-modules.*

4 Geometric Representation Theory on Spectra Algebraic Geometry

For a formal group og height n over a perfect field k with char k=p, we have a spectral sequence

$$E_2^{s,t} \simeq H^s_{cts}(\mathbb{G}_n, \pi_t E_n) \Longrightarrow \pi_{t-s} L_{K(n)} S^0$$

where E_n is the associated Morava E-theory and $\mathbb{G}_n$ is the Morava stabilizer group. So when consider resolutions of Morava E-theories, it is reasonable ot consider Sp_H, the category of spectra admits a H action, where H is a subgroup of $\mathbb{G}_n$ with finite index. By the topological Jacquet-Langlands correspondence, we can consider resolutions in Sp_{GL_n}, then get resolutions of $\mathbb{G}_n$ by correspondence.

This the reason why we need representation theory in the category of E_∞-spectra, spectral schemes, and spectral stacks.

Question 4.1 *What is the right definition of algebraic groups acting on spectra stacks?*

The reason we asked this question is that when we say a spectrum admits a G-action, if G is topological group, it is fine, but when G is a algebraic group, how do we say this action is compatible with the algebraic structure? In spectral algebraic geometry, it is easy to define the action of algebraic group, but how do we pass through this definition to spectra?

Question 4.2 *Does the principal bundle theory holds in spectral algebraic geometry? What good properties does spectral* Bun_G *have in spectral algebraic geometry?*

Just like the classical algebraic geometry, we want the moduli stack of principal bundles over a curve has a good structure, but there are still something unknow in spectral algebraic geometry.

Question 4.3 *What is a good definition of D-modules in spectral algebraic geometry? Is it different from classical algebraic geometry in characteristic 0?*

We know that spectral algebraic geometry also exists for characteristic p. And there is some study of PD operads recently [BCN21], so we may hope have a good representation theory in positive characteristic in the context of spectral algebraic geometry.

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