Example of Model Categories

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Model Categories

Weak Factorisation System

We say i has a lifting property with respect to p, or, equivalently, the P has a lifting property with respect to i, If for any commutative square of the following form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

has a diagonal filler h, make every triangle in the following diagram commutative. (i.e. a morphism h such that $hi = a$ and $ph = b$)

A week factorisation system in a category $\mathcal{C}$ is a couple (A, B), where A and B are class of morphisms satisfying the following properties.

a) both A and B are stable under retracts;

b) $A \subset l(B) (\Leftrightarrow B \subset r(A))$

C) any morphisms $f : X \to Y$ of $\mathcal{C}$ admits a factorisation of the form $f = pi$, with $i \in A$ and $p \in B$.

Model Category

A model category is a locally small category $\mathcal{C}$, endowed with three classes of morphisms, Weak equivalnece, fibration and confibration, and satisfying the following condition.

- $\mathcal{C}$ admits finite limits and finite colimits.
- The class weak equivalnece has two-out-of three property, which says that if there is a communtative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

  Every two of f,g,h are in weak equivalence, so is the third.
- Both couples $(W \cap Coib, Fib)$ and $(Cofib, W \cap Fib)$ are weak factorisation systems.

Cylinder Object and Path Object

For an object A in a model category $\mathcal{C}$, a cylinder of A is a factorisation of $A \coprod A \to A$ into a cofibration followed by a weak equivalence, i.e, there exists a diagram

$$\begin{tikzcd} A \coprod A \arrow[r, "(\partial_0,\partial_1)"] \arrow[rr, bend left, "(1_A,1_A)"] & IA \arrow[r, "\sigma"] & A \end{tikzcd}$$

where $A \coprod A \overset{(\partial_0,\partial_1)}{\to} IA$ is a cofibration and $IA \overset{\sigma}{\to} A$ is a weak equivalence.

Similarly, for a object X of a model category $\mathcal{C}$, a cocylinder(path) object is a factorisation of $X \to X \times X$ in to a weak equivalence followed by a fibration, i.e there exists a diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[rr, bend left, "(1_A,1_A)"] & X^I \arrow[r, "(d^0,d^1)"] & X \times X \end{tikzcd}$$

where $X \overset{s}{\to} X^I$ is a weak equivalence, and $X^I \overset{(d^0,d^1)}{\longrightarrow} X \times X$ is a fibration.

Left Homotpy and Right Homotopy

For two morphism $f_0, f_1 : A \to X$, a left homotopy form f_0 to f_1 is a cylinder object IA of A together with a morphism $h : IA \to X$, making the following diagram commute.

$$\begin{tikzcd} A \coprod A \arrow[r, "(f_0.f_1)"] \arrow[d, "(\partial_0{,}\partial_1)"'] & X \\ IA \arrow[ur, dotted, "h"] & \end{tikzcd}$$

And similarly we can define the right homotopy form f_0 to f_1 is a path object X^I of X together with a morphism $k : I \to X^I$, making the following diagram commute.

$$\begin{tikzcd} X^I \arrow[r, "(d^0{,}d^1)"] & X \times X \\ A \arrow[u, dotted, "k"] \arrow[ur, "(f_0{,}f_1)"'] & \end{tikzcd}$$

Let A be a cofibrant object and X a fibrant object. We define a relation o the set of morphism from A to X by defining $f_0 \sim f_1$, when there exists a left homotopy from f_0 to f_1.
One can prove that this relation is a equivalence relation.
And we denote $[A, X] = \mathrm{Hom}_{\mathcal{C}}(A, X)/\sim$

Quillen Equivalence

Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories, $F : \mathcal{C} \to \mathcal{D}$ be a functor. A left derived $LF : ho(\mathcal{C})$ of F is a right kan extension of F along $\gamma_{\mathcal{C}} : \mathcal{C} \to ho(\mathcal{C})$. Actually one can prove if F preserve trivial cofibration, then the left derived functor exists.If LF is a left derived functor of F, then $\gamma_{\mathcal{D}} \circ LF$ is a left derived functor of $\gamma_{\mathcal{D}} \circ F$. We still denote it by

$$LF : \mathrm{ho}(\mathcal{C}) \to \mathrm{ho}(\mathcal{D})$$

And call it the total left derived functor.
Similarly, if a functor $F : \mathcal{C} \to \mathcal{D}$ preserves trivial fibrations, we denote by

$$RF : \mathrm{ho}(\mathcal{C}) \to \mathrm{ho}(\mathcal{D})$$

the right derived functor of $\gamma_{\mathcal{D}} \circ F$, and call it the total right derived functor.

Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories. A Quillen adjunction is a pair of adjoint functor

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

such that F preserves cofibrations and G preserves fibrations.

One can prove the following conditions are equivalent.

- The pair (F, G) is a Quillen adjunction.
- F preserves cofibrations and trivial cofibrations.
- G preserves fibrations and trivial fibrations.

The Quillen adjoint pair

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is a Quillen equivalence, if for any cofibrant object $X \in C$ and fibrant object $Y \in D$, $FX \to Y$ is a weak equivalence iff the adjoint $X \to GY$ is a weak equivalence.

Any Quillen adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ naturally induces an adjunction of categories:

$$LF : ho(\mathcal{C}) \leftrightarrows ho(\mathcal{D}) : RG$$

When (F, G) is an Quillen equivalence, they induces equivalence on homotopy categories, i.e. the derived functors (LF, RG) are equivalences of categories

Projective Model Structure

Let $\mathcal{C}$ be a category and I a small category. Denote $C^I = \mathrm{Hom}(I, \mathcal{C})$. We define

- $F \to G$ in C^I is a weak equivalence if and only if $F(i) \to G(i)$ is a weak equivalence for every $i \in I$.
- $F \to G$ in C^I is a fibration if and only if $F(i) \to G(i)$ is a fibration for every $i \in I$.

If such model structure exists, we call it the projective model structure.

If the projective model structure exists, then we have a Quillen adjunction

$$\mathrm{colim} : \mathcal{C}^I \leftrightarrows \mathcal{C} : \Delta$$

where $\Delta(X) = X_I$ is the constant funcotr.
And thus define a functor:

$$L\mathrm{colim} : ho(\mathcal{C}^I) \to ho(\mathcal{C}) : R\Delta$$

Homotopy coimit

Homotopy Colimit

An object X is a homotopy colimit of a functor $F : I \to \mathcal{C}$ if there exists a morphism $F \to X_I$ in C^I and the induced morphism $(L\mathrm{colim})(F) \to X$ is an isomorphism in $\mathrm{ho}(\mathcal{C})$.

Homotopy limit

Simliaryl, we have

$$\Delta : \mathcal{C} \leftrightarrows \mathcal{C}^I : \lim$$

And thus define a functor:

$$L\Delta : ho(\mathcal{C}) \to ho(\mathcal{C}^I) : R\lim$$

Homotopy Limit

An object X is a homotopy colimit of a functor $F : I \to \mathcal{C}$ if there exists a morphism $X_I \to F$ in C^I and the induced morphism $(R\lim)(F) \to X$ is an isomorphism in $\mathrm{ho}(\mathcal{C})$.

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Topological Space

Fibration and Extension

We say a map $p : E \to B$ have a **homotopy lifting property** for a space X, if there is a map $f_0 : X \to B$, a homtotpy $f_t : X \to B$ and a lifting $\widetilde{f_0} : X \to E$ lift f_0, s.t. $f_0 = p \circ \widetilde{f_0}$, then there exists homotopy $\widetilde{f_t}$ lifts f_t, i.e. we have following commutative diagram for every $0 \leq t \leq 1$

$$
\begin{tikzcd}
 & E \arrow[d, "p"] \\
X \arrow[ur, dotted, "\widetilde{f_t}"] \arrow[r, "f_t"'] & B
\end{tikzcd}
$$

We say a map $q : A \to X$ have a **homotopy extension property** for a space S, if there is a map $g_0 : A \to S$, a homtotpy $g_t : A \to S$ and a extensionof g_0, $\widetilde{g_0} : X \to S$ l, s.t. $g_0 = \widetilde{g_0} \circ q$, then there exists homotopy $\widetilde{g_t}$ lifts g_t, i.e. we have following commutative diagram for every $0 \leq t \leq 1$

$$
\begin{tikzcd}
A \arrow[r, "g_t"] \arrow[d, "q"'] & S \\
X \arrow[ur, dotted, "\widetilde{g_t}"'] &
\end{tikzcd}
$$

If a map have homotopy lifting property with respect to all space, then we call it a fibration.
If a map have homotopy extension property with respect to all space, then we call it a cofibration.

Serre Fibration

A map $p : E \to B$ is a Serre fibration if it has a right lifiting of all maps $D^n \to D^n \times I$

Fact, A serre fibration has the right lifting property against all $X \to X \times I$ if X is a CW complex.
Let $f : X \to Y$ be a Serre fibration, and $y \in Y$ be a point of Y, define $F_y := f^{-1}(y)$, then there is an exact sequence for any $x \in F_y$,

$$\pi_*(F_y, x) \to \pi_*(X, x) \to \pi_*(Y, y)$$

Model Categories of Topological Space

The three class of morphism, Serre fibration, cofibration and weak equivalence make Top become a model category. (q-model structure.)

Proof: Any morphism $f : X \to Y$ in Top can be factored as a composite

$$X \xrightarrow{i} Z \xrightarrow{p} Y$$

where i is a cofibration and p is a trivial fibration

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Simplicial Set

Definition of Simplicial Set

We denote Δ the category of finite ordered numbers, the object of $\triangle$ are the finite sets

$$[n] = i \in Z | 0 \leq i \leq n = 0, \ldots, n$$

endowed with nature order, and morphisms are the(non strictly) order-preserving maps. It means that if $f : [m] \to [n]$ is a morphism in Δ and $i < j$ in $[m]$, then $f(i) \leq f(j)$

Simplicial Set

A simplicial set is a functor $X : \Delta^{op} \to Set$, where Δ^{op} is the opposite category of Δ and Set is the category of Sets.

For a ordinary category $\mathcal{C}$, a simplicial object in $\mathcal{D}$ is a functor $F : \Delta^{op} \to \mathcal{C}$. If X is the a simplicial set, we denote $X([n])$ by X_n and call it the set of n-simplices. An element of X_n is called a simplex.

Definition of Simplicial Set

There is standard functor:

$$\begin{array}{ccc} \Delta & \to & \mathrm{Top} \\ n & \mapsto & |\Delta^n| \end{array}$$

where $|\Delta^n|$ is the topological standard n-simplex

$$|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$$

Tere are two operations in the category Δ, coface and codegeneracy, defined as follows.

$$\text{Coface} \qquad d_n^i : [n-1] \to [n]$$

$$d_n^i(k) = \begin{cases} k & \textit{if} \quad k < i \\ k+1 & \textit{if} \quad k \geq i \end{cases}$$

$$\text{Codegeneracy} \qquad s_n^j : [n+1] \to [n]$$

$$s_n^j(k) = \begin{cases} k & \textit{if} \quad k \geq j \\ k-1 & \textit{if} \quad k > j \end{cases}$$

So if X is a simplicial set, which is a functor from $\Delta^{op} \to \mathrm{Set}$. Then coface and codegeneracy will be maped to morphisms in the category of *Set*, which we will called face and degeneracy.

$$\text{face} \quad d_i^n : X_n \to X_{n-1}$$

$$\text{degeneracy} \quad s_j^n : X_n \to X_{n+1}$$

And one can easily verify that the face and degeneracy satisfying some equations (the upper index of d and s are n).

$$\begin{cases} d_i d_j = d_{j_1} d_i & \text{if} i < j \\ d_i s_j = s_{j-1} d_i & \text{if} i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & if i > j+1 \\ s_i s_j = s_{j+1} s_i & if i \geq j \end{cases}$$

One can prove that morphisms induced by order-preserving map can always be written as the composition of face map and degeneracy map. So in order to define a simplicial set Y. It suffices to assign each Y_n and face, degeneracy map statisfying the upper condition.
We let Δ^n denote the functor $Hom(\cdot, [n])$ represented by $[n]$, and call it the standard n simplexes.
So by the Yoneda Lemma, we have

$$X_n = X([n]) \simeq Hom(Hom(\cdot, [n]), X) = Hom(\Delta^n, X)$$

If X is a simplicial set, we say that a object of X is a morphism $\triangle^0 \to X$ and a 1-morphism is an element of X_1, i.e a map $f : \Delta^1 \to X$.

For the simplicial set Δ^n, we can define a simplicial set, called the boundary of Δ^n

$$\partial\Delta^n := \bigcup_{E \subsetneq [n]} \Delta^E$$

And define the k-th horn $\wedge_k^n$, for $0 \leq k \leq n$

$$\wedge_k^n := \bigcup_{k \in E \subsetneq [n]} \Delta^E$$

The k-th horn is actually corresponds to the subset of an n-simplex Δ^n in which the j-th face and the interior have been removed, like $\wedge_1^2$ as follows

$$\begin{tikzcd}
 & 1 \arrow[dr] & \\
0 \arrow[ur] & & 2
\end{tikzcd}$$

Nerves

For a category $\mathcal{C}$, we define a simplicial set $N(\mathcal{C}))$ which its value on $[n]$ is $Home([n], C)$. So a n-simplices can be represented by a graph

$$x_0 \to x_1 \to \cdots \to x_{n-1} \to x_n$$

If$X$is a simplicail set.
An object of X is a 0-simplex $x : \delta^0 \to X$.
An arrow (or morphism) of X is a 1-simplex $f : \delta^1 \to X$. So in this notion, a morphism have a source and a target given by .

Realization of a simplicial set

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If X is a simplicial set, then we associate X a topological space $S(X)$, called the realization of X.

Step 1 We define a category $X_{\downarrow}$, whose obiect are map $\Delta^n \to X$ and morphisms are communtative diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "f"] \arrow[d, "i"] & X \\ \Delta^m \arrow[ur, "f'"] & \end{tikzcd}$$

Step 2 We define the geometry realization of X.

$$|X| := \varinjlim_{\Delta^n \to X} |\Delta^n|$$

The realization functor is a left adjont to the singular functor. i.e there is a bijective

$$\mathrm{Hom}_{\mathrm{Top}}(|X|, T) \cong \mathrm{Hom}_{Set_\Delta}(X, Sing(T))$$

where X is a simplicial set and T is a topological space.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Top}}(|X|, T) &\cong \varinjlim_{\Delta^n \to X} (|\Delta^n|, T) \\
&\cong \varinjlim_{\Delta^n \to X} \mathrm{Hom}_S(\Delta^n, Sing(T)) \\
&\cong \mathrm{Hom}_S(X, Sing(T))
\end{aligned}$$

where $S = \mathrm{Set}_\Delta$ is the category of simplicial set.

Model Categories of Simplicial Set

A kan complex is a simplical set which satisfies for $0 \leq k \leq n$ and any morphism $f : \wedge_k^n \to X$, there exists a morphism $f' : \triangle^n \to X$, such that the composition of $i : \wedge_k^n \to \triangle^n$ and f' is equal to f, this means that there exists a commutative diagram

$$\begin{tikzcd} \wedge_l^n \arrow[r, "f"] \arrow[d, "i"'] & X \\ \triangle^n \arrow[ur, dotted, "f'"] & \end{tikzcd}$$

A morphism $f : X \to Y$ of simplicail set is called fibration if for any commutative diagram of the following form, there exists a dot arrow make the entire diagram commute.

$$\begin{tikzcd} \wedge_k^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

So a kan complex(fibrant simplicial set) is a simplicial set X such that the map $X \to *$ is a fibration, where $*$ is the terminal object of the category of simplicial set.

Model Categories of Simplicial Set

Let $f, g : K \to X$ be two morphisms in the category of simplicial set. We say that there is a simplicial homotopy $f \xrightarrow{\simeq} g$ from f to g if there is a commutative diagram.

$$
\begin{tikzcd}
K \times \Delta^0 = K \arrow[d, "1\times d^1"'] \arrow[dr, "f"] & \\
K \times \Delta^1 \arrow[r, "h"] & X \\
K \times \Delta^1 = K \arrow[u, "1\times d^0"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

The realization functor is a left adjont to the singular functor. i.e there is a bijective

$$\mathrm{Hom}_{\mathrm{Top}}(|X|, T) \cong \mathrm{Hom}_{Set_\Delta}(X, Sing(T))$$

where X is a simplicial set and T is a topological space.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Top}}(|X|, T) &\cong \varinjlim_{\Delta^n \to X} (|\Delta^n|, T) \\
&\cong \varinjlim_{\Delta^n \to X} \mathrm{Hom}_S(\Delta^n, Sing(T)) \\
&\cong \mathrm{Hom}_S(X, Sing(T))
\end{aligned}$$

where $S = \mathrm{Set}_\Delta$ is the category of simplicial set.

Dold—Kan Correspondence

In the context of Homological Algebra, the chain complex was most studied. And it looks like the chain complex has a liitle like the simplicial object.

Dold-Kan Correspondence

Let $\mathcal{A}$ be an abelian category and $A = (A_*, d)$ be a nonnegatively graded chain complex with values in $\mathcal{A}$. Then there is a equivalence of categories.

$$N : \mathrm{Fun}(\Delta^{op}, \mathcal{A}) \leftrightarrows \mathrm{Ch}(\mathcal{A})_{\geq 0} : DK$$

where

$$N(X)([n]) := \mathrm{Ker}(X_n \to \otimes_{1 \leq i \leq n} X_{n-1}) = \cap_{1 \leq i \leq n} \mathrm{Ker}(d_i)$$

the $X_n \to X_{n-1}$ is given by d_i, $1 \leq i \leq n$