Derived Level Structures

Xuecai Ma

December 28, 2023

Abstract

We define the derived level structure in the context of spectral algebraic geometry. We prove some results about moduli problems associated with derived level structures.

Contents

1 Introduction

The first application of spectral algebraic geometry in algebraic topology was in [Lur09b]. Lurie used it to study elliptic cohomology and topological modular forms. In [Lur09b] and [Lur18b], Lurie uses spectral algebraic methods give a proof of Goerss-Hopkins-Miller theorem for topological modular forms. Except the application of elliptic cohomology, Lurie also proved the $\mathbb{E}_\infty$ structures of Morava E-theories [Lur18b], which use the spectral version of deformations of formal groups and p-divisible groups, the classical version was studied in [LT66]. The earliest proof of $\mathbb{E}_\infty$ structures of Morava E-theories is due to Goerss, Hopkins and Miller [GH04]. They turned the problem into a moduli problem and developed an obstructions theory. One can finish the proof by compute the Andre-Quillen groups. Comparing with their method, Lurie's proof is more conceptual. There are more and more its application in algebraic topology. Like topological automorphic forms [BL10], Morava E-theories over any F_p-algebra [Lur18b], not only just for a perfect field k, the construction of equivariant topological modular forms[GM20], elliptic Hochschild homology [ST23] and so on.

On the other hand, the moduli problem of deformations of formal groups with level structures is also representable and moduli spaces of different levels forms a Lubin-Tate tower [RZ96],[FGL08]. We know that the universal objects of deformation of formal groups have higher algebra analogues which are the Morava E-theories. A natural question is what the higher categorical analogue of the moduli problem of deformations with level structures is? And can we find a higher categorical analogue of Lubin-Tate tower. Although the $\mathbb{E}_\infty$-structure of topological modular forms with level structures can be get from the spectral elliptic curves, we still hope there exists a derived stack of spectral elliptic with level structures. Except this, in the computation of unstable homotopy groups of sphere, after apply the EHP-spectral sequences and Bousfield-Kuhn functor, we find some term in E_2-page also comes form the universal deformation of isogenies of form groups. They are computed by the Morava E-theories on the classify spaces of symmetric groups [Str97], [Str98]. They can be viewed as sheaves on the Lubin-Tate tower. We hope a more conceptual view about this fact in the higher categorical Lubin-Tate tower.

We now give an outline of this paper. In the second section, we consider the moduli problem of spectral elliptic curves with classical level structures of its underlying ordinary elliptic curves. We prove that its moduli space have a structure of spectral Deligne-Mumford stack.This give us an evidence of representability of more complicated moduli problems.

In the third section, we define the derived isogeny and prove that the kernel of a derived isogeny in some cases have the same phenomenon as the classical case. By an isogeny of spectral abelian varities, we mean a morphism $f : X\, toY$ which is finite flat and geometric surjective. We can find that if the underlying map $f^\heartsuit : X^\heartsuit \to X^\heartsuit$ of a derived isogeny $f : X \to Y$ determine a locally constant discrete sheaf, then fib$f$ is a homotopy locally constant sheaf, see lemma 3.8. This gives us an hint about how to defined derived level structures. Roughly speaking, a derived level structure of a spectral elliptic curves E over an $\mathbb{E}_\infty$-ring R is just a morphism of $\mathbb{E}_\infty$ group like spaces

$$\mathcal{A} \to E(R)$$

satisfying its restriction to the heart is an ordinary level structure and induce a closed immersion $\underline{\mathcal{A}} \to E$ whose the associated ideal sheaf is a line bundle over E. That is, $\mathcal{A}$ is group like $\mathbb{E}_\infty$-space, satisfying $\pi_0\mathcal{A} = A$, usually $A = \mathbf{Z}/N\mathbf{Z}$ or $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$. We prove some results about the derived level structure. This allow us to consider the moduli space of spectral elliptic curves with derived level structures. Our first main theorem is

Theorem A. The moduli problem

$$\begin{aligned} \mathcal{M}_{ell}(\mathcal{A}) \ :\ & \mathrm{CAlg} \to \mathcal{S} \\ & R \longmapsto \mathrm{Ell}(\mathcal{A})(R)^{\simeq} \end{aligned}$$

is representable by a spectral Deligne-Mumford stack, where $\mathrm{Ell}(\mathcal{A})(R)$ is the ∞-categories of spectral elliptic curves over R with derived level structures.

In the last section, we consider the spectral deformations with derived level structures. In [Lur18b], Lurie consider the spectral deformations of a classical formal group. As we have the concept of derived level structure, it is natural to consider the moduli of spectral deformations with derived level structures. Let G_0 be a p-divisible group over a perfect F_p algebra R_0. We consider the following functor

$$\begin{aligned} \mathcal{M}^{or}_{\mathcal{A}} \ :\ & \mathrm{CAlg}^{cn}_{k} \to \mathcal{S} \\ & R \to \mathrm{Def}(G_0, R, \underline{\mathrm{Level}}(\mathcal{A})) \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, \mathcal{A})$ is the ∞-category whose objects are quaternions (G, ρ, η)

1. G is a spectral p-divisible group over R.
2. ρ is a G_0 taggings of R.
3. e is an orientation of the connected component of G.
4. $\eta : \mathcal{A} \to G(R)$ is a derived level structure.

Theorem B. The functor $\mathcal{M}_n$ is representable by a affine spectral Deligne-Mumford Stack $\mathrm{Sp\acute{e}t}\mathcal{JL}$, where $\mathcal{JL}$ is a finite $R^{or}_{G_0}$ algebra.

We call the resulting spectrum Jacquet-Langlands spectrum, this spectrum admits a natural action of $GL_n(Z/p^mZ) \times \mathrm{Aut}G_0$. The π_0 of this spectrum can realize the Jacquet-Langlands correspondence, and we hope to realize a topological Jacquet-Langlands correspondence. This is a different way to realize topological version of Jacquet-Langlands correspondence comparing the way using the degenerating level structures, see [SS23].

Notations

1. For a spectral Deligne-Mumford stack X, we let $X^{\heartsuit}$ denote its underlying ordinary Deligne-Mumford stack.
2. By a spectral Deligne-Mumford stack X over R, we mean a map of spectral Deligne-Mumfrd stack $X \to \mathrm{Sp\acute{e}t}R$.

3. X be a spectral Deligne-Mumford stack over R, let S be an R-algebra. We let $X \times_R S$ denote the product $X \times_{\mathrm{Sp\acute{e}t}R} \mathrm{Sp\acute{e}t}S$.

4. $\mathcal{M}_{ell}$ denote the spectral Deligne-Mumford stack of spectral elliptic curves, which is defined in [Lur18a].

Acknowledgements: This paper is finished during my PhD study. I thanks my advisor Yifei Zhu for his advising and helping in my entire PhD periods, especially for the discussions about ideas of derived level structures. I thanks Jing Liu for his discussion about some knowledge of Algebraic Geometry. I thanks Guozhen Wang, Jack Morgan Davies, Yingxin Li, Qingrui Qu for helpful discussion about ideas in this paper.

2 Spectral Elliptic Curves with Classical Level Structures

The first thing we can consider is the moduli problem of spectral elliptic curves with a classical level structure of its heart. We review the definition of level structures in the classical case.

Let C/S be a smooth commutative group scheme over S of relative dimension one, A be a abstract finite abelian group. A homomorphism of abstract groups

$$\phi : A \to C(S)$$

is said to be an A-Level structure on C/S if the effective Cartier divisor D in C/S defined by

$$D = \Sigma_{a\in A}[\phi(a)]$$

is a subgroup G of C/S.

The following result due to Katz-Mazur [KM85] give the representability of level structures moduli problems.

Proposition 2.1. *([KM85, Proposition 1.6.2]) Let C/S be a smooth commutative group scheme over S of relative dimension one, A be a abstract finite abelian group. Then the functor*

$$A - \underline{\mathrm{Level}} : \mathrm{Sch}_S \to \mathbf{Set}$$

$$T \mapsto \textit{the set of A-level structures on } C_T/T$$

is represented by a closed subscheme of $\underline{\mathrm{Hom}}(A, C) \cong C[N_1] \times_S \cdots \times_S C[N_r]$.

Although we have this representability, but we will follow the Deligne-Rapport's stacky method [DR73],which says that there is a Deligne-Mumford stack parameterize elliptic curves with level structures.

The moduli problem of spectral elliptic curves with classical level structures can be thought as a functor.

$$\begin{aligned} \mathcal{M}^{cl}_{ell}(A) &: \mathrm{CAlg} \to \mathcal{S} \\ & R \longmapsto \mathcal{M}^{cl}_{ell}(A)(R) \end{aligned}$$

where $\mathcal{M}^{cl}_{ell}(A)(R)$ for $R \in \mathrm{CAlg}^{cn}$ is defined by the following diagram:

$$\begin{tikzcd} \mathcal{M}^{cl}_{ell}(A)(R) \arrow[r] \arrow[d] & \mathcal{M}_{ell}(R) \arrow[d] \\ \mathcal{M}^{\heartsuit}_{ell}(A)(R^{\heartsuit}) \arrow[r] & \mathcal{M}^{\heartsuit}_{ell}(R^{\heartsuit}) \end{tikzcd}$$

It is easy to say that object in $\mathcal{M}_{ell}(A)(R)$ is an spectral elliptic curve E with a classical level structure of $E^{\heartsuit}$. We notice that for a map Spét$(R) \to \mathcal{M}^{cl}_{ell}(A)$, it is equivalent to maps Spét$R \to \mathcal{M}_{ell}$ and Spét$R \to \mathcal{M}^{\heartsuit}_{ell}(A)$. That is we have an ordinary elliptic curve E_0 over $R^{\heartsuit}$ and a spectral elliptic curve E over R, which is a lift of E_0 , and we have a level structure $A \to E_0$.

Proposition 2.2. *([Lur18c, Proposition 18.1.1.1]) Let $X : \mathrm{CAlg}^{cn} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a contangent complex. Then the following conditions are equivalent:*

1. *The functor X is a sheaf with respect to the étale topology,*
2. *The functor $X|_{\mathrm{CAlg}^{\heartsuit}}$ is a sheaf with respect to the étale topology.*

Proposition 2.3.

$$\mathcal{M}^{cl}_{ell}(A) : \mathrm{CAlg} \to \mathcal{S}, \quad R \mapsto \mathcal{M}^{cl}_{ell}(A)(R)$$

is an étale sheaf.

Proof. We first claim that $\mathcal{M}^{cl}_{ell}(A)|_{\mathrm{CAlg}^{\heartsuit}}$ is a étale sheaf. This is easy to see, we just notice that for a ordinary ring R_0, there is an equivalence of spaces $\mathcal{M}_{ell}(R_0)$ and $\mathcal{M}^{\heartsuit}_{ell}(R_0)$. So we have an equivalence $\mathcal{M}^{cl}_{ell}(A)(R_0) \simeq \mathcal{M}^{\heartsuit}_{ell}(A)(R_0)$. $\mathcal{M}^{cl}_{ell}(A)|_{\mathrm{CAlg}^{\heartsuit}}$ satisfying the étale descent because $\mathcal{M}^{\heartsuit}_{ell}(A)$ satisfying the étale descent.

1. $\mathcal{M}_{ell}(A) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is nilcomplete.

We need to show that for every connective $\mathbb{E}_\infty$-ring R, the canonical map $\mathcal{M}^{cl}_{ell}(A)(R) \to \varprojlim \mathcal{M}^{cl}_{ell}(A)(\tau_{\leq n}R)$ is an equivalence. This is easy to see, because $\mathcal{M}_{ell}(A)$

is nilcomplete, $\mathcal{M}_{ell}(R) \to \underleftarrow{\lim}\mathcal{M}_{ell}(\tau_{\leq n}R)$, and $\tau_{\leq n}R^{\heartsuit} \simeq R^{\heartsuit}$ naturally, so by the homotopy-pull back diagram

$$\begin{tikzcd}
\mathcal{M}_{ell}^{cl}(A)(R) \arrow[r] \arrow[d] & \mathcal{M}_{ell}(R) \arrow[d] \\
\mathcal{M}_{ell}^{\heartsuit}(A)(R^{\heartsuit}) \arrow[r] & \mathcal{M}_{ell}^{\heartsuit}(R^{\heartsuit})
\end{tikzcd}$$

$\mathcal{M}_{ell}^{cl}(A)(R) \to \underleftarrow{\lim}\mathcal{M}_{ell}^{cl}(A)(\tau_{\leq n}R)$ is an equivalence.

2. $\mathcal{M}_{ell}^{cl}(A) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is infinitesimally cohesive.

We need to prove that for every pullback diagram

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] & R \arrow[d] \\
S' \arrow[r] & S
\end{tikzcd}$$

in CAlg^{cn} such that $\pi_0 R \to \pi_0 S$ and $\pi_0 S' \to \pi_0 S$ are surjective, whose kernels are nilpotent ideals in $\pi_0 R$ and $\pi_0 S'$, the induced diagram

$$\begin{tikzcd}
\mathcal{M}_{ell}^{cl}(A)(R') \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{cl}(A)(R) \arrow[d] \\
\mathcal{M}_{ell}^{cl}(A)(S') \arrow[r] & \mathcal{M}_{ell}^{cl}(A)(S)
\end{tikzcd}$$

is a pull-back square in $\mathcal{S}$.

But we already have pullback diagram

$$\begin{tikzcd}
\mathcal{M}_{ell}(R') \arrow[r] \arrow[d] & \mathcal{M}_{ell}(R) \arrow[d] \\
\mathcal{M}_{ell}(S') \arrow[r] & \mathcal{M}_{ell}(S).
\end{tikzcd} \tag{1}$$

Because every functor $\mathrm{CAlg}^{cn} \to \mathcal{S}$ which is determined by a spectral Deligne-Mumford stack is cohesive [Lur18c, Remark 17.3.1.6]. Similarly, we have a pull-back diagram

$$\begin{tikzcd}
\mathcal{M}_{ell}^{\heartsuit}(A)(R') \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{\heartsuit}(A)(R) \arrow[d] \\
\mathcal{M}_{ell}^{\heartsuit}(A)(S') \arrow[r] & \mathcal{M}_{ell}^{\heartsuit}(A)(S)
\end{tikzcd} \tag{2}$$

So for a point in $\mathcal{M}_{ell}(A)(R')$, that is a spectral elliptic curve $E_{R'}$ over R' with a classical level structure $\phi_{R'} : A \to E_{R'}^{\heartsuit}(R')$. By the diagram (1), this corresponds two spectral elliptic curves E_R over R and $E_{S'}$ over S' which are compatible for base change to S. Because these two diagrams are all over pull-back diagrams

$$\begin{tikzcd} \mathcal{M}_{ell}^{\heartsuit}(R') \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{\heartsuit}(R) \arrow[d] \\ \mathcal{M}_{ell}^{\heartsuit}(S') \arrow[r] & \mathcal{M}_{ell}^{\heartsuit}(S) \end{tikzcd} \tag{3}$$

By the diagram(2), this point corresponds two classical level structures $\phi_R : A \to E_R^{\heartsuit}(R_0)$ over R_0 and $\phi_{S'} : A \to E_{S'}^{\heartsuit}$ over S_0' which are compatible for base change to S_0.

Combining these two, we get a spectral elliptic curve E_R over R with a classical level structure $\phi_R : A \to E_R^{\heartsuit}$ over R_0 over R_0, and a spectral elliptic curve $E_{S'}$ over S' with a classical level structure $\phi_{S'} : A \to E_{S'}^{\heartsuit}$. These two data are compatible for base change to S. That are just a point in $\mathcal{M}_{ell}^{cl}(A)(R)$ and a point in $\mathcal{M}_{ell}^{cl}(A)(S')$, and they are compatible when maps to $\mathcal{M}_{ell}^{cl}(A)(S)$.

So this is a map

$$\mathcal{M}_{ell}(A)(R') \to \mathcal{M}_{ell}(A)(S') \prod_{\mathcal{M}_{ell}(A)(S)} \mathcal{M}_{ell}(A)(R)$$

The pull-pack commutates with pullback, so we get this a equivalence.

3. $\mathcal{M}_{ell}^{cl}(A)$ admits a connective cotangent complex.

This is easy to see. By [Lur18c, Remark 17.2.4.5], if $X : \mathrm{CAlg} \to \mathcal{S}$ is the limit of diagram of functors $\{X_\alpha : \mathrm{CAlg} \to \mathcal{S}\}$. Assume each X_α admits a n-connective cotangent complex, then X admits a n-connective cotangent complex. By the pull-back diagram

$$\begin{tikzcd} \mathcal{M}_{ell}^{cl}(A)(R) \arrow[r] \arrow[d] & \mathcal{M}_{ell}(R) \arrow[d] \\ \mathcal{M}_{ell}^{\heartsuit}(A)(R^{\heartsuit}) \arrow[r] & \mathcal{M}_{ell}^{\heartsuit}(R^{\heartsuit}) \end{tikzcd}$$

$\mathcal{M}_{ell}^{cl}(A)$ admits a connective contangent complex.

∎

Like the classical case, we want the spectral elliptic curves with classical level structures have the structure of spectral Deligne-Mumford stacks. We first recall the following spectral Artin's representability theorem.

Theorem 2.4. *[Lur18c, Theroem 18.3.0.1] Let $X : \mathrm{CAlg}^{cn} \to \mathcal{S}$ be a functor, if we have a natural transformation $f : X \to \mathrm{Spec}R$, where R is a Noetherian $\mathbb{E}_\infty$-ring and $\pi_0 R$ is a Grothendieck ring. For $n \geq 0$, X is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R if and only if the following conditions* *are satisfied:*

1. *For every discrete commutative ring R_0, the space $X(R_0)$ is n-truncated.*
2. *The functor X is a sheaf for the étale topology.*
3. *The functor X is nilcomplete, infinitesimally cohesive, and integrable.*
4. *The functor X admits a connective cotangent complex L_X.*
5. *The natural transformation f is locally almost of finite presentation.*

Proposition 2.5. *The functor $\mathcal{M}^{cl}_{ell}(A)$ is represented by a spectral Deligne-Mumford stack which is locally almost of finite presentation over the sphere spectrum.*

Proof. By the spectral Artin's representability theorem and the proof of étale descent of $\mathcal{M}^{cl}_{ell}(A)$. We need to prove that: (1), For $A \in \mathrm{CAlg}^{\heartsuit}$, the space $\mathcal{M}^{cl}_{ell}(A)$ is n-truncated. (2), $\mathcal{M}^{cl}_{ell}(A)$ is locally almost of finite presentation. (3), $\mathcal{M}^{cl}_{ell}(A)$ is integrable.

1. For $R_0 \in \mathrm{CAlg}^{\heartsuit}$, the space $\mathcal{M}^{cl}_{ell}(R_0)$ is 1-truncated.

   This is easy to see, we just notice that when $R \in \mathrm{CAlg}^{\heartsuit}$, $\mathcal{M}_{ell}(R)$ is 1-truncated, and $\mathcal{M}^{\heartsuit}_{ell}(A)(R)$ is also 1-truncated. So by the definition of $\mathcal{M}^{cl}_{ell}$, $\mathcal{M}^{cl}_{ell}(A)$ is at least 2-truncated.

2. By the definition of locally almost of finite presentation of functors [Lur18c, Definition 17.4.1.1], we need to prove that: $\mathcal{M}^{cl}_{ell}(A) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ commutes with filtered colimits when restricted to $\tau_{\leq n}\mathrm{CAlg}^{cn}$, for each $n \geq 0$. i.e. For a diagram $I \to \mathrm{CAlg}^{cn}, \alpha \mapsto R_\alpha$, we have

   $$\mathcal{M}^{cl}_{ell}(A)(\mathrm{colim} R_\alpha) \simeq \mathrm{colim}\mathcal{M}^{cl}_{ell}(A)(R_\alpha)$$

   A point in $\mathcal{M}^{cl}_{ell}(A)(\mathrm{colim}R_\alpha)$ consists of a spectral elliptic curve $E^{\heartsuit}_{\mathrm{colim}R_\alpha}$ over $\mathrm{colim}R_\alpha$ and a classical level structure $\phi_{\mathrm{colim}R_\alpha} : A \to E^{\heartsuit}_{\mathrm{colim}R_\alpha}(\pi_0(\mathrm{colim}R_\alpha))$. But $\mathcal{M}_{ell}$ and $\mathcal{M}^{\heartsuit}_{ell}(A)$ are all spectral Deligne-Mumford stacks which are locally almost of finite presentation. So this point corresponds to the limit of the diagram $I \to E_{R_\alpha}$ and the limit of $\{A \to E^{\heartsuit}_{R_\alpha}(\pi_0 R_\alpha)\}_{\alpha \in I}$. That is for every $\alpha \in I$, we get a spectral elliptic curve E_α over R_α with a classical level structure $\phi_{R_\alpha} : A \to E_{R^{\heartsuit}_\alpha}$. This is just a point in $\mathrm{colim}\mathcal{M}^{cl}_{ell}(A)(R_\alpha)$. We actually get a map

   $$\mathcal{M}^{cl}_{ell}(A)(\mathrm{colim}R_\alpha) \to \mathrm{colim}\mathcal{M}^{cl}_{ell}(A)(R_\alpha).$$

   This map is an equivalence since we have homotopy pullback commutates with homotopy filtered colimit.

3. $\mathcal{M}^{cl}_{ell}(A) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is integrable.

   We need to prove that for R a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $m \subset \pi_0 R$. Then the inclusion functor induces a homotopy equivalence

$$\mathcal{M}^{cl}_{ell}(A)(R) \simeq \mathrm{Map}_{Fun(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Sp\acute{e}t}R, \mathcal{M}^{cl}_{ell}(A)) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn},\mathcal{S})}(\mathrm{Spf}A, \mathcal{M}^{cl}_{ell}(A)(R/m^n)).$$

   But by [Lur18c, Proposition 17.3.5.1], we only need to prove that

$$\mathcal{M}^{cl}_{ell}(A)(R) \to \varprojlim \mathcal{M}^{cl}_{ell}(A)(R/m^n)$$

   is a homotopy equivalence. For every $n \geq 0$, we have a homotopy-pull back diagram

$$\begin{tikzcd}
\mathcal{M}^{cl}_{ell}(A)(R/m^n) \arrow[r] \arrow[d] & \mathcal{M}_{ell}(R/m^n) \arrow[d] \\
\mathcal{M}^{\heartsuit}_{ell}(A)((R/m^n)^{\heartsuit}) \arrow[r] & \mathcal{M}^{\heartsuit}_{ell}((R/m^n)^{\heartsuit})
\end{tikzcd}$$

   We know that pull-back commute with limit, so we have a pull-back diagram

$$\begin{tikzcd}
\varprojlim \mathcal{M}^{cl}_{ell}(A)(R/m^n) \arrow[r] \arrow[d] & \varprojlim \mathcal{M}_{ell}(R/m^n) \arrow[d] \\
\varprojlim \mathcal{M}^{\heartsuit}_{ell}(A)((R/m^n)^{\heartsuit}) \arrow[r] & \varprojlim \mathcal{M}^{\heartsuit}_{ell}((R/m^n)^{\heartsuit})
\end{tikzcd}$$

   $\mathcal{M}_{ell}, \mathcal{M}^{\heartsuit}_{ell}(A)$ and $\mathcal{M}^{\heartsuit}_{ell}$ are spectral Deligne-Mumford stacks, so they are all integrable. So we have pull-back diagram

$$\begin{tikzcd}
\varprojlim \mathcal{M}^{cl}_{ell}(A)(R/m^n) \arrow[r] \arrow[d] & \mathcal{M}_{ell}(A)(R) \arrow[d] \\
\mathcal{M}^{\heartsuit}_{ell}(A)(R^{\heartsuit}) \arrow[r] & \mathcal{M}^{\heartsuit}_{ell}(A)(R^{\heartsuit})
\end{tikzcd}$$

   By the homotopy uniqueness of pull-back, we get an equivalence

$$\mathcal{M}^{cl}_{ell}(A)(R) \simeq \varprojlim \mathcal{M}^{cl}_{ell}(A)(R/m^n).$$

■

3 Spectral Elliptic Curves with Derived Level Structures

To define derived level structures, the first question is what the higher categorical analogue of finite abelian groups are? We first recall some finiteness conditions in $\mathbb{E}_\infty$-rings context.

Let A be an $\mathbb{E}_\infty$-ring, M be an A-module. We say M is

1. perfect, if it is an compact object of $L\mathrm{Mod}_R$.

2. almost perfect, if there exits a integer k such that $M \in (L\mathrm{Mod}_R)_{\geq k}$ and M is an almost perfect object of $(L\mathrm{Mod}_R)_{\geq k}$.

3. perfect to order n if for every filtered diagram $\{N_\alpha\}$ in $(L\mathrm{Mod}_A)_{\leq 0}$, the canonical map $\underset{\rightarrow\alpha}{\lim}\mathrm{Ext}^i_A(M, N_\alpha) \to \mathrm{Ext}^i_A(M, \underset{\rightarrow\alpha}{\lim}N_\alpha)$ is injective for $i = n$ and bijective for $i \leq n$.

4. finitely n-presented if M is n-truncated and perfect to order (n+1).

5. finite generated, if it is perfect to order 0.

And when we consider the finite condition on algebra. We say a morphism $\phi : A \to B$ of connective $\mathbb{E}_\infty$-rings is

1. finite presentation if B belongs to the smallest full subcategory of Alg^{free}_A and is stable under finite colimits.

2. locally of finite presentation if B is a compact object of Alg_A.

3. almost of finite presentation if A is an almost compact object of Alg_A, that is, $\tau_{\leq n}B$ is a compact object of $\tau_{\leq n}\mathrm{Alg}_A$ for all $n \geq 0$.

4. finite generation to order n if the following conditions holds:

   Let $\{C_\alpha\}$ be a filtered diagram of connective $\mathbb{E}_\infty$-rings over A having colimit C. Assume that each C_α is n-truncated and that each of the transition maps $\pi_n C_\alpha \to \pi_n C_\beta$ is a monomorphism. Then the canonical map

   $$\lim_{\alpha} \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$$

   is a homotopy equivalence.

5. finite type if it is of finite generation to order 0.

6. finite if B is a finitely generated as an A-module.

Proposition 3.1. *[Lur18c, Proposition 2.7.2.1, Proposition 4.1.1.3] Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings.. Then The following conditions are equivalent.*

1. *ϕ is of finite (finite type).*
2. *The commutative ring $\pi_0 B$ is finite (finite type) over $\pi_0 A$.*

Definition 3.2 [Lur18c, Definition 4.2.0.1] Let $f : X \to Y$ be a morphism of spectral Deligne-Mumford Stack. We say that f is locally of finite type, (locally of finite genreration to order n, locally almost of finite presentation, locally of finite presentation) if the following conditions is satisfied: for every commutative diagram

$$\begin{array}{ccc} \mathrm{Sp\acute{e}t}B & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Sp\acute{e}t}A & \longrightarrow & Y \end{array}$$

where the horizontal morphisms are étale, the $\mathbb{E}_\infty$-ring B is finite type (finite generation to order n, almost of finite presentation, locally of finite presentation) over A.

Definition 3.3 [Lur18c, Definition 5.2.0.1] Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of spectral Deligne-Mumford stacks, we say f is finite, if the following conditions hold

1. f is affine.
2. The push-forward is $f_* \mathcal{O}_X$ is perfect to order 0 as a $\mathcal{O}_Y$ module.

Remark 3.4 By the [Lur18c, Example 4.2.0.2], A morphism $f : X \to Y$ of spectral Deligne-Mumford stack is locally of finite type if the underlying map of spectral Deligne-Mumford stacks is locally of finite type in the sense of ordinary algebraic geometry.

And by [Lur18c, 5.2.0.2], A morphism of $f : X \to Y$ is finite if the underlying map $f^\heartsuit : X^\heartsuit \to Y^\heartsuit$ is finite. If X and Y are spectral algebraic spaces, then f is finite is equivalent to $f^\heartsuit$ is finite is the sense of ordinary algebraic geometry.

We recall that a morphism $f : X \to Y$ of spectral Deligne-Mumford stacks is surjective if for every field k and any map $\mathrm{Sp\acute{e}t}k \to Y$, the fiber product $\mathrm{Sp\acute{e}t}k \times_Y X$ is nonempty [Lur18c, Definition 3.5.5.5].

Definition 3.5 Let $f : X \to Y$ be a morphism of spectral abelian varieties over a connective $\mathbb{E}_\infty$-ring R, we say f is an isogeny if it is finite, flat and surjective.

Lemma 3.6. *Let $f : X \to Y$ be a morphism of spectral abelian varieties, then $f^\heartsuit : X^\heartsuit \to Y^\heartsuit$ is an isogeny in the classical sense.*

Proof. In classical abelian varieties, $f^\heartsuit$ isogeny means $f^\heartsuit$ is surjective and $\ker f^\heartsuit$ is finite. But it is equivalent to $f^\heartsuit$ is finite, flat and surjective [Mil86, Proposition 7.1]. And it is easy to see that $f^\heartsuit$ is finite, flat. We only need to prove that $f^\heartsuit$ is surjective.

For every morphism $|\mathrm{Spec}k| \to |Y^\heartsuit|$, this correspond to a morphism $\mathrm{Sp\acute{e}t}k \to Y^\heartsuit$, by the inclusion-truncation adjunction [Lur18c, Proposition 1.4.6.3], this corresponds to a morphism $\mathrm{Sp\acute{e}t}k \to X$. By the definition of surjective, we get a commutative diagram

$$\begin{tikzcd} \mathrm{Sp\acute{e}t}k' \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Sp\acute{e}t}k \arrow[r] & Y \end{tikzcd}$$

The upper horizontal morphism corresponds to a morphism $\mathrm{Sp\acute{e}t}k' \to X^\heartsuit$ by inclusion-truncation adjunction. On the underlying topological space level, this corresponds to a point $|\mathrm{Sp\acute{e}t}k| \to |Y^\heartsuit|$. It is clear that this point in $|Y^\heartsuit|$ is a preimage of $|\mathrm{Sp\acute{e}t}k|$ in $X^\heartsuit$. So $f^\heartsuit$ is surjective. ■

Lemma 3.7. *Let $f : X \to Y$ be an isogeny of spectral elliptic curves over a connective $\mathbb{E}_\infty$-ring R, then* $\mathrm{fib}(f)$ *exists and is a finite and flat nonconnective spectral Deligne-Mumford stack over R.*

Proof. By [Lur18c, Proposition 1.14.1.1], the finite limits of nonconnective spectral Deligne-Mumford stack exists, so we can define fib(f). We consider the following diagram

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \arrow[ddr] & \\ * \arrow[r] \arrow[drr] & Y \arrow[dr, "i"] & \\ & & \mathrm{Sp\acute{e}t}R \end{tikzcd}$$

where the square is a pull-back diagram. We find that $\mathrm{fib}(f)$ is over $\mathrm{Sp\acute{e}t}R$. By [Lur18c, Remark 2.8.2.6], $f' : \mathrm{fib}(f) \to *$ is flat since it is a pull-back of a flat morphism. Obviously $i : * \to \mathrm{Sp\acute{e}t}R$ is flat, so by [Lur18c, Example 2.8.3.12] (Being flat morphism is local on the source with respect to the flat topology), $i \circ f' : \mathrm{fib}(f) \to \mathrm{Sp\acute{e}t}R$ is flat.

Next, we show $\ker f$ is finite over R. Since $*, X, Y$ are all spectral algebraic spaces, so we have $\mathrm{fib} f$ is also a spectral algebraic space. And $\mathrm{Sp\acute{e}t}R$ is an algebraic space [Lur18c, Example 1.6.8.2]. By the above remark 3.4, we only need to prove that the underlying morphism is finite. The truncation functor is a right adjoint , so preserve limits. So we get a pull-back diagram

$$\begin{tikzcd} \mathrm{fib}(f)^\heartsuit \arrow[r] \arrow[d] & X^\heartsuit \arrow[d] \\ * \arrow[r] & Y^\heartsuit \end{tikzcd}$$

So we are reduced to prove that for an isogeny $f : X \to Y$ of ordinary abelian varieties over a commutative ring R. $\ker f$ is finite over R. We consider the map factorisation $\ker f \to * \to R$. $\ker f \to *$ is finite since it is a pull-back of finite morphism. And $* \to \mathrm{Sp\acute{e}t}R$ is quasi-finite. we can choose a field Ω and a morphism $R \to \Omega$ such that $\mathrm{Spec}\Omega \simeq * \to \mathrm{Spec}R$ is closed, so proper , and hence finite. So by composition, we get $\ker f \to \mathrm{Spec}R$ is finite.

■

Lemma 3.8. *Let $f_N : E \to E$ be an isogeny of spectral elliptic curves over R, such that the underline map of ordinary elliptic curve is the multiplication N map, $N : E^{\heartsuit} \to E^{\heartsuit}$. Then $\mathrm{fib} f$ is finite locally free of rank N in the sense of [Lur18c, Definition 5.2.3.1]. And moreover if N is invertible in $\pi_0 R$, then $\mathrm{fib} f$ is a locally constant étale sheaf.*

Proof. By [KM85, Theorem 2.3.1], we know that $N : E^{\heartsuit} \to E^{\heartsuit}$ is locally free of rank N in the classical sense. When N is invertible in $\pi_0 R$, then $\ker N$ is locally constant *é*tale sheaf. $\mathrm{fib}(f_N)$ is a spectral algebraic space which is finite and flat and its underlying map $\mathrm{fib}(f_N)^{\heartsuit} = \ker N$ is locally free of rank N. f_N is finite by the above theorem. We need to prove that $\mathrm{fib} f_N \to \mathrm{Sp\acute{e}t}R$ is locally free of rank N. But $\ker f_N$ is finite and flat, so is affine. We are reduce to prove $f_N : \mathrm{Sp\acute{e}t}S \to \mathrm{Sp\acute{e}t}R$ is locally free, for $\mathrm{Sp\acute{e}t}S$ is an affine substack of $\mathrm{fib} f_N$. This is equivalent to prove that $R \to S$ is locally free of rank N in the sense of [Lur18c, Definition 2.9.2.1]. So we need to prove that

1. S is a locally free of finite rank over R.(By [Lur17, Proposition 7.2.4.20], this is equivalent to say S is a flat and almost perfect R-module.)
2. For every $\mathbb{E}_\infty$-ring maps $R \to k$, the vector space $\pi_0(M \otimes_R k)$ is a N-dimensional k-vector space.

For (1), we know that $\pi_0 S$ is projective $\pi_0 R$-module, and S is a flat R-module, so by [Lur09a, Proposition 7.2.2.18], S is a projective R-module. And since $\pi_0 S$ is a finitely generate R-module, so by [Lur17, Corollary 7.2.2.9], S is a retract of a finitely generated free R-module M, so is locally free of finite rank.

For (2), $\pi_0(k \otimes_R M)$ since R and M are connective, by [Lur17, Corollary 7.2.1.23], we get $\pi_0(k \otimes_R M) \simeq k \otimes_{\pi_0 R} \pi_0 M$ is a rank N k-vector space ($\pi_0 M$ is rank N free $\pi_0 R$ module).

We next show that if N is invertible in $\pi_0 R$, then $\mathrm{fib} f$ is a locally constant sheaf. By the above discussion, $\mathrm{fib} f$ is spectral Deligne-Mumford stack, so the associated functor points $\mathrm{fib}_f : \mathrm{CAlg}_R \to \mathcal{S}$ is nilcomplete and locally of almost finite presentation. By [KM85, Theorem 2.3.1], $\mathrm{fib}_f|_{\mathrm{CAlg}^{\heartsuit}_{\pi_0 R}}$ is a locally constant sheaf, the desired results follows form the following lemma. ■

Lemma 3.9. *Let $\mathcal{F} \in \mathrm{Shv}^{\acute{e}t}(\mathrm{CAlg}_R^{cn})$, and is nilcomplete, locally of almost finite presentation and $\mathcal{F}|_{(\mathrm{CAlg}_R^{cn})^{\heartsuit}}$ is the associated sheaf of constant presheaf valued on A. Then $\mathcal{F}$ is a homotopy locally constant sheaf (i.e., sheafification of a homotopy constant presheaf).*

Proof. Let $R_1 \to R_2$ be an *é*tale morphism in CAlg_R^{cn} such that $\mathrm{Sp\acute{e}t}R_2 \to \mathrm{Sp\acute{e}t}R_1$ is a surjective étale morphism in SpDM. We consider the following diagram

$$\begin{array}{ccc} \tau_{\leq 0}R_1 & \longrightarrow & \tau_{\leq 0}R_2 \\ \downarrow & & \downarrow \\ \tau_{\leq n}R_1 & \longrightarrow & \tau_{\leq n}R_2 \end{array}$$

which is push-out diagram, since R_2 is an étale R_1 algebra. This is a colimit diagram in $\tau_{\leq n}\mathrm{CAlg}_R$. $\mathcal{F}$ is a sheaf of locally of almost finite prsentation, so we get push-out diagram

$$\begin{array}{ccc} \mathcal{F}(\tau_{\leq 0}R_1) & \longrightarrow & \mathcal{F}(\tau_{\leq 0}R_2) \\ \downarrow & & \downarrow \\ \mathcal{F}(\tau_{\leq n}R_1) & \longrightarrow & \mathcal{F}(\tau_{\leq n}R_2) \end{array}$$

We have already know that $\mathcal{F}_{(\mathrm{CAlg}_R^{cn})^{\heartsuit}}$ is a locally constant sheaf, and $\tau_{\leq 0}R_1 \to \tau_{\leq 0}R_2$ is an étale surjection, so the upper horizontal morphism is a homotopy equivalence. Hence $\mathcal{F}(\tau_{\leq n}R_1) \simeq \mathcal{F}(\tau_{\leq n}R_2)$. But $\mathcal{F}$ is a nilcomplete sheaf, that means $\mathcal{F}(R') \simeq \mathrm{colim}\mathcal{F}(\tau_{\leq n}R')$. We get an homotopy equivalence $\mathcal{F}(R_1) \simeq \mathcal{F}(R_2)$. So for étale surjection map f, $F(f)$ is a homotopy equivalence, so $\mathcal{F}$ is a locally constant sheaf. ■

Definition 3.10 For A an abstract abelian group, R an connective $\mathbb{E}_\infty$-ring. We let $\mathrm{Derived}_A$ denote the ∞-subcategory of $\mathbf{CMon}(\mathcal{S})$, consists of those $\mathcal{A}$ which is group like $\mathbb{E}_\infty$-space and $\pi_0\mathcal{A} = A$.

We recall that for an ordinary smooth commutative group scheme X over S of relative dimension 1. Let A be an abstract abelian group, a A-level structure is a group homomorphism

$$\phi : A \to X(S)$$

such that $\Sigma_{a\in A}\phi(a)$ is a subgroup of X. We denote $\underline{\mathrm{Level}}(A, X/S)$ the set of A-level structures of X.

For a locally spectrally topoi $X = (\mathcal{X}, \mathcal{O}_x)$, we can consider its functor of points

$$h_X : \infty\mathbf{Top}^{loc}_{\mathrm{CAlg}} \to \mathcal{S}, \quad Y \mapsto \mathrm{Map}_{\infty\mathbf{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}}(YX)$$

For a fixed space $\mathcal{A}$, we can consider the constant presheaf

$$\underline{\mathcal{A}} : \mathrm{CAlg}^{cn} \to \mathcal{S}, \quad R \mapsto \mathcal{A}$$

its sheafification determine a sheaf, we still denote it as $\underline{\mathcal{A}} : \mathrm{CAlg}^{cn} \to \mathcal{S}$. It is easy to say that this $\mathcal{A}$ can be regard as the functor of points of a spectral Deligne-Mumford stack. We denote it as $\underline{\mathcal{A}} = (\underline{\mathcal{A}}, \mathcal{O}_{\underline{\mathcal{A}}})$

By [Lur18c, Remark 3.1.1.2], the closed immersion of locally spectrally ringed topos $f : X = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to Y = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ corresponds to morphism of sheaves of connective $\mathbb{E}_\infty$-rings $\mathcal{O}_{\mathcal{X}} \to f_* \mathcal{O}_{\mathcal{Y}}$ over $\mathcal{X}$. We consider the the fiber of this map $\mathrm{fib} f$. When $X = \underline{\mathcal{A}}$, we denote $\mathrm{fib}(f)$ as $I(\mathcal{A})$.

In the following, for a nonconnective spectral Deligne-Mumford stack $X = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, we let $\mathscr{P}ic(X)$ denote ∞-category of line bundles over X. We have a functor

$$\mathscr{P}ic(X/S) \to \{\text{closed substack of X}\}, \quad L \mapsto \mathcal{Y}_{L'}$$

defined by sending L to the associated closed stack of L', where L' fits into a cofiber sequence.

$$L \to \mathcal{O}_X \to L'$$

Definition 3.11 Let X be a spectral elliptic over a nonconnective spectral Deligne-Mumford stack S, $X^\heartsuit$ its underlying ellitpic curves over $S^\heartsuit$, A be an abstract finite abelian group . For any $\mathcal{A} \in \mathrm{Derived}_A$, we define the space of derived level structure with level $\mathcal{A}$ of X to be the pull-back of the following diagram

$$\begin{tikzcd}
\underline{\mathrm{Level}}(\mathcal{A}, X/S) \arrow[r] \arrow[d] & \mathscr{P}ic(X/S) \arrow[d] \\
\underline{\mathrm{Level}}(A, X^\heartsuit/S_0) \arrow[r] & \mathrm{Map}_{\mathbf{Ab}}(A, X^\heartsuit(S_0))
\end{tikzcd}$$

Remark 3.12 This definition is easy to understand. In the level of objects, a derived level structure is a morphism of $\mathbb{E}_\infty$-group spaces $\phi : \mathcal{A} \to X(S)$, such that the associated morphism $A \to X^\heartsuit(S_0)$ is a classical level structure and $\underline{\mathcal{A}} \to X$ is a closed immersion of spectrally ringed topoi whose associated ideal sheaf is a line bundle.

Lemma 3.13. *Let E be a spectral elliptic curves over a nonconnective spectral Deligne-Mumford stack S, $\phi_S : \mathcal{A} \to E(S)$ be a derived level structure, $T \to S$ be a morphism of nonconnective spectral Deligne-Mumford stacks, then the induce morphism $\phi_S : \mathcal{A} \to (E \times_S T)(T) \simeq E(T)$ is a derived level structure of E_T/T.*

Proof. we notice that this lemma is true in the classical case. We need to prove that, (1) $\phi_S^\heartsuit : A \to (E \times_S T)^\heartsuit(T_0) = E^\heartsuit(T_0)$ is a classical level structure. But this is just the classical case. (2) $\underline{\mathcal{A}} \mapsto E_T$ determine a line bundle over E_T, this is easy, since we have a decomposition

$$\underline{\mathcal{A}} \to E_T \to E$$

We have have already know that its push-forward on $\mathrm{QCoh}(E)^{cn}$ is a line bundle, hence an invertible object in $\mathrm{QCoh}^{cn}(E)$. So its pushforward on $\mathrm{QCoh}(E_T)^{cn}$ is also a invertible object, hence a line bundle by [Lur18c, Proposition 2.9.4.2]. ■

Lemma 3.14. *Let E/S be a spectral elliptic curve, and D be a closed immersion, such that the associated sheaf is a line bundle over E, and D_0 is an effective Cartier divisor in E_0/S_0. Then there exists a closed spectral Deligne-Mumford substack $Z \subset S$, satisfying the following universal property:*

For any $T \to S$, such that the associated sheaf of D_T is a line bundle over X_T and $(D_T)^\heartsuit$ is a subgroup of $(E_T)^\heartsuit$, then T factor through Z.

Proof. For $T \to S$, it is obvious that the associated sheaf of D_T is a line bundle over X_T. By [KM85, Corollary 1.3.7], we jnow that if $(D_T)^\heartsuit/T_0$ is a subgroup of $(E_T)^\heartsuit/T_0$, we have T_0 must passing through $Z_0 \subset S_0$. So we find that the required closed substack is just $Z_0 \times_{S_0} S$. ■

Fro convenience, in the following, we only consider the base stack is affine, that is $S = \mathrm{Sp\acute{e}t} R$ for a connective $\mathbb{E}_\infty$-ring R. To prove the relative representability, we need the representability of the Picard functor. If we have a map $f : X \to \mathrm{Sp\acute{e}t} R$ of spectral Deligne-Mumford stack, we define a functor

$$\mathscr{P}ic_{X/R} : \mathrm{CAlg}_R^{cn} \to \mathcal{S}, \quad R' \mapsto \mathscr{P}ic(\mathrm{Sp\acute{e}t} R' \times_{\mathrm{Sp\acute{e}t} R} X)$$

If we suppose that f admits a section $x : \mathrm{Sp\acute{e}t} R \to X$. Then pullback along x determines a natural transformation of functors $\mathscr{P}ic_{X/R} \to \mathscr{P}ic_{R/R}$. We denote the fiber of this map by

$$\mathscr{P}ic^x_{X/R} : \mathrm{CAlg}_R^{cn} \to \mathcal{S}$$

Theorem 3.15. *[Lur18c, Theorem 19.2.0.5] Let $f : X \to \mathrm{Sp\acute{e}t} R$ be a map spectral algebraic spaces which is flat, proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. For any section $x : \mathrm{Sp\acute{e}t} R \to X$, the functor $\mathscr{P}ic^x_{X/R}$ is representable by a spectral algebraic space which is quasi-separated and locally of finite presentation of R.*

By the [Lur18c, Remark 19.2.0.3], the functor $\mathscr{P}ic^x_{X/R}$ is independent of the section x, up to canonical equivalence. In the following, we will choose a fixed section x.

Proposition 3.16. *Let E/R be a spectral elliptic curve, then the functor*

$$\begin{aligned} \underline{\mathrm{Level}}(\mathcal{A}, E/R) \quad : \quad & \mathrm{CAlg} \to \mathcal{S} \\ & R' \mapsto \underline{\mathrm{Level}}(\mathcal{A}, E_{R'}/R') \end{aligned}$$

is represented by a closed substack $S(\mathcal{A})$ of $\mathscr{P}ic^x_{X/R}$. Moreover, $S(\mathcal{A})$ is affine and locally of finite presentation over R.

Proof. By definition, the functor $\underline{\mathrm{Level}}(\mathcal{A}, E/R)$ is a subfunctor of the representable functor $\mathscr{P}ic^x_{X/R}$. It is the closed sub-stack of $\mathscr{P}ic^x_{X/R}$ such that the associated divisor of degree $\sharp(\pi_0\mathcal{A})$ in $(E \times_R \mathscr{P}ic^x_{X/R}/\mathscr{P}ic^x_{X/R})^\heartsuit$

$$\Sigma_{a\in\pi_0\mathcal{A}}\phi_{univ}(a)$$

attached to the universal morphism $\phi_{univ}; \mathcal{A} \to E(R)$, is a subgroup, then the assertion follows from lemma 3.14.

To prove the second part, we consider the map $S(\mathcal{A}) \to \mathrm{Sp\acute{e}t}R$, they are all spectral algebraic spaces. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary spectral algebraic space is finite in ordinary algebraic geometry. So we only need to prove $S(\mathcal{A})^\heartsuit$ is finite over $\mathrm{Spec}\pi_0 R$, but this is just the classical case since $S(\mathcal{A})^\heartsuit$ is the relative representable object of the classical level structure, which is finite over R_0 by [KM85, Corollary 1.6.3]. ■

Moduli of Spectral Elliptic Curves with Derived Level Structures

In the classical case, we know that the elliptic curves with level structures has a structure of Deligne-Mumford stack. We already have the definition of derived level structures, can we still have the similar results? In the following, A will still denote an abstract abelian group. And suppose we fix a $\mathcal{A} \in \mathrm{Derived}_A$.

The construction of $X \in \underline{\mathrm{Level}}(\mathcal{A}, X/R)$ determines a functor $Ell(R) \to \mathcal{S}$ which is calssified by a left fibration $\mathrm{Ell}(\mathcal{A})(R) \to Ell(R)$. The objects of $Ell(\mathcal{A})(R)$ can be identified with pairs (X, ϕ), where X is a spectral elliptic curve and $\phi : \mathcal{A} \to E(R)$ is a derived level structures of E.

For every $\mathbb{E}_\infty$-ring R, we can consider all spectral elliptic curves over R with derived level structures. This moduli problem can be thought as a functor

$$\begin{aligned}\mathcal{M}_{ell}(\mathcal{A}) \quad &: \quad \mathrm{CAlg}^{cn} \to \mathcal{S}\\ &\qquad R \longmapsto \mathrm{Ell}(\mathcal{A})(R)^{\simeq}\end{aligned}$$

where $\mathrm{Ell}(A)(R)$ is the ∞-category of spectral elliptic curves E with a derived level structures $\phi : \mathcal{A} \to E$. And $\mathrm{Ell}(\mathcal{A})(R)^{\simeq}$ is its underlying ∞-groupoid.

Proposition 3.17. *The functor $\mathcal{M}^{de}_{ell}(A) : \mathrm{CAlg}^{cn} \mapsto \mathcal{S}$ is an étale sheaf.*

Proof. Let $R \to U_i$ be a étale cover of R, and $U_\bullet$ be the associate check simplicial object. We consider the following diagram

$$\begin{tikzcd} \mathrm{Ell}(\mathcal{A})(R)^{\simeq} \arrow[r, "f"] \arrow[d, "p"] & \lim_{\Delta} \mathrm{Ell}(\mathcal{A})(U_\bullet)^{\simeq} \arrow[d, "q"] \\ \mathrm{Ell}(R)^{\simeq} \arrow[r, "g"] & \lim_{\Delta} \mathrm{Ell}(U_\bullet)^{\simeq} \end{tikzcd}$$

The left map p is a left fibration between Kan complex, so is a Kan fibration [Lur09a, Lemma 2.1.3.3]. And the right vertical map is pointwise Kan fibration . By picking a suit model for the homotopy limit we may assume that q is a Kan fibration as well. We have g is a equivalence by [Lur18a, Lemma 2.4.1]. To prove that f is a equivalence. We only need to prove that for every $E \in \mathrm{Ell}(R)$, the map

$$p^{-1}E \simeq \underline{\mathrm{Level}}(\mathcal{A}, E/R) \to \lim_{\Delta} \underline{\mathrm{Level}}(\mathcal{A}, E \times_R U_\bullet/U_\bullet) \simeq q^{-1}g(E)$$

is an equivalence.

We have the $\underline{\mathrm{Level}}(\mathcal{A}, E)$ as full ∞-subcategory of $\mathrm{Hom}(\mathcal{A}, E(R)) = \mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E(R))$ and $\lim_{\Delta} \underline{\mathrm{Level}}(\mathcal{A}, E \times_R U_\bullet)$ as a full subcategory of

$$\lim_{\Delta} \mathrm{Hom}(\mathcal{A}, E \times_R U_\bullet(U_\bullet)) \simeq \mathrm{Hom}(\mathcal{A}, \lim E(U_\bullet)) \simeq \mathrm{Hom}(\mathcal{A}, E(R))$$

So the functor

$$\underline{\mathrm{Level}}(\mathcal{A}, E/R) \to \lim_{\Delta} \underline{\mathrm{Level}}(\mathcal{A}, E \times_R U_\bullet/U_\bullet).$$

is fully faithful. To prove it is a equivalence, we only need to prove it is essentially surjective.

For any $\{\phi_{U_\bullet} : \mathcal{A} \to E \times_R U_\bullet\}$ in $\lim_{\Delta} \underline{\mathrm{Level}}(\mathcal{A}, E \times_R U_\bullet/U_\bullet)$. It was just $\{\phi_{U_\bullet} : \mathcal{A} \to E(U_\bullet)\}$. Clearly, we can find a morphism $\phi_R : \mathcal{A} \to E(R)$ in $\mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E(R))$ whose image under the equivalence $\mathrm{Hom}(\mathcal{A}, E(R)) \simeq \lim_{\Delta} \mathrm{Hom}(\mathcal{A}, E(U_\bullet))$ is $\{\phi_{U_\bullet} : \mathcal{A} \to E(U_\bullet)\}$. We just need to prove this $\phi_R : \mathcal{A} \to E(R)$ is a derived level structure. This is true since the level structure is determined by its underlying morphism in π_0 and in the classic case, $\underline{\mathrm{Level}}(A, E^{\heartsuit}(R_0)) \simeq \lim_{\Delta} \underline{\mathrm{Level}}(A, E^{\heartsuit}(\tau_{\leq 0}U_\bullet))$. And $I(\mathcal{A})$ is a line bundle over E, since we have the decomposition

$$\underline{\mathcal{A}} \to E \times_R U_\bullet \to E$$

We have $E_{U_\bullet}$ is an étale cover of E, $I(\mathcal{A})$ is a line bundle on $E_{U_\bullet}$, so it is a line bundle over E. ∎

Lemma 3.18. *$\mathcal{M}^{de}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ is a nilcomplete functor, i.e., $\mathrm{Ell}(\mathcal{A})(R)$ is the* *homotopy limit of the following diagram*

$$\cdots \to \mathrm{Ell}(\mathcal{A})(\tau_{\leq m}R) \to \mathrm{Ell}(\mathcal{A})(\tau_{\leq m-1}R) \to \cdots \to \mathrm{Ell}(\mathcal{A})(\tau_{\leq 0}R)$$

Proof. We let $\tau_{\leq n}E$ denote $E \times_R \tau_{\leq n}R$, then there is an obvious functor

$$\theta : \mathrm{Ell}(\mathcal{A})(R) \to \lim_{\leftarrow n} \mathrm{Ell}(\mathcal{A})(\tau_{\leq n}R)$$

define by $(E, A \to E) \mapsto \{(\tau_{\leq n}E, \mathcal{A} \to \tau_{\leq n}E)\}_n$, here we notice that $(\tau_{\leq n}E, \mathcal{A} \to \tau_{\leq n}E)$ is in $\mathrm{Ell}(\mathcal{A})(\tau_{\leq n}R)$. ($\tau_{\leq n}E$ is spectral elliptic curve because flat, prober, locally of almost finite presentation is stable under base change).

First, we prove that θ is essentially surjective. An object in $\lim_{\leftarrow m} \mathrm{Ell}(\mathcal{A})(\tau_{\leq m}R)$ can be written as a diagram

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrrr] & & & & \\
\cdots & E_n \arrow[r] & E_{n-1} \arrow[r] & E_{n-2} \arrow[r] & \cdots \arrow[r] & E_0
\end{tikzcd}$$

where each E_n is spectral elliptic curve over $\tau_{\leq n}R$ and $\mathcal{A} \to E_n$ is a derived level structure. By the nilcompletness of $R \mapsto \mathrm{Ell}(R)^{\simeq}$, we get a spectral elliptic curves E, such that $E \times_R \tau_{\leq n}R \simeq E_n$. And it is obvious that the induced map $\mathcal{A} \to E$ is a derived level structure, because $\mathcal{A} \to E \times_R \tau_{\leq 0}R$ is a level structure. So θ is essentially surjective.

Second, we need to prove that this functor is fully faithful. Unwinding the definitions, we need to prove that for every $(X, \mathcal{A} \to X), (Y, \mathcal{A} \to Y) \in \mathrm{Ell}(\mathcal{A})(R)$, the following map is a homotopy equivalence.

$$\mathrm{Map}_{\mathrm{Ell}(\mathcal{A})(R)}((X, \mathcal{A} \to X), (Y, \mathcal{A} \to Y)) \to \mathrm{Map}_{\mathrm{Ell}(\mathcal{A})(R)}(\lim_{\leftarrow n}(X_n, \mathcal{A} \to X_n), \lim_{\leftarrow n}(Y_n, \mathcal{A} \to Y_n)). \quad (1)$$

where X_n is $\tau_{\leq n}X = X \times_R \tau_{\leq n}R$, and Y similarly.

The left hand side consists of commutative diagrams

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[dr] & X \arrow[d] \\
 & Y
\end{tikzcd}$$

It was sending to

$$\begin{tikzcd}
 & \cdots X_n \arrow[r] \arrow[dd] & X_{n-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & X_0 \arrow[dd] \\
\mathcal{A} \arrow[ur] \arrow[urr] \arrow[urrrr] \arrow[dr] \arrow[drr] \arrow[drrrr] & & & & \\
 & \cdots Y_n \arrow[r] & Y_{n-1} \arrow[r] & \cdots \arrow[r] & Y_0
\end{tikzcd}$$

By [Lur18c, Proposition 19.4.0.2, Proposition 19.4.5.6, Proposition 19.2.4.3] and [Lur18a, Proposition 2.1.5, Theorem 2.4.1], says that functor $R \mapsto \mathrm{Ell}(R)^{\simeq}$ is nicomplete, so we have equivalence $\mathrm{Map}_{\mathrm{Ell}(R)}(X, Y) \simeq \mathrm{Map}_{\mathrm{Ell}(R)}(\lim_{\leftarrow n} X_n, \lim_{\leftarrow n} X_n)$. So we have

$$\mathrm{Map}_{\mathcal{U}}(\mathrm{CAlg}_R^{vn}))(\mathcal{A}, \mathrm{Map}_{\mathrm{Ell}(R)}(X, Y)) \simeq \mathrm{Map}_{\mathcal{U}}(\mathcal{A}, \mathrm{Map}(\lim_{\leftarrow n} X_n, \lim_{\leftarrow n} X_n)) \quad (2)$$

where $\mathcal{U}$ is the the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Shv}_{\acute{e}t}(\mathrm{CAlg}_R^{cn}))$, here we think $\mathcal{A}$ as $\mathcal{A} \xrightarrow{Id} \mathcal{A}$. By this equivalence, we get (1), since if a morphism in one side of (2) is a level structure if and only it does in other side.

■

Proposition 3.19. $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ *is a cohesive functor.*

Proof. For every pullback diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & A \arrow[d] \\
C \arrow[r] & B
\end{tikzcd}$$

of connective $\mathbb{E}_\infty$-ring such that the underlying homomorphisms $\pi_0 A \to \pi_0 B \leftarrow \pi_0 C$ are surjective. We need to prove that

$$\begin{tikzcd}
\mathrm{Ell}(\mathcal{A})(D)^{\simeq} \arrow[r] \arrow[d] & \mathrm{Ell}(\mathcal{A})(A)^{\simeq} \arrow[d] \\
\mathrm{Ell}(\mathcal{A})(C)^{\simeq} \arrow[r] & \mathrm{Ell}(\mathcal{A})(B)^{\simeq}
\end{tikzcd}$$

is a pullback diagrams. But we know that $\mathcal{M}_{ell}$ is a spectral Deligne-Mumford stacks, so we have pull-back diagram

$$\begin{tikzcd}
\mathrm{Ell}(D)^{\simeq} \arrow[r] \arrow[d] & \mathrm{Ell}(A)^{\simeq} \arrow[d] \\
\mathrm{Ell}(C)^{\simeq} \arrow[r] & \mathrm{Ell}(B)^{\simeq}
\end{tikzcd}$$

We define a functor

$$G : \mathrm{Ell}(\mathcal{A})(D)^{\simeq} \longrightarrow \mathrm{Ell}(\mathcal{A})(A)^{\simeq} \times_{\mathrm{Ell}(\mathcal{A})(B)^{\simeq}} \mathrm{Ell}(\mathcal{A})(C)^{\simeq}$$

Given by $(E_D, \phi_D : \mathcal{A} \to E_D) \mapsto (E_D \times_D A, \phi_A : \mathcal{A} \to E_D \times_D A) \otimes (E_D \times_D C, \phi_A : \mathcal{A} \to E_D \times_D C)$, this map is well-defined, we just notice that the induced morphism $\phi_C : \mathcal{A} \to E_D \times_D C$ is still a derived level structure.

The right hand side of G can be thought of triples $((E_A, \phi_A : \mathcal{A} \to E_A), (E_C, \phi_C : \mathcal{A} \to E_C), \alpha)$ where α is an equivalence $E_A \times_A B \simeq E_C \times_C B$ which is compatible with level structures. By the cohesiveness of $\mathcal{M}_{ell} : \mathrm{CAlg}^{cn} \to S$, we get a pull-back, which is spectral elliptic curve E_D over D, such that

$$E_D \otimes_D A \simeq E_A \quad E_D \otimes_D C \simeq E_C$$

when we have two level structures $\phi_A : \mathcal{A} \to E_A$ and $\phi_C : \mathcal{A} \to E_C$, we can get a lift $\phi_D : \mathcal{A} \to E_D$. We need to check that this ϕ_A is a derived level structure. We notice that $\phi_A^{\heartsuit} : A \to E_D^{\heartsuit} \simeq E_C^{\heartsuit} \otimes_{E_B^{\heartsuit}} E_A^{\heartsuit}$. This is a classical level structure, and it is easy to say that the associated ideal sheaf is a line bundle. so we get ϕ_D is a derived level structure.

The construction $((E_A, \phi_A : \mathcal{A} \to E_A), (E_C, \phi_C : \mathcal{A} \to E_C), \alpha) \to (E_D, \phi_D : \mathcal{A} \mapsto E_D)$ determine a functor

$$F : \mathrm{Ell}(\mathcal{A})(A)^{\simeq} \times_{\mathrm{Ell}(\mathcal{A})(B)^{\simeq}} \mathrm{Ell}(\mathcal{A})(C)^{\simeq} \to \mathrm{Ell}(\mathcal{A})(D)^{\simeq}.$$

It is easy to check that G is a left adjoint to G. To prove that G is an equivalence, we only to need to prove that the counit map $v : F \circ G \to Id$ and the unit map $Id \to G \circ F$ are equivalences.

1. $v : F \circ G \to Id$ is an equivalence. Suppose that we have an object $(E_D, \phi_D : \mathcal{A} \to E_D) \in \mathrm{Ell}(\mathcal{A})(D)$. By the cohesive of $\mathcal{M}_{ell} : \mathrm{CAlg} \to \mathcal{S}$, and the proof of [Lur18c, Proposition 16.3.1.1]. The following diagram is a pull-back diagram

$$\begin{tikzcd} E_D \arrow[r] \arrow[d] & E_D \times_D A \arrow[d] \\ E_D \times_D C \arrow[r] & E_D \times_D B \end{tikzcd}$$

And we consider the following pull back diagram

$$\begin{tikzcd} (E'_D, \phi'_D) \arrow[r] \arrow[d] & (E_D \times_D A, \phi_A) \arrow[d] \\ (E_D \times_D C, \phi_C) \arrow[r] & (E_D \times_D B, \phi_B) \end{tikzcd}$$

where ϕ_A, ϕ_B, ϕ_C are induced by ϕ_D. By the pull-back of elliptic curve, we get $E'_D \simeq E_D$. Because of level structure is a morphism, we get $\phi'_D \simeq \phi_D$ in $\mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E_D(D))$. But ϕ'_D is already a level structure, so we get $\phi_D \simeq \phi_{D'}$ in $\underline{\mathrm{Level}}(\mathcal{A}, E_D)$. So we have pull-back diagram

$$\begin{tikzcd} (E_D, \phi_D) \arrow[r] \arrow[d] & (E_D \times_D A, \phi_A) \arrow[d] \\ (E_D \times_D C, \phi_C) \arrow[r] & (E_D \times_D B, \phi_B) \end{tikzcd}$$

So $v : F \circ G \to Id$ is an equivalence.

2. $u : Id \to G \circ F$ is an equivalence. Suppose that we have an object $((E_A, \phi_A), (E_C, \phi_C), \alpha)$ in $\mathrm{Ell}(\mathcal{A})(A)^{\simeq} \times_{\mathrm{Ell}(\mathcal{A})(B)^{\simeq}} \mathrm{Ell}(\mathcal{A})(C)^{\simeq}$. So that we have a pull-back diagram

$$\begin{tikzcd} (E_D, \phi_D) \arrow[r] \arrow[d] & (E_A, \phi_A) \arrow[d] \\ (E_C, \phi_C) \arrow[r] & (E_B, \phi_B) \end{tikzcd}$$

we wish to prove that

$$E_A \to E_D \times_D A, \quad E_C \to E_D \times_D A$$

are equivalecnes, and $\phi'_A : \mathcal{A} \to E_D \times_D A, \phi'_C : \mathcal{A} \to E_D \times_D C, \phi'_B : \mathcal{A} \to E_D \times_D B$ induced by ϕ_D are equivalent to ϕ_A, ϕ_C, ϕ_B. The former statement follows from the proof of [Lur18c, Theorem 16.3.0.1] and the cohesiveness of $\mathcal{M}_{ell} : \mathrm{CAlg} \to S$. And the later is also easy, we just notice that $E_A \simeq E_D \times_D A, \quad E_C \simeq E_D \times_D A$. We have diagram

$$\begin{tikzcd} (E_D, \phi_D) \arrow[r] \arrow[d] & (E_D \times_D A, \phi'_A) \arrow[d] \\ (E_D \times_D C, \phi'_C) \arrow[r] & (E_D \times_D B, \phi'_B) \end{tikzcd}$$

where ϕ'_A, ϕ'_C are induced form ϕ_D. This diagrm is a pull-back diagram(The first pieces is , and morphism is induced). By ϕ_D is pull-back of ϕ_A and ϕ_C over ϕ_B, and $E_A \simeq E_D \times_D A, E_C \simeq E_D \times_D A, E_B \simeq E_D E \times_D B$. So ϕ'_A is equivalent to ϕ_A, ϕ'_B is equivalent to ϕ_B, ϕ'_C is equivalent to ϕ_C.

■

Lemma 3.20. *The functor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \mapsto \mathcal{S}$ admits a cotangent complex $L_{\mathcal{M}^{de}_{ell}}$, moreover $L_{\mathcal{M}^{de}_{ell}}$ is connective and almost perfect.*

Proof. We consider the following natural transformations $\mathcal{M}_{ell}(\mathcal{A}) \to \mathcal{M}_{ell}$ between functors $\mathrm{CAlg}^{cn} \to \mathcal{S}$ defined by sending $(E/R, f)$ to E/R. We have derived level structure is relative representable over the moduli stack of spectral elliptic curves, and $\mathcal{M}_{ell}$: $\mathrm{CAlg}^{cn} \to \mathcal{S}$ admits a connective and almost perfect contangent complex, so we get $\mathcal{M}_{ell}(\mathcal{A})$ admits a contangent complex which is connective and almost perfect. ■

Lemma 3.21. *The functor $\mathcal{M}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \mapsto \mathcal{S}$ is locally almost of finite presentation.*

Proof. Consider the functor $\mathcal{M}_{ell}(\mathcal{A}) \to *$, it is infitesimally cohesive and admits a cotangent complex which is almost perfect, so by [Lur18c, 17.4.2.2], it is locally almost of finite presentation. So $\mathcal{M}_{ell}(\mathcal{A})$ is locally almost of finite presentation, since $*$ is a final object of $\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})$.

■

Actually, there is another proof without using the cotangent complex. By the definition of locally almost of finite presentation of functors [Lur18c, Definition 17.4.1.1], we need to prove that: $\mathcal{M}^{de}_{ell}(\mathcal{A}) : \mathrm{CAlg}^{cn} \to \mathcal{S}$ commutes with filtered colimits when restricted to $\tau_{\leq n}\mathrm{CAlg}^{cn}$, for each $n \geq 0$. i.e. For a filtered diagram $I \to \tau_{\leq n}\mathrm{CAlg}^{cn}, \alpha \mapsto R_\alpha$, we have

$$\mathcal{M}^{de}_{ell}(\mathcal{A})(\mathrm{colim} R_\alpha) \simeq \mathrm{colim} \mathcal{M}^{de}_{ell}(\mathcal{A})(R_\alpha).$$

Let $(E_{\mathrm{colim}}, \phi_{\mathrm{colim}} : A \to E_{\mathrm{colim}})$ be an object of $\mathcal{M}^{de}_{ell}(\mathcal{A})(\mathrm{colim} R_\alpha)$. By [Lur18a, Theorem 2.4.1], we have $\mathcal{M}_{ell}(\mathrm{colim} R_\alpha) \simeq \mathrm{colim} \mathcal{M}_{ell}(R_\alpha)$. We have $E_{\mathrm{colim}} \in \mathcal{M}_{ell}(\mathrm{colim} R_\alpha)$, this corresponds to a family of $\{E_\alpha/R_\alpha\}_{\alpha \in I}$, such that $E_\beta \simeq E_\alpha \times_{R_\alpha} R_\beta$ for $\alpha \to \beta$. The level structure $\phi_{\mathrm{colim}} : A \to E_{\mathrm{colim}}$ is a morphism of space

$$\phi_{\mathrm{colim}} : \mathcal{A} \to E_{\mathrm{colim}}(\mathrm{colim} R_\alpha)$$

since we have $E_\beta \simeq E_\alpha \times_{R_\alpha} R_\beta$, so $E_\beta(R_\beta) \simeq (E_\alpha \times_{R_\alpha} R_\beta)(R_\beta) \simeq E_\alpha(R_\beta)$. So we get

$$E_{\mathrm{colim}}(\mathrm{colim} R_\alpha) \simeq E_0(\mathrm{colim} R_\alpha) \simeq \mathrm{colim} E_0(R_\alpha) \simeq \mathrm{colim} E_\alpha(R_\alpha).$$

So for each $\alpha \in I$, we get a map

$$\phi_\alpha : \mathcal{A} \to E_\alpha(R_\alpha).$$

It is easy to see that this is a derived $\mathcal{A}$-level structure of E_α/R_α. The construction above actually define a functor

$$\Theta : \mathcal{M}^{de}_{ell}(\mathcal{A})(\mathrm{colim} R_\alpha) \to \mathrm{colim} \mathcal{M}^{de}_{ell}(\mathcal{A})(R_\alpha)$$

To prove that Θ is an equivalence of ∞-categories, by [Cis18, Theorem 3.9.7],we need to prove that Θ is fully faithful and essentially surjective.

1. Θ is essentially surjective. For an object $\{(E_\alpha, \phi_\alpha : \mathcal{A} \to E_\alpha)\}_{\alpha\in I} \in \mathrm{colim}\mathcal{M}^{de}_{ell}(\mathcal{A})(R_\alpha)$, by the equivalence $\mathcal{M}_{ell}(\mathrm{colim}R_\alpha) \simeq \mathrm{colim}\mathcal{M}_{ell}(R_\alpha)$, we get a spectral elliptic curve $E_f/\mathrm{colim}R_\alpha$ satisfying $E_\alpha \times_{R_\alpha} \mathrm{colim}R_\alpha \simeq E_f$. And we have a composition map.

$$\phi_f : \mathcal{A} \to E_0(R_0) \to E_0(\mathrm{colim}R_\alpha) \simeq E_f(\mathrm{colim}R_\alpha).$$

It is easy to see that ϕ_f is a derived $\mathcal{A}$-level structure of $E_f/\mathrm{colim}R_\alpha$. We have a find an object $(E_f, \phi_f : \mathcal{A} \to E_f)$ in $\mathcal{M}^{de}_{ell}(\mathcal{A})(\mathrm{colim}R_\alpha)$. We need to show that $\Theta((E_f, \phi_f)) \simeq \mathrm{colim}(E_\alpha, \phi_\alpha)$. But we have already

$$E_f \simeq \mathrm{colim}E_\alpha,$$

and for the level structure

$$E_f(\mathrm{colim}R_\alpha) \simeq E_0(\mathrm{colim}R_\alpha) \simeq \mathrm{colim}E_0(R_\alpha) \simeq \mathrm{colim}E_\alpha(R_\alpha).$$

So, Θ send E_f to $\{E_\alpha\}_{\alpha\in I}$, send $\phi_f : \mathcal{A} \to E_f$ to $\{\phi_\alpha : \mathcal{A} \to E_\alpha(\alpha)\}$. Therefore,

$$\Theta((E_f, \phi_f : \mathcal{A} \to E_f)) \simeq \mathrm{colim}\{(E_\alpha, \phi_\alpha : \mathcal{A} \to E_\alpha)\}_{\alpha\in I}.$$

2. Θ is fully faithful. This is easy we use the same way used in the proof of étaleness. So we only need to prove that for a spectral elliptic curve E, the functor

$$\underline{\mathrm{Level}}(\mathcal{A}, E/(\mathrm{colim}R_\alpha)) \simeq \mathrm{colim}\underline{\mathrm{Level}}(\mathcal{A}, E_\alpha/R_\alpha)$$

is fully faithful. We just notice that left and right hand sides are all full subcategories of

$$\mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E(\mathrm{colim}R_\alpha)) \simeq \mathrm{colim}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E_\alpha(R_\alpha)).$$

Lemma 3.22. *The functor*

$$\begin{aligned} \mathcal{M}_{ell}(\mathcal{A}) \quad : \quad & \mathrm{CAlg} \to \mathcal{S} \\ & R \longmapsto \mathrm{Ell}(\mathcal{A})(R)^{\simeq} \end{aligned}$$

is integrable.

Proof. By [Lur18c, Proposition 17.3.5.1], we need to prove that for R a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $m \subset \pi_0 R$. Then there is an equivalence

$$\mathcal{M}_{ell}(\mathcal{A})(R) \to \underleftarrow{\lim}\mathcal{M}_{ell}(\mathcal{A})(R/m^n).$$

We consider the following diagram

$$\begin{tikzcd} \mathcal{M}_{ell}(\mathcal{A})(R) \arrow[r, "f"] \arrow[d, "p"] & \underleftarrow{\lim}\mathcal{M}_{ell}(\mathcal{A})(R/m^n) \arrow[d, "q"] \\ \mathcal{M}_{ell}(R) \arrow[r, "g"] & \underleftarrow{\lim}\mathcal{M}_{ell}(R/m^n) \end{tikzcd}$$

The left map p is a left fibration between Kan complex, so is a Kan fibration [Lur09a, Lemma 2.1.3.3]. And the right vertical map is pointwise Kan fibration . By picking a suit model for the homotopy limit we may assume that q is a Kan fibration as well. We have g is a equivalence by [Lur18a, Theorem 2.4.1]. To prove that f is a equivalence. We only need to prove that for every $E \in \mathrm{Ell}(R)$, the map

$$p^{-1}E \simeq \underline{\mathrm{Level}}(\mathcal{A}, E/R) \to \underleftarrow{\lim}\underline{\mathrm{Level}}(\mathcal{A}, E \times_R R/m^n/R/m^n) \simeq q^{-1}g(E)$$

is an equivalence.

We have the $\underline{\mathrm{Level}}(\mathcal{A}, E)$ as full ∞-subcategory of $\mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E(R))$ and $\underleftarrow{\lim}\underline{\mathrm{Level}}(\mathcal{A}, E \times_R (R/m^n)/(R/m^n))$ as a full subcategory of

$$\begin{aligned} \underleftarrow{\lim}\mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E \times_R (R/m^n)/(R/m^n)) &\simeq \mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E \times_R R/m^n/R/m^n) \\ &\simeq \mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E(R)) \end{aligned}$$

So the functor

$$\underline{\mathrm{Level}}(\mathcal{A}, E/R) \to \lim_{\Delta} \underline{\mathrm{Level}}(\mathcal{A}, E \times_R R/m^n/R/m^n).$$

is fully faithful. To prove it is a equivalence, we only need to prove it is essentially surjective.

For any $\{\phi_n : \mathcal{A} \to E \times_R (R/m^n)/(R/m^n)\}$ in $\underleftarrow{\lim}\underline{\mathrm{Level}}(\mathcal{A}, E \times_R R/m^n/R/m^n)$. It was just $\{\phi_n : \mathcal{A} \to E(R/m^n)\}$. Clearly, we can find a morphism $\phi_R : \mathcal{A} \to E(R)$ in $\mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E(R))$ whose image under the equivalence $\mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E(R)) \simeq \underleftarrow{\lim}\mathrm{Map}_{\mathbf{CMon}(\mathcal{S})}(\mathcal{A}, E \times_R (R/m^n)/(R/m^n))$ is $\{\phi_n : \mathcal{A} \to E \times_R R/m^n/R/m^n\}$. We just need to prove this $\phi_R : \mathcal{A} \to E(R)$ is a derived level structure. This is true since the level structure is determined by its underlying morphism in π_0 and in the classic case, $\underline{\mathrm{Level}}(A, E^{\heartsuit}(R_0)) \simeq \underleftarrow{\lim}\underline{\mathrm{Level}}(A, (E \times_R R/m^n)^{\heartsuit}(\pi_0(R/m^n)))$.

We next prove that $I_E(\mathcal{A})$ is a line bundle over E. Let E^n denote $E \times_R R/m^n$. We know that

$$\mathcal{A} \to E^n(R/m^n) = E(R/m^n)$$

determine line bundle $I_n(\mathcal{A})$ as a $\mathcal{O}_{E^n}$-module, so as module over $\mathcal{O}_E$ is locally free of rank one. We have $I(\mathcal{A}) = \varprojlim I_n(\mathcal{A})$ if we think $I_n(\mathcal{A})$ as a $\mathcal{O}_E$-module. So $I(\mathcal{A})$ is a line bundle.

■

Theorem 3.23. *The functor*

$$\begin{aligned} \mathcal{M}_{ell}(\mathcal{A}) \quad : \quad & \mathrm{CAlg} \to \mathcal{S} \\ & R \longmapsto \mathrm{Ell}(\mathcal{A})(R)^{\simeq} \end{aligned}$$

is representable by a spectral Deligne-Mumford stack.

Proof. By [Lur18c, Theroem 18.3.0.1], we need to prove that the functor $\mathcal{M}_{ell}(\mathcal{A})$ satisfying the following condition

1. For every discrete commutative ring R_0, the space $\mathcal{M}_{ell}(\mathcal{A})(R_0)$ is n-truncated.
2. The functor $\mathcal{M}_{ell}(\mathcal{A})$ is a sheaf for the étale topology.
3. The functor $\mathcal{M}_{ell}(\mathcal{A})$ is nilcomplete, infinitesimally cohesive, and integrable.
4. The functor $\mathcal{M}_{ell}(\mathcal{A})$ admits a connective cotangent complex $L_{\mathcal{M}_{ell}(\mathcal{A})}$.
5. The functor $\mathcal{M}_{ell}(\mathcal{A})$ is locally almost of finite presentation.

But these follows form the above series of lemmas. ■

4 Derived Deformations with Derived Level Structures

We first recall the classical Lubin-Tate Tower.

Theorem 4.1. *Let H_0 be a p-divisible group over $\bar{F}_p$. F be a functor which sends artin local ring with residue field $\bar{\mathbf{F}}_p$ to the set of isomorphism classes of triples (H, ρ, η) where*

1. *H is a p-divisible group over A.*
2. *$\rho : H_0 \simeq H \otimes_A \bar{\mathbf{F}}_p$.*
3. *$\eta : (p^{-n}\mathbf{Z}/\mathbf{Z})^h \to H[p^n](A)$.*

η is a Drinfeld level structure. This functor is pro-reprsentable by a formal scheme $\mathfrak{X}_n = \mathrm{Spf}(R_n)$.

Proposition 4.2. *For all k , the complete local rings R_k are regular with a system of parameter given by $(\eta(p^{-n}e_1), \ldots, \eta(p^{-n}e_h))$, η be the universal level structure and $(e_1, \ldots, e_h)$ the canonical basis of $\mathbf{Z}^n$.*

We first review some definition and results about derived deformations.

Definition 4.3 Let $f : X \to Y$ be a morphism of non connective spectral Deligne-Mumford stacks. We say that f is *finite flat* (of degree d) , if for every map $\mathrm{Sp\acute{e}t}A \to Y$, the fiber product $X \times_Y \mathrm{Sp\acute{e}t}A$ has the form $\mathrm{Sp\acute{e}t}B$, where B is a finite flat rank d A-module. We let $FF(A)$ denote the full subcategory of SpDM_A^{nc} spanned by the finite flat morphisms $X \to \mathrm{Sp\acute{e}t}A$.

And one can also define the commutative finite flat scheme over A is a grouplike commutative monoid object of the ∞-category $FF(A)$. We let FFG(A) denote the ∞-category of the commutative finite flat group schemes.

Definition 4.4 Let A be an $\mathbb{E}_\infty$-ring and let S be a set of prime numbers. A *S-divisible* group over A is a functor $X : (\mathbf{Ab}^S_{fin})^{\mathrm{op}} \to FFG(A)$ with the following conditions

1. The commutative finite flat scheme $X(0)$ is trivial.

2. For every short exact sequence of finite abelian S-groups, the induced diagram

$$\begin{tikzcd} X(M'') \arrow[r] \arrow[d] & X(M) \arrow[d] \\ X(0) \arrow[r] & X(M') \end{tikzcd}$$

   is an exact sequence of commutative finite flat schemes over A.

3. The S-divisible group has height h, if for every finite abelian S-group M, the commutative finite flat group scheme X(M) has degree $|M|^h$ over A.

When S consists of only one prime p, then we call it p-divisible group over A.

Remark 4.5 By [Lur18a, Proposition 6.5.8], there is another equivalent definition of spectral p-divisible group [Lur18b, Definition 6.0.2]. A spectral p-divisible group over an connective $\mathbb{E}_\infty$-ring R is just a functor

$$G : \mathrm{CAlg}_R^{cn} \to \mathrm{Mod}_{\mathbf{Z}}^{cn}$$

with the following properties:

1. For every $A \in \mathrm{CAlg}_R^{cn}$, the Z-module spectrum $G(A)$ is p-nilpotent, i.e., $G(A)[1/p] \simeq 0$.

2. For every finite Abelian p-group M, the functor

$$\mathrm{CAlg}_R^{cn} \to \mathcal{S}, \quad A \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, G(A))$$

is copresentable by a finite flat R-algebra.

3. The map $p : G \to G$ is locally surjective with respect to the finite flat topology. That is for every object $A \in \mathrm{CAlg}_R^{cn}$ and every element $x \in \pi_0(G(A))$, there exists a finite flat map $A \to B$ for which $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective and the image of x in $\pi_0 G(B)$ is divisible by p.

Let G_0 be a p-divisible group over a commutative ring R_0. A be an $\mathbb{E}_\infty$-ring, G be a p-divisible group over A. A G_0-tagging of G is a triple (I, μ, α), where I is a finitely generated ideal of definition, $\mu : R_0 \to \pi_0 A$ is a ring homomorphism. and $\alpha : (G_0)_{\pi_0(A)/I} \simeq G_{\pi_0 A/I}$ is an isomorphism of p-divisible group over the commutative ring $\pi_0(A)/I$.

Definition 4.6 Let G_0 be a p-divisible group over a commutative ring R_0 and let A be an adic $\mathbb{E}_\infty$-ring. A deformation of G_0 over A is a p-divisible group over A together with an equivalence class of G_0-tagging of G.

The collection of deformations of G_0 over an adic $\mathbb{E}_\infty$-ring can be organized into an ∞-category. The following definition is due to Lurie [Lur18b, Definition 3.1.4].

Definition 4.7 For a classical p-divisible G_0 over a commutative ring R_0. Let A be an adic $\mathbb{E}_\infty$-ring. Then the ∞-category of deformations of G_0 over A is defined as the filtered colimit

$$\operatorname*{colim}_I BT^p(A) \times_{BT(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I).$$

Lemma 4.8. *([Lur18b, lemma 3.1.10]) Let R_0 be a commutative ring and G_0 be a p-divisible group. Let A be an complete adic $\mathbb{E}_\infty$ A, the ∞-category $\mathrm{Def}_{G_0}(A)$ is a Kan complex.*

By this lemma, we have a functor

$$\mathrm{Def}_{G_0} : \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S}.$$

Theorem 4.9. *([Lur18b, Theorem 3.1.15]) If R_0 is Noetherian F_p algebra such that the Frobenius morphism is finite. and G_0 be a nonstationary p-divisible group over R_0. Then*

1. *There exists an universal deformation of G_0. i.e., there exists a complete adic $\mathbb{E}_\infty$ $R_{G_0}^{un}$, and a morphism $\rho : R_{G_0}^{un}$ such that the functor Def_{G_0} is corepresentable by $R_{G_0}^{un}$.*

i.e. , for any complete adic $\mathbb{E}_\infty$-ring A there is a equivalence

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R, A) \to \mathrm{Def}_{G_0}(A).$$

2. *The $\mathbb{E}_\infty$ ring $R^{un}_{G_0}$ is connective and Noetherian.*

3. *The induced map $\pi_0(\rho) : \pi_0(R^{un}_{G_0}) \to R_0$ is surjective, and $R^{un}_{G_0}$ is complete with respect to the ideal $\ker(\pi_0(\rho))$.*

Derived Level Structures of Spectral p-Divisible Groups

Let X be a spectral p-divisible group of height h over an $\mathbb{E}_\infty$-ring R, that is a functor

$$X : \mathbf{Ab}^p_{\mathrm{fin}} \to \mathrm{FFG(R)}.$$

Fro every $p^k \in \mathbf{Ab}^p_{\mathrm{fin}}$, we let $X[p^k]$ denote the image of p^k of X.

Definition 4.10 Let $\mathcal{A} \in \mathrm{Derived}_{(\mathbf{Z}/p^k\mathbf{Z})^h}$, then we define $\underline{\mathrm{Level}}(\mathcal{A}, X)$ is the following pull-back

$$\begin{tikzcd}
\underline{\mathrm{Level}}(\mathcal{A}, X) \arrow[r] \arrow[d] & \mathscr{P}ic(X[p^k]) \arrow[d] \\
\underline{\mathrm{Level}}(A, X[p^k]^\heartsuit/R_0) \arrow[r] & \mathrm{Map}_{\mathbf{Ab}}(A, X[p^k]^\heartsuit(R_0))
\end{tikzcd}$$

Lemma 4.11. *Let X/R be a spectral p-divisible group of height h, for any k, we have a nonconnective spectral Deligne-Mumford stack $X[p^k]$, let D be a closed immersion of $X[p^k]$, such that the associated sheaf is a line bundle over $X[p^k]$, and D_0 is an effective Cartier divisor in $X[p^k]_0/R_0$. Then there exists a $\mathbb{E}_\infty$-ring $S_{X/R}$, satisfying the following universal property:*

For any $R \to R'$ in CAlg^{cn}, such that the associated sheaf of $D_{R'}$ is a line bundle over $X[p^k]_{R'}$ and $(D_R)^\heartsuit$ is a subgroup of $(X[p^k]_R)^\heartsuit$, then $R \to R'$ factor through $S_{X/R}$.

Proof. For $\mathrm{Sp\acute{e}t}R' \to \mathrm{Sp\acute{e}t}R$, it is obvious that that the associated sheaf of $D_{R'}$ is a line bundle over $X[p^k]_{R'}$. And by [KM85, Corollary 1.3.7], if $(D_{R'})^\heartsuit/R'_0$ is a subgroup of $(X[p^k]_{R'})^\heartsuit/R'_0$, we have $\mathrm{Spec}R'_0 \to \mathrm{Spec}R_0$ must passing through a $\mathrm{Spec}Z$, where $\mathrm{Spec}Z$ is a closed subscheme of $\mathrm{Spec}R_0$. So we find that the required closed substack $S_{X/R}$ is just $Z \times_{R_0} R$.

■

Proposition 4.12. *Let X be a spectral p-divisible group of height h over a $\mathbb{E}_\infty$-ring R. Then the following functor*

$$\mathrm{CAlg}_R \to \mathcal{S}; \quad R' \to \underline{\mathrm{Level}}(\mathcal{A}, X_{R'})$$

is representable by an affine spectral Deligne-Mumford stack.

Proof. We first prove the representability. By definition, the functor $\underline{\mathrm{Level}}(\mathcal{A}, X/R)$ is a subfunctor of the representable functor $\mathscr{P}ic^x_{X[p^k]/R}$. It is the closed sub-stack of $\mathscr{P}ic^x_{X[p^k]/R}$ which the associated effective It is the closed sub-stack of $\mathscr{P}ic^x_{X/R}$ such that the associated divisor of degree $\sharp(\pi_0\mathcal{A})$ in $(X[p^k] \times_R \mathscr{P}ic^x_{X[p^k]/R}/\mathscr{P}ic^x_{X[p^k]/R})^{\heartsuit}$

$$\sum_{a\in\pi_0\mathcal{A}} \phi_{univ}(a)$$

attached to the universal morphism $\phi_{univ} : \mathcal{A} \to X[p^k](R)$, is a subgroup, then the assertion follows from the lemma 4.11. We denote this closed substack as $\mathcal{P}_{X/R}$.

For the affine condition, we need to prove that $\mathcal{P}_{X/R}$ is finite in the spectral algebraic geometry. By [Lur18c, Remark 5.2.0.2], a morphism between spectral algebraic spaces is finite if and only if its underlying morphism between ordinary spectral algebraic space is finite in ordinary algebraic geometry. We have $\mathcal{P}_{X/R}$ and SpétR are spectral spaces. So we only need to prove $\mathcal{P}^{\heartsuit}_{X/R}$ is finite over R_0, but this is just the classical case, which is finite by [KM85, Corollary 1.6.3]. ■

We consider the following functor

$$\begin{aligned} \mathcal{M}_{\mathcal{A}} \ : \ & \mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}(G_0, R, \mathcal{A}) \end{aligned}$$

where $\mathrm{DefLevel}(G_0, R, \mathcal{A})$ is the ∞-category whose objects are triples (G, ρ, η)

1. G is a spectral p-divisible group over R.
2. ρ is an equivalence of G_0 taggings of R.
3. $\eta : \mathcal{A} \to G(R)$ is a derived level structure.

Theorem 4.13. *The functor $\mathcal{M}_{\mathcal{A}}$ is representable by a spectral Deligne-Mumford stack* Spét$\mathcal{P}_{\mathcal{A}}$ *where $\mathcal{P}_{\mathcal{A}}$ is an $\mathbb{E}_\infty$-ring which is finite over the unoriented spectral deformation ring of G_0.*

Proof. We let $E_{univ}/R^{un}_{G_0}$ denote the universal spectral deformation of G_0/R_0, for any spectral deformation G of G_0 to R, we get a map of $\mathbb{E}_\infty$-ring $R^{un}_{G_0} \to R$, It is easy to see that $E_{univ} \times_{R^{un}_{G_0}} R \simeq G$. So we have the following equivalence

$$\underline{\mathrm{Level}}(\mathcal{A}, G/R) \simeq \underline{\mathrm{Level}}(\mathcal{A}, E_{univ} \times_{R^{un}_{G_0}} R) \simeq \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R^{un}_{G_0}}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R).$$

The last equivalence comes from Proposition 3.16. Then we consider the following moduli problem

$$\mathrm{CAlg}^{ad}_{cpl} \to \mathcal{S}, \quad R \mapsto \mathrm{Map}_{\mathrm{CAlg}^{ad,cpl}_{R_0}}(\mathcal{P}_{E_{univ}/R^{un}_{G_0}}, R).$$

For $R \in \mathrm{CAlg}_{R_0}^{ad,cpl}$, $Map_{\mathrm{CAlg}_{R_0}^{ad,cpl}}(\mathcal{P}_{E_{univ}/R_{G_0}^{un}}, R)$ can viewed the ∞-categories of pairs (α, f), where

$$\alpha : R_{G_0}^{un} \to R$$

is the classified map of a spectral p-divisible group G, which is a deformation of G_0, that is $\alpha = (G, \rho)$, and $f \in \mathrm{Map}_{\mathrm{CAlg}_{R_{G_0}^{un}}^{ad,cpl}}(\mathcal{P}_{E_{univ}/R_{G_0}^{un}}, R) = \underline{\mathrm{Level}}(\mathcal{A}, E_{univ} \times_{R_{G_0}^{un}} R)$ is a derived level structure of G/R. So we get $\mathrm{Map}_{\mathrm{CAlg}_{R_0}^{ad,cpl}}(\mathcal{P}_{E_{univ}/R_{G_0}^{un}}, R)$ is just the ∞-categories of pairs (G, ρ, η). By lemma 4.12, $\mathcal{P}_{E_{univ}/R_{G_0}^{un}}$ is finite over $R_{G_0}^{un}$. So $\mathcal{JL}_{\mathcal{A}} = \mathcal{P}_{R_{G_0}^{un}}$ is the desired spectrum.

■

Orientation of the Jacquet-Langlands Spectrum

Although we get spectrum come from a conceptual derived moduli problem, but this spectrum may be complicated. In algebraic topology, orientation of an $\mathbb{E}_\infty$-spectrum make E_2 page of Atiyah-Hirzebruch spectral sequence degenerating.

Let G_0 be a p-divisible group over R_{G_0}. We consider the following functor

$$\begin{aligned} \mathcal{M}_{\mathcal{A}}^{or} \quad : \quad & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{DefLevel}^{or}(G_0, R, \mathcal{A}) \simeq \end{aligned}$$

where $\mathrm{DefLevel}^{or}(G_0, R, \mathcal{A})$ is the ∞-category of pairs (G, ρ, e, η), where

1. G is a spectral p-divisible over R.
2. ρ is an equivalence class of G_0 taggings of R.
3. $e : S^2 \to \Omega^\infty G^0(R)$ is an orientation of the G^0, where G^0 is the identity component of G.
4. $\eta : \mathcal{A} \to G(R)$ is a derived level structure.

Proposition 4.14. *The functor $\mathcal{M}_{\mathcal{A}}^{or} : \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S}$ is representable by a affine spectral Deligne-Mumford stack.*

Proof. Let $\mathrm{Def}^{or}(G_0, R)^{\simeq}$ is the ∞-groupoid of pairs (G, ρ, e), where G is a p-divisible of over R, ρ is an equivalence class of G_0-taggings of R. By [Lur18b, Theorem 6.0.3, Remark 6.0.7], the functor

$$\begin{aligned} \mathcal{M}_{\mathcal{A}}^{or} \quad : \quad & \mathrm{CAlg}_{cpl}^{ad} \to \mathcal{S} \\ & R \to \mathrm{Def}^{or}(G_0, R)^{\simeq} \end{aligned}$$

is corepresneted by the orentated deformation ring $R^{or}_{G_0}$, that is we have an equivalence of spaces

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R^{or}_{G_0}, R) \simeq \mathrm{Def}^{or}(G_0, R)^{\simeq}.$$

Let E^{or}_{univ} be the associated deformation of G_0 to $R^{or}_{G_0}$, then it is obvious that $\mathrm{Sp\acute{e}t}\mathcal{P}_{E^{or}_{unive}}$ is the desired affine spectral Deligne-Mumford stack. ■

We call this spectrum Jacquet-Langlands spectrum. It is easy to see that this $\mathcal{JL}$ admit an action of $GL_n(Z/p^mZ) \times \mathrm{Aut}(G_0)$. In the classical algebraic geometry, the Lubin-Tate can be used to realize the Jacquet-Langlands correspondence [HT01]. Is there a topological realization of the Jacquet-Langlands correspondence. Actually, in a recent paper [SS23], they already realized the topological Jacquet-Langlands correspondence. But their method is based on the Goerss-Hopkins-Miller-Lurie sheaf. They actually consider the degenerate level structure such that representing object is étale over representing object of universal deformations. We hope that over derived level structure can also realize the topological Lubin-Tate tower, and is there a relation with the construction of degenerating level structures.

References

[BL10] Mark Behrens and Tyler Lawson. *Topological automorphic forms*. American Mathematical Soc., 2010.

[Cis18] Denis-Charles Cisinski. *Higher Category and Homotopical Algebra*. 2018.

[DJ$^+$] AJ De Jong et al. The stacks project. *URL http://stacks. math. columbia. edu*.

[DR73] Pierre Deligne and Michael Rapoport. Les schémas de modules de courbes elliptiques. In *Modular Functions of One Variable II: Proceedings International Summer School University of Antwerp, RUCA July 17–August 3, 1972*, pages 143–316. Springer, 1973.

[FGL08] Laurent Fargues, Alain Genestier, and Vincent Lafforgue. *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*. Springer, 2008.

[GH04] Paul G Goerss and Michael J Hopkins. Moduli spaces of commutative ring spectra. *Structured ring spectra*, 315(151-200):22, 2004.

[GM20] David Gepner and Lennart Meier. On equivariant topological modular forms. *arXiv preprint arXiv:2004.10254*, 2020.

[HT01] Michael Harris and Richard Taylor. *The Geometry and Cohomology of Some Simple Shimura Varieties.(AM-151), Volume 151*, volume 151. Princeton university press, 2001.

[KM85] Nicholas M Katz and Barry Mazur. *Arithmetic Moduli of Elliptic Curves*. Number 108. Princeton University Press, 1985.

[LT66] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal lie groups. *Bulletin de la Société Mathématique de France*, 94:49–59, 1966.

[Lur09a] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Lur09b] Jacob Lurie. A survey of elliptic cohomology. pages 219–277, 2009.

[Lur17] Jacob Lurie. *Higher Algebra*. 2017.

[Lur18a] Jacob Lurie. Elliptic cohomology I: Spectal abelian varietes. 2018.

[Lur18b] Jacob Lurie. Elliptic cohomology II: Orientations. 2018.

[Lur18c] Jacob Lurie. *Spectral Algebraic Geometry*. 2018.

[Mil86] James S Milne. Abelian varieties. *Arithmetic geometry*, pages 103–150, 1986.

[RZ96] Michael Rapoport and Thomas Zink. *Period spaces for p-divisible groups*. Number 141. Princeton University Press, 1996.

[SS23] Andrew Salch and Matthias Strauch. ℓ-adic topological jacquet-langlands duality. 2023.

[ST23] Nicolò Sibilla and Paolo Tomasini. Equivariant elliptic cohomology and mapping stacks I. *arXiv preprint arXiv:2303.10146*, 2023.

[Str97] Neil P Strickland. Finite subgroups of formal groups. *Journal of Pure and Applied Algebra*, 121(2):161–208, 1997.

[Str98] Neil P Strickland. Morava E-theory of symmetric groups. *arXiv preprint math/9801125*, 1998.