Formal Moduli Problems

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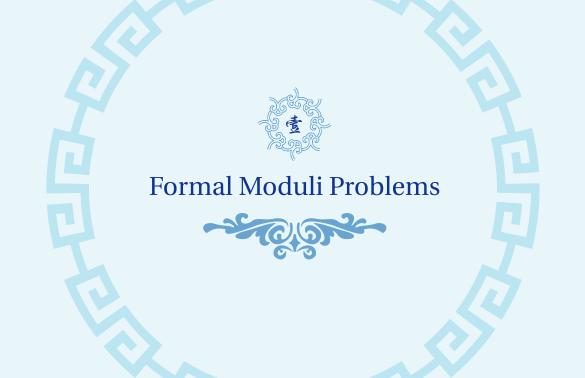


Formal Moduli Problems

Applications

PD Operads and Partition Lie Algebra





Deformation Context

Definition

A deformation context is a pair $(A, \{E_{\alpha}\}_{{\alpha} \in T})$, where A is a presentable ∞ -category with finite limits and E is a set of objects of $\operatorname{Stab}(A)$.

- 1. A morphism in $\mathcal A$ is elementary if it is a pull-back of $* \to \Omega^{\infty n} E_{\alpha}$.
- 2. A morphism in A is small if it can be written as a finite sequence of elementary morphisms.
- 3. An object *A* is artinian(small) if the morphism $A \rightarrow *$ is small.

Example

If C=D(k), which is already stable, in this context , we can consider the spectrum object $E=(k[n+1])_{n\in\mathbb{Z}}$.

Formal Moduli Problems

Definition

Let $(A, \{E_{\alpha}\}_{{\alpha} \in T})$ be a deformation context. A formal moduli problem is a functor $X : A^{\operatorname{art}} \to S$ satisfying the following pair of conditions:

- 1. The space X(*) is contractible.
- 2. Let σ

$$A' \longrightarrow B'$$

$$\downarrow \phi$$

$$A \longrightarrow B$$

be a pullback diagram in $\mathcal{A}^{\mathrm{art}}$ such that ϕ is small, then $X(\sigma)$ is pullback diagram in \mathcal{S} .

Example

Let $B \in \mathcal{A}^{\operatorname{art}}$,

$$\operatorname{Spf}(B): \mathcal{A}^{art} \to \mathcal{S}, \quad A \mapsto \operatorname{Map}_{\mathcal{A}}(B, A)$$



Tangent Complex

Definition

Let $(\mathcal{A}, \{E_{\alpha}\}_{\alpha \in T})$ be a deformation context, $Y: \mathcal{A}^{\operatorname{art}} \to \mathcal{S}$ be a formal moduli problem. For each $\alpha \in T$, the tangent complex of Y at α is the following composite functor

$$\mathcal{S}_*^{fin} \stackrel{E_{\alpha}}{\to} \mathcal{A}^{art} \stackrel{Y}{ o} \mathcal{S}.$$

Proposition

Let $(A, \{E_{\alpha}\}_{{\alpha} \in T})$ be deformation context and let $u: X \to Y$ be a map of formal moduli problems. Suppose that u induces an equivalence of tangent complexes

$$X(E_{\alpha}) \to Y(E_{\alpha})$$

for each $\alpha \in T$. Then u is an equivalence.



Weak Deformation Theory

Definition

A weak deformation theory for a deformation context $(\mathcal{A}, \{E_{\alpha}\})$ is a functor $\mathcal{D}: \mathcal{A}^{op} \to \mathcal{B}$ satisfying the following conditions

- 1. The ∞ -category is presentable.
- 2. The functor admits a left adjoint $D': \mathcal{B} \to \mathcal{A}^{op}$.
- 3. There exists a full subcategory $\mathcal{B}_0 \subset \mathcal{B}$ satisfying the following conditions:
 - **□** For every $K \in \mathcal{B}_0$, the unit map $K \to \mathcal{DD}'K$ is an equivalence.
 - \blacksquare \mathcal{B}_0 contains the initial object $\emptyset \in \mathcal{B}$.
 - For every $\alpha \in T$ and every $n \geq 1$, there exists an object $K_{\alpha,n} \in \mathcal{B}_0$ and an equivalence $\Omega^{\infty n} E_{\alpha} \simeq \mathcal{D}' K_{\alpha,n}$.
 - For every pushout diagram



If K belongs to \mathcal{B}_0 , them K' also belongs to \mathcal{B}_0 .



Let $(A, \{E_{\alpha}\}_{{\alpha} \in T})$ be a deformation context, $\mathcal{D} : A^{op} \to \mathcal{B}$ a weak deformation theory, and $j : \mathcal{B} \to \operatorname{Fun}(\mathcal{B}^{op}, \mathcal{S})$ be the Yoneda embedding. Then

1. For every $B \in \mathcal{B}$, the composition

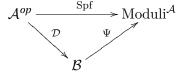
$$\mathcal{A}^{\operatorname{art}} \subset \mathcal{A} \stackrel{\mathcal{D}}{
ightarrow} \mathcal{B}^{op} \stackrel{j(B)}{
ightarrow} \mathcal{S}$$

is a formal moduli problem.

2. The construction $B\mapsto (j(B)\circ \mathcal{D})|_{\mathcal{A}^{\mathrm{art}}}$ determine a functor

$$\Psi: \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$$

3. The diagram



commutes up to homotopy.



Weak Deformation Theory

Proposition

Let $(A, \{E_{\alpha}\}_{{\alpha} \in T})$ be a deformation context and $\mathcal{D}: \mathcal{A}^{op} \to \mathcal{B}$ a weak deformation theory. $\mathcal{B}_0 \subset \mathcal{B}$ be a full subcategory satisfying the condition above, then

- 1. \mathcal{D} carries final objects of \mathcal{A} to initial objects of \mathcal{B} .
- 2. If $A = \mathcal{D}'(K)$ for some $K \in \mathcal{B}_0$. Then the unit map $A \to \mathcal{D}'\mathcal{D}(A)$ is an equivalence in \mathcal{A} .
- 3. If $A \in \mathcal{A}^{\mathrm{art}}$, $\mathcal{D}(A) \in \mathcal{B}_0$ and the unit map $A \to \mathcal{D}'\mathcal{D}(A)$ is an equivalence in \mathcal{A} .
- 4. If we have a pullback diagram σ



in $\mathcal A$ where $A,B\in\mathcal A^{\mathrm{art}}$ and the morphism ϕ is small. Then $\mathcal D(\sigma)$ is a pushout diagram in $\mathcal B$.



Deformation Theory

Lemma

Let $(A, \{E_{\alpha}\}_{\alpha \in T})$ be a deformation context and $\mathcal{D}: A^{op} \to \mathcal{B}$ a weak deformation theory. For each $\alpha \in T$ and each $K \in \mathcal{B}$, the composite map

$$\mathcal{S}_*^{\mathit{fin}} \overset{E_{lpha}}{ o} \mathcal{A} \overset{\mathcal{D}}{ o} \mathcal{B}^{\mathit{op}} \overset{\mathit{j}(\mathit{K})}{ o} \mathcal{S}$$

is reduced and excisive and therefore can be identified with a spectrum which we will denote by $e_{\alpha}(K)$. This determines a functor $e_{\alpha}: \mathcal{B} \to \operatorname{Sp}$.

Definition

A deformation theory for $(\mathcal{A}, \{E_{\alpha}\}_{\alpha \in T})$ is a weak deformation theory $\mathcal{D}: \mathcal{A}^{op} \to \mathcal{B}$ satisfying the following condition: For each $\alpha \in T$, the functor $e_{\alpha}: \mathcal{B} \to \operatorname{Sp}$ preserves small sifted colimits. Morevever, a morphism f in B is an equivalence if and only each $e_{\alpha}(f)$ is an equivalence of spectra.

Formal Moduli Problems

Main Theorem

Given a deformation context $(\mathcal{A}^{op}, \{E_{\alpha}\}_{\alpha \in T})$ and a deformation theory (Koszul duality context)

$$\mathfrak{D}:\mathcal{A}^{op}\leftrightarrows\mathcal{B}:\mathfrak{D}',$$

Then the functor

$$\Psi: \mathcal{B} \to \mathrm{Moduli}^{\mathcal{A}}$$

is an equivalence of ∞ -category.



Sketch of Proof

Lemma

Let $(A, \{E_{\alpha}\}_{\alpha \in T})$ be a deformation context and let $\mathcal{D}: \mathcal{A}^{op} \to \mathcal{B}$ be a deformation theory. For every Artinian object $A \in \mathcal{A}^{art}$, $\mathcal{D}(A)$ is a compact object of the ∞ -category \mathcal{B} .

The functor $\Psi : \mathcal{B} \to \mathrm{Moduli}^A \subset \mathrm{Fun}(\mathcal{A}^{\mathrm{art}}, \mathcal{S})$ is defined by

$$\Psi(K)(A) = \operatorname{Map}_{\mathcal{B}}(\mathcal{D}(A), K)$$

 Ψ preserves small limits. And Ψ preserves filtered colimits and is therefore accessible. So by the ∞ -categorical adjoint functor theorem, Ψ admits a left adjoint Φ . To prove that Ψ is an equivalence, it will suffice to show that

- 1. The functor Ψ is conservative.
- 2. The unit transformation $u : \mathrm{Id} \to \Psi \circ \Phi$ is an equivalence.



Proof of Conservative

Let $f: K \to K'$ in \mathcal{B} , such that $\Psi(f)$ is an equivalence.

$$\operatorname{Map}_{\mathcal{B}}(\mathcal{D}(\Omega^{\infty-n}E_{\alpha}), K) \simeq \Psi(K)\mathcal{D}(\Omega^{\infty-n}E_{\alpha})$$
$$\operatorname{Map}_{\mathcal{B}}(\mathcal{D}(\Omega^{\infty-n}E_{\alpha}), K') \simeq \Psi(K')\mathcal{D}(\Omega^{\infty-n}E_{\alpha})$$

It follows that $e_{\alpha}(K) \simeq e_{\alpha}(K')$. Since the functors are jointly conservative, we conclude that f is an equivalence.



Proof of Equivalence

To prove that $X \to \Psi \circ \Phi(X)$ is an equivalence, by the proposition of tangent complex. it suffice to show that for each $\alpha \in T$, the induced map

$$\theta: X(E_{\alpha}) \to (\Psi \circ \Phi)(X)(E_{\alpha}) \simeq e_{\alpha}(\Phi X)$$

is equivalence of spectra.

Every formal moduli problems admits a smooth hypercovering by "affine" objects.

Proposition

Let $(\mathcal{A}, \{E_{\alpha}\}_{\alpha \in T})$ be a deformation context and $X : \mathcal{A}^{\operatorname{art}} \to \mathcal{S}$ be a formal moduli problem. Then there exists a simplicial objects X_{\bullet} in $\operatorname{Moduli}_{/X}^{A}$ with the following properties:

- 1. Each X_n is prorepresentable.
- 2. For each $n \geq 0$, let $M_n(X_{\bullet})$ denote the matching object of the simplicial object X_{\bullet} . Then the canonical map $X_n \to M_n(X_{\bullet})$ is smooth.

In particular, X is equivalent to the geometric realization $|X_{\bullet}|$ in $\operatorname{Fun}(A^{\operatorname{art}},\mathcal{S})$.

$$\theta: X(E_{\alpha}) \to (\Psi \circ \Phi)(X)(E_{\alpha}) \simeq e_{\alpha}(\Phi X)$$

Choose a simplicial object X_{\bullet} of $\operatorname{Moduli}_{/X}^{A}$ satisfying the above proposition. For each $a \in A^{\operatorname{art}}$.

- 1. $X_{\bullet}(a)$ is an hypercovering of X(A), $|X_{\bullet}(A)| \to X(A)$ is a homotopy equivalence.
- 2. X is a colimit of the diagram X_{\bullet} in the ∞ -category $\operatorname{Fun}(A^{\operatorname{art}}, \mathcal{S})$.
- 3. Similarly, $X(E_{\alpha})$ is equivalent to the geometric realization $|X_{\bullet}(E_{\alpha})|$.
- 4. Since Φ preserves small colimits and e_{α} preserves sifted colimits.

$$e_{\alpha}(\Phi(X)) \simeq e_{\alpha}(\Phi|X_{\bullet}|) \simeq |e_{\alpha}(\Phi X_{\bullet})|.$$

- 5. It follows that θ is a geometric realization of a simplicial morphism $\theta_{\bullet}: X_{\bullet}(E_{\alpha}) \to e_{\alpha}(\Phi X_{\bullet}).$
- 6. It suffices to prove that each θ_n is an equivalence.
- 7. Equivalent to prove that $X_n \to (\Psi \circ \Phi)(X_n)$ is an equivalence.



When X is prorepresentable, since Φ and Ψ both commutes with filtered colimits. We may reduce to the case $X = \operatorname{Spf}(A)$ for some $A \in \mathcal{A}^{art}$. But $\Phi(\operatorname{Spf}(A)) = \mathcal{D}(A)$, it is equivalent to prove that for each $B \in \mathcal{A}^{art}$, the map

$$\operatorname{Map}_{\mathcal{A}(A,B)} \to \operatorname{Map}_{\mathcal{B}}(\mathcal{D}(B),\mathcal{D}(A)) \simeq \operatorname{Map}_{\mathcal{A}}(A,\mathcal{D}'\mathcal{D}(B)).$$

This a consequence form above proposition.





Formal Moduli Problems in Different Graded Algebras

- 1. Cdga_k^{aug} is the ∞ -category of augmented commutative differential graded algebras.
- 2. A morphism in $\operatorname{Cdga}_k^{aug}$ is called elementary if it is a pullback of $k \to k \oplus k[n]$ for some $n \ge 1$, where $k \to k \oplus k[n]$ is the square zero extension of k by k[n].
- 3. A morphism in Cdga_k^{aug} is called small if it is a finite composition of elementary morphisms.
- 4. An object in $\operatorname{Cdga}_k^{aug}$ is called small if the augmentation morphism $\epsilon:A\to k$ is small.

Proposition

An object $\operatorname{Cdga}_k^{aug}$ is small if and only if the following conditions hold:

- 1. $H^n(A) = \{0\}$ for n positive and for n sufficiently negative.
- 2. All cohomology groups $H^n(A)$ are finite dimensional over k.
- 3. $H^0(A)$ is a local ring with maximal ideal m, and the morphism $H^0(A)/m \to k$ is an isomorphism.



Definition

A formal moduli problem is an ∞ -functor $X:(Cdga)_k^{sm}\to \mathcal{S}$ satisfying the following two conditions:

- 1. X(k) is contractible.
- 2. X perserves pull-back along small morphisms.

The second condition means that given a Cartesian diagram

$$\begin{array}{ccc}
N \longrightarrow A \\
\downarrow & & \downarrow \\
M \longrightarrow B
\end{array}$$

in Cdga where $A \rightarrow B$ is small, then

$$\begin{array}{ccc} X(N) & \longrightarrow X(A) \\ & & \downarrow \\ X(M) & \longrightarrow X(B) \end{array}$$



is Cartesian.

The second condition is stable under composition and pullback. We can replace small morphism in the condition by $k \to k \oplus k[n]$.

Theorem

There is a equivalence of ∞ -categories $\operatorname{dgLie}_k \to \operatorname{Moduli}_k$.



Chevalley-Eilenberg Complex

For any differential graded Lie algebra \mathfrak{g} , we can construct the homological and cohomological Chevalley-Eilenberg complex CE_{\bullet}

- 1. As vector space $CE_{\bullet} = S(\mathfrak{g}[1])$ is the graded symmetric algebra of $\mathfrak{g}[1]$. The differential is obtained by extending, as a degree graded coderivation. The complex CE_{\bullet} is actually counital, conilpotent cocommutative coalgebra object in the category of complexes.
- 2. CE^{\bullet} is the linear dual of $CE_{\bullet}(\mathfrak{g})$, it is an augmented cdga.

$$CE_{ullet}(\mathfrak{g}) \simeq k \overset{L}{\otimes}_{U(\mathfrak{g})} k \simeq \operatorname{Tor}_{ullet}^{U(\mathfrak{g})}(k,k)$$

$$CE^{ullet}(\mathfrak{g}) \simeq R\operatorname{Hom}_{U(\mathfrak{g})}(k,k) \simeq \operatorname{Ext}_{U(\mathfrak{g})}^{ullet}(k,k)$$



The Chevalley-Eilenberg construction preserves weak equivalence, thus defining an functor

$$CE^{\bullet}: \operatorname{Lie}_{k}^{op} \to \operatorname{CAlg}_{k}^{aug}$$

This functor commutes with small colimits.

The ∞ -category Lie_k is presentable, so \it CE^{\bullet} admits a left adjoint. We denote this adjoint \mathcal{D} .

$$\mathcal{D}: \mathrm{CAlg}_k^{aug} \leftrightarrows \mathrm{Lie}_k^{op}: \mathit{CE}^{\bullet}$$

We define an ∞ -functor form Lie_k to $\mathrm{Fun}(\mathrm{CAlg}_k^{aug},\mathcal{S})$

$$\Delta(\mathfrak{g}) = \operatorname{Hom}_{\operatorname{Lie}_{\iota}^{op}}(\mathfrak{g}, \mathcal{D}(-)) = \operatorname{Hom}_{\operatorname{Lie}_{\iota}}(\mathcal{D}(-), \mathfrak{g})$$



A differential graded Lie algebra L is good if there exists a finite chain $0 = L_0 \to L_1 \to \cdots \to L_n = L$ such that each of these morphism appears in the pushout diagram

$$\operatorname{free} k[-n_i - 1] \longrightarrow L_i \\
\downarrow \\
\{0\} \longrightarrow L_{i+1}$$

is

Lemma

If $\mathfrak g$ is good, the counit morphism $\mathcal DCE^{\bullet}(\mathfrak g)$ in Lie_k^{op} is an equivalence.



If we have a cartesian diagram

$$\begin{array}{ccc} N & \longrightarrow & k \\ \downarrow & & \downarrow \\ M & \longrightarrow & k \oplus k[n] \end{array}$$

where N and M are small, then

$$\mathcal{D}(N) \longrightarrow \{0\}$$

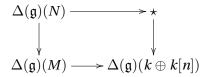
$$\downarrow \qquad \qquad \downarrow$$

$$\mathcal{D}(M) \longrightarrow \mathcal{D}(k \oplus k[n])$$

is also cartesian in $\operatorname{Lie}_{k}^{op}$ and therefore







is also cartesian in sSet. So Δ is an object of FMP $_l$. Hence Δ factor through the category FMP $_k$.



Formal Moduli Problem for Associative Algebras

Assume that k is a field, $X : Alg_k^{art} \to \mathcal{S}$ be a functor. We will say that X is a formal E_1 -moduli problem if it satisfies the following conditions:

- 1. X(k) is contractible.
- 2. For every pullback diagram σ



in $\mathrm{Alg}_k^{\mathrm{art}}$ where the underlying maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. Then $X(\sigma)$ is a pull back square.

Theorem

Let k be a field. Then there is an equivalence of ∞ -categories

$$Alg_k^{aug} \to Moduli_k^{(1)}$$
.



Moduli Problem for \mathbb{E}_n algebras

There is a diagram

$$\cdots \to \operatorname{Alg}_k^{(3)} \to \operatorname{Alg}_k^{(2)} \to \operatorname{Alg}_k^{(1)} \simeq \operatorname{Alg}_k,$$

where $Alg^{(n)}$ denote the ∞ -category of E_n algebras over k.

We say that $A \in Alg_k^{(n)}$ is Artinian if its image in Alg_k is Artinian.

 $X: \mathrm{Alg}_k^{(n),art} \to \mathcal{S}$ be a functor. We will say that X is a formal E_n -moduli problem if it satisfies the following conditions:

- 1. X(k) is contractible.
- 2. For every pullback diagram σ



in $\mathrm{Alg}_k^{(n),\mathrm{art}}$ where the underlying maps $\pi_0R_0\to\pi_0R_{01}\leftarrow\pi_0R_1$ are surjective.

Then $X(\sigma)$ is a pull back square.

Theorem

Let k be a field. Then there is an equivalence of ∞ -categories

$$\operatorname{Alg}_k^{(n),aug} \to \operatorname{Moduli}_k^{(n)}.$$

Morever, the diagram

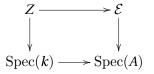
$$\begin{array}{ccc} \operatorname{Alg}_k^{(n),aug} & \xrightarrow{\Psi} \operatorname{Moduli}_k^n \\ & & \downarrow^{m_A} & & \downarrow^{\Sigma^{-n}T} \\ \operatorname{Mod}_k & \longrightarrow \operatorname{Sp} \end{array}$$

commutes up to homotopy.



Deformation as Formal Moduli Problems

Given a smooth scheme Z over k, then formal deformation theory of Z deal with the equivalence classes of Cartesian diagrams



where A is a local artinian algebra with residue field k. This define a deformation functor Def_Z from the category of local artinian algebra to sets.

Theorem

When $A = k[t]/t^2$. There is a bijection between the isomorphism class of X over $\operatorname{Spec}(k[t]/t^2)$ and the cohomology $H^1(Z, T_Z)$.

Let $f: X \to S$ be a scheme, and $t: S \to S'$ be a square zero infinitesimal thickening, which is morphism of scheme with the kernel

$$\mathcal{I} = \ker(\mathcal{O}_{S'} \to \mathcal{O}_S)$$

satisfying $\mathcal{I}^2 = 0$. Given a \mathcal{O}_X -module \mathcal{G} , and a morphism $\mathcal{I} \to \mathcal{G}$ of \mathcal{O}_X module. We ask whether we can find a \mathcal{M} fitting into the following diagram

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{M}(?) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{S'} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

$$(1)$$

and what situation the solution is unique?



Theorem

In the situation above we have

- 1. There is a canonical element $\zeta \in \operatorname{Ext}^2_{\mathcal{O}_X}(L_{X/S},\mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to the above diagram.
- 2. If there exists a solution, then the set of isomorphism classes of solution is principal homogeneous under $\operatorname{Ext}^1_{\mathcal{O}_Y}(L_{X/S},\mathcal{G})$.
- 3. Given a solution X', the set of automorphisms of X' fitting into the diagram is canonically isomorphic to $\operatorname{Ext}^0_{\mathcal{O}_X}(L_{X/S},\mathcal{G})$



Deformation Theory in the Higher Categorical Case

Let k be field, C be a stable k-linear ∞ -category, and $E \in C$

$$Def : Alg^{art} \to S$$

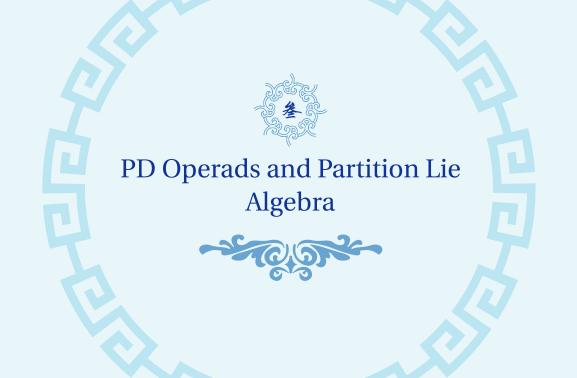
$$B \mapsto (R \operatorname{Mod}_B(\mathcal{C}) \times_{\mathcal{C}} E)^{\simeq}$$

Theorem

Let k be field, \mathcal{C} be a stable k-linear ∞ -category, and $E \in \mathcal{C}$. Let $\Psi : \mathrm{Alg}_k^{aug} \to \mathrm{Moduli}_k^{(1)}$ be the equivalence of ∞ -category of formal moduli problem. Then there is an equivalence of formal \mathbb{E}_1 -moduli problems

$$\mathrm{Def}_E \simeq \Psi(k \oplus \mathrm{End}(E)).$$





Partition Lie Algebra

Definition

The Monad $\operatorname{Lie}_{k,\Delta}^{\pi}$ is defined by the following properties

- 1. If V is a finite dimensional k-vector space, then $\operatorname{Lie}_{k,\Delta}^\pi(V)$ is the linear dual of the algebraic cotangent fiber of $k \oplus V^\vee$, the trivial square-zero extension of k by V^\vee .
- 2. If $V \simeq \operatorname{Tot}(V^{\bullet})$ is represented by a cosimplicial k-vector space V^{\bullet} , then

$$\mathrm{Lie}_{k,\Delta}^{\pi}(V) = \underset{n}{\oplus} \mathrm{Tot}(\widetilde{C}^{\bullet}(\Sigma|\Pi_{n}|^{\diamond},k) \otimes (V^{\bullet})^{\otimes n})^{\Sigma_{n}}.$$

Here $\widetilde{C}^{\bullet}(\Sigma|\Pi_n|^{\diamond},k)$ denote the k-valued cosimplices on the space $\Sigma|\Pi_n|^{\diamond}$, the functor $(-)^{\Sigma}$ takes the strict fixed points, and the tensor product is computed in cosimplicial k-modules.

- 3. The functor $\mathrm{Lie}_{k,\Delta}^\pi$ commuted with filtered colimits and geometric realisations.
- 4. The tangent fiber T_X of any $X \in \text{Moduli}_{k,\Delta}$ has the structure of a $\text{Lie}_{k,\Delta}^{\pi}$ -algebra.



Theorem (Brantner-Mathew, 2019)

If k is a field, there is an equivalence of ∞ -categories

$$\mathrm{Moduli}_{k,\Delta} \simeq \mathrm{Alg}_{\mathrm{Lie}_{k,\Delta}^{\pi}}$$

between formal moduli problems and partition Lie algebra k. k. It sends a formal moduli problem $X \in \mathrm{Moduli}_{k,\Delta}$ to its tangent fibre T_X equipped with a suitable partition Lie algebra structure.

