Chromatic Homotopy Theory

Xuecai Ma

2022.11.22

壹

Tensor-Triangulated Geometry

Stable homotopy category

Brown representability theorem :

$$\text{Generalized cohomology theories of } \mathrm{Top} \longleftrightarrow \text{Spectra}$$

Stable homotopy category (closed symmetric monoidal category)
Models of Spectra: S-Modules, symmetric spectra, orthogonal spectra
Modern approach: ∞-category of spectra, Sp

- ring spectra: $\mathrm{Alg}(\mathrm{Sp})$
- E_∞-ring spectra : $\mathrm{CAlg}(\mathrm{Sp})$
- H_∞-ring spectra : $\mathrm{CAlg}(\mathrm{ho}(\mathrm{Sp}))$

Waldhausen's version of *braver new algebra* of abelian groups: The category Sp of spectra should be thought of as a homotopical enrichment of the derived category $\mathcal{D}_{\mathbb{Z}}$

Local-to-global principle

The Hasse square is a pullback square

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \prod_p \mathbb{Z}_p \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q} \otimes_p \prod_p \mathbb{Z}_p \end{tikzcd}$$

This is the special case of a local-to-global principle for any chain complex $M \in \mathcal{D}_{\mathbb{Z}}$.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \prod_p M_p^{\wedge} \arrow[d] \\ \mathbb{Q} \otimes M \arrow[r] & \mathbb{Q} \otimes_p \prod_p M_p^{\wedge} \end{tikzcd}$$

which is a homotopy pullback square, where $M_p^{\wedge}$ denote the derived p-completion (p-local and $\mathrm{Ext}^i(\mathbb{Q}, M_p^{\wedge}) = 0$, for $i = 0, 1$.)

The Category $\mathcal{D}_{\mathbb{Z}}$

$\mathcal{D}_{\mathbb{Q}}$: The derived category of $\mathbb{Q}$-vector spaces.
$(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$: The category of derived p-complete complexes of abelian groups.

- $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$ is compactly generated by $\mathbb{Z}/p$, any object $X \in (\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$ is trivial if and only if $X \otimes \mathbb{Z}/p$ is trivial.
- The only proper localizing subcategory (triangulated subcategory closed under shifts and colimits) of$(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$ is (0).
- Any object $M \in \mathcal{D}_{\mathbb{Z}}$ can be reassembled from its derived p-completions $M_p^{\wedge} \in (\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$, its rationalization $Q \times M \in \mathcal{D}_{\mathbb{Q}}$, together with the gluing information specified in the pullback square on last page.

$$\{\mathbb{Q} \text{ and } \mathrm{F}_p \text{ for p prime}\} \leftrightarrow \{\mathcal{D}_{\mathbb{Q}} \text{ and } (\mathcal{D}_{\mathbb{Z}})_p^{\wedge} \text{ for p prime}\}$$

Examples of tensor-triangulated categories

1. The category of spectra.
2. The derived category $D(R)$ of a commutative ring R.
3. The ∞-category Mod_R of modules over an E_∞-ring spectrum R.
4. The quasi-coherent shaves complexes over a scheme (algebraic stack).
5. $\mathrm{Fun}(K, \mathcal{C})$ when K is a ∞-category and $\mathcal{C}$ is a tensor-triangulated category . If $K = BG$, then this functor category are those objects in $\mathcal{C}$ with a G-action.
6. Derived category of geometric motives $DM_{gm}(S) \subset DM(S)$ constructed by Voevodsky.
7. $SH^{\mathbb{A}^1}_{gm}(S) \subset SH^{\mathbb{A}^1}(S)$ of the stable $\mathbb{A}^1$ homotopy theory.
8. Homotopy category of Fukaya category $\mathrm{Fuk}(X)$ of a Calabi-Yau manifold X (symmetric tensor is induced by its mirror).
9. $kG - \mathrm{stmod} = \frac{kG-mod}{kG-proj} \simeq \frac{D^b(kG-mod)}{D^{perf}(kG)}$ in modular representation theory, for G a finite group.
10. Tensor-triangulated category of non-commutative motives by Kontsevich.
11. G-equivariant KK-theory (or its stabilization E-theory) of C^*-algebras in Alain Connes's non-commutative geometry.

Tensor-triangulated category

Definition

A tensor-triangulated category, is a triangulated category $\mathcal{K}$ together with a symmetric monoidal category structure

$$\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$$

which is exact in each variable.

- A thick subcategory $\mathcal{J} \subset \mathcal{K}$ is a triangular subcategory closed under direct summands: if $X \oplus Y \in \mathcal{J}$, then $X, Y \in \mathcal{J}$.
- $\mathcal{J} \subset \mathcal{K}$ is a tensor-triangular ideal if $\mathcal{K} \otimes \mathcal{J} \subset \mathcal{J}$.

Definition

A prime $\mathcal{P} \subset \mathcal{K}$ is a proper tensor-triangular ideal such that $X \otimes Y \in \mathcal{P}$ implies $X \in \mathcal{P}$ or $Y \in \mathcal{P}$.

Balmer's Spectrum

Definition

For $\mathcal{K}$ a tensor-triangular category, we define

$$\mathrm{Spc}(\mathcal{K}) = \{\mathcal{P} \subset \mathcal{K} | \mathcal{P} \text{is prime}\},$$

$$\mathrm{Supp}(X) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) | X \notin \mathcal{P}\}.$$

The Supp has the following properties:

1. $\mathrm{Supp}(0) = \emptyset$ and $\mathrm{Supp}(\mathbb{I}) = \mathrm{Spc}(\mathcal{K})$.
2. $\mathrm{Supp}(a \oplus b) = \mathrm{Supp}(a) \cup \mathrm{Supp}(b)$, for every $a, b \in \mathcal{K}$
3. $\mathrm{Supp}(\Sigma a) = \mathrm{Supp}(a)$ for every $a \in \mathcal{K}$.
4. $\mathrm{Supp}(c) \subset \mathrm{Supp}(a) \cup \mathrm{Supp}(b)$ for every distinguished triangle $a \to b \to c \to \Sigma a$.
5. $\mathrm{Supp}(a \otimes b) = \mathrm{Supp}(a) \cap \mathrm{Supp}(b)$ for every $a, b \in \mathcal{K}$.

We define a topology on $\mathrm{Spc}(\mathcal{K}) : \{\mathrm{Supp}(X)\}_{X \in \mathcal{K}}$ as a basis of closed subsets.

Ideal-Thomason Subset

Definition

For every subset $V \subseteq \mathrm{Spc}(\mathcal{K})$, we can associate a tensor-triangular ideal

$$\mathcal{K}_V = \{X \in \mathcal{K} | \mathrm{Supp}(X) \subseteq V\}.$$

A subset $V \subseteq \mathrm{Spc}(\mathcal{K})$ is called a Thomason subset if it is the union of the complements of a collection of quasi-compact open subsets $V = \cup_\alpha V_\alpha$ where each V_α is closed with quasi-compact complement.

Theorem

The assignment $V \to \mathcal{K}_V$ defines a order-preserving bijection between the Thomason subsets $V \subset \mathrm{Spc}(\mathcal{K})$ and the tensor-triangular ideal.

Examples: stable homotopy category

There is a map $\phi : S^0 \to \tau_{\leq 0} S^0 \simeq H\mathbb{Z}$,

$$\mathrm{Sp} \simeq \mathrm{Mod}_{S^0}(\mathrm{Sp}) \xrightarrow{\phi^*} \mathrm{Mod}_{H\mathbb{Z}}(\mathrm{Sp}) \simeq \mathcal{D}_{\mathbb{Z}}$$

$$\mathrm{Spc}(\mathcal{D}_{\mathbb{Z}}) \xrightarrow{\mathrm{Spc}(\phi^*)} \mathrm{Spc}(\mathrm{Sp}) \xrightarrow{\rho} \mathrm{Spec}(\mathbb{Z})$$

Question: What is the inverse image of the irreducible building block $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$? Answer: There are infinitely many blocks in Sp between (0) and $(\mathcal{D}_{\mathbb{Z}})_p^{\wedge}$

The Balmer's Spectrum of classical stable homotopy category (Hopkins-Smith ,1988-1996) is the following topological space.

$$
\begin{array}{cccc}
\mathcal{P}_{2,\infty} & \mathcal{P}_{3,\infty} & \cdots & \mathcal{P}_{3,\infty} \cdots \\
\vdots & \vdots & & \vdots \\
\mathcal{P}_{2,n+1} & \mathcal{P}_{3,n+1} & \cdots & \mathcal{P}_{p,n+1} \cdots \\
\mathcal{P}_{2,n} & \mathcal{P}_{3,n} & \cdots & \mathcal{P}_{p,n} \cdots \\
\vdots & \vdots & & \vdots \\
\mathcal{P}_{2,2} & \mathcal{P}_{3,2} & \cdots & \mathcal{P}_{p,2} \cdots \\
 & & \mathcal{P}_{0,1} &
\end{array}
$$

- $\mathcal{P}_{0,1} = \ker(SH^c \to SH^c =\cong D^b(\mathbb{Q}))$, $\mathcal{P}_{n.\infty} = \ker(SH^c \to SH^c_{(p)})$.
- $\mathcal{P}_{p,n} = \ker(SH^c \to SH^c_{(p)} \to \mathbb{F}_p[v_{n-1}^{\pm 1}] - grmod)$ of localization at p and (n-1) Morava K-theory $K_{p,n-1}$.

- The higher point belongs to the closure of the lower one.
- A closed subset i s either empty, or the whole $\mathrm{Spc}(SH^c)$, or a finite union of closed points $\{\mathcal{P}_{p,\infty}\}$ and of columns

$$\overline{\{\mathcal{P}_{p,m_p}\}} = \{\mathcal{P}_{p,n} | m_p \leq n \leq \infty\}$$

Examples

Theorem(Thomason， 1997)

Let X be a quasi-compact and quasi-separated scheme. Then there is a homeomorphism of topological space

$$|X| \xrightarrow{\cong} \mathrm{Spc}(D^{perf}(X))$$

$$x \longmapsto \mathcal{P}(X)$$

where $\mathcal{P}(x) = \{Y \in D^{perf}(X) | Y_x \cong 0\}$

Corollary

Let A be a commutative ring, $K^b(A - proj) \cong D^{perf}(A)$. Then we have

$$\mathrm{Spec}(K^b(A - proj)) \cong \mathrm{Spec}(A).$$

Examples

Theorem (Benson-Carlson-Richard, 1997)

Let G be a finite group, then there is a homeomorphism

$$\mathrm{Spc}(kG - \mathrm{stmod}) \cong \mathrm{Proj}(H^{\bullet}(G, k)).$$

Theorem (Balmer-Sanders, 2017)

Let G be a finite group. Then every tensor triangular prime in $SH(G)^c$ is of the form $\mathcal{P}(H, p, n)$ for a unique subgroup $H \subset G$ up to conjugation, where

$$\mathcal{P}(H, p, n) \cong (\Phi^H)^{-1}(\mathcal{P}_{p,n})$$

is the preimage under geometric H-fixed points $\Phi^H : SH(G)^c \to SH^c$.
If $K \lhd H$ is a normal subgroup of index $p > 0$, then $\mathcal{P}(K, p, n+1) \subset \mathcal{P}(H, p, n)$.

贰

Chromatic Homotopy Theory

Formal Groups

Let R be a complete local ring with residue filed characteristic $p > 0$, C_R denote the category of local Noetherian R-algebras. We define

$$\hat{\mathbb{A}}^1(A) := C_R(R[[t]], A)$$

A commutative one-dimensional formal group over R is a functor

$$G : C_R \to \mathrm{Ab}$$

which is isomorphic to $\hat{\mathbb{A}}^1$.

$$\mathcal{O}_G \to \mathcal{O}_{G\times G} \cong \mathcal{O}_G \otimes \mathcal{O}_G$$

$\mathcal{O}_G$ is just $R[\![X]\!]$ and $\mathcal{O}_G \otimes \mathcal{O}_G$ is $R[\![X]\!] \otimes_R R[\![Y]\!] = R[\![X, Y]\!]$.

$$\begin{array}{cccc} \phi : & R[\![X]\!] & \to & R[\![X, Y]\!] \\ & X & \to & f(X, Y) \end{array}$$

Formal Group Laws

Definition

Formal group law : $F \in R[\![x_1, x_2]\!]$

- $F(x, 0) = F(0, x) = x$ (Identity)
- $F(x_1, x_2) = F(x_2, x_1)$ (Commutativity)
- $F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$ (Associativity)

There exists a ring L and $F_{univ}(x, y) \in L[\![x, y]\!]$

$$\{\text{Formal Group Law over R}\} \longleftrightarrow \{L \to R\}$$

such that $F(x, y) \in R[\![x, y]\!]$ over R,

$$f^*(F_{univ}(x, y)) = F(x, y).$$

Lazard's Theorem

$L \cong \mathbb{Z}[t_1, t_2, \cdots]$

Heights of Formal Groups

Let $f(x, y) \in R[\![x, y]\!]$

1. If $n = 0$,we set $[n](t) = 0$.
2. If $n > 0$, we set $[n](t) = f([n-1](t), t)$.

P-series $p[t]$ is either 0 or equals $\lambda t^{p^n} + O(t^{p^n+1})$ for some $n > 0$.

Definition

Let v_n denote th coefficient of t^{p^n} in the p-series, f has height $\leq\ n$ if $v_i\ =\ 0$ fro $i <\ n$, f has height exactly n if it has height $\leq\ n$ and v_n is invertible.

Examples

- Formal multiplicative group $f(x, y) = x + y + xy$, $[n](t) = (1 + t)^n - 1$. If $p = 0$ in R, then $[p](t) = (1 + t)^p - 1 = t^p$, so f has height 1.
- Formal additive group $f(x, y) = x + y$, if $p = 0$ in R. Then $[p](t) = 0$, so f has infinite height.

Complex Oriented Cohomology Theories

Definition (Complex Orientation)

Let E be cohomology theory. Then a complex orientation of E is a choice $x \in E^2(\mathbb{C}P^\infty)$ which restricts to 1 under the composite

$$E^2(\mathbb{C}P^\infty) \to E^2(\mathbb{C}P^1) = E^2(S^2) \cong E^0(*)$$

$$E^*(\mathbb{CP}^\infty) \cong E^*(*)[\![t]\!] = (\pi_* E)[[t]]$$

$$(\pi_* E)[[t]] \cong E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong (\pi_* E)[[x, y]]$$

$$\{\text{complex oriented cohomology theory} E\} \to \text{Fromal Groups} G_E = \mathrm{Spf} E^0(\mathbb{C}P^\infty).$$

$$E \longrightarrow G_E = \mathrm{Spf} E^0(\mathbb{C}P^\infty).$$

Theorem (Quillen , 1969)

MU is the universal complex oriented cohomology theory, $L \cong \pi_*\mathrm{MU}$. For E complex oriented, $MU \to E$, induce $L = \pi_* MU \to \pi_* E$.

The Landweber Exact Functor Theorem

If we already have a ring map $L \to R$, can we construct a complex oriented cohomology theory E such that $R = \pi_* E$?

$$E_*(X) = MU_*(X) \otimes_{\pi_* MU} R = MU_*(X) \otimes_L R$$

Landweber's Exact Functor Theorem, 1976

Let M be a module over the Lazard ring L. Then M is flat over $\mathcal{M}_{FG}$ if and only if for every prime number p, the elements $v_0 = p, v_1, v_2, \cdots \in L$ form a regular sequence for M

Lubin-Tate Theory

Deformation of formal groups: Let G_0 be a formal group over a perfect field k with characteristic p, then a deformation of G_0 to R is a triple (G, i, Ψ) satisfying

1. G is a formal group over R,
2. There is a map $i : k \to R/m$,
3. There is an isomorphism $\Psi : \pi^* G \cong i^* G_0$ of formal groups over R/m.

Lubin-Tate's Theorem, 1966

There is a universal formal group G over $R_{LT} = W(k)[[v_1, \cdots, v_n - 1]]$ in the following sense: for every infinitesimal thickening A of k, there is a bijection

$$\mathrm{Hom}_{/k}(R_{LT}, A) \to \mathrm{Def}(A).$$

Morava E-theories and Morava K-theories

Using Landweber exact functor theorem, there is a even periodic spectrum $E(n)$

$$\pi_* E(n) = W(k)[\![v_1, \cdots, v_{n-1}]\!][\beta^{\pm 1}]$$

Theorem (Goerss-Hopkins-Miller)

The spectrum E(n) admits a unique E_∞-ring structure.

$M(k)$ denote the cofiber of the map $\sum^{2k} MU_{(p)} \to MU_{(p)}$ given by the multiplication by t_k.
Let $K(n)$ denote the smash product

$$MU_{(p)}[v_n^{-1}] \otimes_{MU_{(p)}} \bigotimes_{k \neq p^n - 1} M(k).$$

This spectrum $K(n)$ is called **Morava K-theory**. The homotopy groups of $K(n)$ is

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}]/(t_0, t_1, \cdots t_{p^n-2}, t_{p^n}, \cdots) \cong \mathbb{F}_p[v_n^{\pm 1}]$$

Properties of Morava K-theories

- A commutative evenly graded ring is a graded field every nonzero homogeneous element is invertible. Equivalently, R is a field or $R \simeq k[\beta^{\pm}]$.
- We say a homotopy associative ring spectrum is a field if $\pi_* E$ is a graded filed.

Example

For every prime p and every integer n, $K(n)$ is a field.

Proposition

- If E is an field such that $E \otimes K(n)$ is nonzero, then E admits a structure of $K(n)$-module.
- Let E be complex-oriented ring spectrum of height n and $\pi_* E \simeq \mathbb{F}_p[v_n^{\pm 1}]$. Then $E \simeq K(n)$.

Localization

Let S be a set of prime numbers, for example $S = (p)$.

- A ring R is S-local, if all prime numbers not in S is invertible in R.
- A group G is said to be S-local if the p^{th} power map $G \to G$ is a bijection for $p \notin S$.
- If G is abelian,

1. G is S-local;
2. G admits a structure of Z_S-module (necessarily unique);

Definition

A spectrum X is called S-local if its homotopy groups are S-local abelian groups.

The S-localization can be constructed as the Bousfield localization of spectra with respect to the Moore spectrum $M(\mathbb{Z}_S)$

Localization

The general idea of localization at a spectrum E is to associate to any spectrum X the "part of X that E can see", denoted by $L_E X$. L_E is a functor with the following equivalent properties:

- $E \wedge X \simeq * \Rightarrow L_E X \simeq *$.
- If $X \to Y$ induces an equivalence $E \wedge X \to E \wedge Y$ then $L_E X \to L_E Y$.

Bousfield Localization

Let $\mathcal{C}$ be a full subcategory of Sp, which is closed under shifts and homotopy colimits, and can be generated by small subcategory under homotopy colimits.
If X is a spectrum, define $G(X)$ to be the homotopy colimit of all $Y \in \mathcal{C}$ with a map to X.
We have a counit map $v : G(X) \to X$, and we let $L(X)$ denote the cofiber of v, then we have a cofiber sequence

$$G(X) \to X \to L(X).$$

A spectrum is $\mathcal{C}$-local if every may $Y \to X$ is nullhomotopic when $Y \in \mathcal{C}$. We denote the category of $\mathcal{C}$-local spectra as $\mathcal{C}^{\perp}$

Bousfield localization

Let G_E the collection of E-acyclic spectra. We say that a spectrum is E-local if every map for every $Y \in G_E$, the map $Y \to X$ is nullhomotopic.
We have a cofiber sequence

$$G_E(X) \to X \to L_E(X).$$

where $G_E(X)$ is E acyclic and $L_E(X)$ is E-local. This functor is called Bousfield localization with respect to E.
The map $X \to L_E(X)$ is characterized up to equivalence by two properties.

1. The spectrum $L_E(X)$ is E-local.
2. The map $X \to L_E(X)$ is an E-equivalence.

Theorem

A spectrum X is E-local if and only if for each E-equivalence $S \to T$, the induced map $[T, X] \to [S, X]$ is an isomorphism.

Moore Spectrum

For G an abelian group, then the Moore spectrum MG of G is the spectrum characterized by having the following homotopy groups:

1. $\pi_{<0}MG = 0$;
2. $\pi_0(MG) = G$;
3. $H_{>0}(MG, Z) = \pi_{>0}(MG \wedge HZ) = 0$.

A basic special case of E-Bousfield localization of spectra is given by $E = MA$ the Moore spectrum of an abelian group A.

1. For $A = Z_{(p)}$, this is p-localization.
2. For $A = F_p$, this is p-completion
3. For $A = \mathbb{Q}$, this is the rationalization .

Examples of Localization

Theorem

p-Localization is a smashing localization:

$$L_{MZ_{(p)}}X \simeq MZ_{(p)} \wedge X$$

We denote this as $L_{MZ_{(p)}}X \simeq X_{(p)}$, which is called the Bousfield p-localization

A spectrum E is p-complete, if $\pi_* E$ is a (p)-adic complete ring. Bousfield localization at the Moore spectrum MF_p is p-completion to p-adic homotopy theory.

Theorem

The localization of spectra at the Moore spectrum $MF_p$is given by the mapping spectrum out of $\Omega M\mathbb{Z}/p^\infty$:

$$L_p = L_{MF_p}X \simeq [\Omega M\mathbb{Z}/p^\infty, X]$$

where $\mathbb{Z}/p^\infty = \mathbb{Z}[1/p]/\mathbb{Z}$. We denote this spectrum $L_p = L_{MF_p}X$ as $X_p^\wedge$

Examples of Localization

Theorem

$L_{M\mathbb{Q}}X = X \wedge L_{\mathbb{Q}}S^0 = X \wedge M\mathbb{Q} = X \wedge H\mathbb{Q}$ is smashing, we call this as the rationalization of X, denote it as $L_{\mathbb{Q}}X$.

Examples

Localization with respect to $E(n)$ and $K(n)$.

- $L_{E(n)}$, behaves like restriction to the open substack $\mathcal{M}_{FG}^{\leq n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbb{Z}_{(p)}$.
- $L_{K(n)}$, behaves like completion along the locally closed substack $\mathcal{M}_{FG}^{n} \subset \mathcal{M}_{FG} \times \mathrm{Spec}\mathbb{Z}_{(p)}$.

Localization with respect to E(n) and K(n)

Lemma

The Spectrum E(n) is Bounsfield equivalent to $E(n)\times K(n)$. Here $E(0) = H\mathbb{Q}[\beta^{\pm}]$ which is Bounsfield equivalent to $H\mathbb{Q}$.

So a spectrum is $E(n)$-acyclic if and only if it is both $E(n)$-acyclic and $K(n)$-acyclic.

$$L_{E(n)}(X) \cong L_{K(0)\vee K(1)\cdots K(n)}(X).$$

There is pullback square

$$\begin{tikzcd}
L_{E(n)}X \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\
L_{E(n-1)}X \arrow[r] & L_{E(n-1)}(L_{K(n)}X)
\end{tikzcd}$$

This come from $L_{E(n-1)}X$ is K(n)-acyclic and the following Lemma

Lemma

Let E, F, X be spectra with $E_* L_F X = 0$. Then there is a homotopy pullback square.

$$\begin{tikzcd} L_{E\vee F}X \arrow[r] \arrow[d] & L_E X \arrow[d] \\ L_F X \arrow[r] & L_F(L_E X) \end{tikzcd}$$

So we have the following **Suillivan arithmetic square** for $E = \bigvee_p M(Z/p)$, $F = H\mathbb{Q}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p L_p X \arrow[d] \\ L_{\mathbb{Q}} X \arrow[r] & L_{\mathbb{Q}}(\prod_p L_p X) \end{tikzcd}$$

In chromatic homotopy, we often cares the Bousfield localization with respect to the Morava E-theories and Morava K-theories.

Nilpotence

We say that a collection of ring spectra $\{E^\alpha\}$ detect nilpotence if for any p-local ring spectra R, $x \in \pi_m R$ is send to zero in $E_0^\alpha R$ for all α, then x is nilpotent in $\pi_* R$.

Nilpotence Theorem (Devinatz-Hopkins-Smith, 1988)

For any ring spectrum R, the kernel of the map $\pi_* R \to MU_* R$ consists of nilpotent elements. In particular, the single MU detects nilpotence.

Theorem

The spectra $\{K(n)\}_{0 \leq n \leq \infty}$ detect nilpotence.

Let E be a nonzero p-local ring spectrum, then $E \otimes K(n)$ is nonzero for some $0 \leq n \leq \infty$. If not, every element of $\pi_0 E$ is nilpotent, so $\mathbb{I} \in \pi_0 E$ is nilpotent, so that $E \simeq 0$.

Thick Subcategories

Let $\mathcal{C}$ be a full subcategory of finite p-local spectra. We say that $\mathcal{C}$ is **thick** if it contains 0 , closed under fiber and cofibers, and every retract of a spectrum belong to $\mathcal{C}$ also belongs to $\mathcal{C}$.

Lemma

Let X be a finite p-local spectrum, if $K(n)_*(X) \simeq 0$ for some $n > 0$. Then $K(n-1)_*(X) = 0$.

We say that a p-local finite spectrum has type n if $K(n)_*(X) \neq 0$ and $K(m)_*(X) = 0$ for $m < n$. X has type 0 if $H_*(X, \mathbb{Q}) \simeq 0$.
We let $\mathcal{C}_{\geq n}$ be the category of p-local spectra which has type $\geq n$.

Thick Subcategory Theorem

Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. Then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $0 \leq n \leq \infty$.

Different Localizations

We have an adjunction

$$\text{inclusion} : G_E = \{E - \text{acyclic}\} \leftrightarrows \text{Sp} : G_E$$

Localization with respect to E means localization with respect to G_E.

$$G_E \hookrightarrow \text{Sp} \xrightarrow{L_E} E - \text{local} = (G_E)^{\perp}$$

$$G_E(X) \longrightarrow X \longrightarrow L_E(X)$$

We know $E(n)$ acyclic means $E(n-1)$ acyclic and $K(n)$-acyclic, but $\ker L_E = G_E = \{E(n) - \text{acyclic}\}$, so we get inclusions

$$0 = \ker(id) \subset \ker(L_{E(\infty)}) \cdots \subset \ker(L_{E(n)}) \subset \ker(L_{E(n-1)}) \cdots \ker(L_{E(0)}) \subset \text{Sp}$$

by taking orthocomplement, we get

$$0 \subset E(1)\text{-local Sp} \subset \cdots \subset E(n-1)\text{-local Sp} \subset E(n)\text{-local Sp} \subset \cdots$$

Different Localization

We have $K(n)_*(X) = 0 \Rightarrow K(n-1)_*(X) = 0$.

$$\mathcal{C}_{\geq n} = \{X \in \mathrm{Sp}_{(p)} | X \text{ has type } \geq n, i.e., K(m)_*X = 0, m < n\}$$

So we have sequence

$$(0) \subset \cdots \subset \mathcal{C}_{\geq n+1} \subset \mathcal{C}_{\geq n} \subset \cdots \subset \mathcal{C}_{\geq 0} = \mathrm{Sp}$$

by taking orthocomplement, we get

$$\mathcal{C}_{\geq 0} \text{ local spectra} \subset \cdots \subset \mathcal{C}_{\geq n} \text{ local spectra} \subset \mathcal{C}_{\geq n+1} \text{ local spectra} \subset \cdots$$

Telescope Localization

The telescope localization L_n^t: Localization with respect to $\mathcal{C}_{\geq n+1}$.

$$C(X) \to X \to L_n^t(X).$$

where $C(X)$ is a filtered colimit of object in $\mathcal{C}_{\geq n+1}$

Different Localizations

Definition

We say a localization functor L is a smash localization if $L(X) = K \wedge X$ for a K.

The following conditions are equivalent

1. L preserves homotopy colimits.
2. $C^{\perp} \subset \mathrm{Sp}$ is stable under homotopy colimits
3. G preserves homotopy colimits.
4. $L(X) = K \wedge X$.

Examples

- $L_{E(n)}$ is a smash localization.
- L_n^t is a smash localization.
- Rationalization and p-localization is a smash localization.

For any smashing localization L

$$\ker(L_n^t) \subset \ker(L) \subset \ker(L_{E(n)})$$

So there is a comparison

$$L_n^t \to L \to L_{E(n)}$$

Telescope Conjecture

$$L_n^t \simeq L_{E(n)}$$

The periodicity theorem: find a type n spectrum

Consider the cofiber sequence

$$\Sigma^k X \xrightarrow{f} X \to X/f$$

If we have X has type $\leq n$, we hope X/f has type $\leq n+1$

Definition

Let X be finite p-local spectrum, a v_n self map is a map $f : \Sigma^q X \to X$ and satisfying the following,

1. f induces an isomorphism $K(n)_*(X) \to K(n)_*X$.
2. The induced map $K(m)_*(X) \to K(m)_*(X)$ is nilpotent, for $m \neq n$.

Theorem

Let X be a finite p-local spectrum of type $\geq n$, then X admits a v_n-self map.

Telescopic Localization

$$X \xrightarrow{f} \Sigma^{-k}(X) \xrightarrow{f} \Sigma^{-2k}(X) \xrightarrow{f} \cdots$$

Let $X[f^{-1}]$ denote the colimit of this sequence.

Proposition

1. If $X \in \mathcal{C}_{\geq n}$, then $L_n^t(X) \simeq X[f^{-1}]$.
2. There is a fiber sequence

$$\varinjlim_{k_0,\cdots,k_n} \Sigma^{-n} X/(v_0^{k_0}, \cdots, v_n^{k_n}) \to X \to L_n^t(X).$$

Monochromatic

Let $L_n(X) = L_{E(n)}(X)$, then we have the following chromatic tower.

$$\begin{tikzcd}
 & M_n(X) \arrow[d] & & M_2(X) \arrow[d] & M_1(X) \arrow[d] & M_0(X) = H\mathbb{Q} \wedge X \arrow[d] \\
\cdots \arrow[r] & L_n(X) \arrow[r] & \cdots \arrow[r] & L_2(X) \arrow[r] & L_1(X) \arrow[r] & L_0(X) = H\mathbb{Q} \wedge X
\end{tikzcd}$$

where the monochromatic layers $M_n(X)$ are defined by the fiber sequence.

$$M_n(X) \to L_n(X) \to L_{n-1}(X)$$

The following is the chromatic convergence theorem proved by Hopkins- Ravenel.

Chromatic Convergence Theorem

Then Canonical Map $X \to \lim_n L_n X$ is an equivalence for a p local finite spectrum X.

Definition

Monochromatic A spectrum X is monochromatic of height n if it is $E(n)$-local and $E(n-1)$- acyclic.

We let $\mathcal{M}_n$ denote the category of all spectra which are monochromatic of height n.

Theorem

There is a equivalence of category between the homotopy category of monochromatic spectra of height n and the homotopy category of $K(n)$-local spectra, which is given by the functor

$$L_{K(n)} : \mathcal{M}_n \rightleftharpoons \text{K(n) local spectra} : M_n$$

K(n)-Local Spectra

1. $\mathrm{Sp}_{K(n)}$ is compactly generated by $L_E(n)F$, for any type n spectrum F, an object $X \in \mathrm{Sp}_{K(n)}$ is trivial if an donly $X \wedge K(n)$ is trivial.
2. The only proper localizing subcategory of $\mathrm{Sp}_{K(n)}$ is (0).
3. A spectrum $X \in \mathrm{Sp}_{E(n)}$ can be reassembled form $L_{K(n)}X$, $L_{E(n-1)}X$, together with the gluing information.

$$\begin{tikzcd}
L_{E(n)}X \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\
L_{E(n-1)}X \arrow[r] & L_{E(n-1)}(L_{K(n)}X)
\end{tikzcd}$$

The chromatic approach to $\pi_* S^0_{(p)}$:

1. Compute $\pi_* L_{K(n)} S^0$ for each n.
2. Understanding the gluing of above square.
3. Using chromatic convergence $\lim_n \pi_* L_{E(n)} S^0 \cong \pi_* S_{(p)}$

叁

From Algebra to Algebraic Topology

How do we detect topological structure from algebraic information?

- E_* module structure with symmetry $\Longrightarrow$ Fixed point spectral sequence.
- (E_*, E_*E) module structure $\Longrightarrow$ Adams spectral sequence

Morava Stabilizer Groups

We let G_0 denote a formal group of height n over a perfect field $k/\mathbb{F}_p$
The small Morava stabilizer group $\mathrm{Aut}_k(G_0)$ is the group of automorphism of G_0 with coefficients in k,

$$\mathrm{Aut}(G_0) = \{f(x) \in k[[x]] : f(G_0(X, Y)) = G_0(f(x), f(y)), f'(0) \neq 0\}$$

Since G_0 is defined over k, the Galois group $\mathrm{Gal} = \mathrm{Gal}(k/\mathbb{F}_p)$ act on G_0 by acting on the coefficients.

The Morava stabilizer group $\mathbb{G}_n$ is defined by

$$\mathbb{G}_n = \mathrm{Gal}(k/\mathbb{F}_p) \ltimes \mathrm{Aut}(G_0)$$

Morava Stabilizer Groups

$$(G_0, k) \longrightarrow \text{Morava E-theory} E(G_0, k)$$

Does the action $\mathbb{G}_n$ lifts to $E(G_0, k)$?

Theorem (Devinatz-Hopkins , Goerss-Hopkins-Miller)

The Morava stabilizer group acts on E_n, and it givens essential all automorphisms of $E(n)$

$$E(n)^{h\mathbb{G}_n} \simeq L_{K(n)} S^0$$

Example

When p is odd and n=1, $L_{K(1)}(S)$ is the spectrum $\widehat{KU}^{\psi^g=1}$

Homotopy fixed point spectral sequence

If we E_* module structure with an action of Morava stabilizer group $\mathbb{G}_n$, how can we get $L_{K(n)}S^0$?

$$\mathrm{Sp}_{K(n)} \longrightarrow \{\text{ Morava Modules } : E_* \text{ module structure with action of } \mathbb{G}_n\}$$

Proposition

There is a homotopy fixed point spectral sequence (descent spectral sequence)

$$E_2^{s,t} = H^s_{gp}(G; \pi_t(X)) \Longrightarrow \pi_{t-s}(X^{hG})$$

similarly for X_{hG}, X^{tG}.

We have $E(n)^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$, so

$$E_2^{s,t} \cong H^s_{gp}(\mathbb{G}, E(n)_t) \Longrightarrow \pi_{t-s}L_{K(n)}S^0$$

The structure of Morava stabilizer group

For f a formal group law over $\bar{\mathbb{F}}_p$.

$$\mathrm{End} f = \{g(t) \in tR[\![t]\!] \quad | \quad f(g(x), g(y)) = gf(x,y)\}$$

Proposition

End(f) is a noncommutative local ring: The collection non-invertible elements is the left ideal generated by $\pi(t) = \nu(t^p)$, where $\nu f^p(x,y) = f(\nu(x), \nu(y))$.

Let $D = \mathbb{Q} \otimes \mathrm{End}(f)$.

Lemma

D is a central division algebra over $\mathbb{Q}_p$. And $\mathrm{End}(f) = \{x \in D : v(x) \geq 0\}$.

Morava Stabilizer Group

$$\det : \mathbb{G}_n \to \mathbb{Z}_p^\times \qquad \det : \mathbb{S}_n \to \mathbb{Z}_p^\times$$

Composition with $\mathbb{Z}_p^\times/\mu \cong \mathbb{Z}_p$.

$$\zeta_n : \mathbb{G}_n \to \mathbb{Z}_p.$$

Let $\mathbb{G}_n^1 = \ker \zeta_n$, we have

$$\mathbb{G}_n \cong \mathbb{G}_n^1 \rtimes \mathbb{Z}_p, \qquad \mathbb{S}_n \cong \mathbb{S}_n^1 \rtimes \mathbb{Z}_p.$$

As a consequence of $\mathbb{G}_n/\mathbb{G}_n^1 \rtimes \mathbb{Z}_p$, there is a equivalence $L_{K(n)}S^0 \simeq (E_n^{h\mathbb{G}_n^1})^{h\mathbb{Z}_p}$.

$$L_{K(n)}S^0 \longrightarrow E_n^{h\mathbb{G}_n^1} \xrightarrow{\psi-1} E_n^{h\mathbb{G}_n^1} \xrightarrow{\delta} \Sigma L_{K(n)}S^0.$$

The action of Morava stabilizer group

Let F_n be the universal deformation over $(E_n)_0$ of G_0. If we have $\alpha = (f, \sigma) \in \mathbb{G}_n$. The universal property of F_n implies that there is ring isomorphism $\alpha_* : (E_n)_0 \to (E_n)_0$ and an isomorphism of formal group laws $f_\alpha : \alpha_* F_n \to F_n$.
And the action can extends to $(E_n)_* \cong \mathbb{W}_n[\![u_1, \cdots, u_{n-1}]\!][u^{\pm 1}]$

1. $\alpha = (id, \sigma)$ for $\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)$. Then the action is action of Galois group on $\mathbb{W}_n$.
2. If $\omega \in \mathbb{S}_n$ is a primitive $(p^n - 1)$-th root of the unity, then $\omega_*(u_i) = \omega^{p^i - 1} u_i$ and $\omega_*(u) = \omega u$.
3. $\Psi \in \mathbb{Z}_p^\times \subset \mathbb{S}_n$ is thee center, then $\psi_*(u_i) = u_i$ and $\Psi_* u = \Psi u$.

Theorem (Devinatz-Hopkins)

Let $1 \le i \le n-1$ and $f = \sum_{j=0}^{n-1} f_j \xi_j \in \mathbb{S}_n$, where $f_j \in \mathbb{W}_n$. Then modulo $(p, u_1, \cdots u_{n-1})^2$,

$$f_*(u) \equiv f_0 u + \sum_{j=1}^{n-1} f_{n-j}^{\sigma^j} u u_j \qquad f_*(u u_i) \equiv \sum_{j=1}^{i} f_{i-j}^{\sigma^j} u u_j + \sum_{j=i+1}^{n} p f_{n+i-j}^{\sigma^j} u u_j$$

Stable Homotopy Groups of Sphere

Lemma

The K(1)-local sphere $L_{K(1)}S$ is given by the homotopy fiber of the map $\Psi^g - 1 : \widehat{KU} \to \widehat{KU}$.

$$\pi_{2n}(\widehat{KU}^{\Psi^g-1}) \simeq 0$$

$$\pi_{2n-1}(\widehat{KU}^{\Psi^g-1}) \simeq \mathbb{Z}^p/(g^n - 1).$$

By this theorem, we can compute the homotopy group of $L_{K(1)}S$

$$\pi_n L_{K(1)}S = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Q}_p/\mathbb{Z}_p & n = -2 \\ Z/p^{k+1}Z & n+1 = (p-1)p^k m, p \nmid m \\ 0 & \text{otherwise} \end{cases}$$

Let $im(J)_n$ denote the image of the composition map

$$\pi_n(O) \to \pi_n(S) \to \pi_n(S_{(p)})$$

The relation of image of J and the $L_{(K(1))}S$ is described as

Theorem

For $n > 0$, the Bousfield Localization at E(1), $S_{(p)} \to L_{E(1)}S$ induces an isomorphism

$$im(J)_n = \pi_n(L_{E(1)}S)$$

In particular, $\pi_n S_{(p)} \to \pi_n L_{E(1)}S$ is surjective.

By this theorem and the computation of $L_{(E(1))}S$, we can get

$$\pi_{2n}(KU) \to \pi_{2n-1}(U) \xrightarrow{J} \pi_{2n-1}(S) \to \pi_{2n-1}(\widehat{KU}^{\Psi^g-1})$$

is surjective, and for $n > 0$,

$$im(\pi_* J)_{(p)} = \begin{cases} \mathbb{Z}/p^{k+1} & n = (p-1)p^k m, p \nmid m \\ 0 & (p-1) \nmid n. \end{cases}$$

Adams Spectral Sequence

There is an equivalence

$$D(R) \cong \mathrm{Mod}_{HR}(\mathrm{Sp})$$

Homology forget the $\mathcal{A}_p$-module structure.

$$\begin{tikzcd} & \mathrm{Mod}^{\mathrm{graded}}_{\mathcal{A}_p} \arrow[d, "\mathrm{forget}"] \\ \mathrm{Sp}^{op} \arrow[ur, dotted, "H^*(-{,}\mathbb{F}_p)"] \arrow[r, dotted, "H^*(-{,}\mathbb{F}_p)"'] & \mathrm{Mod}^{\mathrm{graded}}_{\mathbb{F}_p} \end{tikzcd}$$

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*Y, H^*X) \Longrightarrow [X, Y_p^\wedge]_{t-s}$$

1. $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*(\mathbb{S})_p$

E based Admas spectral sequence

There exists a cohomological spectral sequence $E_*^{*,*}$ such that

$$E_2^{s,t} = Ext_{E^*E}^{s,t}(E^*Y, E^*X) \Longrightarrow [X, \Sigma^{t-s}Y]_E$$

where $[X, \Sigma^{t-s}Y]_E$ is the set of stable homotopy class form X to Y in an E-localization.

Power Operations

Suppose $\mathcal{C}$ is a tensor triangulated category (presentable stable symmetric momoidal ∞ category), then the functor

$$\pi_0 : \mathrm{CAlg}(\mathcal{C}) \longrightarrow \mathrm{Set},\, R \mapsto \pi_0 \mathrm{Map}_{\mathcal{C}}(\mathbb{I}, R)$$

is represented by the free commutative algebra on a copy of the unit, $\mathbb{I}\{t\}$.
We can define the power operation on $\pi_0 R$ which is given by the elements of $\pi_0 \mathbb{I}\{t\} = \pi_0 \bigoplus \mathbb{I}^{\otimes}_{h\Sigma_s} \cong \pi_0 \bigoplus \mathbb{I}_{h\Sigma_s}$.

Definition

To each object $P \in \pi_0 \mathbb{I}_{h\Sigma_r}$, we define the power operation of weight r by sending a class $x \in \pi_0 R = [\mathbb{I}, R]$ to be the composite

$$\mathbb{I} \xrightarrow{P} \mathbb{I}_{h\Sigma_r} \hookrightarrow \oplus_s \mathbb{I}_{h\Sigma_s} \cong \mathbb{I}\{t\} \xrightarrow{t \mapsto x} R.$$

Power Operations

If E is a structured commutative ring spectra(ie, a commutative S-algebra), we have have a map $E^*(X) \to E^*(X^m)$ given by $x \to x^{\times m}$,then there is **total m-th power operation**

$$P_m : E^0(X) \to E^0(X \times B\Sigma_m)$$

If h^* is a multiplicative cohomology theory, that is, we have map: $h^p(X) \otimes h^q(X) \to h^{p+q}(X)$. Then we have the m-th power map

$$h^q(X) \to h^{mq}(X) : \quad x \mapsto x^m.$$

Let R be a commutative S-algebra in the context of EKMM category , and M is an R-module, then we can define a free commutative R-algebra on M:

$$\mathbb{P}_R M = \bigvee_{m \geq 0} \mathbb{P}_R^m(M) \cong \bigvee_{m \geq 0} (M \wedge_R \cdots \wedge_R M)_{h\Sigma m}$$

And if A is commutative R-algebra A, then we have a map

$$\mu : \mathbb{P}_R A \to A.$$

If A is a commutative R -algebra.

1. We can choose a $\alpha : R \to \mathbb{P}_R^m(R) \cong R \wedge B\Sigma^+$
2. For any element $x \in \pi_0 A$ which is represented by $f_x : R \to A$.
3. We define a element $Q_\alpha(x) \in \pi_0 A$ which is represented by the following composite
$$R \xrightarrow{\alpha} \mathbb{P}_R^m(R) \xrightarrow{\mathbb{P}_R^m(f_x)} \mathbb{P}_R^m(A) \subset \mathbb{P}_R(A) \xrightarrow{\mu} A$$

So we have define a map $Q_\alpha : \pi_0 A \to \pi_0 A$. And we can also define $Q_\alpha : \pi_q A \to \pi_{q+r} A$ if

$$\alpha : \Sigma^{q+r} R \to \mathbb{P}_R^m(\Sigma^q R) \cong R \wedge B\Sigma_m^{qV_m}.$$

Example of Power Operations

Let $H = H\mathbb{F}_2$ is the mod 2 Maclane spectrum, if A is a commutative H-algebra spectrum, then $\pi_* A$ is a graded commutative $\mathbb{F}_2$-algebra. $Q^r : \pi_q A \to \pi_{q+r} A$

- $Q^r(x+y) = Q^r(x) + Q^r(y)$.
- $Q^r(xy) = \sum Q^i(x) Q^{r-i}(y)$.
- $Q^r Q^s(x) = \epsilon^{i,j}_{r,s} Q^i Q^j(x)$ if $r > 2s$, where $i \leq 2j$.

if $A = \mathrm{Fun}(\Sigma^\infty X, H\mathbb{F}_2)$, then the power operations are Steenrod operations on $H^*(X, \mathbb{F}_2)$.

Power Operations in K-theory

If K is the complex K-theory spectrum, and A is a p-complete K-algebra. $\psi^p : \pi_0 A \to \pi_0 A$.

- $\psi^p(x+y) = \psi^p(x) + \psi^p(y)$.
- $\psi^p(x) \equiv x^p \mod p$.
- $\psi(xy) = \psi(x)\psi(y)$.

Power Operation in Morava E-theories

Theorem (Rezk)

There exists a monad $\mathbb{T}$ on the category of discrete E_0-modules whose categroy of algebras $\mathrm{Alg}_{\mathbb{T}}$ is the image of the functor $\pi_0(-)$ on commutative E-algebras.

$$\begin{tikzcd} & \mathrm{Alg}_{\mathbb{T}} \arrow[d, "U_{\mathbb{T}}"] \\ \mathrm{CAlg}_E^{\wedge} \arrow[ur, dotted, "\pi_0"] \arrow[r, "\pi_0"] & \mathrm{CRing}_{E_0} \end{tikzcd}$$

In the case $n = 1$ and $E = E(\mathbb{F}_p, \mathbb{G}_m) = KU_p$. $\mathrm{Alg}_{\mathbb{T}}$ can be identified with the category CRing_δ-rings. If R is a $T(1)$-local commutative KU_p algebra, then there is a operation $\delta : \pi_0(R) \to \pi_0(R)$ which act as a p-derivation

$$\psi(x) = x^p + p\delta(x)$$

For formal reasons, the forgetful functor $U_{\mathbb{T}} : \mathrm{Alg}_{\mathbb{T}} \to \mathrm{CRing}_{E_0}$ admits both left and right adjoint

$$U_{\mathbb{T}} : \mathrm{Alg}_{\mathbb{T}} \rightleftarrows \mathrm{CRing}_{E_0} : W_{\mathbb{T}}$$
$$F_{\mathbb{T}} : \mathrm{CRing}_{E_0} \rightleftarrows \mathrm{Alg}_{\mathbb{T}} : U_{\mathbb{T}}$$

In the case of $\mathrm{Alg}_{\mathbb{T}} = \mathrm{CRing}_{\delta}$ at height 1, we have $W_{\mathbb{T}}(A) = W(A) = \pi_0 E(A)$. By composing with the adjunction

$$(-/p)^{\sharp} : \mathrm{CRing} \rightleftarrows \mathrm{Perf}_{\mathbb{F}_p} : Incl$$

We obtain an adjunction

$$(U(-)/p)^{\sharp} : \mathrm{CRing}_{\delta} \rightleftarrows \mathrm{Perf}_{\mathbb{F}_p} : \pi_0 E(-)$$

This adjunction can be generalize to any height.

Theorem (Burklund-Schlank-Yuan， 2022)

There is an adjunction

$$(U(-)/m)^{\sharp} : \mathrm{Alg}_{\mathbb{T}} \rightleftarrows \mathrm{Perf}_{k} : \pi_0 E(-)$$

where the right adjoint $\pi_0 E(-)$ is fully faithful.

Theorem (Rezk)

Let A be a K(n)-local E-Algebra, then the power operation of the homotopy group of A has the structure of an amplified Γ-ring.

We say that a graded Γ-algebra B satisfies thee congruence condition if for all $x \in B_0$,

$$x\sigma \equiv x^p modpB.$$

Theorem

An object $B \in \mathrm{Alg}_\Gamma^*$ which is p-torsion free, then B admits the structure of a $\mathbb{T}$-algebra if and only B satisfies the congruence condition.

Sheaves on the Categories of Deformations

Let R be complete local ring whose residue has characteristic p. Let $\phi : R \to R, x \mapsto x^p$ be the Frobenius map.

The **Frobenius isogeny** $\mathrm{Frob} : G \to \phi^* G$ is induced by the relative Frobenius map on rings.
We write $\mathrm{Frob}^r : G \to (\phi^r)^* G$ which is the composition $\phi^*(\mathrm{Frob}^{r-1}) \circ \mathrm{Frob}$

Let (G, i, α) and $(G', i'\alpha')$ be two deformation of G_0 to R. A deformation of Frob^r is a homomorphism $f : G \to G'$ of fromal groups over R which satisfying

1. $i \circ \phi^r = i'$ and $i^*(\phi^r)^* G_0 = (i')^* G_0$.

$$\begin{tikzcd} k \arrow[r, "i'"] \arrow[d, "\phi^r"'] & R/m \\ K \arrow[ur, "i"] & \end{tikzcd}$$

2. the square

$$\begin{tikzcd} i^* G_0 \arrow[r, "i^*(\mathrm{Frob}^r)"] \arrow[d, "\alpha"'] & i^*(\phi^r)^* G_0 \arrow[d, "\alpha'"] \\ \pi^* G \arrow[r, "\pi^*(f)"] & \pi^* G' \end{tikzcd}$$

of homomorphisms of formal groups over R/m commutes.

We let Def_R denote the category whose objects are deformations fo G_0 to R, and whose morphisms are homomorphsim which are deformation of Frob^r for some $r \geq 0$. Say that a morphism in Def_R has **height** r, if it is a deformation of Frob^r.

Proposition

Let G be deformation of G_0 to R, then the assignment $f \rightarrow \mathrm{Ker} f$ is a one-to-one correspondence between the morphisms in Sub^r_R with source G and the finite subgroup of G which have rank p^r.

For the following, Let $G_E = G_{univ}/E_0$ be the universal deformation of G_0.

Deformation of Frobenius

Theorem (Strickland, 97)

Let G_0/k be a formal group of height h over a perfect field k. For each $r > 0$, there exists a complete local ring A_R which carries a universal height r morphism $f^r_{univ} : (G_s, i_s, \alpha_s) \to (G_t, i_t, \alpha_t) \in Sub^r(A_r)$. That is the operation $f^r_{univ} \to g^*(f^r_{univ})$ define a bijective relation from the set of local homomorphism $g : A_r \to R$ to the set Sub^r_R. Furthermore, we have:

1. $A_0 \approx W(k)[[v_1, \cdots, v_{h-1}]]$.
2. Under the map $s : A_0 \to A_r$ which classifiers the source of the universal height r map, i.e. $G_s = i^* G_E$, and A_r is finite and free as an A_0 module.
3. Under the map $t : A_0 \to A_r$ which classifies the target of the universal height r map, i.e. $G_t = t^* G_E$

So there is a bijection

$$\{g : A_r \to R\} \to Sub^r(R)$$

$$g \mapsto g^*(f^r_{univ})(g^* G_s \to g^* G_t)$$

Thus, $Sub = \coprod Sub^r$ is a affine graded-category scheme. In particular, there are ring maps:

$$s = s_k, t = t_k : A_0 \to A_k,$$

which is induced by E^0 cohomology on $B\Sigma \to *$

$$\mu = \ mu_{k,l} : A_{k+l} : A_{k+l} \to A_k{}^s\otimes_{A_0}{}^t A_l$$

which classifying the source,target, and composite of morphisms.

Theorem (Strickiand , 1998)

The ring $A[r]$ in the universal deformation of Frobenuis is isomorphic to $E^0(B\Sigma_{p^r})/I$,i.e,

$$A[r] \cong E^0(B\Sigma_{p^r})/I$$

where I is transfer ideal.

So for the power operation

$$R^k(X) \to R^k(X \times B\Sigma_m)$$

For $x = *$, we have $\pi_0 R \to E^0(B\Sigma_{p^r})/I \otimes \pi_0 R = A[r] \otimes \pi_0 R$. This make $\pi_0 R$ becomes a Γ-module.

Andre-Quillen Cohomology Groups

Let A be a commutative ring, B be an A-algebra, and M be a B-module. The André-Quillen cohomology groups are the derived functors of the derivation functor $Der_A(B, M)$.

Morphisms of commutative rings $A \to B \to C$ and a C-module M, there is a three-term exact sequence of derivation modules:

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$$

Let P be a simplicial cofibrant A-algebra resolution of B. André notates the qth cohomology group of B over A with coefficients in M by $H^q(A, B, M)$, while Quillen notates the same group as $D^q(B/A, M)$. The q-th André-Quillen cohomology group is:

$$D^q(B/A, M) = H^q(A, B, M) \overset{\mathrm{def}}{=} H^q(\mathrm{Der}_A(P, M))$$

Let $L_{B/A}$ denote the relative cotangent complex of B over A. Then we have the formulas:

$$D^q(B/A, M) = H^q(\mathrm{Hom}_B(L_{B/A}, M))$$

$$D_q(B/A, M) = H_q(L_{B/A} \otimes_B M)$$

In general , let C be an operad, A is an C -algebra, M is an Module. The square zero extension $M \rtimes A$ is a new A -algebra
We have definitions of derivation

$$\mathcal{D}\rceil\nabla_{C}(X, M) := \mathrm{Alg}_{C/A}(X, M \rtimes A)$$

We can form the simplicial module $K(M, n)$ over A whose normalization $NK(M, n) \cong M$. And define $K_A(M, n) = K(M, n) \rtimes A$.
We define the Andre-Quillen Cohomology of X with coefficients in M by the formula

$$D_C^n(X, M) = [X, K_A(M, m)]_{s\mathrm{Alg}/A} \cong \pi_0 \mathrm{Map}_{s\mathrm{Alg}/A}(X, K_A(M, n))$$

$$D_C^n(X, M) \cong \pi_{-n} hom_{s\mathrm{Alg}/A}(X, K_A M)$$

Lemma

$$D_C^n(X, M) = H^n N(\mathcal{D}\rceil\nabla_{C}(Y, M))$$

where Y is some cofibrant model for X and N is some normalization functor from comsimplicial k-module to cochain complex.
Let $X \to Y$ be a morphism of $\mathcal{F}$-algebra in spectra. There is second quadrant spectral sequence with E_2 term

$$E_2^{0,0} = \mathrm{Hom}_{E_*\mathcal{F}}(E_*X, Y*)$$

and

$$E_2^{s,t} = D^s_{E_*T}(E_*X, \Omega^t Y_*)$$

converge to

$$\pi_{t-s}(\mathrm{Map}_{Alg_F}(X, Y), \phi)$$

Goerss-Hopkins Obstruction Theory

Goerss-Hopkins Obstruction Theory

Let R and S be E -local E_∞-rings, and let $A = E_* R$ and $B = E_* S$. Given a map ϕ $A \to B$ of commutative algebras in $E_* E$ -comodules, there exists an inductively defined sequence of obstructions valued in

$$\mathrm{Ext}^{n+1,n}_{\mathrm{Mod}_A(Comod_{E_* E})}(L_{A/E_*}, B)$$

which vanishes iff there is an E_∞-ring map $\widetilde{\phi} : R \to S$ such that $E_*(\widetilde{\phi}) = \phi$.

Elliptic Cohomology

An elliptic cohomology consists of

- An even periodic spectrum E .
- An elliptic curve C over $\pi_0 E$.
- $\phi : G_E \cong \hat{C}$

We denote this data as (E, C, ϕ)

Theorem(Goerss-Hopkins-Miller-Lurie)

There is a sheaf $\mathcal{O}_{tmf}$ of E_∞-ring spectra over the stack $\overline{\mathcal{M}}_{ell}$ for the *étale* topology. For any *étale* morphism $f : \mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$, there is a natural structure of elliptic spectrum $(\mathcal{O}_{tmf}(f), C_f, \phi)$, satisfying $\pi_0\mathcal{O}_{tmf}(f) = R$, and C_f is a generalized elliptic curve over R classified by f.

$Tmf = \mathcal{O}_{tmf}(\overline{\mathcal{M}}_{ell} \to \overline{\mathcal{M}}_{ell})$, topological modular forms.

Topological Automorphic Forms

Theorem

let M^n_{pd} denote the moduli stack of one dimensional height n p-divisible group, then there is a sheaf of E_∞-ring space, $\mathcal{O}^{top}$ on the etale site. such that for any

$$E := \mathcal{O}^{top}(\mathrm{Spec} R \xrightarrow{G} M^n_{pd})$$

we have

$$F_E = G^0$$

where G_0 is the formal part of the p-divisible group G.

The main issue of this construction is that fro a general n-dimensional abelian variety, their associated p-divisible group are not 1-dimensional.
PEL Shimura stacks are moduli stacks of abelian varieties with the extra structure of Polarization, Endomorphisms, and Level structure . A class of PEL Shimura stacks (associated to a rational form of the unitary group U(1, n1)) whose PEL data allow for the extraction of a 1-dimensional p-divisible group satisfying the hypotheses of above theorem.

肆

Orientations

Obstructions to H_∞-maps

$$\begin{tikzcd}
H_\infty \mathcal{C} \arrow[r] \arrow[d] & (\text{ formal groups with descent data}) \arrow[d] \\
\text{homogeneous spectra } \mathcal{C} \arrow[r] & (\text{formal groups})
\end{tikzcd}$$

Theorem (Ando-Hopkins-Strickland, 2004)

The rule which associates a level structure

$$l : A \to i^* G(R)$$

to a map $\psi_l^E : \mathrm{Spf} R \to S_E$ given by the ring map $\pi_0 E \xrightarrow{D_A} \pi_0 E^{BA_+^*} \to R$ and the isogeny

$$\psi_l^{G/E} : i^* G \to \psi_l^* G$$

is descent data for level structure on the formal group G over S_E.

$\mathcal{L}$ is a line bundle over G. Given a subset $I \subset \{1, \cdots, k\}$, $\sigma_I : G_S^k \to G$ defined by $\sigma_I(a_1, \ldots, a_k) = \Sigma_{i \in I} a_i$.
We define a line bundle over G_S^k by

$$\Theta^k(\mathcal{L}) = \bigotimes_{I \subset \{1,\ldots,k\}} (\mathcal{L}_I)^{(-1)^{|I|}}$$

And set $\Theta^0(\mathcal{L}) = \mathcal{L}$.

$$\begin{aligned}
\Theta^0(\mathcal{L})_a &= \mathcal{L}_a \\
\Theta^1(\mathcal{L})_a &= \frac{\mathcal{L}_0}{\mathcal{L}_a} \\
\Theta^2(\mathcal{L})_{a,b} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b}}{\mathcal{L}_a \otimes \mathcal{L}_b} \\
\Theta^3(\mathcal{L})_{a,b,c} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b} \otimes \mathcal{L}_{a+c} \otimes \mathcal{L}_{b+c}}{\mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c \otimes \mathcal{L}_{a+b+c}}
\end{aligned}$$

Definition

A Θ^k structure on a line bundle $\mathcal{L}$ over a group G is a trivialization s of the line bundle $\Theta^k(\mathcal{L})$ such that

1. For $k > 0$, s is a rigid section.
2. s is symmetric,i.e., for each $\sigma \in \Sigma_k$, we have $\xi_\sigma \pi_\sigma^* s = s$.
3. The section
   $s(a_1, a_2, \dots) \otimes s(a_0 + a_1, a_2, \dots)^{-1} \otimes s(a_0, a_1 + a_2, \dots) \otimes s(a_0, a_1, \dots)^{-1} \otimes$
   corresponds to 1.

If $g : MU\langle 2k \rangle \to E$ is an orientation, then the composition

$$((\mathbb{C}P^\infty)^k)^V \to MU\langle 2k \rangle \to E$$

represents a rigid section s of $\Theta^k(I_G(0))$

Theorem

For $0 \le k \le 3$, the maps of ring spectra $MU\langle 2k \rangle \to E$ are in one to one correspondence with Θ^k-structures on $\mathcal{I}(0)$ over G_E.

Theorem (Ando-Hopkins-Strickland, 2004)

Let $g : MU\langle 2k\rangle \to E$ be a homotopy multiplicative map, $s = s_g$ be the section of $\Theta^k(I_G(0))$ as before. If the map g is H_∞, then for each level structure

$$A \xrightarrow{l} i^* G,$$

the section s satisfy the identity

$$\widetilde{N}_{\psi_l^{G/E}} s = (\psi_l^E) i^* s$$

And if $k \leq 3$, the converse is true.

Using this theorem, they proved the σ orientation of an elliptic spectrum is an H_∞ map. Zhu (2020) proved that the map $MU\langle 0\rangle \to E$ coming from a coordinate of $\mathrm{Spf} E^0(\mathbb{C}^\infty)$ is a H_∞ map, since the map satisfying the condition above, which is called norm coherence.

Obstructions to E_∞-maps

Hopkins-Lawson obstruction theory (2018): There exists a diagram of E_∞-ring spectra

$$\mathbb{S} \to MX_1 \to MX_2 \to MX_3 \to \cdots$$

such that the following hold:

1. $\lim MX_n \to MU$ is an equivalence.
2. $\mathrm{Map}_{E_\infty}(MX_1, E) \simeq Or(E)$ for each E_∞-ring E.
3. Given $m > 0$ and an E_∞-ring E, there is a pull back square

$$\begin{tikzcd}
\mathrm{Map}_{E_\infty}(MX_m, E) \arrow[r] \arrow[d] & \mathrm{Map}_{E_\infty}(MX_{m-1}, E) \arrow[d] \\
\{*\} \arrow[r] & \mathrm{Map}_*(F_m, \mathrm{Pic}(E))
\end{tikzcd}$$

   where F_m is a certain pointed space.

4. $MX_{m-1} \to MX_m$ is a rational equivalence if $m > 1$, a p-local equivalence if m is not a power of p, and a $K(n)$-local equivalence if $m > p^n$.
5. Let E denote an E_∞ such that $\pi_* E$ is p-local and p-torsion free. Then an E_∞-map $MX_1 \to E$ extends to an E_∞ map $MX_P \to E$ if and only if the corresponding complex orientation of E satisfies the Ando criterion.

Theorem (Senger, 2022)

Let E denote a height ≤ 2 Landweber exact E_∞-ring whose homotopy groups is concentrated in even degrees. Then any complex orientation $MU \to E$ which satisfies the Ando criterion lifts uniquely up to homotopy to an E_∞-ring map $MU \to E$.

The proof of Senger's theorem was based on E-cohomology of some certain spaces. We have the following pullback square.

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \prod_p E_p^{\wedge} \arrow[d] \\ E_{\mathbb{Q}} \arrow[r] & (\prod_p E_p^{\wedge})_{\mathbb{Q}} \end{tikzcd}$$

$\mathrm{Map}_{E_\infty}(MU, R) \simeq Or(R)$ for a rational E_∞-ring R, and
$\pi_1 \mathrm{Map}_{E_\infty}(MU, R) \cong \pi_1 Or(R) \cong 0$, if R is concentrated in even degrees.

$$\begin{tikzcd} \pi_0 \mathrm{Map} E_\infty(MU, R) \arrow[r] \arrow[d] & \pi_0 \mathrm{Map}_{E_\infty}(MU, \prod_p E_P^{\wedge}) \arrow[d] \\ \pi_0 \mathrm{Or}(E_{\mathbb{Q}}) \arrow[r] & \pi_0 \mathrm{Or}((\prod_p E_p^{\wedge})_{\mathbb{Q}}) \end{tikzcd} \qquad \begin{tikzcd} \pi_0 \mathrm{Or}(E) \arrow[r] \arrow[d] & \pi_0 \mathrm{Or}(\prod_p E_p^{\wedge}) \arrow[d] \\ \pi_0 \mathrm{Or}(E_{\mathbb{Q}}) \arrow[r] & \pi_0 \mathrm{Or}((\prod_p E_p^{\wedge})_{\mathbb{Q}}) \end{tikzcd}$$

It suffices to lift the induced complex orientation of $E_p^\wedge$.
Assume that E is p-complete. So we only need to prove

$$\pi_0 \mathrm{Map}_{E_\infty}(MX_{p^2}, E) \to \pi_0 \mathrm{Map}_{E_\infty}(MX_p, E)$$

is surjective.
There is a cofiber sequence.

$$\mathrm{Map}_{E_\infty}(MX_{p^2}, E) \to \mathrm{Map}_{E_\infty}(MX_p, E) \to \mathrm{Map}_*(F_{p^2}, Pic(E))$$

and a equivalence

$$\mathrm{Map}_{E_\infty}(F_m, Pic(E)) \simeq \mathrm{Hom}(\Sigma^\infty F_m, pic(E)) \simeq \mathrm{Hom}(\Sigma^\infty F_m, \Sigma E).$$

It suffices to show that

$$E^1(\Sigma^\infty F_{p^2}) \simeq 0$$

Lemma (Senger, 2022)

$$E^{2n}(F_p) \cong E^{2n+1}(F_{p^2}) \cong 0.$$

Let L_m denote the nerve of the poset of proper direct sum decomposition of $\mathbb{C}^m$, and $(L_m)^\diamond$ is its unreduced suspension.

$$F_m \simeq ((L_m)^\diamond \wedge S^{2m})_{hU(m)}.$$