Analytic Geometry and Homotopy Groups of the $K(n)$-Local Spheres

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2024.06.20

Our Goal

Theorem(Barthel-Schlank-Stapleton-Weinstein， 2024)

There is an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathbb{Q} \otimes \pi_* L_{K(n)} S^0 \cong \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \cdots \zeta_n),$$

where the latter is the exterior $\mathbb{Q}_p$-algebra with generators ζ_i in degree $1 - 2i$.

Rationalization of the K(n)-Local Sphere

Analytic Geometry

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Rationalization of the K(n)-Local Sphere

Morava E-theories and Morava K-theories

Let G_0 be a formal group over a perfect field k with characteristic p, then a deformation of G_0 to R is a triple (G, i, Ψ), where G is a formal group over R, $i : k \to R/m$, $\Psi : \pi^* G \cong i^* G_0$ is an isomorphism of formal groups over R/m.

Theorem (Lubin-Tate , 1966)

There is a universal formal group G over $R_{LT} = W(k)[[v_1, \cdots, v_n - 1]]$ in the following sense: for every infinitesimal thickening A of k, there is a bijection

$$\mathrm{Hom}_{/k}(R_{LT}, A) \to \mathrm{Def}(A).$$

There is a spectrum E_n called **Morava E-theory**, whose homotopy group is

$$\pi_* E_n = W(k)[\![v_1, \cdots, v_{n-1}]\!][\beta^{\pm 1}],$$

This is a even spectrum $K(n)$ called **Morava K-theory**, whose homotopy groups is

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}]/(t_0, t_1, \cdots t_{p^n-2}, t_{p^n}, \cdots) \cong \mathbb{F}_p[v_n^{\pm 1}]$$

Morava Stabilizer Groups

We let G_0 denote a formal group of height n over a perfect field $\overline{\mathbb{F}}_p/\mathbb{F}_p$
The small Morava stabilizer group $\mathrm{Aut}_{\overline{\mathbb{F}}_p}(G_0)$ is the group of automorphism of G_0 with coefficients in $\overline{\mathbb{F}}_p$,

$$\mathrm{Aut}(G_0) = \{f(x) \in \overline{\mathbb{F}}_p[[x]] : f(G_0(X, Y)) = G_0(f(x), f(y)), f'(0) \neq 0\}$$

Since G_0 is defined over $\overline{\mathbb{F}}_p$, the Galois group $\mathrm{Gal} = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ act on G_0 by acting on the coefficients. The Morava stabilizer group $\mathbb{G}_n$ is defined by

$$\mathbb{G}_n = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \ltimes \mathrm{Aut}(G_0)$$

Theorem (Devinatz-Hopkins , Goerss-Hopkins-Miller)

The Morava stabilizer group acts on E_n, and it givens essential all automorphisms of E_n

$$E_n^{h\mathbb{G}_n} \simeq L_{K(n)} S^0$$

Stable Homotopy Groups of Sphere

Lemma

The K(1)-local sphere $L_{K(1)}S$ is given by the homotopy fiber of the map $\Psi^g - 1 : \widehat{KU} \to \widehat{KU}$.

$$\pi_{2n}(\widehat{KU}^{\Psi^g - 1}) \simeq 0$$

$$\pi_{2n-1}(\widehat{KU}^{\Psi^g - 1}) \simeq \mathbb{Z}^p/(g^n - 1).$$

By this theorem, we can compute the homotopy group of $L_{K(1)}S$

$$\pi_n L_{K(1)}S = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Q}_p/\mathbb{Z}_p & n = -2 \\ Z/p^{k+1}Z & n+1 = (p-1)p^k m, p \nmid m \\ 0 & \text{otherwise} \end{cases}$$

Homotopy fixed point spectral sequence

Proposition

There is a homotopy fixed point spectral sequence (descent spectral sequence)

$$E_2^{s,t} = H^s_{gp}(G; \pi_t(X)) \Longrightarrow \pi_{t-s}(X^{hG})$$

similarly for X_{hG}, X^{tG}.

We have $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$, then we get

$$E_2^{s,t} \cong H^s_{cts}(\mathbb{G}, \pi_t E_n) \Longrightarrow \pi_{t-s}L_{K(n)}S^0.$$

The structure of Morava stabilizer group

For f a formal group law over $\overline{\mathbb{F}}_p$.

$$\mathrm{End} f = \{g(t) \in tR[\![t]\!] \quad | \quad f(g(x), g(y)) = gf(x, y)\}$$

Proposition

End(f) is a noncommutative local ring: The collection non-invertible elements is the left ideal generated by $\pi(t) = \nu(t^p)$, where $\nu f^p(x, y) = f(\nu(x), \nu(y))$.

Let $D = \mathbb{Q} \otimes \mathrm{End}(f)$.

Lemma

D is a central division algebra over $\mathbb{Q}_p$. $\mathrm{End}(f) = \{x \in D : v(x) \geq 0\}$.

Morava Stabilizer Group

$$\det : \mathbb{G}_n \to \mathbb{Z}_p^\times \qquad \det : \mathbb{S}_n \to \mathbb{Z}_p^\times$$

Composition with $\mathbb{Z}_p^\times/\mu \cong \mathbb{Z}_p$.

$$\zeta_n : \mathbb{G}_n \to \mathbb{Z}_p.$$

Let $\mathbb{G}_n^1 = \ker \zeta_n$, we have

$$\mathbb{G}_n \cong \mathbb{G}_n^1 \rtimes \mathbb{Z}_p, \quad \mathbb{S}_n \cong \mathbb{S}_n^1 \rtimes \mathbb{Z}_p.$$

As a consequence of $\mathbb{G}_n/\mathbb{G}_n^1 \rtimes \mathbb{Z}_p$, there is a equivalence $L_{K(n)}S^0 \simeq (E_n^{h\mathbb{G}_n^1})^{h\mathbb{Z}_p}$.

$$L_{K(n)}S^0 \longrightarrow E_n^{h\mathbb{G}_n^1} \xrightarrow{\psi-1} E_n^{h\mathbb{G}_n^1} \xrightarrow{\delta} \Sigma L_{K(n)}S^0.$$

The action of Morava stabilizer group

Let F_n be the universal deformation over $(E_n)_0$ of G_0. If we have $\alpha = (f, \sigma) \in \mathbb{G}_n$. The universal property of F_n implies that there is ring isomorphism $\alpha_* : (E_n)_0 \to (E_n)_0$ and an isomorphism of formal group laws $f_\alpha : \alpha_* F_n \to F_n$. The action can extends to $(E_n)_* \cong \mathbb{W}_n[\![u_1, \cdots, u_{n-1}]\!][u^{\pm 1}]$

1. $\alpha = (id, \sigma)$ for $\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)$. Then the action is action of Galois group on $\mathbb{W}_n$.
2. If $\omega \in \mathbb{S}_n$ is a primitive $(p^n - 1)$-th root of the unity, then $\omega_*(u_i) = \omega^{p^i - 1} u_i$ and $\omega_*(u) = \omega u$.
3. $\psi \in \mathbb{Z}_p^\times \subset \mathbb{S}_n$ is the center, then $\psi_*(u_i) = u_i$ and $\psi_* u = \psi u$.

Theorem (Devinatz-Hopkins)

Let $1 \leq i \leq n-1$ and $f = \sum_{j=0}^{n-1} f_j \zeta^j \in \mathbb{S}_n$, where $f_j \in \mathbb{W}_n$. Then modulo $(p, u_1, \cdots u_{n-1})^2$,

$$f_*(u) \equiv f_0 u + \sum_{j=1}^{n-1} f_{n-j}^{\sigma^j} u u_j \qquad f_*(u u_i) \equiv \sum_{j=1}^{i} f_{i-j}^{\sigma^j} u u_j + \sum_{j=i+1}^{n} p f_{n+i-j}^{\sigma^j} u u_j$$

Local-to-global Principle

The Hasse square is a pullback square

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \prod_p \mathbb{Z}_p \arrow[d] \\
\mathbb{Q} \arrow[r] & \mathbb{Q} \otimes_p \prod_p \mathbb{Z}_p
\end{tikzcd}$$

This is the special case of a local-to-global principle for any chain complex $M \in \mathcal{D}_{\mathbb{Z}}$.

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \prod_p M_p^{\wedge} \arrow[d] \\
\mathbb{Q} \otimes M \arrow[r] & \mathbb{Q} \otimes_p \prod_p M_p^{\wedge}
\end{tikzcd}$$

which is a homotopy pullback square, where $M_p^{\wedge}$ denote the derived p-completion (p-local and $\mathrm{Ext}^i(\mathbb{Q}, M_p^{\wedge}) = 0$, for $i = 0, 1$.)

Rationalization of the $K(n)$-Local Sphere

Theorem(Barthel-Schlank-Stapleton-Weinstein， 2024)

There is an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathbb{Q} \otimes \pi_* L_{K(n)} S^0 \cong \Lambda_{\mathbb{Q}_p}(\zeta_1, \zeta_2, \cdots \zeta_n),$$

where the latter is the exterior $\mathbb{Q}_p$-algebra with generators ζ_i in degree $1 - 2i$.

Lemma

For all $t \neq 0$ and all $s \in \mathbb{Z}$, we have $H^s_{cts}(\mathbb{G}_n, , \mathbb{Q} \otimes \pi_t E_n) = 0$.

Proof: There is a short exact sequence

$$1 \to \mathcal{O}_D^\times \to \mathbb{G}_n \cong \mathcal{O}_D^\times \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1$$

where $\mathcal{O}_D^\times$ is isomorphic to the automorphism group of our choose formal group law $\mathbb{G}_n$ over $\overline{\mathbb{F}}_p$. The center of $\mathcal{O}_D^\times$ is isomorphic to $\mathbb{Z}_p^\times$. The central subgroup $\mathbb{Z}_p \subset \mathbb{Z}_p^\times \subset \mathcal{O}_D^\times$ which can be generated by the element $1 + p \in \mathbb{Z}_p^\times$. We have the convergent Lydon-Hochschild-Serre spectral sequence

$$H^p(\mathcal{O}_D^\times/\mathbb{Z}_p, H^q_{cts}(\mathbb{Z}_p, \mathbb{Q} \otimes \pi_t E_n)) \Longrightarrow H^{p+q}_{cts}(\mathcal{O}_D^\times, \mathbb{Q} \otimes \pi_t E_n))$$

The generator acts on $\mathbb{Q} \otimes \pi_t E_n$ by multiplication by $(1+p)^t$. Consider the complex

$$\mathbb{Q} \otimes \pi_t E_n \xrightarrow{(1+p)^t - 1} \mathbb{Q} \otimes \pi_t E_n$$

Since $\mathbb{Q}_p \otimes \pi_t E_n$ is a $\mathbb{Q}_p$-vector space, when $t \neq 0$ the action by $(1+p)^t - 1$ is invertible, so the complex is acyclic, $H^q_{cts}(\mathcal{O}_D^\times, \mathbb{Q} \otimes \pi_t E_n)) = 0$ for $t \neq 0$.

We continue to consider the spectral sequence

$$H^p(\mathbb{G}_n/\mathcal{O}_D^\times, H^q_{cts}(\mathcal{O}_D^\times, \mathbb{Q}\otimes\pi_t E_n)) \Longrightarrow H^{p+q}_{cts}(\mathbb{G}_n, \mathbb{Q}\otimes\pi_t E_n)),$$

we get $H^s_{cts}(\mathbb{G}_n, \mathbb{Q}\otimes\pi_t E_n) = 0$ for all $t \neq 0$.

Cohomology of Morava Stabilizer Group

Proposition

For every integer $s \geq 0$, the natural map $W = W(\overline{\mathbb{F}}_p) \to \pi_0 E_n = W[\![u_1, \ldots, u_{n-1}]\!]$ induces a split injection

$$H^s_{cts}(\mathbb{G}_n, W) \hookrightarrow H^s_{cts}(\mathbb{G}_n, \pi_0 E_n)$$

whose complement killed by a power of p. In particular,

$$H^s_{cts}(\mathbb{G}_n, W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^s_{cts}(\mathbb{G}_n, \pi_0 E_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

Proof:

The cohomology groups $H^i_{cts}(\mathcal{O}_D^\times, A^c)$ and $H^i_{cts}(\mathbb{G}_n, A^c)$ are p-power torsion.

$$\begin{tikzcd}
 & \mathcal{X} \arrow[dl, "\mathrm{GL}_n(\mathbb{Z}_p)"'] \arrow[dr, "\mathcal{O}_D^\times"] & \\
\mathrm{LT}_K & & \mathcal{H}_K
\end{tikzcd}$$

This diagram induces an isomorphism in $D(\mathrm{Solid})$:

$$R\Gamma(\mathrm{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathcal{O}_D^\times} \cong R\Gamma(\mathcal{H}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathrm{GL}_n(\mathbb{Z}_p)}$$

We have

$$H^*(R\Gamma(\mathcal{H}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathrm{GL}_n\mathbb{Z}_p}) \otimes_W K \cong \Lambda_K(y_1, y_3, \ldots, y_{2n-1})[\epsilon]$$

$$H^*(R\Gamma(\mathrm{LT}_{K,\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}})^{h\mathcal{O}_D^+}) \otimes_W K \cong \Lambda_K(x_1, x_3, \ldots, x_{2n-1})[\epsilon] \oplus ((A^c)^{h\mathcal{O}_D^\times} \otimes_W K)[\epsilon].$$

We then have $H^*_{cts}(\mathcal{O}_D^\times, A^c) \otimes_W K = 0$, using the Hochschild-Serre spectral sequence combined with the fact that the cohomological dimension $\mathbb{G}_n/\mathcal{O}_D^\times \cong \widehat{\mathbb{Z}}$ is 1, we get $H^i_{cts}(\mathbb{G}_n, A^c)$ is also p-power torsion.

Galois Cohomology of Witt Rings

Lemma

Let $W = W(\overline{\mathbb{F}}_p)$ and $K = W[1/p]$.

1. $H^i_{cts}(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), W)$ is $\mathbb{Z}_p$ if $i = 0$, and is 0 otherwise.
2. Let $\mathbb{G}_n$ action on K through its quotient $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. There is an isomorphism of graded $\mathbb{Q}_p$-algebras:

$$H^*_{cts}(\mathbb{G}_n, K) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1}).$$

Proof:

1. $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \widehat{Z}$, so it is enough to prove in degree 1. W is p-adically complete, this is further reducing that $H^1_{cts}(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), \overline{\mathbb{F}}_p) = 0$, this is true because $x \mapsto x^p - x$ is surjective on $\overline{\mathbb{F}}_p$.
2. Consider the spectral sequence

$$H^i_{cts}(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), H^j_{cts}(\mathcal{O}_D^\times, K)) \Longrightarrow H^{i+j}_{cts}(\mathbb{G}_n, K)$$

Consider the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on $H^j_{cts}(\mathcal{O}_D^\times, K) = H^j_{cts}(\mathcal{O}_D^\times, \mathbb{Q}_p) \otimes_{\mathbb{Z}_p} W$.

The action on the first factor is trivial by the following lemma, and the action on the factor has no higher cohomology by 1. Therefore

$$H^*_{cts}(\mathbb{G}_n, K) \cong H^*_{cts}(\mathcal{O}_D^\times, \mathbb{Q}_p),$$

then again apply the following lemma.

Lemma

Let G be either of the group $GL_n(\mathbb{Z}_p)$ or $\mathcal{O}_D^\times$. Consider the trivial action of G on $\mathbb{Q}_p$. There is an isomorphism of graded $\mathbb{Q}_p$-algebras:

$$H^*_{cts}(G, \mathbb{Q}_p) \cong H^*(\mathrm{Lie}G, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \ldots, x_{2n-1}).$$

In the case of $G = \mathcal{O}_D^\times$, the outer morphism $\mathrm{ad}\Pi$ (where Π is a uniformizer of $D^\times$) act as the identity on $H^*_{cts}(G, \mathbb{Q}_p)$.

Proof of the Main Theorem

The Devinatz-Hopkins spectral sequence

$$E_2^{s,t} \cong H^s_{cts}(\mathbb{G}, \pi_t E_n) \Longrightarrow \pi_{t-s} L_{K(n)} S^0$$

converges strongly and collapses on a finite page with a horizontal vanishing line. Tensor with $\mathbb{Q}$, we get a convergent spectral sequence

$$\mathbb{Q} \otimes E_2^{s,t} \cong H^s_{cts}(\mathbb{G}, \mathbb{Q} \otimes \pi_t E_n) \Longrightarrow \mathbb{Q} \otimes \pi_{t-s} L_{K(n)} S^0$$

By above lemmas, the E_2 page of the rationalization of the Devinatz-Hopkins spectral sequence only have one nonvansihing line, which is $t = 0$ in the (s, t) coordinate system. So we get an isomorphism

$$H^*_{cts}(\mathbb{G}, \mathbb{Q} \otimes \pi_0 E_n) \cong \mathbb{Q} \otimes \pi_* L_{K(n)} S^0$$

By the computation of the cohomology groups of Morava stabilizer groups, the left hand side equals to

$$H^*_{cts}(\mathbb{G}, \mathbb{Q} \otimes \pi_0 E_n) \cong H^*_{cts}(\mathbb{G}, \mathbb{Q} \otimes W) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_2, \ldots, x_n).$$

with x_i in cohomological degree $2i - 1$.

Analytic Geometry

The Langlands correspondence in number theory (Langlands 67) is a conjectural correspondence (a bijection subject to various conditions) between

1. n-dimensional complex linear representations of the Galois group $\mathrm{Gal}(\bar{F}/F)$ of a given number field F
2. certain representations-called automorphic representations of the n-dimensional general linear group $GL_n(\mathbb{A}_F)$ with coefficients in the ring of adeles of F, arising within the representations given by functions on the double coset space $GL_n(F) \setminus GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$.

moduli spaces of shtukas	Shimura varieties
moduli spaces of local shtukas	local Shimura varieties
Drinfled's upper half spaces	Lubin-Tate towers

Shtukas over Function Fields

Definition

- Let $S/\mathbb{F}_p$ be a scheme. A shtuka of rank n with legs $x_1, \ldots, x_m \in X(S)$ is a rank n vector bundle $\mathcal{E}$ over $S \times_{\mathbb{F}_p} X$ together with an isomorphism

$$\phi_{\mathcal{E}} : (\mathrm{Frob}_S)^* \mathcal{E}|_{S \times_{\mathbb{F}_p} X \setminus \cup_i \Gamma_{x_i}} \cong \mathcal{E}|_{S \times_{\mathbb{F}_p} X \setminus \cup_i \Gamma_{x_i}}$$

  on $S \times_{\mathbb{F}_p} X \setminus \cup_i \Gamma_{x_i}$, where $\Gamma_{x_i} \subset S \times_{\mathbb{F}_p} X$ is the graph of x_i.
- Let $\widehat{X}$ be the formal completion of X at one of its $\mathbb{F}_p$ rational points, so that $\widehat{X} \cong \mathrm{Spf}\mathbb{F}_p[\![T]\!]$. A local shtuka of rank n over an adic space $S/\mathbb{F}_p$ with legs $x_1, \ldots, x_m \in \widehat{X}(S)$ is a rank n vector bundle $\mathcal{E}$ over $S \times_{\mathbb{F}_p} \widehat{X}$ together with an isomorphism

$$\phi_{\mathcal{E}} : (\mathrm{Frob}_S)^* \mathcal{E}|_{S \times_{\mathbb{F}_p} \widehat{X} \setminus \cup_i \Gamma_{x_i}} \cong \mathcal{E}|_{S \times_{\mathbb{F}_p} \widehat{X} \setminus \cup_i \Gamma_{x_i}}$$

  over $S \times_{\mathbb{F}_p} \widehat{X} \setminus \cup_i \Gamma_{x_i}$

Suppose that we are given a shtuka $(\mathcal{E}, \phi_{\mathcal{E}})$ of rank n over $\mathrm{Spec} k$, where k is an algebraically closed. Then it can be described by the following data:

1. The collection of points $x_1, \dots, x_m \in X(k)$ where $\phi_{\mathcal{E}}$ is undefined. We call these points legs of the shtuka.
2. For each $i = 1, \dots, m$ a conjugacy class μ_i of cocharacters $G_m \to GL_n$, encoding the behaviour of $\phi_{\mathcal{E}}$ near x_i.

Now we explain the second item. Let $x \in X(k)$ be a leg of shtuka, and let $t \in \mathcal{O}_{X,x}$ be a uniformizing parameter at x. We have the complete stacks $(\mathrm{Frob}_S^* \mathcal{E})_x^\wedge$ and $\mathcal{E}_x^\wedge$. These two are free rank modules over $\mathcal{O}_{X,x}^\wedge \cong k[\![t]\!]$, whose generic fibers are identified using $\phi_{\mathcal{E}}$. That is we have two $k[\![t]\!]$ lattices in the same n dimensional k((t)) vector space.

- By the theory of elementary divisors, there exists a basis $e_1, \ldots, e_n$ of $\mathcal{E}_x^\wedge$ such that $t^{k_1}e_1, \ldots, t^{k_n}e_n$ is a basis of $(\mathrm{Frob}_S^*\mathcal{E})_x^\wedge$, where $k_1, \ldots, k_n$. These integers depend only on the shtuka. Another way to package of this data is as conjugacy class μ of cocharacters $G_m \to GL_n$ via $\mu(t) = \mathrm{diag}(t^{k_1}, \ldots, t^{K_n})$.
- Thus there are some discrete data attach to a shtuka: the number of legs m and the ordered collection of cocharacterss $(\mu_1, \ldots, \mu_m)$. Fixing these, we can define a moduli space $\mathrm{Sht}_{GL_n,\{\mu_1,\ldots,\mu_m\}}$ whose k-points classify the following data:
    1. An m-tuple of points $(x_1, \ldots, x_m)$ of $X(k)$.
    2. A shtuka $(\mathcal{E}, \phi_{\mathcal{E}})$ of rank n with legs $x_1, \ldots x_m$, for which the relative position of $\mathcal{E}_{x_i}^\wedge$ and $(\mathrm{Frob}_S^*\mathcal{E})_{x_i}^\wedge$ is bounded by the cocharacter μ_i for all $i = 1, \ldots, m$.

- It can be proved that $\mathrm{Sht}_{GL_n,\{\mu_1,\dots,\mu_m\}}$ is representable by a Deligne-Mumford stack. We have a structure map

$$f : \mathrm{Sht}_{GL_n,\{\mu_1,\dots,\mu_m\}} \to X^m$$

by sending a shtuka to its m-tuple of legs.

- We can add level structures to these spaces of shtukas, parametrized by finite closed subscheme $N \subset X$. A level N-structure on $(\mathcal{E}, \phi_{\mathcal{E}})$ is then a trivialization of the pullback of $\mathcal{E}$ to N which is compatible with $\phi_{\mathcal{E}}$. By this additional structure, we can get a family of shtukas $\mathrm{Sht}_{GL_n,\{\mu_1,\dots,\mu_m\}}$ and morphisms

$$f_N : \mathrm{Sht}_{GL_n,\{\mu_1,\dots,\mu_m\},N} \to (X/N)^m.$$

- The stack $\mathrm{Sht}_{GL_n,\{\mu_1,\dots,\mu_m\},N}$ carries an action of $GL_n(\mathcal{O}_N)$, by altering the trivialization of $\mathcal{E}$ on N. The inverse limit $\varprojlim_N \mathrm{Sht}_{GL_n,\{\mu_1,\dots,\mu_m\},N}$ admits an action of $GL_n(\mathbb{A}_K)$, via the Hecke correspondences. Assume the relative dimension of f is d. We consider the cohomology $R^d(f_N)_!\overline{\mathbb{Q}}_l$, this an $\overline{\mathbb{Q}}_l$ étale sheaf on X^m.

Passing to the limit over N, one gets a big representation of $\mathrm{GL}_n(A_K) \times \mathrm{Gal}(\overline{K}/K) \times \cdots \mathrm{Gal}(\overline{K}/K)$ on $R^d(f_N)_!\overline{\mathbb{Q}}_l$. Roughly, we expects this space to decompose is as follows

$$\varinjlim_N R^d(f_N)_!\overline{\mathbb{Q}}_l = \bigoplus_\pi \pi \otimes (r_1 \circ \sigma(\pi)) \otimes \cdots \otimes (r_m \circ \sigma(\pi))$$

- π run over cuspidal automorphic representations of $\mathrm{GL}_n(K)$,
- $\sigma(\pi) : \mathrm{Gal}(\bar{(}K)/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is the corresponding L-parameter,
- $r_i : \mathrm{GL}_n \to \mathrm{GL}_{n_i}$ is an algebraic representation corresponding to μ_i.

Drinfeld (1980, n=2) and L. Lafforgue (general n, 2002) considered the case of $m = 2$, with μ_1 and μ_2 corresponding to the n-tuples $(1, 0, \ldots, 0)$ and $(0, \ldots, 0, -1)$ respectively. V. Lafforgue considered general reductive group G in place of $\mathrm{GL_n}$.

Shimura Varieties

A Shimura datum is a pair (G, μ), where G is a reductive group over $\mathbb{Q}$, and $\mu : C^{\times} \to G(R)$ is a morphism of real groups, such that the conjugacy class $\mathcal{H}_{\mu}$ of μ is a complex manifold. The tower of Shimura varieties is

$$\mathrm{Sh}(G, \mu)_K = G(\mathbb{Q}) \setminus (\mathcal{H}_{\mu}) \times G(\mathrm{A}_f)/K$$

where K runs over all compact open subgroups of $G(\mathrm{A}_f)$. The l-adic cohomology of the tower admits an action $G(\mathrm{A}_f) \times \mathrm{Gal}(\overline{E}/E)$. Let

$$H^i(\xi) = \lim_{\longrightarrow K} H^i(Sh(G, \mu)_{K, \overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_l)$$

$$H^*(\xi) = \sum_i (-1)^i H^i(\xi)$$

Conjecture

$$H^*(\xi) = \sum_{\pi} a(\pi, \xi) \pi_f \otimes (R_{\mu} \circ \phi_{\pi})|_{\mathrm{Gal}(\overline{\mathbb{Q}}/E)}$$

Here π runs over cuspidal automorphic representations of G, $R_{\mu} : {}^L G \to GL_n$ is the representation of highest weight μ, and $a(\pi, \xi)$ is a integer.

Adic Spaces

Definition

- A Huber ring is a topological ring A, such that there exists an open subring $A_0 \subset A$ and a finitely generated ideal $I \subset A_0$ such A has the I-adic topology.
- A Huber ring A is Tate if it continuous a topologically nilpotent unit. Such an element is called a pseudo-uniformizer
- A subset S of a topological ring A is bounded if for all open neighborhoods U of 0, there exists an open neighborhood V of 0 such that $V \cdot S \subset U$.
- An element $f \in A$ is power-bounded if $\{f^N\} \subset A$ is bounded. Let A° be the subset of power-bounded elements. If A is linearly topologized (for instance if A is Huber) then $A^\circ \subset A$ is a subring.
- A Huber ring A is uniform if $A^\circ \subset A$ is bounded.

Definition

- Let A be a Huber ring. A subring $A^+ \subset A$ is a ring of integral elements if it is open and integrally closed and $A^+ \subset A^\circ$. A Huber pair is a pair (A, A^+), where A is a Huber and $A^+ \subset A$ is ring of integral elements.
- Given a Huber pair, we let $\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$ be the subset of continuous valuations x for which $|f| \leq 1$ for all $f \in A^+$. Write $\mathrm{Spa}A$ for $\mathrm{Spa}(A, A^\circ)$.

Example

$A = \mathbb{Q}_p\langle T\rangle$ and $A^+ = A^\circ = \mathbb{Z}_p\langle T\rangle$, we define

$$A^{++} = \{\sum_{n=0}^{\infty} a_n T^n \in A^+ | |a_n| < 1 \text{ for all } n \geq 1\}$$

We have $A^{++} \subset A^+$, so $\mathrm{Spa}(A, A^+) \subset \mathrm{Spa}(A, A^{++})$.

Topology of Adic Spaces

The topology of an adic spectrum $X = \mathrm{Spa}(A, A^+)$ is generated by *rational sets* of the form

$$U = U(\frac{f_1, \ldots, f_r}{g}) = \{v \in \mathrm{Spa}(A, A^+) | v(f_i) \leq v(g) \neq 0, i = 1, \ldots, r\}$$

For $U = U(f_i/g)$ a rational set

- $\mathcal{O}_X(U)$, the completion of $A[f_i/g]$.
- $\mathcal{O}_X^+(U)$, the completion of the integer closure of $A^+[f_i/g]$ in $A[f_i/g]$.

Definition

An adic space is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ which is locally isomorphic to an affinoid adic space $\mathrm{Spa}(A, A+)$.

Rigid Analytic Spaces

Definition

A rigid affinoid is an algebra A which has form T_n/I, where $T_N = K\langle z_1, \ldots, z_n \rangle$ is the subring of the of all power series $K[[z_1, \ldots, z_n]]$ consisting of the power series $\Sigma_\alpha c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in K[[z_1, \ldots, z_n]]$ satisfying $\lim c_\alpha = 0$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index.

One define the **Gauss norm** on T_n by

$$\|\Sigma_\alpha c_\alpha\| = \max |c_\alpha|.$$

Further $T_n^o := \{f \in T_n | \|f\| \leq 1\}$ and $T_n^{oo} := \{f \in T_n | \|f\| < 1\}$.
Rigid analytic spaces are adic spaces, locally are $\mathrm{Spa}(T_n, T_n^\circ)$.

Perfectoid Spaces

Definition

A ring R is perfectoid if R is a complete Tate ring R which is uniform and there exits a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p | p$ holds in R°, and such that the p-th power Frobenius map

$$\phi : R^\circ / \varpi \to R^\circ / \varpi^p$$

is an isomorphism.

Definition

A perfectoid field is a perfectoid Tate ring R which is a nonarchmedean field. That is is a complete non-archimedean field K of residue characteristic p, equipped with a non-discrete valuation of rank 1, such that the Frobenius map $\theta : \mathcal{O}_K / p \to \mathcal{O}_K / p$ is surjective, where $\mathcal{O}_K \subset K$ is the subring of elements of norm ≤ 1.

Definition

- A perfectoid space is an adic space that may be covered by affinoids of the form $\mathrm{Spa}(A, A^+)$, where A is perfectoid.
- Let A be a perfectoid ring. We define its tilt to be

$$A^\flat := \varprojlim_{x \to x^p} A$$

- A diamond is a pro-étale sheaf $\mathcal{D}$ on Perf such that one can write D = X/R as a quotient of a perfectoid space X of characteristic p by an equivalence relation $R \subset X \times X$ such that R is a perfectoid space with $s, t : R \to X$ proétale.

v-Topology

Definition

In Perfd, $\{f_i : X_i \to Y\}_{i\in I}$ is a cover if and only if for all quasi-compact open subsets $V \subset Y$ there is some finite subset $I_V \subset I$ and quasicompact open $U_i \subset X_i$ for $i \in I_U$ such that $V = \cup_{i\in I_U} f_i(U_i)$.

Definition

An Artin v-stack is a small v-stack X such that the diagonal map $\Delta_X : X \to X \times X$ is representable in locally spatial diamonds, and there is some surjection map $f : U \to X$ from a locally spatial diamond U such that f is separated and cohomologically smooth.

Theorem (Fargues-Scholze , 2021)

The stack Bun_G is a cohomologically smooth Artin v-stack of l-dimension 0.

Mix-Characteristic Shtukas

Theorem

Let $S \in \mathrm{Perf}$. The following sets are naturally identified:

1. Sections of $(S \dot{\times} \mathrm{Spa}\mathbb{Z}_p)^{\diamond} \to S$,
2. Morphisms $S \to \mathrm{Spd}\mathbb{Z}_p$,
3. Untilts $S^{\sharp}$ of S.

Definition

Let S be a perfectoid space in characteristic p. Let $x_1, \ldots, x_m : S \to \mathrm{Spd}\mathbb{Z}_p$ be a collection of morphism; We let $\Gamma_{x_i} : S_i^{\sharp} \to S \dot{\times} \mathrm{Spa}\mathbb{Z}_p$ be the corresponding closed Cartier divisor. A mixed-characteristic shtuka of rank n over S with legs $x_1, \ldots, x_m$ is a rank n vector bundle $\mathcal{E}$ over $S \dot{\times} \mathrm{Spa}\mathbb{Z}_p$ together with an isomorphism

$$\phi_{\mathcal{E}} : \mathrm{Frob}_S^* \mathcal{E}|_{S \dot{\times} \mathrm{Spa}\mathbb{Z}_p \backslash \cup_i \Gamma_{x_i}} \cong \mathcal{E}|_{S \dot{\times} \mathrm{Spa}\mathbb{Z}_p \backslash \cup_i \Gamma_{x_i}}$$

that is meromorphic along $\cup_i \Gamma_{x_i}$.

Theorem (Scholze-Weinstein, 2020)

The following categories are equivalent:

1. Shtukas over $\mathrm{Spa}\, C^\flat$ with one leg at $\phi^{-1}(x_C)$, i.e., vector bundles $\mathcal{E}$ on $\mathcal{Y}_{[o,\infty]}$ together with an isomorphism $\phi_{\mathcal{E}} : (\phi^*\mathcal{E})|_{\mathcal{U}_{[0,\infty)}\backslash\phi^{-1}(x_C)} \cong \mathcal{E}|_{\mathcal{U}_{[0,\infty)}\backslash\phi^{-1}(x_C)}$.
2. Pairs (T, E), where T is a finite free $\mathbb{Z}_P$-module, and $E \subset T \otimes_{\mathbb{Z}_p} B_{dR}$ is a B_{dR}^+ lattice.
3. Quadruples $(\mathcal{F}, \mathcal{F}, \beta, T)$, where $\mathcal{F}$ and $\mathcal{F}'$ are vector bundles on the Fargues-Fontaine curve X_{FF} such that $\mathcal{F}$ is trivial, $\beta : \mathcal{F}|_{X_{FF}\backslash\{\infty\}} \cong \mathcal{F}'|_{X_{FF}\backslash\{\infty\}}$ is an isomorphism, and $T \subset H^0(X_{FF}, \mathcal{F})$ is a $\mathbb{Z}_p$ lattice.
4. Vector bundles $\widetilde{\mathcal{E}}$ on $\mathcal{Y}$ together with an isomorphism $\phi_{\widetilde{\mathcal{E}}} : (\phi^*\widetilde{\mathcal{E}})|_{\mathcal{Y}\backslash\phi^{-1}(x_C)} \cong \widetilde{\mathcal{E}}|_{\mathcal{Y}\backslash\phi^{-1}(x_C)}$.
5. Breuil-Kisin-Fargues modules over $A_{\inf}$, i.e., finite free $A_{\inf}$-modules M together with isomorphism $\phi_M : (\phi^* M)[\frac{1}{\phi(\xi)}] \cong M[\frac{1}{\phi(\xi)}]$.

Local Mix-Characteristic Shtukas

Let k be a discrete algebraically closed field, and $L = W(k)[\frac{1}{p}]$.
Let $(\mathcal{G}, b, \{\mu_i\})$ be a triple consisting of a smooth group scheme $\mathcal{G}$ with reductive generic fiber G and connected special fiber, and element $b \in G(L)$, and a collection $\mu_1, \ldots, \mu_m$ of conjugacy class of cocharacters $G_m \to G_{\overline{\mathbb{Q}}_p}$. For $i = 1, \cdots, m$, let $E_i/\mathbb{Q}_p$ be the field of definition of μ_i, and let $\breve{E}_i = E_i \cdot L$.
For any perfectoid space $S = \mathrm{Spa}(R, R^+)$ over k, a shtuka associated with $(\mathcal{G}, b, \{\mu_i\})$ is a quadruples $(\mathcal{P}, \{S_i^\sharp\}, \phi_{\mathcal{P}}, \iota_r)$, where:

1. $\mathcal{P}$ is a $\mathcal{G}$-torsor on $S \dot{\times} \mathrm{Spa}\mathbb{Z}_p$,
2. $S_i^\sharp$ is a an untilt of $S \to \breve{E}_i$, for $i = 1, \cdots, m$,
3. $\phi_{\mathcal{P}}$ is an isomorphism

$$\phi_{\mathcal{P}} : \mathrm{Frob}_S^* \mathcal{P}|_{S \dot{\times} X \backslash \cup_i \Gamma_{x_i}} \cong \mathcal{P}|_{S \dot{\times} X \backslash \cup_i \Gamma_{x_i}},$$

4. ι_r is an isomorphism

$$\iota_r : \mathcal{P}|_{\mathcal{Y}_{[r,\infty]}(S)} \to G \times \mathcal{Y}_{[r,\infty]}(S)$$

for large enough r, under which $\phi_{\mathfrak{P}}$ gets identified with $b \times \mathrm{Frob}_S$.

By the definition of local shtukas, we can define a moduli functor

$$\begin{aligned} \mathrm{Shtuka}_{\mathcal{G},b,\mu} \quad : \quad & \mathrm{Perf}_k \to \mathrm{Set} \\ & S \to \{(\mathcal{P}, \{S_i^\sharp\}, \phi_{\mathcal{P},\iota_r})\} \end{aligned}$$

Theorem

The moduli space $\mathrm{Shtuka}_{\mathcal{G},b,\mu_\bullet}$ is a locally spatial diamond.

Theorem (Scholze-Weinstein , 2013)

There is a natural isomorphism

$$\mathcal{M}_{\mathbb{X},\breve{Q}_p} \cong \mathrm{Shtuka}_{(GL_n,b,\mu)}$$

as diamonds over $\mathrm{Spf}\breve{\mathbb{Q}}_p$.

Definition

A local Shimura datum is a triple (G, b, μ) consists a reductive group G over $\mathbb{Q}_p$, a conjugacy class μ of minuscule cocharacters $G_m \to G_{\bar{\mathbb{Q}}_p}$, and $b \in B(G, \mu^{-1})$, that is $\nu_b \leq (\mu^{-1})^{\diamond}$ and $\kappa(b) = -\mu^{\natural}$.

There is a étale map

$$\pi_{GM} : \mathrm{Shtuka}_{G,b,\mu,K} \to Gr_{G,\mathrm{Spd}\breve{E},\leq\mu}.$$

By the construction of diamonds, there exists a unique smooth rigid space $\mathcal{M}_{G,b,\mu,K}$ over $\breve{E}$ with an étale map towards $\mathscr{F}_{G,\mu,\breve{E}}$.

Definition

The local Shimura variety associated with (G, b, μ) is the tower

$$(\mathcal{M}_{G,b,\mu,K})_{K\subset G(\mathbb{Q}_p)}$$

of smooth rigid space over $\breve{E}$, together with its étale period map to $\mathscr{F}_{G,\mu,\breve{E}}$

Condensed Mathematics

Definition

1. We define $*_{proét}$ as the proétale site of a point, which is the category of profinite sets S, with finite jointly surjective families of maps as covers.

A condensed set /group/ring, ...is a functor

$$T : \{\textbf{profinte sets}\}^{op} \to \{\textbf{sets/rings/groups/ ...}\}$$
$$S \mapsto T(S)$$

satisfies $T(\emptyset) = *$ and satisfying the following condition

1. For any profinte set S_1, S_2, the natural map
$$T(S_1 \cup S_2) \to T(S_1) \times T(S_2)$$
is a bijection.
2. For any surjection $S' \to S$ of profinte sets with the fibre product $S' \times_S S'$ and its projection p_1, p_2 to S', the map
$$T(S) \to \{x \in T(S) | p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$
is a bijection.

Solid Abelian Groups

Definition

1. For a profinite set $S = \varprojlim_i S_i$, we define the condensed abelian group

$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim_i \mathbb{Z}[S_i].$$

   There is a natural map $S = \varprojlim_i S_i \to \mathbb{Z}[S]^{\blacksquare}$, inducing a map $\mathbb{Z}[S] \to \mathbb{Z}[S]^{\blacksquare}$.

2. A solid abelian group is a condensed abelian group A such that for all profinite set S and all maps $f : S \to A$, there is a unique map $\widetilde{f} : \mathbb{Z}[S]^{\blacksquare} \to A$ extending f.

3. A complex $C \in D(\mathrm{Cond}(\mathrm{Ab}))$ of condensed abelian groups is solid if for all profinite sets, the natural map

$$R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) \to R\Gamma(S, C) = R\mathrm{Hom}(\mathbb{Z}[S], C)$$

   is an isomorphism.

- Consider the functor of fixed points $\mathrm{Solid}_G \to \mathrm{Solid}$ defined by $\mathcal{M} \to \mathcal{M}^G$,which is right adjoint to the trivial action functor $\mathrm{Solid} \to \mathrm{Solid}_G$. Let $\mathcal{C} \to R\Gamma(G, \mathcal{C})$ be its derived functor $D(\mathrm{Solid}_G) \to D(\mathrm{Solid})$.
- If M is an abelian group which is separated and complete for a linear topology, and G act continuously on M, then

$$H^i(R\Gamma(G, M)) \cong \underline{H^i_{cts}(G, M)}.$$

- Let G be a profinite group, write

$$D(\mathrm{Solid}_G) \to D(\mathrm{Solid})$$
$$\mathcal{C} \to \mathcal{C}^{hG}$$

for the functor $R\Gamma(G, C)$.

Proétale Cohomology of Rigid Analytic Spaces

Definition

Let X be a rigid-analytic space over K, the object of the pro-étale site $X_{\text{proét}}$ are formal limits $U = \varprojlim U_i$, where i runs over a filtered index set and U_i are rigid analytic spaces which are étale over X.

Let $\widehat{\mathcal{O}}^+ = \varprojlim \mathcal{O}^+/p^n$.

Proposition

We have an isomorphism in $D(\mathrm{Cond}(\mathrm{Ab}))$:

$$R\Gamma_{\mathrm{cond}}(X_{\text{proét}}, \widehat{\mathcal{O}}^+) \cong R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}}).$$

Let $Y \to X$ be a pro-étale G-torsor. There is an isomorphism in $D(\mathrm{Solid})$:

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}}) \cong R\Gamma(Y_{\text{proét}}, \widehat{\mathcal{O}}^+_{\mathrm{cond}})^{hG}$$

The Proétale Cohomology of of LT_K and $\mathcal{H}$

Theorem

- There is a morphism of differential graded solid W-algebras, which is equivariant for the action of $\mathbb{G}_n$:

$$A[\epsilon] \to R\Gamma(LT_{K.\text{proét}}, \mathcal{O}^+_{\text{cond}}).$$

- There is a morphism of differential graded solid $\mathbb{Z}_p$-algebras, which is equivariant for the action of $\mathrm{GL}_n(\mathbb{Z}_p)$:

$$\mathbb{Z}_p[\epsilon] \to R\Gamma(\mathcal{H}_{\text{proét}}, \widehat{\mathcal{O}}^+_{\text{cond}}).$$

Let A be the cofiber of either of the above morphism. Then $H^i(A) = 0$ for $i \leq 0$, and all $H^i(A)$ for $i \geq 1$ are annihilated by a single power of p.

Thanks for Listening !