

BAYESIAN LEARNING - LECTURE 7

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LECTURE OVERVIEW

- ▶ Monte Carlo simulation and random number generation
- ▶ Gibbs sampling
- ▶ Data augmentation
 - ▶ Mixture models
 - ▶ Probit regression
- ▶ Regularized regression revisited

Monte Carlo Sampling

- ▶ If $\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(N)}$ is an **iid sequence** from a distribution $p(\theta)$, then

$$\frac{1}{N} \sum_{t=1}^N \theta^{(t)} \rightarrow E(\theta)$$
$$\frac{1}{N} \sum_{t=1}^N g(\theta^{(t)}) \rightarrow E[g(\theta)]$$

where $g(\theta)$ is some well-behaved function.

- ▶ Easy to compute **tail probabilities** $\Pr(\theta \leq c)$ by letting

$$g(\theta) = I(\theta \leq c)$$

and

$$\frac{1}{N} \sum_{t=1}^N g(\theta^{(t)}) = \frac{\# \theta\text{-draws smaller than } c}{N}.$$

DIRECT SAMPLING BY THE INVERSE CDF METHOD

- ▶ How to **simulate** from a distribution?
- ▶ Let $f(x)$ be the density function of a stochastic variable. CDF: $F(x)$.
Inverse CDF method:
 1. Generate u from the uniform distribution on $[0, 1]$.
 2. Compute $x = F^{-1}(u)$.
- ▶ Example 1: **Exponential distribution:**

$$u = F(x) = 1 - \exp(-\lambda x)$$

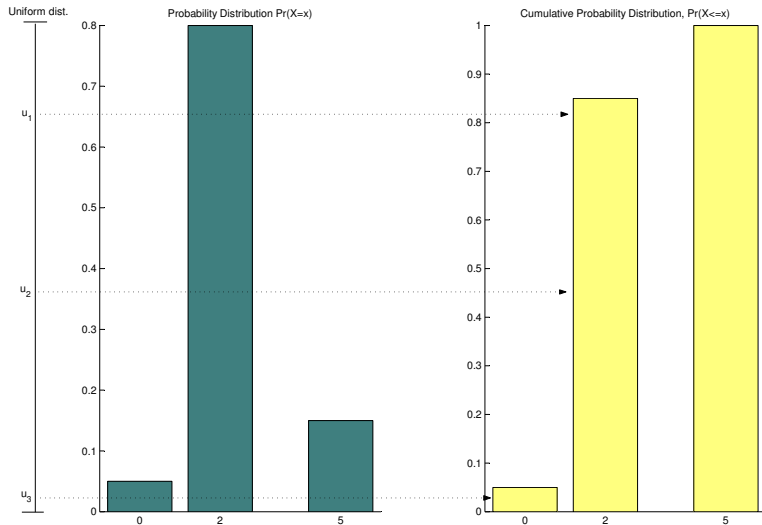
Inverting gives

$$x = -\ln(1 - u) / \lambda$$

But $1 - u$ is also uniformly distributed on $[0, 1]$. So:

- ▶ If $x = -(\ln u) / \lambda$ where $u \sim \text{Unif}(0, 1)$, then $x \sim \text{Expon}(\lambda)$.

INVERSE CDF METHOD, DISCRETE CASE



DIRECT SAMPLING BY THE INVERSE CDF METHOD

- ▶ Example 2: **Cauchy distribution**:

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$
$$u = F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x)$$

Inverting ...

$$x = \tan[\pi(u - 1/2)].$$

- ▶ We can also use relations between distribution to sample from distributions.
- ▶ Cauchy-example, cont. If y and z are independent $N(0, 1)$ variables, then $z = \frac{y}{z} \sim \text{Cauchy}$.
- ▶ Example: **Chi-square**. If $x_1, \dots, x_v \stackrel{iid}{\sim} N(0, 1)$, then $y = \sum_{i=1}^v x_i^2 \sim \chi_v^2$.

GIBBS SAMPLING

- ▶ Easily implemented methods for **sampling from multivariate distributions**, $p(\theta_1, \dots, \theta_k)$.
- ▶ Requirements: Easily sampled **full conditional posteriors**:
 - ▶ $p(\theta_1|\theta_2, \theta_3, \dots, \theta_k)$
 - ▶ $p(\theta_2|\theta_1, \theta_3, \dots, \theta_k)$
 - ▶ \vdots
 - ▶ $p(\theta_k|\theta_1, \theta_2, \dots, \theta_{k-1})$
- ▶ Started out in the early 80's in the image analysis literature.
- ▶ Gibbs sampling is a **special case of Metropolis-Hastings** (see Lecture 8)
- ▶ Metropolis-Hastings is a Markov Chain Monte Carlo (MCMC) algorithm.

THE GIBBS SAMPLING ALGORITHM

- A:** Choose initial values $\theta_2^{(0)}, \theta_3^{(0)}, \dots, \theta_k^{(0)}$.
- B:** B_1 Draw $\theta_1^{(1)}$ from $p(\theta_1 | \theta_2^{(0)}, \theta_3^{(0)}, \dots, \theta_k^{(0)})$
 B_2 Draw $\theta_2^{(1)}$ from $p(\theta_2 | \theta_1^{(1)}, \theta_3^{(0)}, \dots, \theta_k^{(0)})$
 \vdots
 B_n Draw $\theta_k^{(1)}$ from $p(\theta_k | \theta_1^{(1)}, \theta_2^{(1)}, \dots, \theta_{k-1}^{(1)})$
- C:** Repeat Step B N times.

GIBBS SAMPLING, CONT.

- ▶ The Gibbs draws $\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(N)}$ are **dependent** (autocorrelated), but **arithmetic means converge to expected values**

$$\frac{1}{N} \sum_{t=1}^N \theta_j^{(t)} \rightarrow E(\theta_j)$$

$$\frac{1}{N} \sum_{t=1}^N g(\theta^{(t)}) \rightarrow E[g(\theta)]$$

- ▶ $\theta^{(1)}, \dots, \theta^{(N)}$ **converges in distribution** to the target $p(\theta)$.
- ▶ $\theta_j^{(1)}, \dots, \theta_j^{(N)}$ converge to the marginal distribution of θ_j , $p(\theta_j)$.
- ▶ **Dependent** draws \rightarrow **less efficient** than iid sampling.
- ▶ Compare sampling from:
 - ▶ $x_t \stackrel{iid}{\sim} N(0, \sigma^2)$
 - ▶ $x_t = 0.9x_{t-1} + \varepsilon_t$ with $\varepsilon_t \stackrel{iid}{\sim} N(0, \sigma^2)$.

GIBBS SAMPLING MULTIVARIATE NORMAL

- ▶ Bivariate normal:

- ▶ Joint distribution

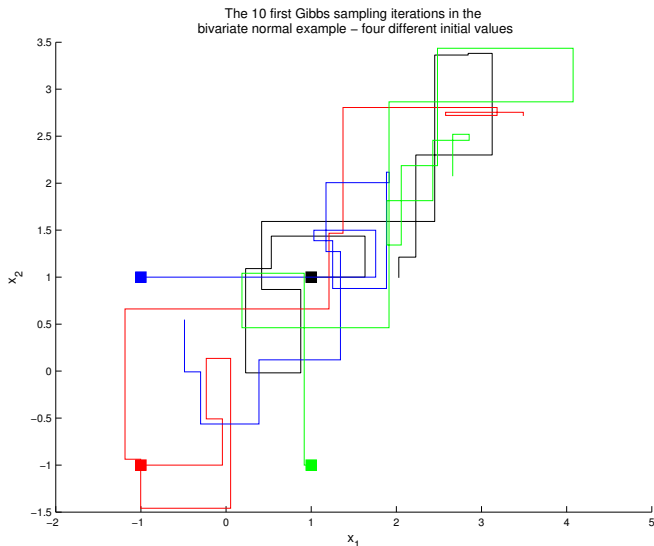
$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \sim \mathcal{N}_2 \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right]$$

- ▶ Full conditional posteriors:

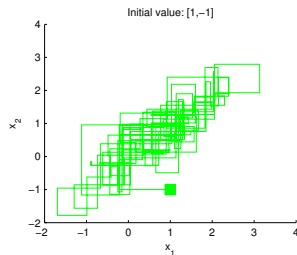
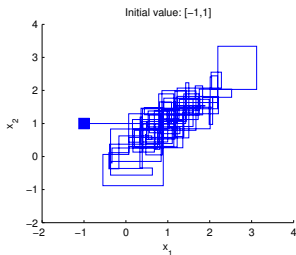
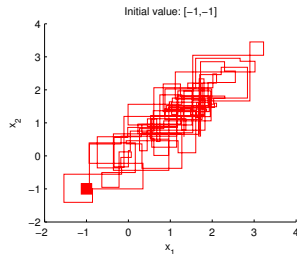
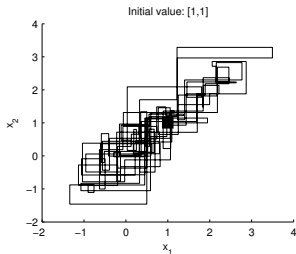
$$\theta_1 | \theta_2 \sim \mathcal{N}[\mu_1 + \rho(\theta_2 - \mu_2), 1 - \rho^2]$$

$$\theta_2 | \theta_1 \sim \mathcal{N}[\mu_2 + \rho(\theta_1 - \mu_1), 1 - \rho^2]$$

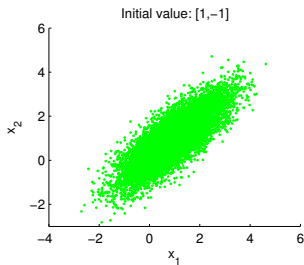
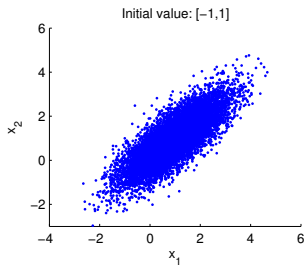
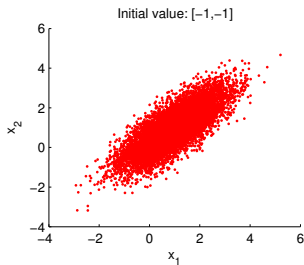
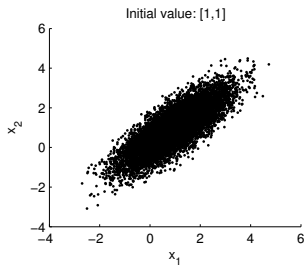
GIBBS SAMPLING - BIVARIATE NORMAL



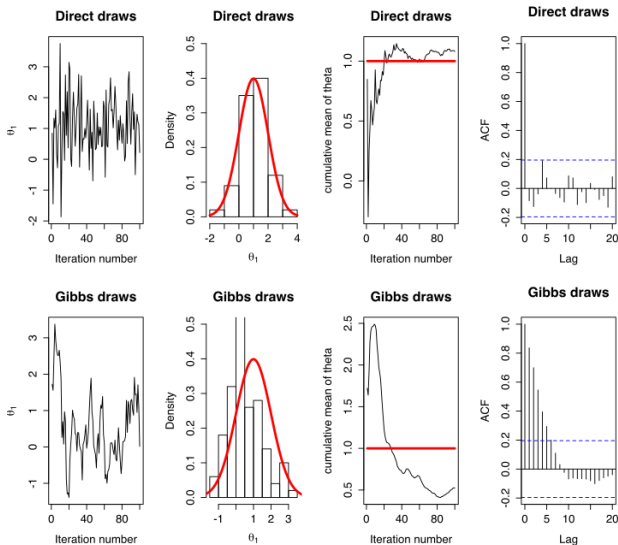
GIBBS SAMPLING - BIVARIATE NORMAL



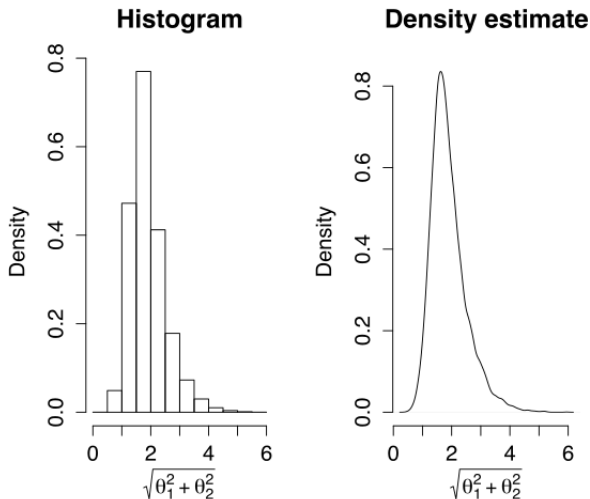
GIBBS SAMPLING - BIVARIATE NORMAL



DIRECT SAMPLING VS GIBBS SAMPLING



ESTIMATING THE DENSITY OF $g(\theta_1, \theta_2) = \sqrt{\theta_1^2 + \theta_2^2}$

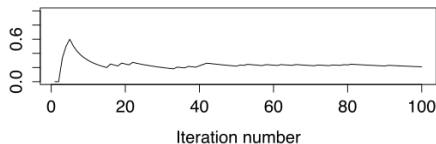


ESTIMATING $Pr(\theta_1 > 0, \theta_2 > 0)$

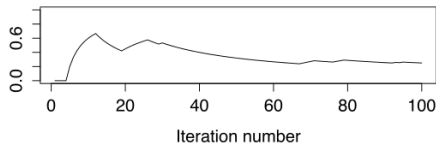
- We can estimate a joint probability by counting:

$$Pr(\theta_1 > 0, \theta_2 > 0) \approx N^{-1} \sum_{i=1}^N 1(\theta_1^{(i)} > 0, \theta_2^{(i)} > 0)$$

Direct draws



Gibbs draws



GIBBS SAMPLING FOR NORMAL MODEL WITH NON-CONJUGATE PRIOR

- ▶ Normal model with semi-conjugate prior

$$\begin{aligned}\mu &\sim N(\mu_0, \tau_0^2) \\ \sigma^2 &\sim \text{Inv} - \chi^2(\nu_0, \sigma_0^2)\end{aligned}$$

- ▶ Conditional posteriors

$$\begin{aligned}\mu | \sigma^2, x &\sim N(\mu_n, \tau_n^2) \\ \sigma^2 | \mu, x &\sim \text{Inv} - \chi^2 \left(\nu_n, \frac{\nu_0 \sigma_0^2 + \sum_{i=1}^n (x_i - \mu)^2}{n + \nu_0} \right)\end{aligned}$$

with μ_n and τ_n^2 defined the same as when σ^2 is known (Lecture 1).

GIBBS SAMPLING FOR AR PROCESSES

► AR(p) process

$$x_t = \mu + \phi_1(x_{t-1} - \mu) + \dots + \phi_p(x_{t-p} - \mu) + \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} N(0, \sigma^2).$$

► Let $\phi = (\phi_1, \dots, \phi_p)'$.

► Prior:

- $\mu \sim \text{Normal}$
- $\phi \sim \text{Multivariate Normal}$
- $\sigma^2 \sim \text{Scaled Inverse } \chi^2$.

► The **posterior** can be simulated by Gibbs sampling:

- $\mu | \phi, \sigma^2, x \sim \text{Normal}$
- $\phi | \mu, \sigma^2, x \sim \text{Multivariate Normal}$
- $\sigma^2 | \mu, \phi, x \sim \text{Scaled Inverse } \chi^2$

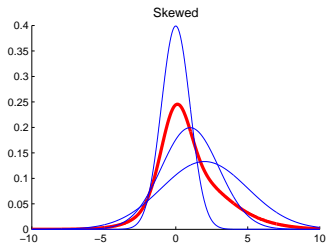
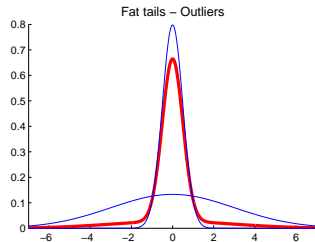
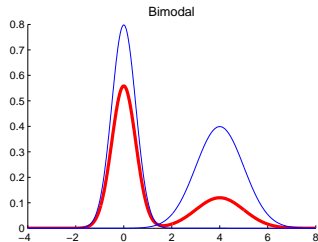
DATA AUGMENTATION - MIXTURE DISTRIBUTIONS

- ▶ Let $\phi(x|\mu, \sigma^2)$ denotes the **PDF** of a **normal** variable $x \sim N(\mu, \sigma^2)$.
- ▶ **Two-component mixture of normals** [MN(2)]

$$p(x) = \pi \cdot \phi(x|\mu_1, \sigma_1^2) + (1 - \pi) \cdot \phi(x|\mu_2, \sigma_2^2)$$

- ▶ **Simulate** from a MN(2):
 - ▶ Simulate an indicator $I \in \{1, 2\}$: $I \sim \text{Bern}(\pi)$.
 - ▶ If $I = 1$, simulate x from $N(\mu_1, \sigma_1^2)$
 - ▶ If $I = 2$, simulate x from $N(\mu_2, \sigma_2^2)$.

ILLUSTRATION OF MIXTURE DISTRIBUTIONS



MIXTURE DISTRIBUTIONS, CONT.

- ▶ Not easy to estimate directly - the likelihood is a product of sums.
- ▶ **Assume** that we knew which of the two densities each observation came from.

$$I_i = \begin{cases} 1 & \text{if } x_i \text{ came from Density 1} \\ 2 & \text{if } x_i \text{ came from Density 2} \end{cases}.$$

- ▶ Armed with knowledge of I_1, \dots, I_n it is now easy to estimate $\pi, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ by separating the sample according to the I 's.
- ▶ But we do **not** know I_1, \dots, I_n !

GIBBS SAMPLING FOR MIXTURE DISTRIBUTIONS

- ▶ Prior: $\pi \sim \text{Beta}(\alpha_1, \alpha_2)$. Conjugate prior for (μ_j, σ_j^2) , see Lecture 5.
- ▶ Define: $n_1 = \sum_{i=1}^n (I_i = 1)$ and $n_2 = n - n_1$.
- ▶ **Gibbs sampling:**
 - ▶ $\pi \mid \mathbf{l}, \mathbf{x} \sim \text{Beta}(\alpha_1 + n_1, \alpha_2 + n_2)$
 - ▶ $\sigma_1^2 \mid \mathbf{l}, \mathbf{x} \sim \text{Inv-}\chi^2(\nu_{n_1}, \sigma_{n_1}^2)$ and $\mu_1 \mid \mathbf{l}, \sigma_1^2, \mathbf{x} \sim N\left(\mu_{n_1}, \frac{\sigma_1^2}{\kappa_{n_1}}\right)$
 - ▶ $\sigma_2^2 \mid \mathbf{l}, \mathbf{x} \sim \text{Inv-}\chi^2(\nu_{n_2}, \sigma_{n_2}^2)$ and $\mu_2 \mid \mathbf{l}, \sigma_2^2, \mathbf{x} \sim N\left(\mu_{n_2}, \frac{\sigma_2^2}{\kappa_{n_2}}\right)$
 - ▶ $I_i \mid \pi, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \mathbf{x} \sim \text{Bern}(\theta_i), i = 1, \dots, n,$

$$\theta_i = \frac{(1 - \pi)\phi(x_i; \mu_2, \sigma_2^2)}{\pi\phi(x_i; \mu_1, \sigma_1^2) + (1 - \pi)\phi(x_i; \mu_2, \sigma_2^2)}.$$

GIBBS SAMPLING FOR MIXTURE DISTRIBUTIONS

- ▶ **K -component mixture of normals**

$$p(\mathbf{x}) = \sum_{k=1}^K \pi_k \phi(\mathbf{x}; \mu_k, \sigma_k^2),$$

where $\sum_{k=1}^K \pi_k = 1$.

- ▶ **Multi-class indicators:** $I_i = k$ if observation i comes from density k .

- ▶ **Gibbs sampling** with

- ▶ $(\pi_1, \dots, \pi_K) \mid \mathbf{I}, \mathbf{x} \sim \text{Dirichlet}(\alpha_1 + n_1, \alpha_2 + n_2, \dots, \alpha_K + n_K)$
- ▶ $\sigma_k^2 \mid \mathbf{I}, \mathbf{x} \sim \text{Inv-}\chi^2$ and $\mu_k \mid \mathbf{I}, \sigma_k^2, \mathbf{x} \sim \text{Normal}$, for $k = 1, \dots, K$,
- ▶ $I_i \mid \pi, \mu, \sigma^2, \mathbf{x} \sim \text{Multinomial}(\theta_{i1}, \dots, \theta_{iK})$, for $i = 1, \dots, n$,

$$\theta_{ij} = \frac{\pi_j \phi(\mathbf{x}_i; \mu_j, \sigma_j^2)}{\sum_{r=1}^K \pi_r \phi(\mathbf{x}_i; \mu_r, \sigma_r^2)}.$$

- ▶ Gibbs sampling is very powerful for **missing data** problems.
Semi-supervised learning.

DATA AUGMENTATION - PROBIT REGRESSION

- **Probit** model:

$$\Pr(y_i = 1 \mid x_i) = \Phi(x_i^T \beta)$$

- **Random utility formulation** of the probit:

$$\begin{aligned} u_i &\sim N(x_i^T \beta, 1) \\ y_i &= \begin{cases} 1 & \text{if } u_i > 0 \\ 0 & \text{if } u_i \leq 0 \end{cases} . \end{aligned}$$

- Check: $\Pr(y_i = 1 \mid x_i) = \Pr(u_i > 0) = 1 - \Pr(u_i \leq 0) = 1 - \Pr(u_i - x_i^T \beta < -x_i^T \beta) = 1 - \Phi(-x_i^T \beta) = \Phi(x_i^T \beta)$.
- If $u = (u_1, \dots, u_n)$ were observed, then β could be analyzed by traditional linear regression. But, u is **not observed**. Gibbs sampling to the rescue!

GIBBS SAMPLING FOR THE PROBIT REGRESSION

- ▶ Simulate from joint posterior $p(u, \beta|y)$ iterating between the **full conditional posteriors**:
 - ▶ $p(\beta|u, y)$, which is multivariate normal (this is just a linear regression)
 - ▶ $p(u_i|\beta, y)$, $i = 1, \dots, n$.
- ▶ The full conditional posterior distribution of u_i is:

$$\begin{aligned} p(u_i|\beta, y) &\propto p(y_i|\beta, u_i)p(u_i|\beta) \\ &= \begin{cases} N(u_i|x_i'\beta, 1) & \text{truncated to } u_i \in (-\infty, 0] \text{ if } y_i = 0 \\ N(u_i|x_i'\beta, 1) & \text{truncated to } u_i \in (0, \infty) \text{ if } y_i = 1 \end{cases} \end{aligned}$$

- ▶ Collect the β -draws. A histogram of these draws approximates $p(\beta|y) = \int p(u, \beta|y)du$.

REGULARIZED REGRESSION WITH GIBBS

- ▶ Recap: The joint posterior of β , σ^2 and λ is

$$\beta|\sigma^2, \lambda, \mathbf{y}, \mathbf{X} \sim N(\mu_n, \Omega_n^{-1})$$

$$\sigma^2|\lambda, \mathbf{y}, \mathbf{X} \sim \text{Inv} - \chi^2(\nu_n, \sigma_n^2)$$

$$p(\lambda|\mathbf{y}, \mathbf{X}) \propto \sqrt{\frac{|\Omega_0|}{|\mathbf{X}'\mathbf{X} + \Omega_0|}} \left(\frac{\nu_n \sigma_n^2}{2}\right)^{-\nu_n/2} \cdot p(\lambda)$$

where $p(\lambda)$ is the $\text{Inv} - \chi^2$ prior for λ .

- ▶ This is the **conditional-marginal decomposition**

$$p(\beta, \sigma^2, \lambda|\mathbf{y}, \mathbf{X}) = p(\beta|\sigma^2, \lambda, \mathbf{y}, \mathbf{X})p(\sigma^2|\lambda, \mathbf{y}, \mathbf{X})p(\lambda|\mathbf{y}, \mathbf{X})$$

- ▶ **Gibbs sampling** can instead be used:

- ▶ Sample $\beta|\sigma^2, \lambda, \mathbf{y}, \mathbf{X}$ from Normal
- ▶ Sample $\sigma^2|\beta, \lambda, \mathbf{y}, \mathbf{X}$ from $\text{Inv} - \chi^2$
- ▶ Sample $\lambda|\beta, \sigma^2, \mathbf{y}, \mathbf{X}$ from $\text{Inv} - \chi^2$

- ▶ Note that λ is now **easy** to simulate **once we condition** on β and σ^2 .

IMPROVING THE EFFICIENCY OF THE GIBBS SAMPLER

- ▶ *Efficient blocking*. Correlated parameters should ideally be included in the same updating block.
- ▶ *Reparametrization*. Convergence can improve dramatically in alternative parametrizations.
- ▶ *Data augmentation*. Bring in latent (unobserved) variables that make the full conditional posteriors more easily sampled (Probit, Mixture models etc). Downside: Typically increases the autocorrelation between draws.
- ▶ *Parameter expansion*. Introducing (non-sense) parameters in the model may break the dependence between the original parameters (Example probit).