Bayesian Learning - Lecture 3

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Lecture overview

- **Multiparameter** models
- **Marginalization**
- **Normal model with unknown variance**
- Bayesian analysis of **multinomial data**
- Bayesian analysis of **multivariate normal data**

MARGINALIZATION

- Models with multiple parameters $\theta_1, \theta_2, ...$
- Examples: $x_i \overset{iid}{\sim} N(\theta, \sigma^2)$; multiple regression ...
- **Joint posterior distribution**

$$p(\theta_1, \theta_2, ..., \theta_p | y) \propto p(y | \theta_1, \theta_2, ..., \theta_p) p(\theta_1, \theta_2, ..., \theta_p).$$

... or in vector form:

$$p(\theta | y) \propto p(y | \theta) p(\theta).$$

- Complicated to graph the joint posterior.
- Some of the parameters may not be of direct interest (**nuisance**).
- Integrate out (**marginalize**) all nuisance parameters.
- Example: $\theta = (\theta_1, \theta_2)'$, θ_2 is a nuisance. **Marginal posterior** of θ_1

$$p(\theta_1 | y) = \int p(\theta_1, \theta_2 | y) d\theta_2 = \int p(\theta_1 | \theta_2, y) p(\theta_2 | y) d\theta_2.$$

Normal model with unknown variance - Uniform prior

- **Model**

$$x_1, ..., x_n \stackrel{iid}{\sim} N(\theta, \sigma^2)$$

- **Prior**

$$p(\theta, \sigma^2) \propto (\sigma^2)^{-1}$$

- **Posterior**

$$\theta|\sigma^2, \mathbf{x} \sim N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$
$$\sigma^2|\mathbf{x} \sim \mathrm{Inv} - \chi^2(n-1, s^2),$$

where

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

is the usual sample variance.

Normal model with unknown variance - Uniform prior

- **Simulating** the posterior of the normal model with non-informative prior:
  1. Draw $X \sim \chi^2(n-1)$
  2. Compute $\sigma^2 = \frac{(n-1)s^2}{X}$ (this a draw from $\text{Inv-}\chi^2(n-1, s^2)$)
  3. Draw a θ from $N\left(\bar{x}, \frac{\sigma^2}{n}\right)$ conditional on the previous draw σ^2
  4. Repeat step 1-3 many times.
- The sampling is implemented in the R program `NormalNonInfoPrior.R`
- We may derive the marginal posterior analytically as

$$\theta|\mathbf{x} \sim t_{n-1}\left(\bar{x}, \frac{s^2}{n}\right).$$

Multinomial model with Dirichlet prior

- *Data*: $y = (y_1, ...y_K)$, where y_k counts the number of observations in the kth category. $\sum_{k=1}^{K} y_k = n$. Example: brand choices.
- **Multinomial model**:

$$p(y|\theta) \propto \prod_{k=1}^{K} \theta_k^{y_k}, \text{ where } \sum_{k=1}^{K} \theta_k = 1.$$

- ***Conjugate prior***: $\mathrm{Dirichlet}(\alpha_1, ..., \alpha_K)$

$$p(\theta) \propto \prod_{k=1}^{K} \theta_k^{\alpha_k - 1}.$$

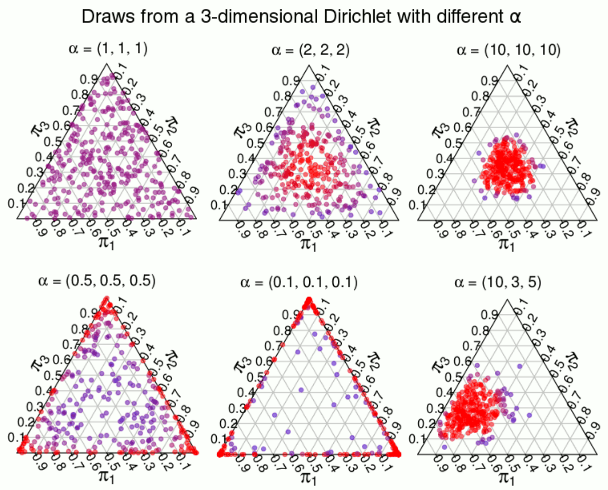
- Moments of $\theta = (\theta_1, ..., \theta_K)' \sim \mathit{Dirichlet}(\alpha_1, ..., \alpha_K)$

$$\mathrm{E}(\theta_k) = \frac{\alpha_k}{\sum_{j=1}^{K} \alpha_j}$$

$$\mathrm{V}(\theta_k) = \frac{\mathrm{E}(\theta_k)\left[1 - \mathrm{E}(\theta_k)\right]}{1 + \sum_{j=1}^{K} \alpha_j}$$

- Note that $\sum_{k=1}^{K} \alpha_k$ is a precision parameter.

Dirichlet distribution



Multinomial model with Dirichlet prior

- 'Non-informative': $\alpha_1 = ... = \alpha_K = 1$ (uniform and proper).
- **Simulating** from the Dirichlet distribution:
  - Generate $x_1 \sim Gamma(\alpha_1, 1), ..., x_K \sim Gamma(\alpha_K, 1)$.
  - Compute $y_k = x_k / (\sum_{j=1}^{K} x_j)$.
  - $y = (y_1, ..., y_K)$ is a draw from the $\mathrm{Dirichlet}(\alpha_1, ..., \alpha_K)$ distribution.
- ***Prior-to-Posterior updating***:

$$\begin{aligned} Model: \quad & y = (y_1, ...y_K) \sim \mathrm{Multin}(n; \theta_1, ..., \theta_K) \\ Prior: \quad & \theta = (\theta_1, ..., \theta_K) \sim \mathrm{Dirichlet}(\alpha_1, ..., \alpha_K) \\ Posterior: \quad & \theta | y \sim \mathrm{Dirichlet}(\alpha_1 + y_1, ..., \alpha_K + y_K). \end{aligned}$$

Example: market shares

- A recent survey among consumer smartphones owners in the U.S. showed that among the 513 respondents:
  - 180 owned an iPhone
  - 230 owned an Android phone
  - 62 owned a Blackberry phone
  - 41 owned some other mobile phone.
- Previous survey: iPhone 30%, Android 30%, Blackberry 20% and Other 20%.
- Pr(Android has largest share | Data)
- Prior: $\alpha_1 = 15, \alpha_2 = 15, \alpha_3 = 10$ and $\alpha_4 = 10$ (prior info is equivalent to a survey with only 50 respondents)
- Posterior: $(\theta_1, \theta_2, \theta_3, \theta_4)|\mathbf{y} \sim \text{Dirichlet}(195, 245, 72, 51)$

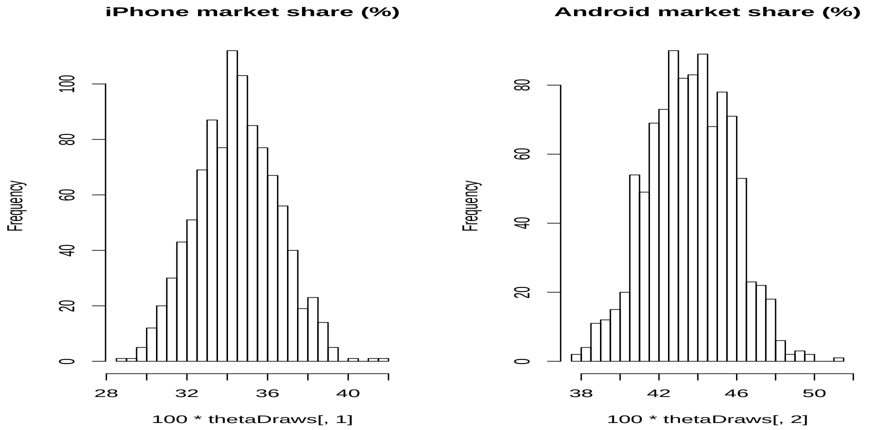
R CODE FOR MARKET SHARE EXAMPLE

```r
# Setting up data and prior
y <- c(180,230,62,41) # The cell phone survey data (K=4)
alpha <- c(15,15,10,10) # Dirichlet prior hyperparameters
nIter <- 1000 # Number of posterior draws

# Defining a function that simulates from a Dirichlet distribution
SimDirichlet <- function(nIter, param){
  nCat <- length(param)
  thetaDraws <- as.data.frame(matrix(NA, nIter, nCat)) # Storage.
  for (j in 1:nCat){
    thetaDraws[,j] <- rgamma(nIter,param[j],1)
  }
  for (i in 1:nIter){
    thetaDraws[i,] = thetaDraws[i,]/sum(thetaDraws[i,])
  }
  return(thetaDraws)
}

# Posterior sampling from Dirichlet posterior
thetaDraws <- SimDirichlet(nIter,y + alpha)
```

R CODE FOR MARKET SHARE EXAMPLE, CONT

```r
# Posterior mean and standard deviation of Androids share (in %)
message(mean(100*thetaDraws[,2]))

## 43.6037640347044

message(sd(100*thetaDraws[,2]))

## 2.12799854134713

# Computing the posterior probability that Android is the largest
PrAndroidLargest <- sum(thetaDraws[,2]>apply(thetaDraws[,c(1,3,4)],1,max))/nIter
message(paste('Pr(Android has the largest market share) = ', PrAndroidLargest))

## Pr(Android has the largest market share) =  0.993
```

R code for market share example, cont



Multivariate normal - known Σ

- **Model**

$$y_1, ..., y_n \overset{iid}{\sim} N_p(\mu, \Sigma)$$

where Σ is a known covariance matrix.

- **Density**

$$p(y|\mu, \Sigma) = |2\pi\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right)$$

- **Likelihood**

$$\begin{aligned} p(y_1, ..., y_n|\mu, \Sigma) &\propto |\Sigma|^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)'\Sigma^{-1}(y_i-\mu)\right) \\ &= |\Sigma|^{-n/2} \exp\left(-\frac{1}{2} tr\Sigma^{-1}S_\mu\right) \end{aligned}$$

where $S_\mu = \sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)'$.

Multivariate normal - known Σ

- **Prior**

$$\mu \sim N_p(\mu_0, \Lambda_0)$$

- **Posterior**

$$\mu | y \sim N(\mu_n, \Lambda_n)$$

where

$$\begin{aligned} \mu_n &= (\Lambda_0^{-1} + n\Sigma^{-1})^{-1}(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y}) \\ \Lambda_n^{-1} &= \Lambda_0^{-1} + n\Sigma^{-1} \end{aligned}$$

- Note how the posterior mean is (matrix) weighted average of prior and data information.
- **Noninformative prior**: let the precision go to zero: $\Lambda_0^{-1} \to 0$.