Contents

1. Outline

My study of mathematics thus far brings me to the following category theory 101 view: 'a thing is profitably, if not completely, determined by its setting'. The most famous example of this is the Yoneda embedding: a (locally small) category is perfectly encoded by assigning to each object the set of arrows to that object.

Not every object of study is given to us as an object in a god-given category. When choosing an analysis of an object, a category of discourse suggests itself. In this relative point of view, the conceit is: we should take that category seriously. For my oral exam I take convergence seriously as a means of studying compact Hausdorff spaces.

In conversation with my advisor John Terilla, the Stone-Cech compactification came up as case study. Stated categorically, the result is very concise: the forgetful functor from compact Hausdorff spaces into spaces has a left adjoint. Unpacked, the content is the familiar statement: any map f from X to a compact Hausdorff space K factors uniquely as $X \xrightarrow{\widehat{\mathrm{id}}} \beta X \xrightarrow{\widehat{f}} K$ e.g. we get commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\widehat{\mathrm{id}}"'] & K \\ \beta X \arrow[ur, "\exists! \widehat{f}"'] & \end{tikzcd}$$

Proofs of this fact are easy to find. A general one given these days is to map $X \to [0,1]^{\mathsf{Top}(X,[0,1])}$ by the evaluation $x \mapsto (fx)_{f \in \mathsf{Top}(X,[0,1])}$ and define βX to be the closure of the image. Tychonoff says that βX is compact. It is also Hausdorff. Then one does some work to verify that this βX has the universal property.

In the presence some separation properties there are other constructions. For completely regular spaces, the Stone-Cech compactification of X can be realized as the maximal ideals of $C_b(X, \mathbb{R})$.

John challenged me to identify the Stone-Cech compactification as a categorical construction intrinsically in terms of the underlying spaces, and in particular, do so without using products of the interval. I was not completely successful. In the effort, I rediscovered some interesting features of Stone Duality which fit nicely into the overarching theme of studying a thing in terms of the categories in which it resides.

Studying compactifications by way of convergence is suggested by the generalized Bolzano-Weierstrass theorem: 'A space is compact iff every maximal filter converges', and the following version of the Hausdorff property 'A space is Hausdorff iff a proper filter converges to at most one point.'

From this perspective, I found a characterization of the Zariski topology of a commutative (semi)ring $\mathcal{L}$ in terms of a canonical retraction. Corollary to this, is the fact that $\operatorname{spec} \mathcal{L}$ is compact. Taken together, these observations indicate a persistent theme in arguments for the Stone-Cech compactification. Generally, they proceed as follows:

(i) Find a functor $\mathcal{L}\colon$ Spaces $\to$ (Semi)Rings, such that $\star \cong \operatorname{spec}\mathcal{L}\star$.

(ii) Note that $\operatorname{spec}\mathcal{L}x\colon \star \to \operatorname{spec}\mathcal{L}X$ where $x \in X$ is viewed as the map $\star \to X$. So we have a map

$$\mathrm{eval}\colon X \to \operatorname{spec}\mathcal{L}X \qquad \text{via} \qquad x \mapsto \operatorname{spec}\mathcal{L}x(\star)$$

and by construction notice $\operatorname{spec}\mathcal{L}f\big(\operatorname{spec}\mathcal{L}x(\star)\big) = \operatorname{spec}\mathcal{L}fx(\star)$ so:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{eval}"'] & Y \arrow[d, "\mathrm{eval}"] \\ \operatorname{spec}\mathcal{L}X \arrow[r, "\operatorname{spec}\mathcal{L}f"'] & \operatorname{spec}\mathcal{L}Y \end{tikzcd}$$

commutes.

(iii) Then the argument resolves to questions about eval. Specifically, one proves eval is continuous for all spaces and an embedding for compact Hausdorff spaces. Typically, eval maps into the maximal spectrum by some version of a principal filter.

This is the roadmap suggested by thinking categorically about Stone-Cech by way of convergence. In my Oral exam, I'd like to walk through the process of extracting this roadmap, characterize the Zariski topology, and present an application drawn from: either the Wallman compactification of a normal space, where $\mathcal{L}X := \{C \text{ closed }\}$, or the completely regular case, where $\mathcal{L}X := C_b(X, \mathbb{R})$.

2. Points: Spatial to Algebraic

Throughout we'll need the flexibility to talk about open sets and closed sets in a topology. As such, we introduce generic notation: $\mathcal{C}$ for closed sets in a space and $\mathcal{T}$ for the open sets. Parsimony compels us to further denote neighborhoods by decoration of these symbols: $\mathcal{T}_x$ denotes the open neighborhoods of a point x and $\mathcal{C}_x$ the closed neighborhoods.

Given a topological space X a point $x \in X$ may be thought of as the corresponding map $(\star \xrightarrow{x} X) \in \mathsf{Top}$. Identifying the topology on the one point space $\big(\varnothing \xrightarrow{\subseteq} \{\star\}\big)$ with the category $2 := \big(0 \to 1\big)$, this gives a map of opens: $\mathcal{T} \xrightarrow{\mathcal{T}_-} 2$. Here, a subtle point presents itself. We also get a map of the closed neighborhoods: $\mathcal{C} \xrightarrow{\mathcal{C}_-} 2$ sending a closed set C to 1 if and only if $x \in C$. Which map is better suited to stand-in for the point x?

Both are relevant. Much of the surrounding theory is fragile and complicated by the differences between these. Separation Axioms typically come into play to interpolate between filters of opens and filters of closeds. For the purposes of convergence, it is more natural to think in terms of filters of opens. For the Zariski topology, filters of closeds.

Because we are working with topologies, the map $\mathcal{T}_-$ is a frame homomorphism and $\mathcal{C}_-$ is a co-frame homomorphism.

Definition 1. A *frame* is a lattice $\mathcal{L}$ satisfying the infinite distributive law:

$$\alpha \wedge (\vee_\bullet \beta_\bullet) = \vee_\bullet(\alpha \wedge \beta_\bullet) \text{ for all } \alpha, \beta \in \mathcal{L}$$

Dually, a *co-frame* is lattice $\mathcal{L}$ satisfying the dual infinite distributive law:

$$\alpha \vee (\wedge_\bullet \beta_\bullet) = \wedge_\bullet(\alpha \vee \beta_\bullet) \text{ for all } \alpha, \beta \in \mathcal{L}$$

i.e a co-frame is a frame in the opposite order.

Frames form a category Frm with morphisms given by maps preserving finite meets and arbitrary joins. Likewise, coframes form a category coFrm with maps preserving arbitrary meets and finite joins.

I'll not belabor the specifics of frames and coframes. Suffice it to say that these are convenient categorical generalizations of 'opens' and 'closeds' respectively. In the main, I need the fact that (co)frame-homs into 2 are in particular monotone, hence completely determined by the structure of their values on 1 (or equivalently 0). These fibers have familiar names. Let $\varphi \in \mathsf{Frm}(\mathcal{T}, 2)$ and consider $\varphi^{-1}1$:

(i) Nonempty: empty meet $= X \longmapsto 1 =$ empty meet
(ii) Downward Directed: $U,\ V \mapsto 1 \implies U \cap V \mapsto 1 \wedge 1 = 1$
(iii) Upward Closed: $V \supseteq U \longmapsto 1 \implies$ If $V \longmapsto 0$ then $U = U \cap V \longmapsto 1 \wedge 0 = 0$ a contradiction.
(iv) Proper: empty join $= \varnothing \longmapsto 0 =$ empty join
(v) Completely Prime: $\cup_\bullet U_\bullet \longmapsto 1 \implies$ some $U_\circ \longmapsto 1$

Weakening the assumptions somewhat to consider lattice homs we see that (v) becomes

(v′) Prime: $\cup_1^n U_i \longmapsto 1 \implies$ some $U_i \longmapsto 1$

Indeed weakening even further to consider $\wedge$-preserving maps, we retain only (i)-(iii). Together these properties define filters.

Definition 2. A nonempty downward directed upward closed subset $\mathcal{F}$ of a meet-semilattice is a *filter*. For the additional properties we can modify this: e.g. we can speak of *proper filters*, *prime filters*, etc.

Just as we did with frames of opens above, we can do with coframes of closeds. For example, a meet homomorphisms $\varphi \in \wedge\mathsf{-hom}(\mathcal{C}, 2)$ corresponds uniquely to $\varphi^{-1}1$ which is a filter of closed sets. Complementing each element, we have the dual: an ideal of open sets. Indeed, these are ideals in the semiring $(\mathcal{T}, \cup, \cap)$. We arrive at notation:

$$\begin{array}{rclcrcr} \operatorname{spec}(\mathcal{L}) & := & \mathsf{Lat}(\mathcal{L}, 2) & \cong & \text{prime filters in } \mathcal{L} & \cong & \text{prime ideals in } \mathcal{L}^* \\ \operatorname{filter}(\mathcal{L}) & := & \wedge\mathsf{-hom}(\mathcal{L}, 2) & \cong & \text{filters in } \mathcal{L} & \cong & \text{ideals in } \mathcal{L}^* \end{array}$$

where $\mathcal{L}^*$ is the dual of $\mathcal{L}$ and in the case of lattices of sets we interpret $\mathcal{L}^*$ as the lattice of complements of members of $\mathcal{L}$.

3. Filters capture convergence

Filters are relevant to continuity and serve as an appropriate general tool for studying convergence. We'll soon explore this in some depth. The key motivation is sequences in metric spaces.

Example 3. Given a sequence $x \colon \mathbb{N} \to X$ a metric space (or a net in any space) we can associate a filter:

$$\mathcal{F}_x := \{A \subseteq X \mid \exists N : \forall n \geq N\ x_n \in A\}$$

the *eventuality filter* of the sequence. Then $x_n \to x$ if and only if $\forall \varepsilon > 0\ \exists N : \forall n \geq N\ x_n \in \mathcal{B}_\varepsilon(x)$. Equivalently, $\mathcal{T}_x \subseteq \mathcal{F}_x$. So convergence is captured by filters.

Definition 4. Given $\varphi \in \wedge\mathsf{-hom}(-, 2)$, we write $\varphi \to x$ and say the filter $\mathcal{T}$*-converges* to x provided $\mathcal{T}_x \leq \varphi$.

The takeaway: studying filter and spec is a means of studying convergence and in turn continuity and this directly generalizes the study of sequences in metric spaces.

Notice, this definition can be applied to any meet semilattice of subsets of X. As defined, the eventuality filter corresponds to an element of $\wedge\mathsf{-hom}(\mathcal{P}X, 2)$.

4. Maximal Filters

The open filters arising from actual points seem like principal filters and consequently like maximal filters: not properly contained in any proper filter. They needn't be. Being tied up with the topology makes this so.

Example 5. The filter of open neighborhoods of $0 \in \mathbb{R}$ is not maximal. Observe that $(0, \varepsilon)$ nontrivially intersects every neighborhood of 0. Finite intersections of members of $\mathcal{T}_0 \cup \{(0, \varepsilon)\}$ form a larger proper filter.

However, because points are closed in $\mathbb{R}$, the closed neighborhoods of 0 form a maximal filter. In fact, this filter generates a convergent filter in any meet semilattice containing both $\mathcal{C}$ and $\mathcal{T}$. We could ask whether there is a map $\text{filter}(\mathcal{C}) \to \text{filter}(\langle \mathcal{C} \cup \mathcal{T} \rangle) \to \text{filter}(\mathcal{T})$ respecting $\mathcal{T}$-convergence. This is more or less how separation axioms enter the discussion.

Definition 6. A filter $\mathcal{F}$ is *maximal* or an *ultrafilter* provided $U \notin \mathcal{F} \iff \exists V \in \mathcal{F} :\ U \cap V = \varnothing$. The easy to find ultrafilters are the *principal ultrafilters* which are defined as $P_x := \{A \subseteq X \mid x \in A\}$.

Alternatively, noticing that $\wedge\!-\!\mathsf{hom}(\mathcal{T}, 2)$ is partially ordered by the pointwise operation $\varphi \leq \psi \iff \forall U \in \mathcal{T} \varphi U \leq \psi U$ a filter φ is maximal if it is maximal as a morphism.

The three most important things to know about maximal filters are: (i) every filter is contained in an ultrafilter, (ii) non-principal ultrafilters (usually) exist, (iii) Every maximal filter is prime.

The first of these is an easy consequence of Zorn's Lemma, and in and of itself constitutes a weak Axiom of Choice. For example, it is equivalent to Tychonoff's theorem for Hausdorff spaces.

Theorem 7 (Ultrafilter Theorem)**.** Every proper filter in may be extended to an ultrafilter.

Proof. Chains of proper filters containing $\mathcal{F}$ are bounded above by the join (see next section). Apply Zorn's Lemma. □

The next property follows as an easy corollary:

Corollary 8. The powerset of any infinite set has a non-principal ultrafilter.

Proof. Consider the Fréchet filter $\mathcal{F} := \{A \subseteq X \mid X \setminus A \text{ finite}\}$ and appeal to the Ultrafilter Theorem to extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$. Were $\mathcal{U}$ to contain any finite set, it would contain its (cofinite) complement, hence $\varnothing \in \mathcal{U}$ clearly contradicting that $\mathcal{U}$ is a proper filter. □

The third fact, is an example of a common theorem in mathematics. Maximal things are usually prime.

Proposition 9. Maximal filters in distributive lattices are prime.

Proof. Suppose $\mathcal{F}$ is an ultrafilter and fails to be prime i.e. there is $\bigvee_{i=1}^{n} \alpha_i \in \mathcal{F}$ but no $\alpha_i \in \mathcal{F}$. Then

$$\begin{array}{ccccc}
\forall i\ \alpha_i \notin \mathcal{F} & \iff & & \forall i\ \exists \beta_i \in \mathcal{F} :\ \alpha_i \wedge \beta_i = \bot & \\
& & & \Downarrow & \\
& & \left(\bigvee_{i=1}^{n} \alpha_i\right) \wedge \left(\bigwedge_{j=1}^{n} \beta_j\right) & = & \bigvee_{i=1}^{n} \left(\alpha_i \wedge (\bigwedge_{j=1}^{n} \beta_j)\right) \\
\bigvee_{i=1}^{n} \alpha_i \notin \mathcal{F} & \iff & & = & \bigvee_{i=1}^{n} \bot \\
& & & = & \bot
\end{array}$$

a clear contradiction. $\square$

5. filter($\mathcal{T}$) in relation to Top

The filters in a given topology are significantly more structured than simply a set. We've already seen that filter($\mathcal{T}$) is partially ordered; in fact, it is a frame when equipped with the following operations:

(i) $\varphi \leq \psi \iff \forall U \in \mathcal{T}\ \varphi U \leq \psi U$
(ii) $(\varphi \wedge \psi)U := \varphi U \wedge \psi U$
(iii) $\bigvee_{\bullet} \varphi_{\bullet} \colon U \mapsto 1 \iff \exists \{(\varphi_i, U_i)\}_1^n : \bigcap_1^n U_i \subseteq U$ and $\varphi_i \colon U_i \mapsto 1$

Clearly, this order aligns with the other operations as $\varphi \leq \psi \iff \varphi^{-1}1 \subseteq \psi^{-1}1$. The meet is clearly a $\wedge-\mathsf{hom}$ and whenever $\gamma \leq \varphi,\ \psi\ \gamma^{-1}1 \subseteq \varphi^{-1}1,\ \psi^{-1}1$ hence contained in $\varphi^{-1}1 \cap \psi^{-1}1 = (\varphi \wedge \psi)^{-1}1$ so the meet is correctly defined. Likewise $\vee_{\bullet}\varphi_{\bullet}$ is the free $\wedge-\mathsf{hom}$ generated by $\cup_{\bullet}\varphi_{\bullet}^{-1}1$ and is therefore minimal among $\wedge-\mathsf{hom}$s larger than all the $\varphi_{\bullet}$.

It remains to show that the infinite distributive law holds for these operations. However, this follows from the definitions. Observe:

$$
\begin{array}{rclcl}
\left[\varphi \wedge (\vee_{\bullet}\psi_{\bullet})\right]U = 1 & \iff & \varphi U = 1 \quad \text{and} \quad \exists \{U_i \in \psi_i^{-1}1\}_1^n & : & \bigcap_i U_i \subseteq U \\
\left[\bigvee_{\bullet}(\varphi \wedge \psi_{\bullet})\right]U = 1 & \iff & \exists \{V_j \in (\varphi \wedge \psi_j)^{-1}1\}_1^m & : & \bigcap_j V_j \subseteq U
\end{array}
$$

That (2) implies (1) is obvious. Use the V_j. For the reverse, note that $\psi_1 U_1 = 1 \implies \psi_1 U = 1$ so $(\varphi \wedge \psi_1)U = 1$ and thus $\left[\bigvee_{\bullet}(\varphi \wedge \psi_{\bullet})\right]U = 1$.

Given any $\left((X, \mathcal{T}_X) \xrightarrow{f} (Y, \mathcal{T}_Y)\right) \in \mathsf{Top}$ there is a frame hom $\mathcal{T}_X \xleftarrow{f^*} \mathcal{T}_Y$ given by the inverse image. For a $\psi \in \text{filter}(\mathcal{T}_Y)$ we can similarly pullback the open sets in ψ^{-1} to opens in $\mathcal{T}_X$ to get a filter $f^*\psi \in \text{filter}(\mathcal{T}_X)$. It is important to note that THIS IS NOT the precomposition functor it is a bonus functor which exists specifically because we are considering maps into $\mathsf{2}$ and know these are uniquely determined by filters viewed as fibers over 1. There is likewise, a pushforward functor defined by the image and together these form an adjoint pair:

$$
\left(f^* \dashv f_!\right) : \ \text{filter}\,\mathcal{T}_X \underset{f^*}{\overset{f_!}{\rightleftarrows}} \text{filter}\,\mathcal{T}_Y \qquad
\begin{array}{rcl}
\mathcal{F} & \longmapsto & \{V \in \mathcal{T}_Y \mid \exists U \in \mathcal{F} \ : \ fU \subseteq V\} \\
\{U \in \mathcal{T}_X \mid \exists V \in \mathcal{G} \ : \ f^*V \subseteq U\} & \longleftarrow & \mathcal{G}
\end{array}
$$

and these constructions are functorial. That is, $-_! \colon \mathsf{Top} \to \mathsf{Frm}$ and $-^* \colon \mathsf{Top}^{\mathrm{op}} \to \mathsf{Frm}$.

First we must show that these are frame homs. It is tedious. Here are the details for f^*.

$$
\begin{array}{lrcl}
\wedge - \text{preserving} & f^*(\varphi \wedge \psi) \colon U \mapsto 1 & \iff & \exists V \in (\varphi \wedge \psi)^{-1}1 : \ f^*V \subseteq U \\
 & & \iff & \exists V \in \varphi^{-1}1 \cap \psi^{-1}1 : \ f^*V \subseteq U \\
 & & \implies & f^*\varphi \wedge f^*\psi \colon U \mapsto 1 \\
 & & & \\
 & f^*\varphi \wedge f^*\psi \colon U \mapsto 1 & \iff & \exists V_1 \in \varphi^{-1}1 \ : f^*V_1 \subseteq U \qquad \text{and} \quad \exists V_2 \in \psi^{-1}1 \ : f^*V_2 \subseteq U \\
 & & \implies & f^*(V_1 \cup V_2) = f^*V_1 \cup f^*V_2 \subseteq U \quad \text{and} \quad V_1 \cup V_2 \in \varphi^{-1}1 \cap \psi^{-1}1 \\
 & & \implies & f^*(V_1 \cup V_2) \colon U \mapsto 1 \\
 & & & \\
\vee - \text{preserving} & f^* \bigvee_{\bullet} \varphi_{\bullet} \colon U \mapsto 1 & \iff & \exists V \in \left(\bigvee_{\bullet} \varphi_{\bullet}\right)^{-1} 1 \ : f^*V \subseteq U \\
 & & \iff & \exists \{V_i \in \varphi_i^{-1}1\}_1^n : \ \cap_1^n f^*V_i = f^* \cap_1^n V_i \subseteq f^*V \subseteq U \\
 & & & \\
 & \bigvee_{\bullet} f^*\varphi_{\bullet} \colon U \mapsto 1 & \iff & \exists \{W_j \in f^*\varphi_j^{-1}1\}_1^m : \ \cap_1^m W_j \subseteq U \\
 & & & \text{but } W_j \xmapsto{\varphi_j} 1 \text{ if and only if } \exists V_j \in \varphi_j^{-1}1 : \ f^*V_j \subseteq W_j \\
 & & \iff & \exists \{V_j \in \varphi_j^{-1}1\}_1^m : \ \cap_1^m f^*V_j \subseteq U
\end{array}
$$

Finally, we get to demonstrate that $f^* \dashv f_!$. Recall for posets adjunctions (or Galois correspondence) are necessarily simple; since all the hom-sets are empty or singletons, the natural isomorphism of hom-sets becomes existence of a dual. Specifically,

$$(f^* \dashv f_!) \quad \text{is equivalent to} \quad f^*\psi \leq \varphi \iff \psi \leq f_!\varphi$$

Let's begin.
$[\implies]$ Suppose $f^*\psi \leq \varphi$. Let $V \in \psi^{-1}$. Then $f^*V \in \varphi^{-1}1$ by hypothesis and hence $V \supseteq ff^*V \in (f_!\psi)^{-1}1$ and consequently, $V \in (f_!\psi)^{-1}1$ by definition.
$[\impliedby]$ Suppose $\psi \leq f_!\varphi$. Let $U \in (f^*\psi)^{-1}1$. That is, assume $\exists V \in \psi^{-1}1 : \ f^*V \subseteq U$. By hypothesis, $V \in (f_!\varphi)^{-1}1$ hence $\exists W \in \varphi^{-1}1 : \ f_!W \subseteq V$. Note then that $W \subseteq f^*V \subseteq U$ so $U \in \varphi^{-1}1$.

I'm leaving any remaining arguments out. Everything else that needs to be said is easy or directly analogous to the arguments I've written above.

6. Continuity

A bit of a recap: our 'weak models of points' have lots of algebraic structure summarized by last section's functorial relationship with Top and they support a generic definition of convergence. The purpose of this section is to leverage the $(f^* \dashv f_!)$ adjunction on filters to give equivalent conditions for continuity of $(X \xrightarrow{f} Y)$. Fortunately, we've done enough legwork to make that easy.

Proposition 10. $\big((X, \mathcal{T}_X) \xrightarrow{f} (Y, \mathcal{T}_Y)\big) \in \mathsf{Top}$ if and only if $\varphi \to x \implies f_!\varphi \to fx$.

Proof. First, we'll prove that continuity is equivalent to having the adjunct of continuity at a point for every x. Specifically, we'll show that f is continuous if and only if $\mathcal{T}_{fx} \leq f_!\mathcal{T}_x$ for all $x \in X$.
$[\implies]$ Suppose f is continuous. Our adjunction proves that $\mathcal{T}_{fx} \leq f_!\mathcal{T}_x \iff f^*\mathcal{T}_{fx} \leq \mathcal{T}_x$ i.e. f is continuous at all x. But of course $f^*\mathcal{T}_{fx} \colon U \mapsto 1 \iff \exists V \in \mathcal{T}_{fx} : \ f^*V \subseteq U$ and continuity implies $f^*V \in \mathcal{T}_x$ so $U \in \mathcal{T}_x$.
$[\impliedby]$ Suppose $\mathcal{T}_{fx} \leq f_!\mathcal{T}_x$ for all x and let $V \in \mathcal{T}_Y$. Notice that $x \in f^*V \iff fx \in V$ and our hypothesis ensures $V \in (f_!\mathcal{T}_x)^{-1}1$ so $\exists U_x \in \mathcal{T}_x : \ fU_x \subseteq V$. Consequently, $f^*V = \cup_{x \in f^*V} U_x \in \mathcal{T}_X$ and f is continuous.

Now suppose $\varphi \in \wedge\text{−}\mathsf{hom}(\mathcal{T}, 2)$ in any $\wedge$-semilattice containing $\mathcal{T}_X$...that is, somewhere $\mathcal{T}_X$-convergence makes sense. We'll show that $(\varphi \to x \implies f_!\varphi \to fx)$ is equivalent to continuity of f.
$[\implies]$ Suppose $(\varphi \to x \implies f_!\varphi \to fx)$. Equivalently, $(\mathcal{T}_x \leq \varphi^{-1}1 \implies \mathcal{T}_{fx} \leq (f_!\varphi)^{-1}1)$. In particular, $\mathcal{T}_x \to x$ so $\mathcal{T}_{fx} \leq f_!\mathcal{T}_x$ which implies f is continuous.
$[\impliedby]$ Now suppose f is continuous and $\varphi \to x$, then $\mathcal{T}_x \leq \varphi$. Continuity implies $\mathcal{T}_{fx} \leq f_!\mathcal{T}_x$ and monotonicity of $f_!$ implies $f_!\mathcal{T}_x \leq f_!\varphi$. Thus, $f_!\varphi \to fx$ as desired. $\square$

Moving on, we can prove some easy results about convergence in compact spaces and in Hausdorff spaces. These are the generalizations of some familiar results from Calculus. There should be result analogous to "compact if and only if every sequence has a convergent subsequence" and a "space is Hausdorff if and only if limits of sequences are unique" but more widely applicable.

Notice that passing to a subsequence relaxes the membership condition for the eventuality filter. It is easier to satisfy a '$\forall$' when you reduce the size of the domain. For us, this means that extensions of filters play the role of subsequences.

Proposition 11. $(X, \mathcal{T})$ is compact $\implies \forall\varphi \in \operatorname{spec}(\mathcal{T})\ \exists x : \ \varphi \to x$.

Proof. Suppose $\varphi \in \operatorname{spec}(\mathcal{T})$ and $\forall x\ \varphi \not\to x$. Equivalently, $\exists U_x \in \mathcal{T}_x - \varphi^{-1}1$ for each $x \in X$. Then $\{U_x\}_x$ is an open cover. By compactness, choose a finite subcover

$$U_{x_1} \cup \cdots \cup U_{x_n} = X \overset{\varphi}{\longmapsto} 1$$

However, φ is prime so $\exists i:\ U_{x_i} \overset{\varphi}{\longmapsto} 1$ a clear contradiction. □

Proposition 12. $(X,\mathcal{T})$ is compact $\Longleftarrow$ ultrafilters converge in all regular topologies refining $\mathcal{T}$.

Proof. Suppose X is not compact. Take a collection $\mathcal{V}$ of closed sets with the finite intersection property (FIP) and an empty intersection. Notice that for each $x \in X$ there exists $V_x \in \mathcal{V}$ such that $x \notin V_x$.

By assumption, we're working with a regular topology $\mathcal{T}' \supseteq \mathcal{T}$. In particular, $\exists U_x,\ W_x \in \mathcal{T}'$ with $U_x \cap W_x = \varnothing$, $x \in W_x$, and $V_x \subseteq U_x$. The collection $\{U_x\}_x$ inherits the FIP. As such, it extends to a proper filter of $\mathcal{T}'$-open sets which in turn extends to an ultrafilter m by the ultrafilter lemma.

However, $m \not\to x$ for any x by construction since $W_x \in m^{-1}1$ implies $\varnothing = W_x \cap U_x \longmapsto 1$ a clearly contradicting that m is proper. □

Corollary 13. X is compact $\iff \forall \varphi \in \operatorname{spec}(\mathcal{P}X)\ \exists x \in X:\ \varphi \to x$

Proof. The discrete topology is regular. □

Proposition 14. $(X,\mathcal{T})$ is Hausdorff $\iff$ every proper filter which $\mathcal{T}$-converges, does so to a unique x.

Proof.

$$\begin{array}{ccc}
\begin{array}{l}\text{Suppose } \mathcal{T}_x,\mathcal{T}_y \le \varphi \\ \text{for some } x \neq y \in X\end{array} & \Longrightarrow & \begin{array}{l}\forall U \in \mathcal{T}_x,\ V \in \mathcal{T}_y \quad U \cap V \overset{\varphi}{\longmapsto} 1 \\ \text{hence, } U \cap V \neq \varnothing\end{array} \\
\Uparrow & & \Downarrow \\
\begin{array}{l}\exists x \neq y:\ \forall U \in \mathcal{T}_x,\ V \in \mathcal{T}_y \\ U \cap V \neq \varnothing \text{ and so } \mathcal{T}_x \cup \mathcal{T}_y \\ \text{extends to a proper filter } \varphi\end{array} & \Longleftarrow & X \text{ is not hausdorff}
\end{array}$$

□

Here it is tempting to surmise that compact Hausdorff topologies on X are equivalent to retractions r of the principal map $x \longmapsto P_x \in \operatorname{spec}(\mathcal{P}X)$. Afterall, such an r serves as a putative 'lim' operator on filters of subsets of X and the previous two results have as a corollary:

Corollary 15. $(X,\mathcal{T})$ is compact Hausdorff $\iff$ for all $\varphi \in \operatorname{spec}(\mathcal{P}X)\ \exists!\ \lim\varphi \in X:\ \varphi \to \lim\varphi$.

In other words, we have a retraction

$$\begin{tikzcd}
X \arrow[loop left, "\mathrm{Id}"] \arrow[r, "\mathcal{T}_-"'] & \operatorname{spec}\mathcal{T} \arrow[r, "\mathrm{Id}_!"'] & \operatorname{spec}\mathcal{P}X \arrow[ll, bend right, "\lim"']
\end{tikzcd}$$

Of course this is diagnostic of compact Hausdorff topologies but a stronger result holds when a space is compact Hausdorff. In such case, we have that the 'lim' operator is a lift from one at $\operatorname{spec}(\mathcal{T})$.

NEED TO WRITE A BIT ON THIS

7. A characterization of the Zariski Topology

making a 'bad' habit of referring to elements of the fiber at one as 'elements' of the morphisms i.e. $U \in \varphi$ when I mean $U \in \varphi^{-1}1$...not sure how I feel about this vestige of dealing with filters.

Suppose $\mathcal{L}$ is a distributive lattice. Then we have a picture:

$$\begin{tikzcd}
 & \mathcal{L} \arrow[dd, "\beta"] & & & \\
\operatorname{spec}\mathcal{L} \arrow[rr, hook] \arrow[dd, "\beta"'] & & \mathcal{P}\mathcal{L} & & \\
 & \operatorname{spec}\mathcal{P}\mathcal{L} \arrow[rr, hook] & & \mathcal{P}^2\mathcal{L} & \\
\operatorname{spec}\mathcal{P}\operatorname{spec}\mathcal{L} \arrow[rrrr, hook, bend right] & & \mathcal{P}\operatorname{spec}\mathcal{P}\mathcal{L} \arrow[uu, "\beta^*"] & & \mathcal{P}^3\mathcal{L} \arrow[ll]
\end{tikzcd}$$

where rightward arrows are inclusions and β's are principal maps

$$\beta\colon \square \to \operatorname{spec}\mathcal{P}\square \qquad \text{via} \qquad \circ \mapsto \{\bullet \in \mathcal{P}\square \mid \circ \in \bullet\}$$

The central observation here is that the inverse image function gets us close to a map $\operatorname{spec}(\mathcal{P}\operatorname{spec}\mathcal{L}) \to \operatorname{spec}(\mathcal{L})$.

Proposition 16. Restricted to $\operatorname{spec}(\mathcal{P}\operatorname{spec}\mathcal{L})$, the image of β^* is in $\operatorname{spec}\mathcal{L}$, and $\beta^*\colon \operatorname{spec}(\mathcal{P}\operatorname{spec}\mathcal{L}) \longrightarrow \operatorname{spec}\mathcal{L}$ is a retraction of β.

Proof. First we need to demonstrate that $\beta^*(\operatorname{spec}\mathcal{P}\operatorname{spec}\mathcal{L}) \subseteq \operatorname{spec}\mathcal{L}$

$$\beta^*\varphi = \{A \in \mathcal{L} \mid \beta A \in \varphi\}$$

Notice that elements of elements of of φ are in $\operatorname{spec}\mathcal{L}$. In particular,

$$\beta A \in \varphi \implies \beta A = \{\gamma \in \operatorname{spec}\mathcal{L} \mid A \in \gamma\}$$

and so

$$\beta^*\varphi = \{A \in \mathcal{L} \mid \{\gamma \in \operatorname{spec}\mathcal{L} :\ A \in \gamma\} \in \varphi\}$$

Now observe that $\beta^*\varphi$ is a filter since inverse maps preserve set operations. Indeed, $\beta^*\varphi$ is prime as claimed. If $A \cup B \in \beta^*\varphi$ then $\beta(A \cup B) \in \varphi$. And for $\gamma \in \beta(A \cup B)$, that is, γ prime and containing $A \cup B$, we have $A \in \gamma$ or $B \in \gamma$. Consequently, $\beta(A \cup B) \subseteq \beta(A) \cup \beta(B)$. By upward closure, $\beta(A) \cup \beta(B) \in \varphi$. Since φ is prime, $\beta(A)$ or $\beta(B)$ is in φ which implies A or $B \in \beta^*\varphi$. Finally, consider the computation

$$\begin{array}{rcl}
\beta\beta^*\varphi & = & \{A \in \mathcal{L} \mid \beta A \in \beta\varphi\} \\
 & = & \{A \in \mathcal{L} \mid \varphi \in \beta A\} \\
 & = & \{A \in \mathcal{L} \mid \varphi \in \{\gamma \in \operatorname{spec}\mathcal{L} :\ A \in \gamma\}\} \\
 & = & \{A \in \mathcal{L} \mid A \in \varphi\} \\
 & = & \varphi
\end{array}$$

□

We've seen retractions arising as lim operators for compact Hausdorff topologies. Pretending that β^* is such a lim operator is profitable. Notice that any topology for which β^* is to be the limit operator must have

$$\mathcal{T}_\psi \subseteq \bigcap\{\varphi \mid \beta^*\varphi = \psi\} := \mathcal{Y}_\psi$$

More to the point: every filter β^* sends to φ should contain every set that contains a neighborhood of φ we're working with a blunt instrument. Luckily, this time we get something fantastic. We have a closed form expression for β^*

$$\psi = \beta^* \varphi \iff \Big(A \in \psi \iff \{\gamma \in \operatorname{spec} \mathcal{L} : \ A \in \gamma\} \in \varphi \} \Big)$$

Immediately, then we have that $\mathcal{Y}_\psi$ is the intersection of all prime φ extending the filter $\mathcal{F} := \{\{\gamma : A \in \gamma\} \mid A \in \psi\}$. In fact, they are equal $\mathcal{Y}_\psi = \mathcal{F}$.

Suppose the contrary that $\exists \mathcal{H} \in \mathcal{Y}_\psi - \mathcal{F}$. Then there is an $\mathcal{A} \in \mathcal{F} : \ \mathcal{A} \cap \mathcal{H}^c = \varnothing$. Otherwise, $\mathcal{F} \cup \{\mathcal{H}^c\}$ has the FIP and extends to a maximal filter, which by hypothesis contains $\mathcal{H}$ and by construction $\mathcal{H}^c$ contradicting properness. But $\mathcal{A} \cup \mathcal{H}^c = \varnothing$ means $\mathcal{A} \subseteq \mathcal{H}$. By upward closure of $\mathcal{F}$, we have $\mathcal{H} \in \mathcal{F}$ contrary to our hypothesis.

Recall, closed sets in the Zariski topology are parameterized by ideals η:

$$V(\eta) := \{\rho \in \operatorname{spec} \ \mid \eta \le \rho\}$$

That these form a topology follows from two calculations:

$V(\eta) \cup V(\eta') = V(\eta \wedge \eta')$:

> *Proof.* We get $\subseteq$ directly from the defining property of the meet. For the reverse, we use primality. Suppose $\gamma \in V(\eta \wedge \eta')$ and not in $V(\eta) \cup V(\eta')$. Then $\exists U, V : \ U \in \eta, \ V \in \eta'$ and $U, V \notin \gamma$. Then $(U \cup V) \in (\eta \wedge \eta') \le \gamma$. As γ is prime, U or $V \in \gamma$ a clear contradiction. $\square$

$\bigcap_\bullet V(\eta_\bullet) = V(\vee_\bullet \eta_\bullet)$:

> *Proof.* Here, we get $\supseteq$ by definition of the join and use primality to get the reverse. Suppose $\gamma \in \bigcap_\bullet V(\eta_\bullet)$ and not in $V(\vee_\bullet \eta_\bullet)$. Then $\eta_\bullet \le \gamma$ for all $\bullet$ and $\exists U : \ U \in (\vee \eta_\bullet), \ U \notin \gamma$. So $\exists \{U_i \in \eta_i\}_{i=1}^n : \ \cap U_i \subseteq U$. But $U_i \in \eta_i \implies U_i \in \gamma$ so $\cap U_i \in \gamma$ and by upward closure $U \in \gamma$ contradicting our hypothesis. $\square$

In this notation, we have $\mathcal{Y}_\psi = \{V(\langle A \rangle)\}_{A \in \psi}$. In words, $\mathcal{Y}_\psi$ is the closed principal neighborhoods of ψ. But any filter is the join over it's principal filters. In sum, we have

Proposition 17. The $\{\mathcal{Y}_\psi\}_{\psi \in \operatorname{spec} \mathcal{L}}$ are a closed base for the Zariski topology

Corollary 18. $\forall \varphi \in \operatorname{spec} \mathcal{P} \operatorname{spec} \mathcal{L} : \ \varphi \to \beta^* \varphi$ in the Zariski topology. That is, equipped with the Zariski topology, $\operatorname{spec} \mathcal{L}$ is compact.

Proof. The Zariski open sets are: $V(\eta)^c = \{\gamma : \ \eta \not\le \gamma\}$. Thus, the neighborhoods of $\beta^* \varphi$ given as:

$$\begin{aligned}
\mathcal{T}_{\beta^* \varphi} &= \{V(\eta)^c \ni \beta^* \varphi\} \\
&= \{V(\eta)^c \mid \eta \not\le \beta^* \varphi\} \\
&= \{V(\eta)^c \mid \exists A : \ A \in \eta \text{ and } A \notin \beta^* \varphi\} \\
&= \{V(\eta)^c \mid \exists A : \ \langle A \rangle \le \eta, \ V(\langle A \rangle) \notin \varphi\} \\
&= \{V(\eta)^c \mid V(\eta) \notin \varphi\} && \text{since } \langle A \rangle \le \eta \implies V(\eta) \subseteq V(\langle A \rangle) \\
&= \{V(\eta)^c \mid V(\eta)^c \in \varphi\}
\end{aligned}$$

$\square$

Corollary 19. $\max \mathcal{L}$ is a compact subspace of $\operatorname{spec} \mathcal{L}$.

Proof. Essentially, the idea is as follows: compactness is equivalent to convergence of every $\Phi \in \operatorname{spec} \mathcal{P} \max \mathcal{L}$. Each such Φ generates a $\langle \Phi \rangle := i_! \Phi = \{\mathcal{A} \subseteq \operatorname{spec} \mathcal{L} \mid \exists \mathcal{B} \in \Phi : \; \mathcal{B} \subseteq \mathcal{A}\} \in \operatorname{spec} \mathcal{P} \operatorname{spec} \mathcal{L}$. We know that this converges to $\varphi := \beta^* \Phi \in \operatorname{spec} \mathcal{L}$.

But every open neighborhood of any maximal extension $\hat{\varphi} \geq \varphi$ is an open neighborhood of φ:

$$\hat{\varphi} \in V(\eta)^c \iff \eta \not\leq \hat{\varphi} \implies \eta \not\leq \varphi \iff \varphi \in V(\eta)^c$$

Thus, $\mathcal{T}_{\hat{\varphi}} \subseteq \mathcal{T}_{\varphi}$ and convergence implies convergence to every maximal extension. In particular, such extensions exist by the ultrafilter lemma. Pick one and observe that Φ converges to $\hat{\varphi}$: for every neighborhood $V \in \mathcal{T}_{\hat{\varphi}}$ there exists a $\mathcal{B} \in \Phi : \; \mathcal{B} \subseteq V$. □

8. Wallman

In this section we construct the Stone-Cech compactification of normal Hausdorff spaces. Briefly, recall that a normal Hausdorff space is characterized by saying (i) points are closed and (ii) disjoint closed sets are separated by disjoint opens. Notice, that this is in fact, a statement comparing the open neighborhoods of a point to the closed neighborhoods. Given any sufficiently large (nonempty interior) closed neighborhood C of x we get opens $V \in \mathcal{T}_x$ and $W \supseteq (C^\circ)^c = \overline{C^c}$ such that $V \cap W = \varnothing$. Alternatively, given an open neighborhood U of x, we get opens $V' \in \mathcal{T}_x$ and $W' \supseteq U^c$ such that $V' \cap W' = \varnothing$. In words, large enough closed neighborhoods mutually refine open neighborhoods.

As a topological condition, this ultimately grants us tools to realize the stone-cech compactification of normal Hausdorff spaces X by considering $\operatorname{spec} \mathcal{C} X$. We follow our roadmap. Let := Hausdorff normal spaces together with continuous maps. Note every compact Hausdorff space $K \in \mathsf{Space}$.

We need two technical results before continuing.

Lemma 20. If K is compact Hausdorff, then maximal filters of closed sets are all principal.

Proof. Suppose $\eta \in \operatorname{spec} \mathcal{C} K$ is maximal. Then the elements of η have the FIP. Thus there is an $x \in \cap \eta$. Therefore, $\langle \{x\} \rangle \geq \eta$, so $\eta = \langle \{x\} \rangle$ by maximality. □

Lemma 21. If X is normal Hausdorff, then the maximal filters $\max \mathcal{C} X$ are a Hausdorff space.

Proof. Suppose $\varphi \neq \psi \in \max \mathcal{C} X$. Then $\exists B \in \psi - \varphi$. In particular, $B \notin \varphi$ is equivalent to $\exists A \in \varphi : \; A \cap B = \varnothing$. By normality, these closed sets are separated by disjoint opens $U \supseteq A$ and $V \supseteq B$. But then $U^c \cup V^c = X$ and

$$\begin{aligned} U^c \supseteq B &\quad\implies\quad U^c \in \psi \\ V^c \supseteq A &\quad\implies\quad V^c \in \varphi \end{aligned}$$

In turn, this gives that $\psi \in V(\langle U^c \rangle) \not\ni \varphi$ and similarly $\psi \notin V(\langle V^c \rangle) \ni \varphi$, and

$$V(\langle U^c \rangle) \cup V(\langle V^c \rangle) = V(\langle U^c \rangle \wedge \langle V^c \rangle) = V(\varnothing) = \operatorname{spec} \mathcal{C} X$$

This is the closed set equivalent of the Hausdorff property. □

should double check this, also probably restate it for not just $\mathcal{C}X$.

Proposition 22. We have a functor $\operatorname{spec} \mathcal{C} \colon \mathsf{Space} \to \mathsf{Space}$ defined by pushforward $f_! := \operatorname{spec} \mathcal{C} f$ and $\star \cong \operatorname{spec} \mathcal{C} \star$.

Proof. Let $(X \xrightarrow{f} Y) \in \mathsf{Space}$ be given. Notice that $f_!$ is continuous in the Zariski topology $\iff$ the $V(\langle B \rangle) \subseteq \operatorname{spec} \mathcal{C}Y$ pullback to closed sets in $\operatorname{spec} \mathcal{C}X$. Directly, we have:

$$\begin{aligned}(f_!)^* V\langle B \rangle) &= \{\varphi \in \operatorname{spec} \mathcal{C}X \mid \exists A \in \varphi : \ fA \subseteq B\} \\ &= \{\varphi \in \operatorname{spec} \mathcal{C}X \mid \exists A \in \varphi : \ A \subseteq f^* B\} \\ &= V(\langle f^* B \rangle)\end{aligned}$$

□

Thus, $f_!$ is continuous. Further, note $\operatorname{spec} \mathcal{C}\star = \big\{\{\star\}\big\}$. Indeed, eval is the principal map:

$$\operatorname{eval}\colon x \mapsto (\star \to X) \mapsto (\{\{\star\}\} \xrightarrow{x_!} \operatorname{spec} \mathcal{C}X)$$

sends $x \mapsto \{B \in \mathcal{C}X \mid \{x\} \subseteq B\} = \langle \{x\} \rangle$ since points are closed.

Proposition 23. The map eval is continuous with a dense image in $\max(\mathcal{C}X)$, and—when applied to a compact Hausdorff space—an embedding.

Proof. As Zariski closed sets are defined in terms of the closed sets of X, continuity is immediate:

$$\operatorname{eval}^* \big(V(\langle B \rangle)\big) = \{x \in X \mid B \in \langle \{x\} \rangle\} = \{x \in X \mid x \in B\} = B$$

To demonstrate $\operatorname{eval}(X)$ as dense in $\max(\mathcal{C}X)$, we must show that every open set containing a maximal filter contains a principal filter. To that end, let $\mathfrak{m} \in V(\eta)^c$ be maximal. Then $\eta \not\leq \mathfrak{m}$ so there exists $A \in \eta : \ A \notin \mathfrak{m}$. Equivalently, by maximality, $\exists B \in \mathfrak{m} : \ A \cap B = \varnothing$. Normality yields disjoint opens $U \supseteq A$ and $V \supseteq B$.

Notice, if $U^c \in \gamma$ then $A \notin \gamma$ since $U^c \cap A = \varnothing$. Thus, $\eta \not\leq \gamma$ and so $V(\langle U^c \rangle) \subseteq V(\eta))^c$. And, for any $x \in B \subseteq U^c$, the principal filter $\langle \{x\} \rangle \in V(\langle U^c \rangle) \subseteq V(\eta)^c$. As promised, principal filters are dense in $\max(\mathcal{C}X)$.

For the embedding, first note that eval is clearly injective. Now suppose K is compact Hausdorff, let C be a closed set, and observe that

$$\operatorname{eval}(C) = \{\langle \{x\} \rangle \mid x \in C\} = \max \mathcal{C}K \cap V(\langle C \rangle)$$

as all maximal filters of closed sets in a compact Hausdorff space are principal. Thus, $\operatorname{eval}(C)$ is a closed subset of the compact Hausdorff subspace $\max \mathcal{C}K$. □

Corollary 24. The space of maximal filters $\beta X := \max \mathcal{C}X$ is the stone-cech compactification of a normal Hausdorff space X.

Proof. At this point we already have the existence of a factorization through βX. Given $(X \xrightarrow{f} K) \in \mathsf{Space}$ where K is compact Hausdorff:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\operatorname{eval}"'] & K \\ \beta X \arrow[ur, "f_!"'] & \end{tikzcd}$$

Moreover, βX is compact Hausdorff. We need only show that the factorization is unique. But $\operatorname{eval}(X)$ is dense and K is Hausdorff: any map that agrees with $f_!$ on $\operatorname{eval}(X)$ equals $f_!$. □