Algebra for N00bs

By Max

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1. First Order Logic // you already know this
2. Words and Languages // aka easy algebra
3. Finite Deterministic Automata
4. Regular Expressions & Languages
5. Monoids
6. Metric Spaces // the secret ingredient to ML

// Plus a short note at the end about how Pin denotes functions …

1st Order Logic

Variables: x_0, x_1, x_2, …

max-is-smart,max-is-dumb

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Operators: ~, ^, V, =	Input: Properties Output: Properties
max-is-dumb v max-is-smart	

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are-siblings(Max, Sam)	

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Quantifier: ∃	∀xP(x) means ~∃x~P(x)
∃ x jacob-thinks(x)	

Predicates: $P_i(y_0,…,y_n)$	Input: Variables Output: Properties
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Input: Variables
Output: Properties

Equality: “=”

This ingredient is different in HoTT

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Input: Variables
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Functions: $f_i(y_0, \dots, y_n) \mapsto y_k$

father-of(Max) = Frank

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Predicates: $P_i(y_0, …, y_n)$

are-siblings(Max, Sam)

Input: Variables
Output: Properties

Theory: a finite set Γ of variables, assumed to be *true*. ⊢ is entailment.

… more on this in a moment.

Functions: $f_i(y_0, …, y_n) \mapsto y_k$

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1st Order Logic <<Surprise Gödel>>

Theory: a finite set Γ of predicates, assumed to be *true*.

```
    // For example, Peano Arithmetic:
    /* { nat(0),
         ∀x(x=x),
         ∀x,y(x=y ⇒ y=x),
         ∀x,y,z (x=y ^ y=z) ⇒ x=z,
         ∀x,y (nat(x) ^ x=y ) ⇒ nat(y),
         ∀x nat(x) ⇒ nat(S(x)),
         ∀x,y (x = y) ⇔ (S(x) = S(y)),
         ∀x ~(S(x)=0) } */
```

1st Order Logic <<Surprise Gödel>>

Theory: a finite set Γ of predicates, assumed to be *true*.

Syntactic Entailment: Γ ⊢ φ.
    // Does there exist a proof, which assumes only Γ, and concludes φ?

1st Order Logic <<Surprise Gödel>>

Theory: a finite set Γ of predicates, assumed to be *true*.

Syntactic Entailment: $\Gamma \vdash \varphi$.

- If x in Γ then Γ⊢x // Is it an axiom?

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  - If x is in Γ it's called a non-logical axiom.

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  - If $\{\ \} \vdash x$ then x is a “tautology”, or a logical axiom.

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    - I'm not getting into logical axioms right now because I don't want to talk about modus ponens …

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- If x in Γ then $\Gamma \vdash x$
- If $\Gamma \vdash x$ and $\Gamma \vdash y$ then $\Gamma \vdash x \wedge y$ // Can we prove this and that?

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- If $\Gamma, y \vdash z$ and $\Gamma, y \vdash \sim z$ then $\Gamma \vdash \sim y$ // Proof by way of contradiction

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Semantic Entailment: $\Gamma \vDash \varphi$ if, for every theory Π containing Γ, $\Pi \nvdash \sim\varphi$.
// The property is compatible with all similar theories.

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Consistency: $\Gamma \vdash \varphi$ if and only if $\Gamma \nvdash \sim\varphi$. // Don't prove a contradiction.

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Consistency: $\Gamma \vdash \varphi$ if and only if $\Gamma \nvdash \sim\varphi$.

Soundness: If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$. // Don't prove anything that can be disproven by a compatible theory.

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Completeness: If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$. // All theories w/ these axioms allow φ

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Thm 1: You can't have both consistency & completeness.

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Completeness: If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.

Thm 1: You can't have both consistency & completeness.

Thm 2: Γ can't prove its own consistency.

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// Plus a short note at the end about how Pin denotes functions …

Words and Languages

Alphabet: a non-empty finite set.

A = { 0, 1, 2, …, 9 }

Words and Languages

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69420

Free Monoid: The set of all words over A.

A* = ℕ (with leading 0s)

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69420

Alternatively, any element of A*.

1 denotes “”.

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Language over A: any subset of A*

$\mathbb{N}$, positives, even numbers, odd numbers, binary, binary programs that terminate, binary programs that were written by Microsoft engineers, prime numbers, real-world RSA moduli, ternary encodings of Max’s passwords …

Words and Languages

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Initial

Words and Languages

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Ending

Words and Languages

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(Proper) prefix or left factor

Words and Languages

69420

(Proper) postfix or right factor

Words and Languages

69420

(Proper) subword

Words and Languages

$|69420| = 5$

Words and Languages

$$|69420| = 5$$

$$|69420|_6 = 1$$

Words and Languages

$$|69420| = 5$$

$$|69420|_6 = 1$$

$$|69666|_6 = 4$$

Words and Languages

$|69420| = 5$

$|69420|_6 = 1$

$|69666|_6 = 4$

W is called **multilinear** if $|w|_x \leq 1$ for all x in A

Words and Languages

69 * 420 = 69420

* is associative, non-commutative

*'s identity is 1

420^3 = 420420420

Words and Languages

$\{69\} * \{666, 314, 420, 000\} = \{69666, 69314, 69420, 69000\}$

* is associative, non-commutative

*'s identity is $\mathbb{1} = \{ 1 \} = \{ \text{“”} \}$

$\{ 22, 33 \}^2 = \{ 2222, 2233, 3322, 3333 \}$

Words and Languages

$\{6,9\} \setminus \{6,300\} = \{9\}$

Words and Languages

$\{6,9\} \setminus \{6,300\} = \{9\}$

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Words and Languages

$\{6,9\} \setminus \{6,300\} = \{9\}$

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$\{6,9\}^c$ = Everything in A^* except for 6 and 9

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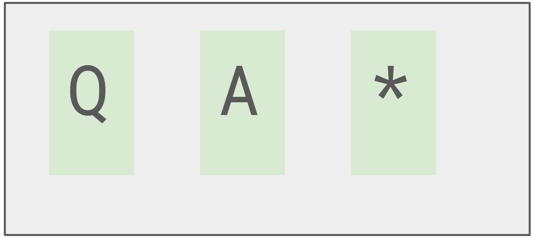
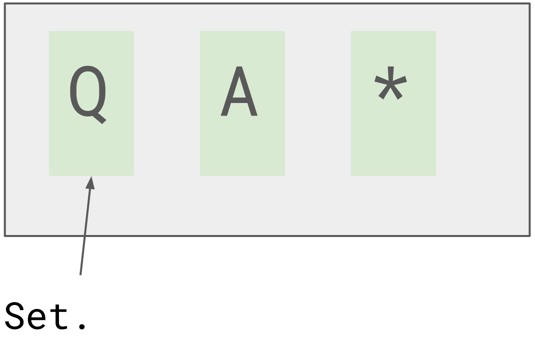


Finite Deterministic Automata



Finite Deterministic Automata

Q A *

… how Pin defines things

Set.

S Σ δ

… how CS people usually define things

Finite Deterministic Automata

Q A *

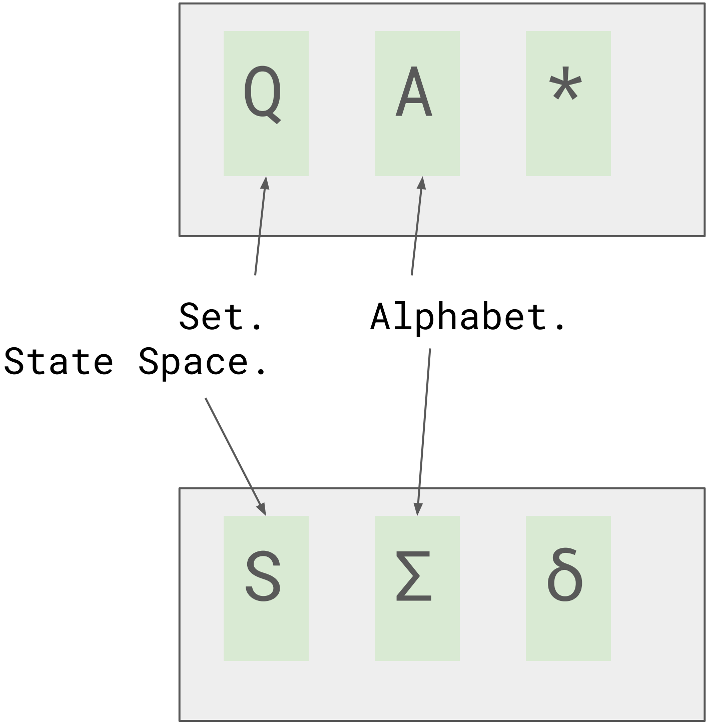
… how Pin defines things

Set.
State Space.

S Σ δ

… how CS people usually define things

Finite Deterministic Automata



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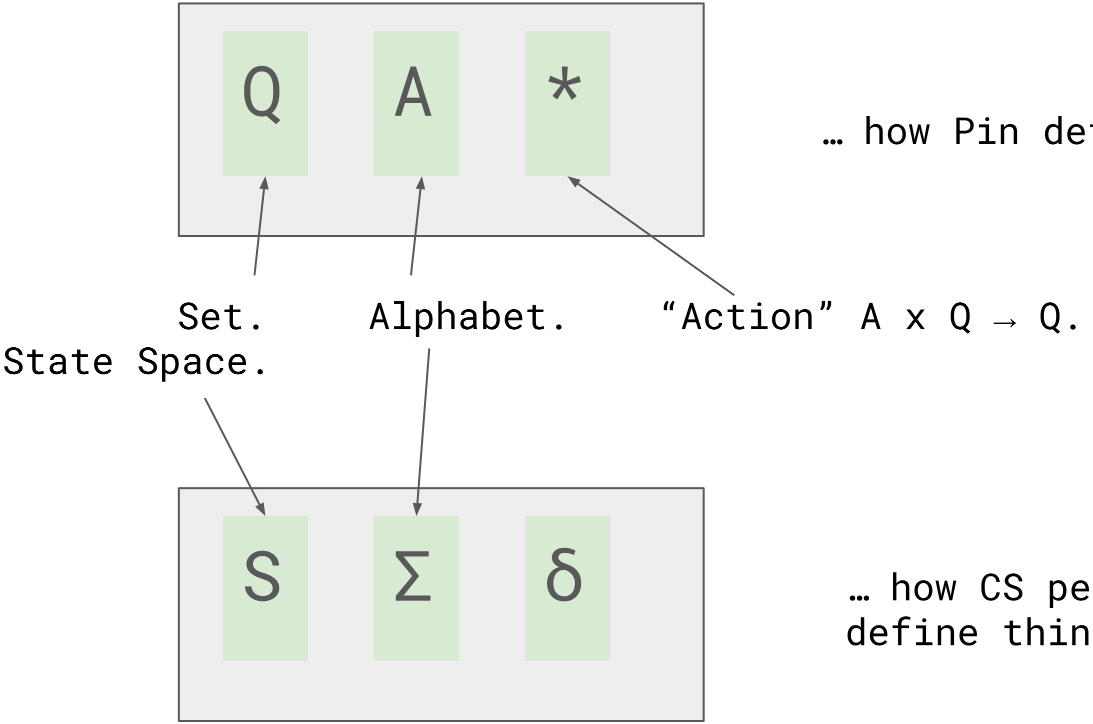
Finite Deterministic Automata



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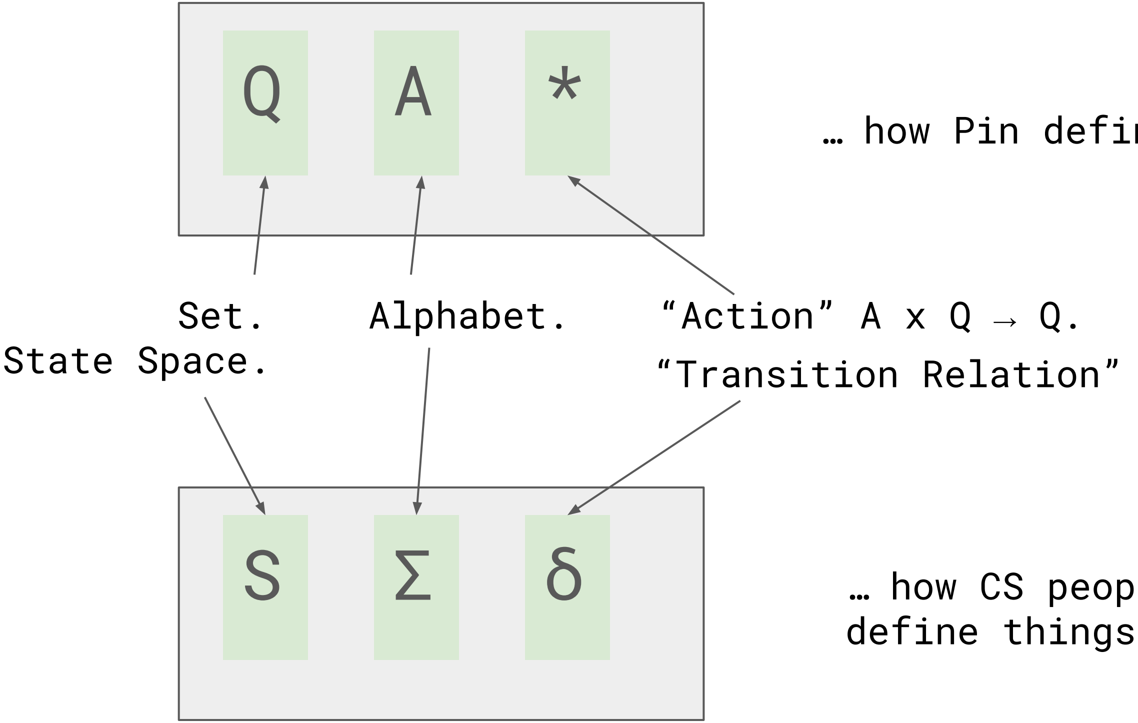
Finite Deterministic Automata



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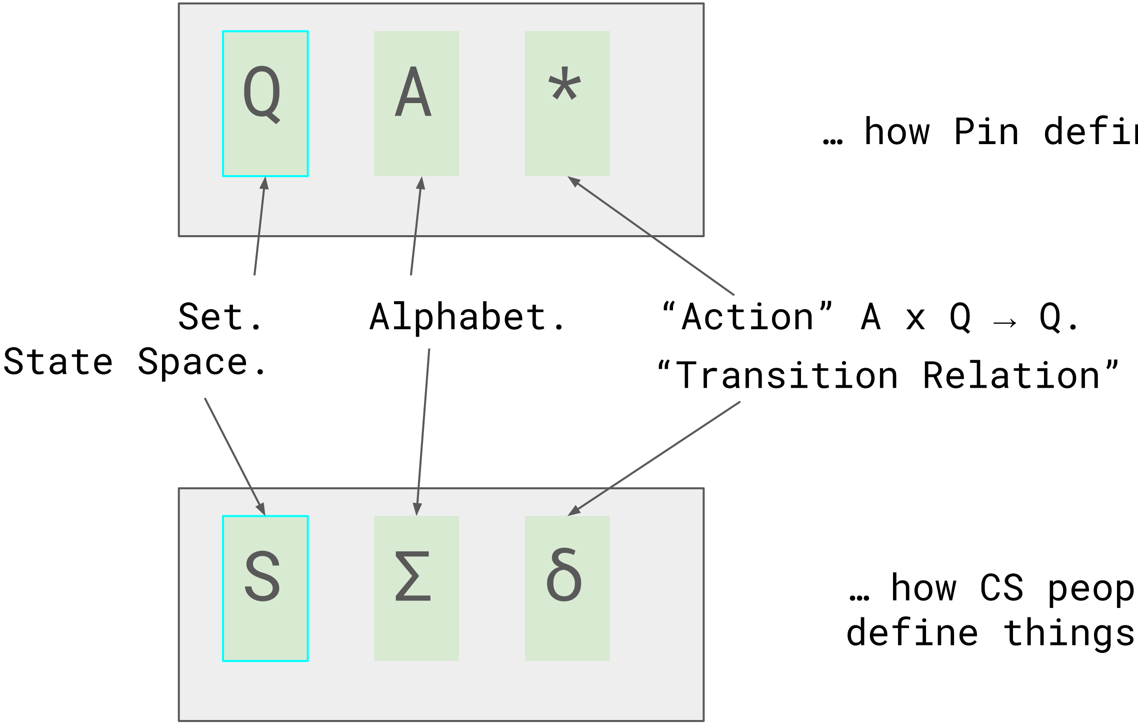

… how Pin defines things

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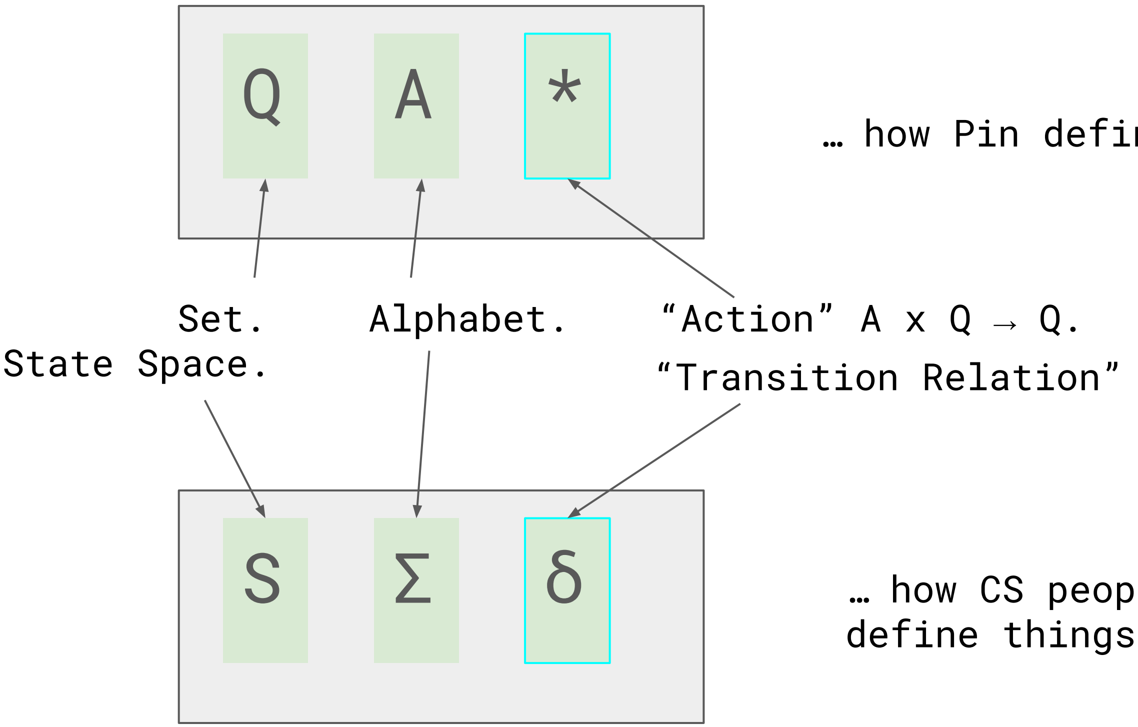
Finite if Q (aka Σ) is finite.

Finite Deterministic Automata



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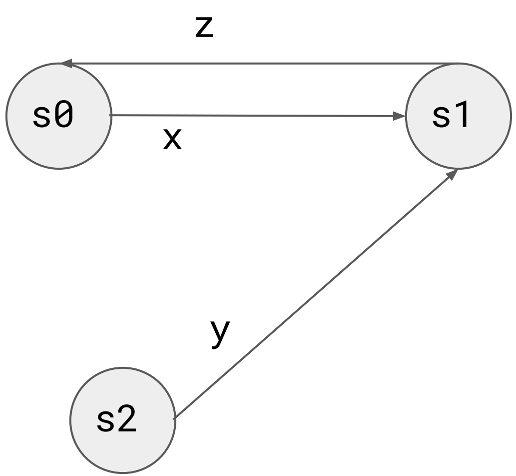
Finite if Q (aka Σ) is finite.

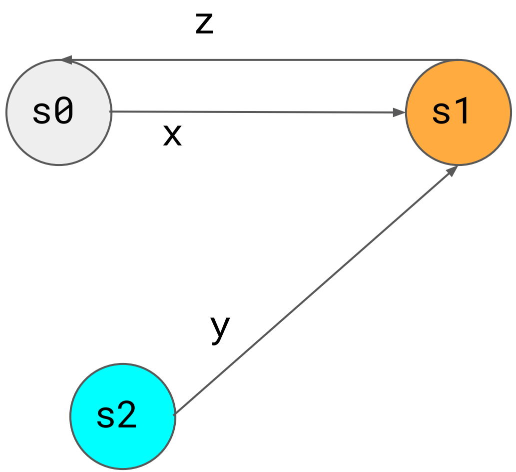
Deterministic if * (aka δ) is a function.

Finite Deterministic Automata



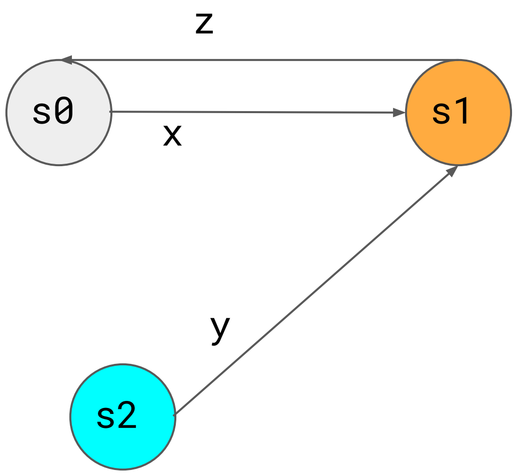

```
Q = { s0, s1, s2 } // aka S
A = { x, y, z }    // aka Σ

s0 * x = s1 // δ(s0,x)=s1
s1 * z = s0 // δ(s1,z)=s0
s2 * y = s1 // δ(s2,y)=s1
```

Finite Deterministic Automata



We can extend our definition to include one or more **initial states** and **final states**.

Finite Deterministic Automata



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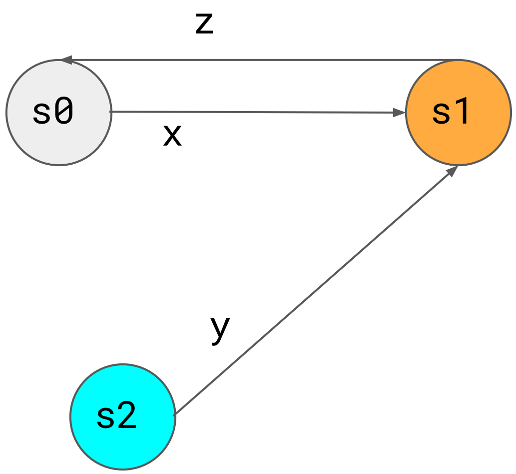
A **path** = a sequence of consecutive transitions.

Finite Deterministic Automata



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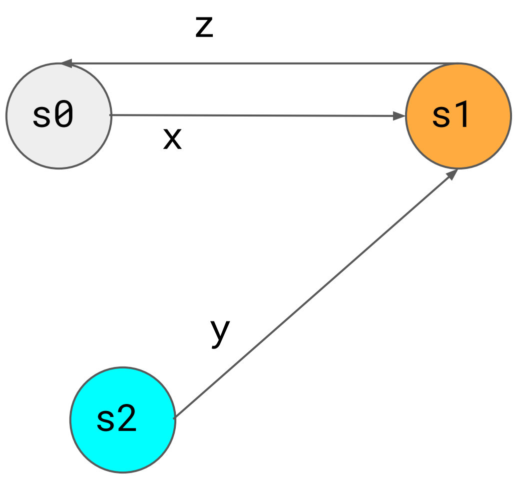

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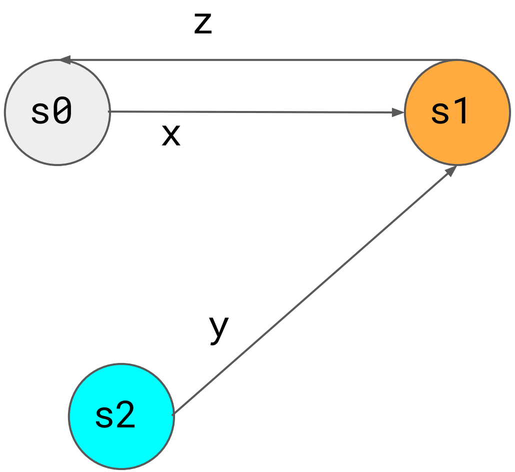
A **path** = a sequence of consecutive transitions.

- "Final" if ends in a final state.

Finite Deterministic Automata

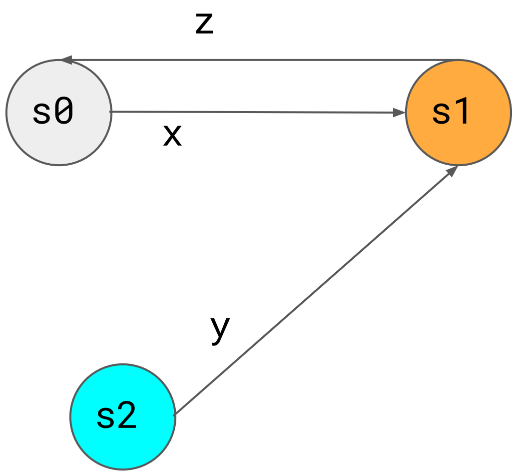


We can extend our definition to include one or more **initial states** and **final states**.

A **path** = a sequence of consecutive transitions.

- “Final” if ends in a final state.
- “Initial” if starts in an initial state.

Finite Deterministic Automata



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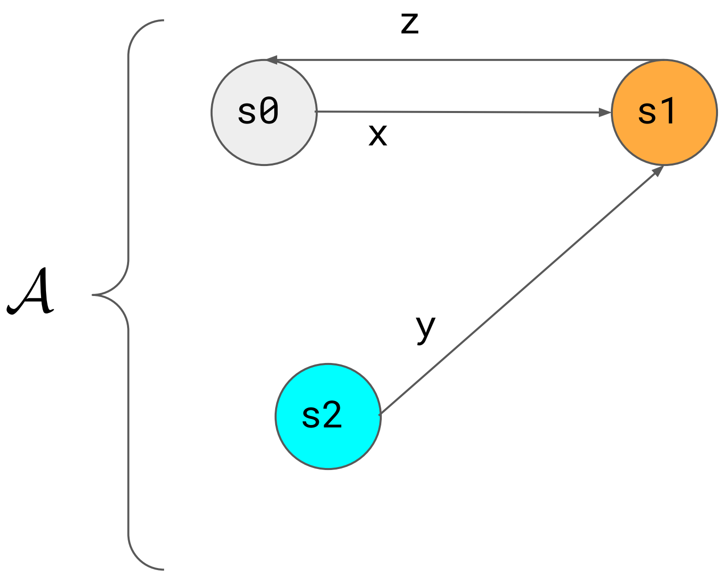
A **path** = a sequence of consecutive transitions.

- "Final" if ends in a final state.
- "Initial" if starts in an initial state.
- "Successful" if both Final and Initial.

Finite Deterministic Automata



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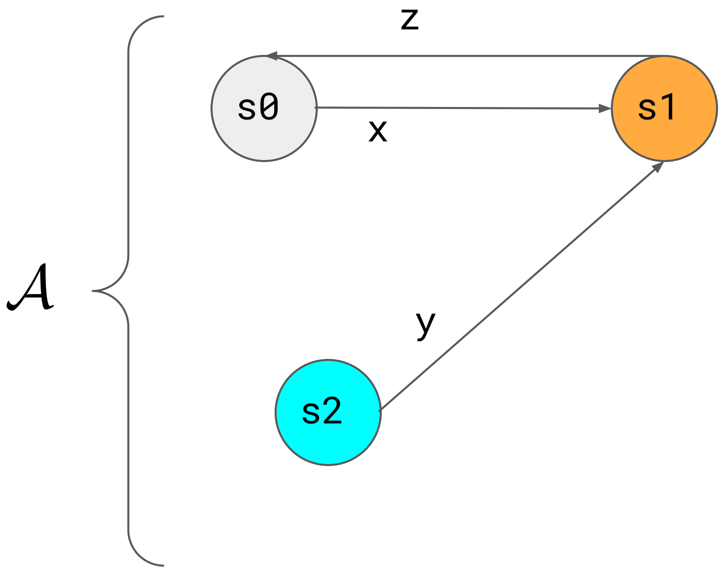

We can extend our definition to include one or more **initial states** and **final states**.

$\mathscr{A}$ accepts the language L iff there exists initial states I and final states F s.t. L equals all of the successful paths from I to F.

{ y, yzx, yzxzx, yzxzxzx, … }

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Let A be a finite alphabet. Rat(A*), is the smallest subset of A* satisfying all the following:

…

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- Rat(A*) contains { }
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Aka “Rational”.

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Let A be a finite alphabet. Rat(A*), is the smallest subset of A* satisfying all the following:

- Rat(A*) contains { }
- If a is in A then { a } is in Rat(A*)
- If X and Y are in Rat(A*) then so are X Y, X U Y, and X*.

Aka “Rational”.

Theorem: Every finite language is regular.

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A*abaA* is the “regular expression” defining the “regular language” { $a_0 \dots a_k$ aba $a_{k+1} \dots a_{k+t}$ | a_i in A and t, k naturals }

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6. Metric Spaces // the secret ingredient to ML

// Plus a short note at the end about how Pin denotes functions …

Monoids

M * 1

Monoids

M   *   1

Set.

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Operator M x M → M satisfying
(associativity) (n * m) * k = n * (m * k)

Monoids

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Identity: $m * 1 = 1 * m = m$

Set.

Operator $M \times M \rightarrow M$ satisfying
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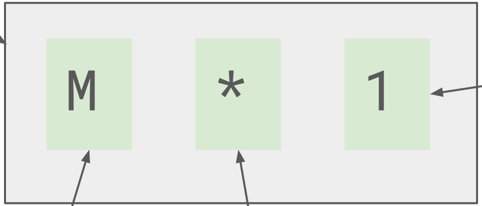
Monoids



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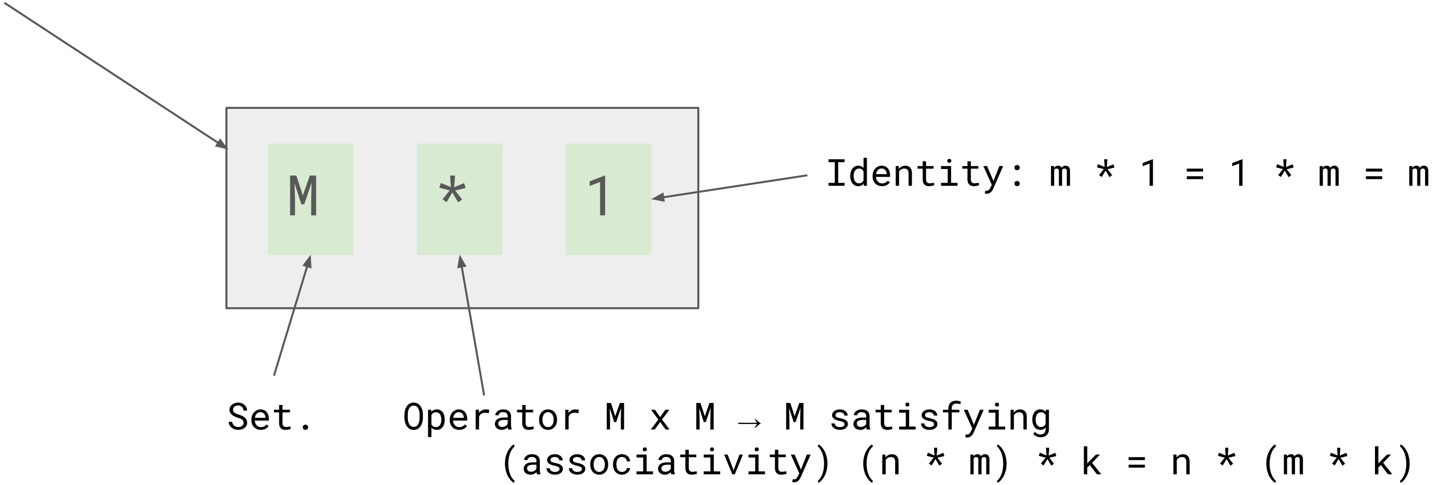

A* is the “free monoid” over A, with * = concatenation and 1 = “”.

Monoids

Suppose that φ is a function …

$$\varphi : \boxed{M \quad * \quad 1} \longrightarrow \boxed{N \quad @ \quad e}$$

… s.t. $\varphi(1)=e$, and $\varphi(mk)=\varphi(m)\varphi(k)$.

Then we say φ is a **morphism**.

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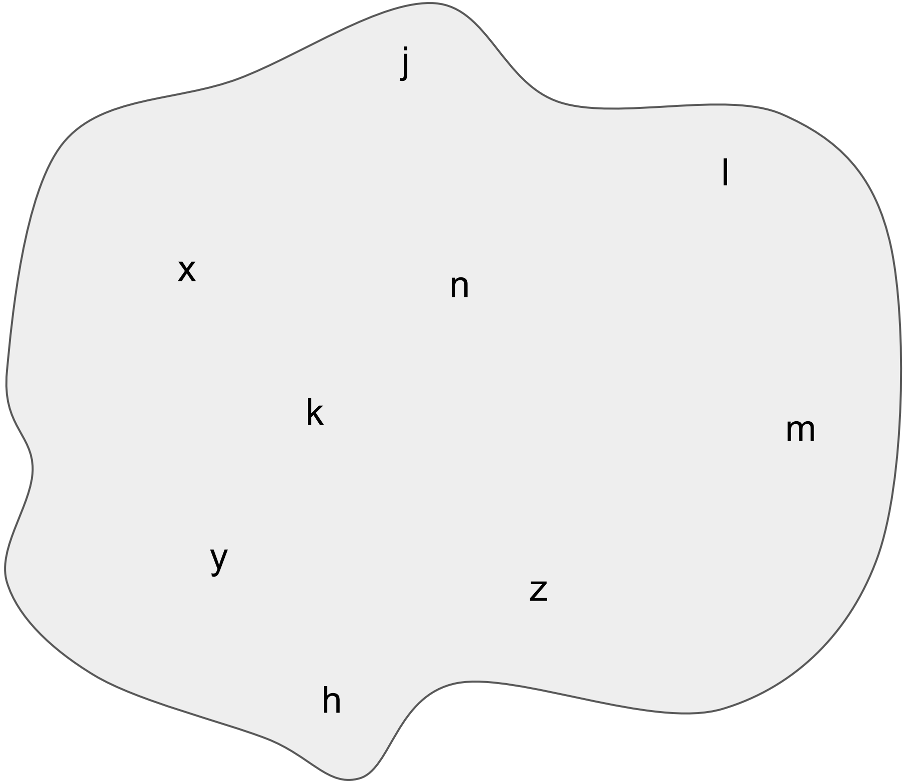
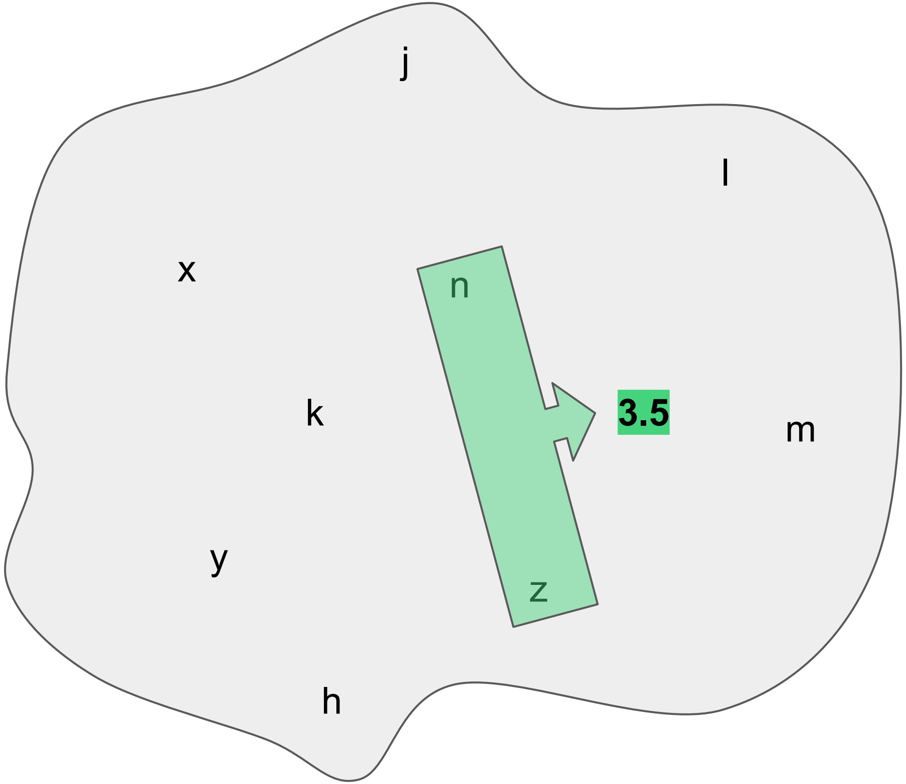


X is a Set.

Metric Spaces



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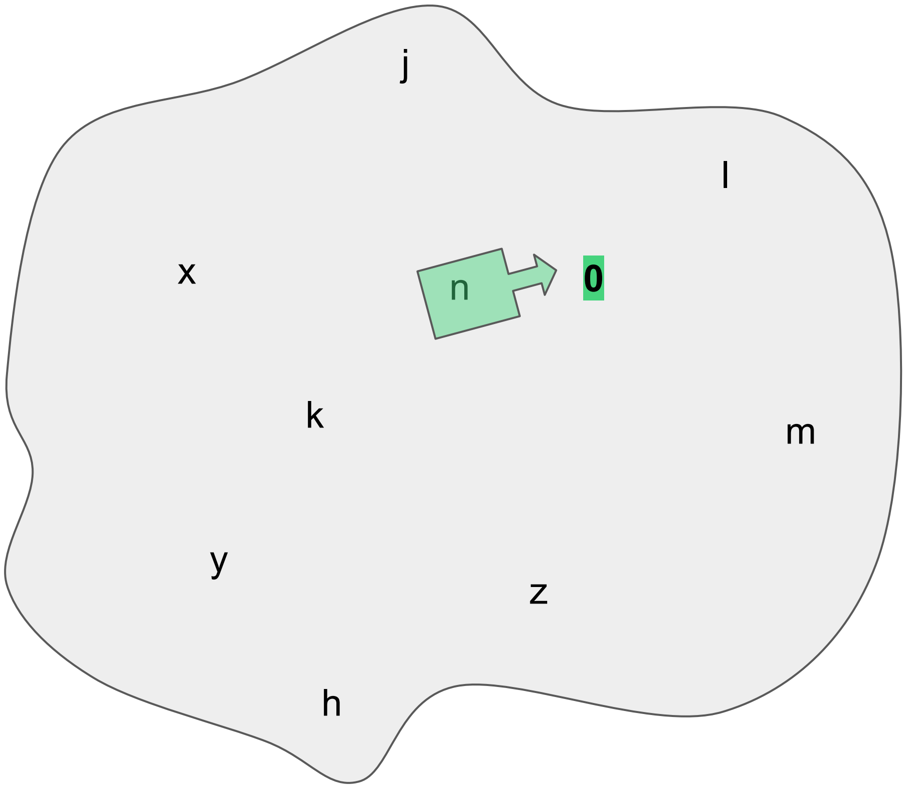

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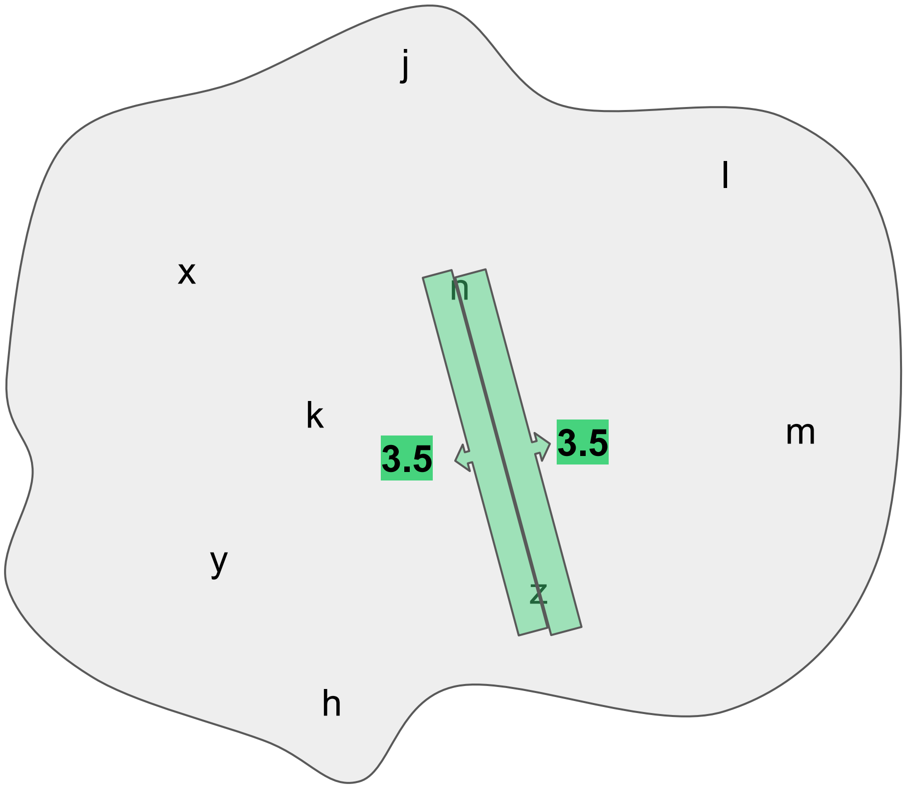

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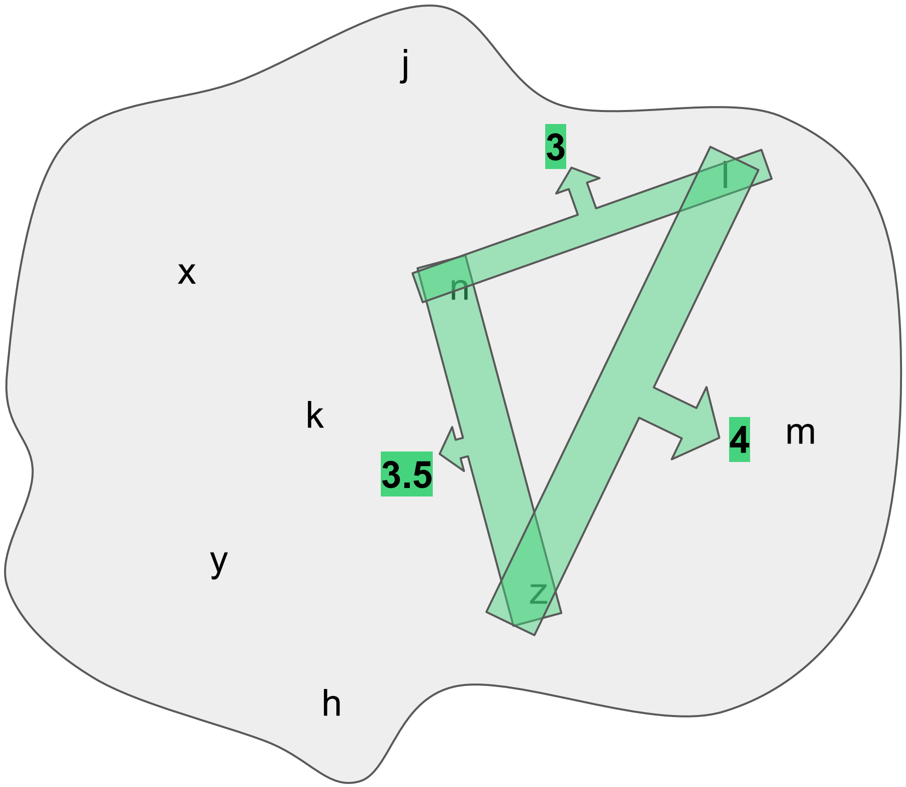

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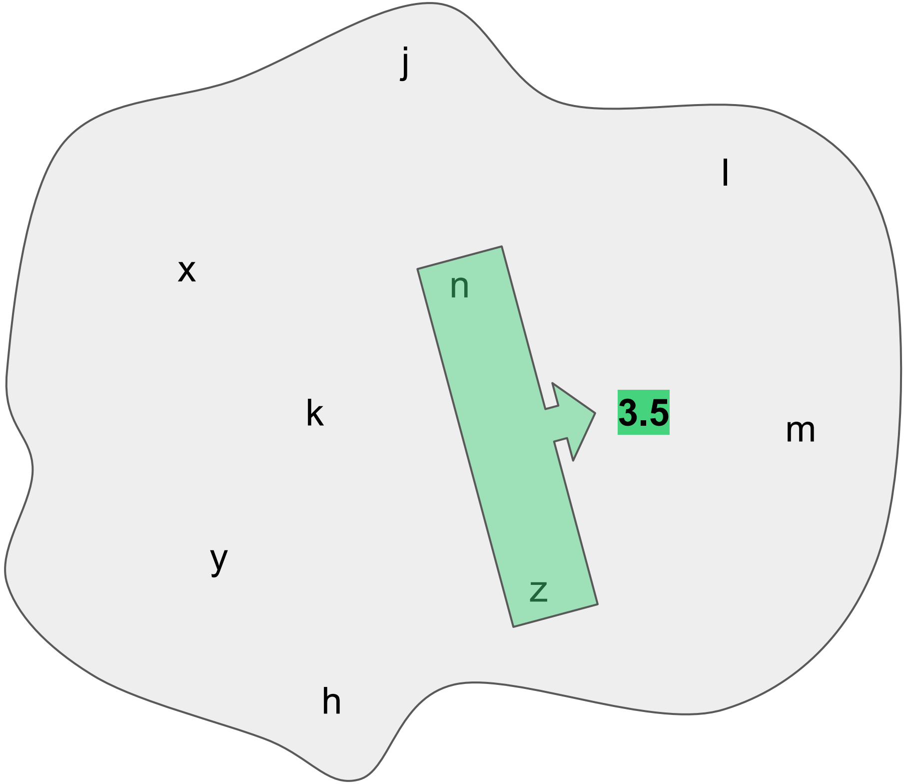

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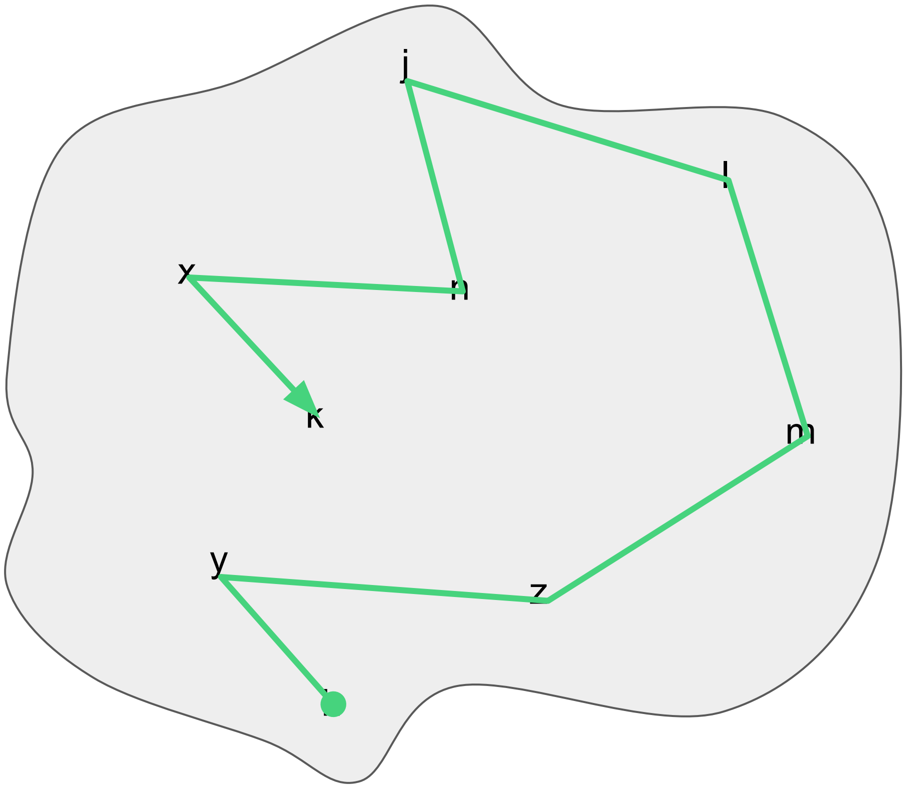
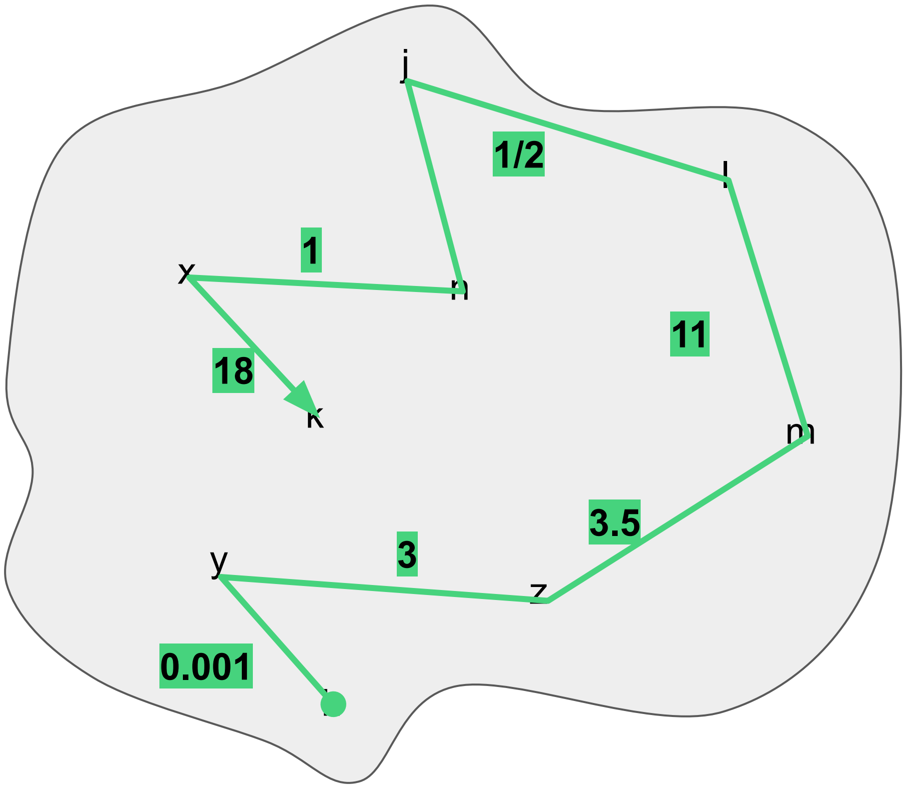
Then d is a “metric” on X, and (X, d) is a “metric space”.

Metric Spaces



Consider a sequence through the space.

Metric Spaces



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We can use our metric to measure the distance of every segment in the sequence.

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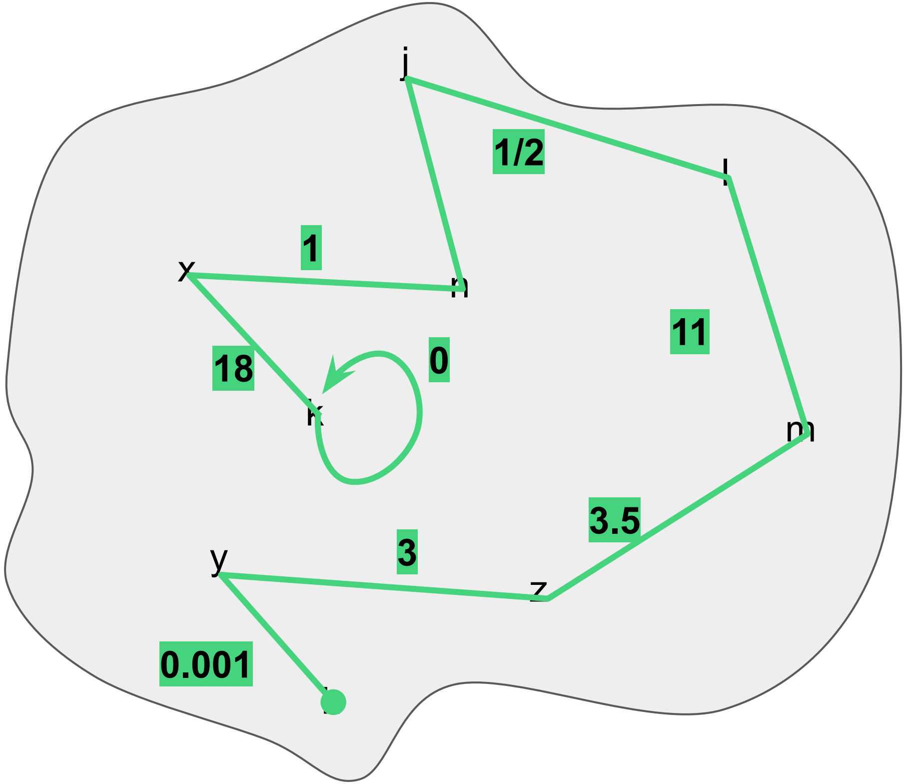

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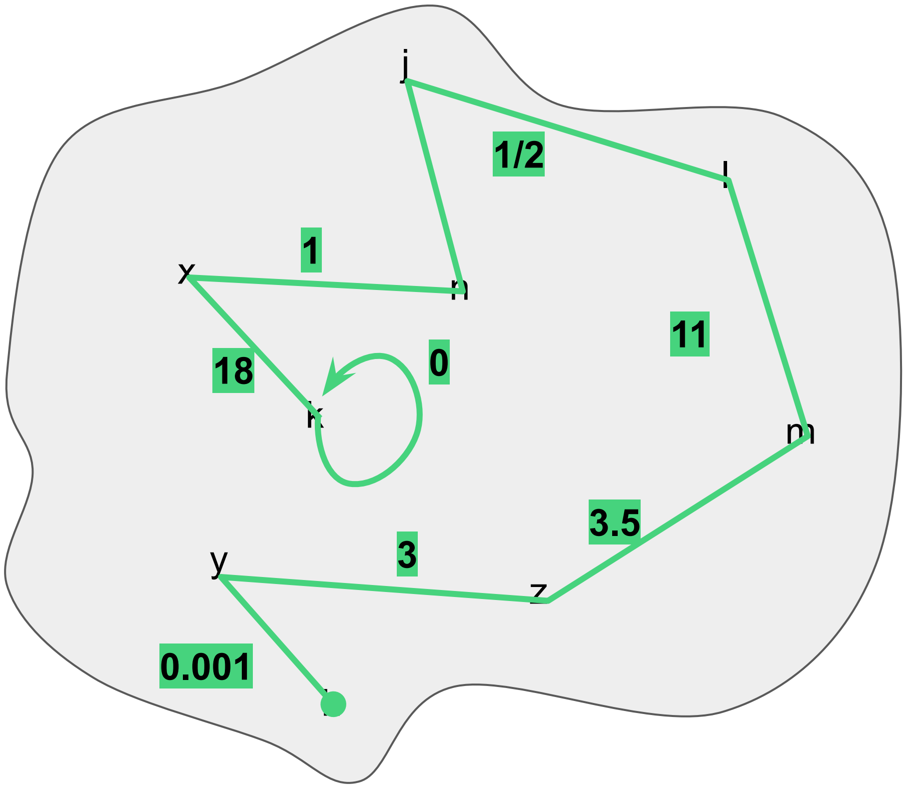

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If the sequence is infinite, and the distances converge to zero, then the sequence is “cauchy”.

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OBSERVE: A Cauchy sequence might *not* converge in the space.

Metric Spaces

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

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Suppose (X, d) is a metric space, X is a subset of Z.

We say X is *dense* in Z if for all z in Z, either z is in X, or there exists a Cauchy sequence in X which converges to z.

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QUESTION If (X, d) is *not* complete, can I “fill in the holes” to make it complete?

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Then ([C(x)], [D]) is a metric space called the “completion” of (X, d).

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The End.

Questions?