The Tiling Method

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1 Problem Statement

Consider a two-party communication problem, in which the participants

(a) Alice

and

(b) Bob

participate to compute a function:

$$f : \underbrace{\mathbb{B}^n}_{\text{Alice's input}} \times \underbrace{\mathbb{B}^n}_{\text{Bob's input}} \to \underbrace{\mathbb{B}}_{\text{global output}}$$

The players can come up with a *protocol* $\Pi = (p_1, \ldots, p_t)$, namely, for some natural $t \in \mathbb{N}$, a sequence of t-many functions $p_i : \mathbb{B}^* \to \mathbb{B}^*$ such that the communication between the players looks like this:

Alice is given input x.

Hi Bob. I'm not divulging x, but, $p_1(x) = \texttt{P1}$.

Bob is given input y.

Thanks Alice. I'm not divulging y, but, $p_2(y, \texttt{P1}) = \texttt{P2}$.

Thanks Bob. Don't tell anyone, but: $p_3(x, \texttt{P1}, \texttt{P2}) = \texttt{P3}$.

Is that so? Well, $p_4(y, \texttt{P1}, \texttt{P2}, \texttt{P3}) = \texttt{P4}$.

Wicked. In that case, $p_5(x, \texttt{P1}, \texttt{P2}, \texttt{P3}, \texttt{P4}) = \texttt{P5}$.

...yada yada yada...

$p_t(x, \texttt{P1}, \texttt{P2}, \texttt{P3}, \texttt{P4}, \ldots, \texttt{P(t-1)}) = \texttt{Pt}$, and TTFN!

Critically, the functions p_i can be *anything* so long as they are well-defined. For example, p_2 could be the function that asks if $\langle y, \texttt{P2} \rangle$ is a word in A_{TM}.

Definition 1 (Communication Complexity)**.** *Suppose* Π *is a protocol for* f *in which at most* t *bits are communicated between Alice and Bob. Then the* communication complexity *of* Π*, denoted* $C(\Pi)$*, is* t*. The communication complexity of* f*, denoted* $C(f)$*, is the minimum communication complexity achieved by any protocol for* f*.*

Given some such function f, it would be nice if we could automatically compute a reasonable lower bound on its communication complexity.

2 The Tiling Method

One way to do this is with the *tiling method*. We will give the method immediately, and in tandem, we will illustrate the method using the function $f(x,y) = x < y$ where x,y are integers in $\{0,1,2,3\}$, encoded in $\mathbb{B}$oolean.

Definition 2 ($M(f)$)**.** *The* matrix of f*, denoted* $M(f)$*, is the* $2^n \times 2^n$ *matrix whose* (x,y)*th entry is the value* $f(x,y)$*. An example is given in Table 1.*

	000 = 0	001 = 1	010 = 2	011 = 3	100 = 4
000 = 0	0	1	1	1	1
001 = 1	0	0	1	1	1
010 = 2	0	0	0	1	1
011 = 3	0	0	0	0	1
100 = 4	0	0	0	0	0

Table 1: The matrix $M(<)$ for inputs $x,y \in \{0,1,2,3\}$. Values of x are given in the rows, while values of y are given in the columns. False (i.e. 0) values are marked red for clarity.

Definition 3 (Combinatorial Rectangle)**.** *A* combinatorial rectangle *in* $M(f)$ *is any submatrix of* M*. We say a rectangle* $A \times B$ *in* $M(f)$ *is* monochromatic *if for all* x,x' *in* A *and* y,y' *in* B*,* $M_{x,y} = M_{x',y'}$*. Examples are given in Figure 2.*

And now for the punchline!

Theorem 1 (The Tiling Method)**.** *Denote by* $\chi(f)$ *the minimum number of rectangles in any monochromatic tiling of* $M(f)$*. Then* $\log_2 \chi(f) \le C(f) \le 16\left(\log_2 \chi(f)\right)^2$.

	0	1	2	3	4
0	0	1	1	1	1
1	0	0	1	1	1
2	0	0	0	1	1
3	0	0	0	0	1
4	0	0	0	0	0

(a) Colored 0.

	0	1	2	3	4
0	0	1	1	1	1
1	0	0	1	1	1
2	0	0	0	1	1
3	0	0	0	0	1
4	0	0	0	0	0

(b) Colored 0 & 1.

	0	1	2	3	4
0	0	1	1	1	1
1	0	0	1	1	1
2	0	0	0	1	1
3	0	0	0	0	1
4	0	0	0	0	0

(c) Colored 0 & 1.

Figure 2: Some example rectangles of $M(<)$. The first rectange, in purple, is monochromatically colored 0. The second rectangle, in orange, illustrates the flexibility of our rectangle definition, namely, that the rectangle does not actually need to be connected in the original matrix. Neither the second nor third rectangle is monochromatic.