Communication Complexity Examples

Parity

$f(x,y) = \bigoplus xy$

π

Player 1	Player 2
$x=101$	$y=110$
$P_1(101)=\bigoplus 101=1 \rightarrow$	$P_2(110,1)=(\bigoplus 110)\oplus 1$
$p_2=1 \leftarrow$	

$C(\pi)=2$

$C(f)$ is <u>at least</u> 2 because f depends on both x and y so there must be some communication.

Thus $C(f) \geq 2$ via π

Halting

Function $H: \underset{\substack{\uparrow \\ x}}{\{0,1\}^n} \times \underset{\substack{\uparrow \\ y}}{\{0,1\}^x} \rightarrow \{0,1\}$

$x=1^n$ $\quad y=\langle M\rangle$

H returns 1 if M halts on x

π

Player 1	Player 2
$x=1^{10}$	$y=\langle M_{accept}\rangle$
$P_1(1^{10})=1 \longrightarrow$	$P_2(\langle M\rangle,1) = M(1^{\lvert\langle M\rangle\rvert}) = M(1^{10}) = 1$
$P_2 = 1 \longleftarrow$	

In communication complexity problems, both players have <u>unlimited</u> computational power. This allows Player 2 to solve the halting problem.

This property allows us to focus on soley the communication between players

Multi-party Communication

So far, we've dealt with functions computed by the communication of 2 parties.

This can be generalized to multi-party functions.

Problem Setup:

In 2 party problems, Player 1 knew nothing about Player 2's input and Player

2 know nothing about Player 1's input

Howver, in n-party problems:

Player i knows the input of every player except itself.

<u>And</u>

Messages are broadcasted to all other players

What does that mean?

Consider a "real-life" example.

All of us place a sticky note with some value on our heads (we don't know what's on our sticky note) and we stand in front of a whiteboard.

The only way we're allowed to communicate is by writing on the whiteboard for all to see.

The goal is for one player, after some amount of communication, to write the output of function f on the board.

Let's look at a real example:

$$f(x_1, x_2, x_3) = \bigoplus_{i=1}^{n} maj(x_{1i}, x_{2i}, x_{3i})$$

$f(1101, 1001, 1011) =$

$\_\ \_\ \_\ \_$

$1\ \_\ \_\ \_$

$1\ 0\ \_\ \_$

$1\ 0\ 0\ \_$

$\oplus 1001 = 0$

Π

Player 1	Player 2	Player 3
$x_2 = 1001$	$x_1 = 1101$	$x_1 = 1101$
$x_3 = 1011$	$x_3 = 1011$	$x_2 = 1001$
$\oplus 10\_1$	$\oplus 1\_\_1$	$\oplus 1\_01$
$p_1 = 0$	$p_2 = 0$	$p_3 = 0$

$\oplus p_1 p_2 p_3 = \oplus 000 = 0$

$C(\Pi) = 3$

By the same argument as 2-party

parity, f depends on all 3 inputs
thus the lowest $C(f)=3$.

By showing that Π exists for f,
$C(f) \geq 3$

Nondeterministic Protocols

"Nondeterministic communication complexity"

Defined akin to NP

Each players are given their input along with some z of length m that may depend on the given inputs.

z is a nondeterministic guess

Otherwise the protocol is deterministic

$f(x,y)=1$ iff $\exists z$ that makes the players output 1
$C(F)= m+$ communication

Inequality / intersection example?

coNP is defined similarly

$g(x,y) = 1 - f(x,y)$

However, it can be shown that in terms of communication complexity:

$NP \cap coNP = P$

This is shown by relating the communication complexities of $f \in NP$ and $\bar{f} \in coNP$.

$C(f) = k$

$C(\bar{f}) = 10k\ell$ for some complexity ℓ