Communication Complexity

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Communication Complexity

If Alice knows x, and Bob knows y, how many bits of information must they communicate, in order for both Alice and Bob to know $f(x, y)$?

(Max)

Consider a two-party communication problem, in which the participants

(a) Alice

and

(b) Bob

participate to compute a function:

$$f : \underbrace{\mathbb{B}^n}_{\substack{\text{Alice's} \\ \text{input}}} \times \underbrace{\mathbb{B}^n}_{\substack{\text{Bob's} \\ \text{input}}} \to \underbrace{\mathbb{B}}_{\substack{\text{global} \\ \text{output}}}$$

(Max)

The players can come up with a *protocol* $\Pi = (p_1, \ldots, p_t)$, namely, for some natural $t \in \mathbb{N}$, a sequence of t-many functions $p_i : \mathbb{B}^* \to \mathbb{B}^*$ such that the communication between the players looks like this ...

(Max)

Alice is given input x.

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Hello Bob. I can't reveal x, but $p_1(x)$ is `p1`.

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> Hello Bob. I can't reveal x, but $p_1(x)$ is $\texttt{p1}$.

Bob is given input y.

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Hello Bob. I can't reveal x, but $p_1(x)$ is $\mathtt{p1}$.

Bob is given input y.

Thanks Alice. I can't reveal y, but $p_2(y, \mathtt{p1})$ is $\mathtt{p2}$.

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Pleasure doing business with you Bob. My final clue for you is that $p_{n-1}(x, \texttt{p1}, \ldots, \texttt{pn-2})$ is $\texttt{pn-1}$.

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Rad. Then $p_n(y, \mathtt{p1}, \ldots, \mathtt{pn-1})$ is $\mathtt{pn}$. TTFN!

(Max)

- The functions p_i can be *anything* so long as they are well-defined. E.g., could solve the Halting Problem.
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Suppose Π is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the *communication complexity* of Π, denoted $C(\Pi)$, is t.

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Definition ($C(f)$)

The communication complexity of f, denoted $C(f)$, is the minimum communication complexity achieved by any protocol for f.

Parity (Jake)

Example (Are the number of 1s in xy even (0), or odd (1)?)

$$f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B} \text{ is the function } (x, y) \mapsto \bigoplus xy$$

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Example protocol Π:

P1 = parity(x).

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$\mathrm{P1} = \mathrm{parity}(x)$.

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Now both Alice and Bob know $f(x, y) = \mathrm{P2}$. $C(f) \leq 2$ because $C(\Pi) = 2$ and Π implements f. But $C(f) \geq 2$ because f depends on x and y. Hence $C(f) = 2$.

Halting (Jake)

Example 2: Halting

Function $H : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

$x = 1^n$ and $y =< M >$

H returns 1 if M halts on x

Protocol Π

Player 1	Player 2
$x = 1^{10}$	$y =< M_{accept} >$
$P_1(1^{10}) = 1 \longrightarrow$	
	$p_1 = 1$
	$P_2(y, p_1) = M_{accept}(1^{\|<M_{acce}$
	$\longleftarrow M(1^{10}) = 1$
$p_2 = 1$	

In communication complexity problems, both players have unlimited computation power. This allows Player 2 to solve the

Methods

TODO

Fooling Set Method (Jake)

TODO

Tiling Method (Max)

TODO

2-Party Discrepency Method (Max)

TODO

Multi-Party Generalization (Jake)

TODO

Methods

TODO

Multi-Party Discrepency Method (Max)

TODO

Non-Deterministic (Jake)

TODO

Randomized (Max)

TODO