

Communication Complexity

Jake Kinsella
and Max von Hippel

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2-Party Discrepancy
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Randomized

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If Alice knows x , and Bob knows y , how many bits of information must they communicate, in order for both Alice and Bob to know $f(x, y)$?

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Consider a two-party communication problem, in which the participants



(a) Alice

and



(b) Bob

participate to compute a function:

$$f : \underbrace{\mathbb{B}^n}_{\text{Alice's input}} \times \underbrace{\mathbb{B}^n}_{\text{Bob's input}} \rightarrow \underbrace{\mathbb{B}}_{\text{global output}}$$

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The players can come up with a *protocol* $\Pi = (p_1, \dots, p_t)$, namely, for some natural $t \in \mathbb{N}$, a sequence of t -many functions $p_i : \mathbb{B}^* \rightarrow \mathbb{B}^*$ such that the communication between the players looks like this ...

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Alice is given input x .

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Alice is given input x .

Hello Bob. I can't reveal x , but $p_1(x)$ is p_1 .

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Alice is given input x .

Hello Bob. I can't reveal x , but $p_1(x)$ is p_1 .

Bob is given input y .

Thanks Alice. I can't reveal y , but $p_2(y, p_1)$ is p_2 .

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Pleasure doing business with you Bob. My final clue for you is that $p_{n-1}(x, p_1, \dots, p_{n-2})$ is p_{n-1} .

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Pleasure doing business with you Bob. My final clue for you is that $p_{n-1}(x, p_1, \dots, p_{n-2})$ is p_{n-1} .

Rad. Then $p_n(y, p_1, \dots, p_{n-1})$ is p_n . TTFN!

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- The functions p_i can be *anything* so long as they are well-defined. E.g., could solve the Halting Problem.
- After the final message, *both parties* must know $f(x, y)$.

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Definition (Communication Complexity)

Suppose Π is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the *communication complexity* of Π , denoted $C(\Pi)$, is t .

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Definition (Communication Complexity)

Suppose Π is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the *communication complexity* of Π , denoted $C(\Pi)$, is t .

Definition ($C(f)$)

The communication complexity of f , denoted $C(f)$, is the minimum communication complexity achieved by any protocol for f .

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Example (Are the number of 1s in xy even (0), or odd (1)?)

$f : \mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{B}$ is precisely $(x, y) \mapsto \bigoplus xy$.

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Example protocol Π :

$$P1 = \text{parity}(x).$$

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Example protocol Π :

$$P1 = \text{parity}(x).$$

$$P2 = \text{parity}(y) \oplus P1$$

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Example (Are the number of 1s in xy even (0), or odd (1)?)

$f : \mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{B}$ is precisely $(x, y) \mapsto \bigoplus xy$.

Example protocol Π :

$$P1 = \text{parity}(x).$$

$$P2 = \text{parity}(y) \oplus P1$$

Now both Alice and Bob know $f(x, y) = P2$. $C(f) \leq 2$ because $C(\Pi) = 2$ and Π implements f . But $C(f) \geq 2$ because f depends on x and y . Hence $C(f) = 2$.

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Example (A_{TM})

$H : \mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{B}$ is precisely $\langle M, x \rangle \mapsto 1$ if M halts on x else 0.

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Example protocol Π :

$P1 = y.$

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Example (A_{TM})

$H : \mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{B}$ is precisely $\langle M, x \rangle \mapsto 1$ if M halts on x else 0.

Example protocol Π :

$P1 = y.$

M does/doesn't accept $y.$

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Example (A_{TM})

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Example protocol Π :

$P1 = y.$

M does/doesn't accept $y.$

Both players have unlimited computation power. We are only interest in communication complexity.

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