The Tiling Method

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Consider a two-party communication problem, in which the participants

(a) Alice

and

(b) Bob

participate to compute a function:

$$f : \underbrace{\mathbb{B}^n}_{\substack{\text{Alice's} \\ \text{input}}} \times \underbrace{\mathbb{B}^n}_{\substack{\text{Bob's} \\ \text{input}}} \rightarrow \underbrace{\mathbb{B}}_{\substack{\text{global} \\ \text{output}}}$$

The players can come up with a *protocol* $\Pi = (p_1, \ldots, p_t)$, namely, for some natural $t \in \mathbb{N}$, a sequence of t-many functions $p_i : \mathbb{B}^* \to \mathbb{B}^*$ such that the communication between the players looks like this:

Alice is given input x.

Hi Bob. I'm not divulging x, but, $p_1(x) = \texttt{P1}$.

Bob is given input y.

Thanks Alice. I'm not divulging y, but, $p_2(y, \texttt{P1}) = \texttt{P2}$.

Thanks Bob. Don't tell anyone, but: $p_3(x, \texttt{P1}, \texttt{P2}) = \texttt{P3}$.

Is that so? Well, $p_4(y, \texttt{P1}, \texttt{P2}, \texttt{P3}) = \texttt{P4}$.

Wicked. In that case, $p_5(x, \texttt{P1}, \texttt{P2}, \texttt{P3}, \texttt{P4}) = \texttt{P5}$.

...yada yada yada...

In that case, the last thing you need to know is that $p_t(x, \texttt{P1}, \texttt{P2}, \texttt{P3}, \texttt{P4}, \ldots, \texttt{P(t-1)}) = \texttt{Pt}$.

Suppose that there is a protocol Π for f consisting of t messages, but, there does not exist any protocol Π' for f consisting of fewer than t messages. Then we say t is the *communication complexity* of f, and we write $C(f) = t$.

Given some such function f, it would be nice if we could automatically compute a reasonable lower bound on its communication complexity. One way to do this is with the *tiling method*. We will give the method immediately, and in tandem, we will illustrate the method using the function $f(x,y) = x < y$ where x, y are integers in $\{0,1,2,3\}$, encoded in $\mathbb{B}$oolean. First, let $M(f)$ be the *matrix of* f, namely, the $2^n \times 2^n$ matrix whose (x,y)th entry is the value $f(x,y)$.

	0	1	2	3	4
0	0	1	1	1	1
1	0	0	1	1	1
2	0	0	0	1	1
3	0	0	0	0	1
4	0	0	0	0	0

Table 1: The matrix $f(<)$ for inputs $x, y \in \{0,1,2,3\}$. Values of x are given in the rows, while values of y are given in the columns. False (i.e. 0) values are marked red for clarity.

TODO...