## Communication Complexity

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Fooling Set Method
Tiling Method
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Other Variants

Non-Determinist Randomized

## Communication Complexity

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## Communication Complexity

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If Alice knows x, and Bob knows y, how many bits of information must they communicate, in order for both Alice and Bob to know f(x,y)?

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Discrepency Method
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Randomized

Consider a two-party communication problem, in which the participants





(a) Alice

and (b) Bob

participate to compute a function:

$$f: \underline{\mathbb{B}^n} \times \underline{\mathbb{B}^n} \to \underline{\mathbb{B}}$$
Alice's Bob's global input input output

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namely, for some natural  $t \in \mathbb{N}$ , a sequence of t-many functions  $p_i: \mathbb{B}^* \to \mathbb{B}^*$  such that the communication between the players looks like this ...

The players can come up with a protocol  $\Pi = (p_1, ..., p_t)$ ,

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Alice is given input x.

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Alice is given input x.

Hello Bob. I can't reveal x, but  $p_1(x)$  is p1.

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## Alice is given input x.

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Bob is given input y.

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Non-Determinis Randomized

### Alice is given input x.

Hello Bob. I can't reveal x, but  $p_1(x)$  is p1.

Bob is given input y.

hanks Alice. I can't reveal y, but  $p_2(y, p1)$  is p2

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... yada yada yada ...

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### Alice is given input x.

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Bob is given input y.

Thanks Alice. I can't reveal y, but  $p_2(y,p1)$  is p2.

... yada yada yada ...

Pleasure doing business with you Bob. My final clue for you is that  $p_{n-1}(x, p1, ..., pn-2)$  is pn-1.

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## Alice is given input x.

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Thanks Alice. I can't reveal y, but  $p_2(y,p1)$  is p2.

... yada yada yada ...

Pleasure doing business with you Bob. My final clue for you is that  $p_{n-1}(x, p_1, ..., p_{n-2})$  is  $p_{n-1}(x, p_1, ..., p_{n-2})$ 

Rad. Then  $p_n(y, p1, ..., pn-1)$  is pn. TTFN!

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- The functions  $p_i$  can be anything so long as they are well-defined. E.g., could solve the Halting Problem.
- After the final message, both parties must know f(x, y).

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## Other Variant Non-Deterministic

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## Definition (Communication Complexity)

Suppose  $\Pi$  is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the communication complexity of  $\Pi$ , denoted  $C(\Pi)$ , is t.

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## Definition (C(f))

The communication complexity of f, denoted C(f), is the minimum communication complexity achieved by any protocol for f.

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Non-Deterministi Randomized Example (Are the number of 1s in xy even (0), or odd (1)?)

 $f: \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$  is precisely  $(x, y) \mapsto \bigoplus xy$ .

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## Example protocol $\Pi$ :

P1 = 
$$parity(x)$$
.

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## Example (Are the number of 1s in xy even (0), or odd (1)?)

 $f: \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$  is precisely  $(x, y) \mapsto \bigoplus xy$ .

## Example protocol $\Pi$ :

P1 = 
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 $P2 = parity(y) \oplus P1$ 

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## Example (Are the number of 1s in xy even (0), or odd (1)?)

 $f: \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$  is precisely  $(x, y) \mapsto \bigoplus xy$ .

## Example protocol $\Pi$ :

P1 = 
$$parity(x)$$
.

 $P2 = parity(y) \oplus P1$ 

Now both Alice and Bob know f(x,y) = P2.  $C(f) \le 2$  because  $C(\Pi) = 2$  and  $\Pi$  implements f. But  $C(f) \ge 2$  because f depends on x and y. Hence C(f) = 2.

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## Example $(A_{TM})$

 $H: \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$  is precisely  $\langle M, x \rangle \mapsto 1$  if M halts on x else 0.

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## Example $(A_{TM})$

 $H: \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$  is precisely  $\langle M, x \rangle \mapsto 1$  if M halts on x else 0.

## Example protocol $\Pi$ :

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## Example $(A_{TM})$

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Example protocol  $\Pi$ :

$$y_1 = y$$
.

P2= 
$$(M \text{ does/doesn't accept } y)$$
.

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Example protocol  $\Pi$ :

$$P1 = y$$
.

P2= 
$$(M \text{ does/doesn't accept } y)$$
.

Both players have unlimited computation power. We are only interest in communication complexity.

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If we find a protocol  $\Pi$ , then we know C(f) is at most  $C(\Pi)$ .

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If we find a protocol  $\Pi$ , then we know C(f) is at most  $C(\Pi)$ . What if we don't know any protocol  $\Pi$ ?

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If we find a protocol  $\Pi$ , then we know C(f) is at most  $C(\Pi)$ . What if we don't know any protocol  $\Pi$ ?

■ Could we upper-bound C(f) without knowing  $\Pi$ ?

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What if the only protocols we find seem really lousy?

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- Could we upper-bound C(f) without knowing  $\Pi$ ?
- What if the only protocols we find seem really lousy?
  - Could we lower-bound C(f) without finding a better protocol?

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■ Could we upper-bound C(f) without knowing  $\Pi$ ?

What if the only protocols we find seem really lousy?

■ Could we lower-bound C(f) without finding a better protocol?

TL;DR: yup.

## Fooling Set Method (Jake)

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