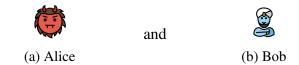
## The Tiling Method

Max von Hippel March 30, 2021

## 1 Problem Statement

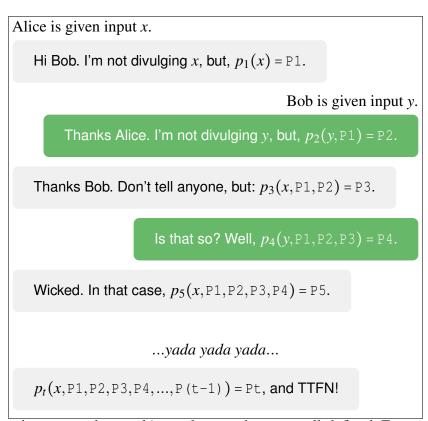
Consider a two-party communication problem, in which the participants



participate to compute a function:

$$f: \underbrace{\mathbb{B}^n}_{\text{Alice's Bob's global}} \rightarrow \underbrace{\mathbb{B}}_{\text{output}}$$
input input output

The players can come up with a *protocol*  $\Pi = (p_1, ..., p_t)$ , namely, for some natural  $t \in \mathbb{N}$ , a sequence of *t*-many functions  $p_i : \mathbb{B}^* \to \mathbb{B}^*$  such that the communication between the players looks like this:



Critically, the functions  $p_i$  can be *anything* so long as they are well-defined. For example,  $p_2$  could be the function that asks if  $\langle y, P2 \rangle$  is a word in  $A_{TM}$ .

**Definition 1** (Communication Complexity). Suppose  $\Pi$  is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the communication complexity of  $\Pi$ , denoted  $C(\Pi)$ , is t. The communication complexity of f, denoted C(f), is the minimum communication complexity achieved by any protocol for f.

Given some such function f, it would be nice if we could automatically compute a reasonable lower bound on its communication complexity.

## 2 The Tiling Method

One way to do this is with the *tiling method*. We will give the method immediately, and in tandem, we will illustrate the method using the function f(x,y) = x < y where x,y are integers in  $\{0,1,2,3\}$ , encoded in  $\mathbb{B}$ oolean.

**Definition 2** (M(f)). The matrix of f, denoted M(f), is the  $2^n \times 2^n$  matrix whose (x,y)th entry is the value f(x,y).

**Definition 3** (Combinatorial Rectangle). A combinatorial rectangle in M(f) is any submatrix of M. We say a rectangle  $A \times B$  in M(f) is monochromatic if for all x, x' in A and y, y' in B,  $M_{x,y} = M_{x',y'}$ .

Each message-send event in a protocol  $\Pi$  splits M(f) into two or more combinatorial rectangles of still-possible values for f(x,y). An example is given below, using the LEASTSIGNFICANTBIT protocol for <, with Alice sending the first message.

				000	001	010	011	100			
			000	0	1	1	1	1			
			001	0	0	1	1	1			
			010	0	0	0	1	1			
			011	0	0	0	0	1			
			100	0	0	0	0	0			
	A	lice: "ɔ	$c = \underline{}$			Alice: " <i>x</i> =1"					
	000	001	010	011	100		1000	γ 1 001	010	011	100
000	0	1	1	1	1		000		. 010	011	100
010	0	0	0	1	1	001		0	1	1	1
100	0	0		0	0	013	0	0	0	0	1
100	U	U	U	U	U						

Figure 2: The matrix M(<) for inputs  $x, y \in \{0, 1, 2, 3\}$ . Values of x are given in the rows, while values of y are given in the columns. False (i.e. 0) values are marked red for clarity. We show how in the LEASTSIGNIFICANTBIT protocol, M(<) can be partitioned into two rectangles depending on the substance of the initial message sent by Alice.

Since every protocol has finite length, a run of a protocol can only split M(f) finitely many times. Hence each protocol  $\Pi$  of f induces a tree of combinatorial rectangles, rooted at M(f), where

the leaves represent the matrix of possible values of f(x,y) once the protocol has concluded. By definition, a protocol must conclude with both participants knowing the value f(x,y). Therefore the leaves of each such tree must be monochromatic.

Now we get to the punchline.

**Definition 4** (Monochromatic Tiling). A monochromatic tiling of M(f) is a partition of M(f) into disjoint monochromatic rectangles.

## It's thinking time.

Notice that if  $\Pi$  is a protocol for f, then the leaves of the tree induced by  $\Pi$  and rooted at M(f) clearly form a monochromatic tiling of M(f).

*Recall* from eons ago, when you were an undergrad and had to know useful things, that the number of leaves in a binary tree can be used to upper-bound its depth.

*Realize* that the depth of the binary tree induced by  $\Pi$  is exactly  $C(\Pi)$ .

*Observe* that although we made the math easy by assuming bit-sized messages, this idea clearly generalizes.

Let  $\chi(f)$  denote the minimum number of rectangles in any monochromatic tiling of M(f).

Theorem 1 (The Punchline).

$$log_2\chi(f) \le C(f) \le 16 \left(log_2\chi(f)\right)^2$$

*Proof.* We need to show the following.

- (a)  $\log_2 \chi(f) \le C(f)$
- (b) and,  $C(f) \le 16(\log_2 \chi(f))^2$ .

We prove (a) then (b). Suppose that f has communication complexity C(f). Then there exists a protocol  $\Pi$  in which at most C(f) bits are communicated between the two participants. For simplicity suppose each bit communicated is an individual message. Then  $\Pi$  induces a tree whose maximum depth is C(f), and which induces a monochromatic partition of M(f). Since every monochromatic partition of M(f) requires at least  $\chi(f)$  rectangles, clearly the tree induced by  $\Pi$  must have at least  $\chi(f)$  leaves. But we're talking about a binary tree so it immediately follows that the tree must have at least  $\log_2 \chi(f)$  depth. So (a) holds and we are done.

Now let's prove (b). Consider the function  $f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ . There are, by definition, at most  $\chi(f) \ge 1$  distinct values f(x, y).

If  $\chi(f) = 1$ , then there is only one possible value of f(x,y), and so C(f) = 0. In this case we get  $C(f) = 0 \le 0 = 16\log_2^2 1 = 16\log_2^2 \chi(f)$ , so (b) holds and we are done.

If  $\chi(f) = 2$ , then there are 2 possible values of f(x,y), say,  $\alpha$  and  $\beta$ . We can partition  $\mathbb{B}^n$  into the spaces X,Y where if  $x \in X$  then  $f(x,y) = \alpha$  if and only if  $y \in Y$ . Then Alice could send a single bit indicating if  $x \in X$ , and Bob could reply with a single bit indicating if  $y \in Y$ , at which point both Alice and Bob would immediately know the value of f(x,y). This protocol communication complexity 2, as only 2 bits need to be communicated. We conclude that if  $\chi(f) = 2$ , then  $2 = C(f) \le 16(\log_2 \chi(f))^2 = 16(\log_2 2)^2 = 16(1)^2 = 16$ , so (b) holds and we are done.

For the inductive step, suppose that whenever  $\chi(f) = k$  for some integer  $k \ge 2$ , it immediately follows that  $C(f) \le 16 \left(\log_2 \chi(f)\right)^2$ . We want to show the same holds for k+1. Suppose  $\chi(f) = k+1$ . Choose some possible value z = f(x,y) arbitrarily from the  $\le k+1$  options. Partition the space  $\mathbb{B}^n$  into sets X,y such that whenever  $x \in X$ , f(x,y) = z if and only if  $y \in Y$ . Have Alice send a single bit b to begin the protocol, indicating if  $x \in X$ , and have Bob reply with a bit b' indicating if  $y \in Y$ . At that point if f(x,y) = z the protocol is over, otherwise it reduces to a protocol over k possible colors and, by our inductive assumption, (b) holds.

Notice that the tiling method directly relates to the fooling set method. Specifically, if f has a fooling set with m pairs, then  $\chi(f) \ge m$ .

*Proof.* Suppose  $(x,y),(\bar{x},\bar{y})$  are two of the pairs in the fooling set. Then they cannot be in a monochromatic rectangle, since by definition, the following set contains at least two distinct values:

$$\{f(x,y), f(x,\bar{y}), f(\bar{x},y)f(\bar{x},\bar{y})\}$$