Communication Complexity

Jake Kinsella and Max von Hippel

Northeastern University

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If Alice knows x, and Bob knows y, how many bits of information must they communicate, in order for both Alice and Bob to know $f(x, y)$?

Introduction (Max)

Consider a two-party communication problem, in which the participants


(a) Alice

and


(b) Bob

participate to compute a function:

$$f : \underbrace{\mathbb{B}^n}_{\text{Alice's input}} \times \underbrace{\mathbb{B}^n}_{\text{Bob's input}} \to \underbrace{\mathbb{B}}_{\text{global output}}$$

Introduction (Max)

The players can come up with a *protocol* $\Pi = (p_1, \ldots, p_t)$, namely, for some natural $t \in \mathbb{N}$, a sequence of t-many functions $p_i : \mathbb{B}^* \to \mathbb{B}^*$ such that the communication between the players looks like this ...

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Thanks Alice. I can't reveal y, but $p_2(y, \mathtt{p1})$ is $\mathtt{p2}$.

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Pleasure doing business with you Bob. My final clue for you is that $p_{n-1}(x, \texttt{p1}, \ldots, \texttt{pn-2})$ is $\texttt{pn-1}$.

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Rad. Then $p_n(y, \texttt{p1}, \dots, \texttt{pn-1})$ is $\texttt{pn}$. TTFN!

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Definition ($C(f)$)

The communication complexity of f, denoted $C(f)$, is the minimum communication complexity achieved by any protocol for f.

Parity (Jake)

Example (Are the number of 1s in xy even (0), or odd (1)?)

$f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is precisely $(x, y) \mapsto \bigoplus xy$.

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Now both Alice and Bob know $f(x, y) = \texttt{P2}$. $C(f) \leq 2$ because $C(\Pi) = 2$ and Π implements f. But $C(f) \geq 2$ because f depends on x and y. Hence $C(f) = 2$.

Halting (Jake)

Example (A_{TM})

$H : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is precisely $\langle M, 1^n \rangle \mapsto 1$ if M halts on 1^n else 0.

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Both players have unlimited computation power. We are only interested in communication complexity.

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TL;DR: yup.

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Lemma (Communication Equality is Image Equality)

If Alice and Bob exchange the same sequence of messages when Alice gets x *and Bob gets* y *as they do when Alice gets* x' *and Bob gets* y'*, then* $f(x, y) = f(x', y')$.

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Proof.

Π is deterministic and f is a function. $\square$

Fooling Set Method (Jake)

Definition (Fooling Set)

If $f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is a function, a *fooling set* for f is a set $S \subseteq \mathbb{B}^n \times \mathbb{B}^n$ such that for some choice $b \in \mathbb{B}$ $f(S) = \{b\}$ but, for all distinct $(x, y), (x', y') \in S$, either $f(x, y') \neq b$ or $f(x', y) \neq b$

Basically, a fooling set is a group of inputs that go to the same output, but which is *brittle* to argument-swapping. In some sense these *brittle* sets lower-bound the difficulty in grouping like inputs.

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Lemma (Fooling Set Method)

If f has a size-M fooling set, then $C(f) \geq \log_2(M)$.

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Proof.

For a contradiction suppose a protocol Π exists for f s.t. $C(\Pi) < \log_2(M)$.

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Then (x, y') must yield the same communication pattern as (x, y) as Bob cannot possibly tell the difference. The argument is symmetric for (x', y) and Alice. One of the two must yield a contradiction and we are done. $\square$

Fooling Set Method (Jake)

Example (Set-Disjointness)

$\textsc{Disj} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is the function that maps (A, B) to 1 if $A \cap B = \varnothing$ else 0.

How many fooling sets does $\textsc{Disj}$ have?

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$$\therefore C(\textsc{Disj}) \geq \log_2(2^n) = n$$

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Example ($M(\vee)$)

	00	01	10	11
00	00	01	10	11
01	01	01	11	11
10	10	11	10	11
11	11	11	11	11

- The green cells are Alice's possible inputs x.
- The blue cells are Bob's possible inputs y.
- The uncolored cells are the matrix $M(f)$.

Tiling Method (Max)

Definition (Combinatorial Rectangle)

A *combinatorial rectangle* in $M(f)$ is any submatrix of M. We say a rectangle $A \times B$ in $M(f)$ is *monochromatic* if for all x, x' in A and y, y' in B, $M_{x,y} = M_{x',y'}$.

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Let's see an example ...

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Example (Π = LeastSignificantBit, $f =<$)

- $f : \mathbb{B}^3 \times \mathbb{B}^3 \to \mathbb{B}$ is the function that maps (x, y) to 1 if $x < y$ else 0.
- Π = LeastSignificantBit is the naïve protocol where Alice and Bob read off their bits from right to left.

	000	001	010	011	100
000	0	1	1	1	1
001	0	0	1	1	1
010	0	0	0	1	1
011	0	0	0	0	1
100	0	0	0	0	0

Alice: "$x =$ __0"

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Alice: "$x =$ __1"

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Theorem (The Punchline)

Let $\chi(f)$ denote the minimum number of rectangles in any monochromatic tiling of $M(f)$.

$$log_2 \chi(f) \leq C(f) \leq 16\left(log_2 \chi(f)\right)^2$$

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NTS: $\log_2 \chi(f) \leq C(f)$.

Proof.

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Assume $C(f)$. Then $\exists$ a protocol Π in which $\leq C(f)$ bits are communicated between the 2 participants. For simplicity suppose each bit is communicated individually. Then Π induces a tree whose max depth is $C(f)$, whose leaves form a monochromatic partition of $M(f)$.

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Assume $C(f)$. Then $\exists$ a protocol Π in which $\leq C(f)$ bits are communicated between the 2 participants. For simplicity suppose each bit is communicated individually. Then Π induces a tree whose max depth is $C(f)$, whose leaves form a monochromatic partition of $M(f)$. Every m.c. partition / $M(f)$ requires $\geq \chi(f)$ rectangles, so the tree induced by Π has $\geq \chi(f)$ leaves.

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Tiling Method (Max)

NTS: $C(f) \leq 16\left(\log_2 \chi(f)\right)^2$. [?aho1983notions]

Proof.

- Let $M_1, \ldots, M_{\chi(f)}$ be a monochromatic partitioning of $M(f)$ known ahead of time to both Alice (on the "left") and Bob (on the "right"). Each rectangle M_i can alternatively be written $X_i \times Y_i$.
- Let G_L, G_R be graphs whose nodes are $\{1, \ldots, \chi(f)\}$. There is an edge $i \to j$ in G_L (G_R resp.) if M_i and M_j have a row (column resp.) in common.
- Let $\deg_L(u)$ (resp. $\deg_R(u)$) denote the degree of the node u in the graph G_L (resp. G_R.)
- Let x be Alice's input and y Bob's input.

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Proof.

- We'll describe the protocol in "rounds". During the rounds, Alice keeps track of Y (a set containing y) and Bob keeps track of X (a set containing x), both of which are initially $\mathbb{B}^n$.
- Both sides know the graphs G_L, G_R and the rectangles M_i ahead of time.

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Proof.

Each stage proceeds as follows.

1. Alice looks for a rectangle $M_i = X_i \times Y_i$ s.t. $x \in X_i$ and $\deg_L(i) \leq 3\chi(f)/4$.
   1. If she finds some such rectangle then she sends i to Bob.
      1. Bob replies to indicate if $y \in M_i$.
      2. If so then the protocol ends because $f(x, y)$ is the color of M_i.
      3. Otherwise $X := X \cap X_i$, and each rectangle $M_\alpha = X_\alpha \cap Y_\alpha$ is replaced with $(X_i \cap X_\alpha) \times Y_\alpha$.
   2. Otherwise she replies that she found no such rectangle. In this case Bob does what Alice just attempted, symmetrically, with a small caveat ...

Tiling Method (Max)

NTS: $C(f) \leq 16\big(\log_2 \chi(f)\big)^2$. [?aho1983notions]

Proof.

s

- If neither Alice nor Bob could find any M_i with low-enough degree, then they both know that every node i in $G := G_L \cap G_R$ for which $(x, y) \in M_i$ has degree $\geq (3\chi(f)/4)^2 = 9\chi(f)/16 > \chi(f)/2$.
- Let i, j both have degree $\geq \chi(f)/2$ in G. Then some node z is adjacent to i and j in G, by the Pigeonhole Principle. Hence $M_i \cap M_z \neq \varnothing$ and $M_j \cap M_z \neq \varnothing$. But the rectangles are monochromatic, hence, M_i and M_j are the same color. So Alice needs to find an M_i containing x and some $y \in Y$ whose degree in G is at least $\chi(f)/2$; and Bob's procedure is symmetric.

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Proof.

- In the worst case for each stage, the first participant sends "nothing found" (1 bit), the second participant sends some i ($\leq \log_2(\chi(f))$ bits), and the first participant replies with some j ($\leq \log_2(\chi(f))$ bits). So in the worst case each round requires $\leq 1 + 2\log_2(\chi(f))$ bits.
- The protocol ends after at most n rounds where $(3\chi(f)/4)^n \approx 1$, i.e., after $\log_{(4/3)}(\chi(f))$ rounds.
- So total communication complexity is $\leq \log_{(4/3)}(\chi(f)) * (1 + 2\log_2(\chi(f)))$.

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Proof.

For $\chi(f) \geq 2$:

$$\begin{aligned}
C(f) &\leq \log_{(4/3)}(\chi(f)) * (1 + 2\log_2(\chi(f))) \\
&= \frac{\log_2(\chi(f))}{\log_2(4/3)} * (1 + 2\log_2(\chi(f))) \\
&< 2.5 * \log_2(\chi(f)) * (1 + 2\log_2(\chi(f))) \\
&\leq 2.5 * \log_2(\chi(f)) * 3\log_2(\chi(f))) \\
&= 7.5\log_2^2(\chi(f)) \\
&\leq 16\log_2^2(\chi(f))
\end{aligned}$$

I'm almost certainly missing a factor of 2, which would explain the choice of 16. -Max. $\square$

2-Party Discrepency Method (Max)

Recall that $\chi(f)$ induces both lower and upper bounds on $C(f)$. So if any bound on $\chi(f)$ induces a bound on $C(f)$. We are about to prove the following lower-bound on $\chi(f)$:

$$\mathrm{Disc}(A \times B) = \frac{1}{2^n * 2^n} \Big| \sum_{a \in A, b \in B} (-1)^{M_{a,b}} \Big| \leq \chi(f)$$

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Proof.

Suppose $\chi(f) \leq K$ for some integer K. If $\chi(f) = K$ then $\exists$ a partioning of $M(f)$ into K m.c. rects, in which case at least 1 must have size $\geq |M(f)|/K$, i.e., $2^n * 2^n/K$.

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$$\sum_{a \in A, b \in B} (-1)^{M_{a,b}} = \begin{cases} -1 * |A \times B| & \text{if it's colored 1} \\ +1 * |A \times B| & \text{if it's colored 0} \end{cases}$$

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So if we wrap an absolute value above our sum, we get:

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But we already assumed that $A \times B$ has at least $2^n * 2^n/K$ entries, hence:

$$|\sum_{a \in A, b \in B} (-1)^{M_{a,b}}| \geq 2^n * 2^n/K$$

2-Party Discrepency Method (Max)

Let's divide both size by $2^n * 2^n$, for fun and profit.

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Mathematicians like to name things.

Definition (Discrepency)

The *discrepency* of a rectangle $A \times B$ of $M(f)$ is exactly the following.

$$\mathrm{Disc}(A \times B) = \frac{1}{2^n * 2^n} \left| \sum_{a \in A, b \in B} (-1)^{M_{a,b}} \right|$$

The *discrepency of* $M(f)$ is the max disc among its rectangles.

2-Party Discrepency Method (Max)

Now that we've named this thing, let's re-write our inequality.

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Now that we've named this thing, let's re-write our inequality.

$$\text{Disc}(A \times B) \geq 1/K$$

Taking inverses:

$$\frac{1}{\text{Disc}(A \times B)} \leq K$$

Certainly $\chi(f) \leq \chi(f)$, so supplanting $\chi(f)$ for K in the statement, we get:

Lemma (2-Party Discrepency Method)

$$\frac{1}{Disc(A \times B)} \leq \chi(f)$$

Multi-Party Problem (Jake)

- There are k of us.
- We all place a sticky note with some value $b \in \mathbb{B}^n$ on our heads.
- Without talking, we must compute some predetermined function via a predetermined protocol. All communication must be done through the whiteboard in front of us.
- The goal is for one player, after some amount of communication, to write the value $f(\text{sticky-note}_1, \ldots, \text{sticky-note}_k)$ on the whiteboard.

Multi-Party Problem (Jake)

Example (Majority Parity)

$\textsc{MajPar} : \mathbb{B}^n \times \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is precisely $\langle x_1, x_2, x_3 \rangle \mapsto 1$ if $\bigoplus_{i=1}^{n} \text{majority}(x_{1i}, x_{2i}, x_{3i})$ else 0.

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Example protocol Π:

Player 1	Player 2	Player 3
$x_2 = 1001$ $x_3 = 1011$	$x_1 = 1101$ $x_3 = 1011$	$x_1 = 1101$ $x_2 = 1001$
parity(10_1) $p_1 = 0$	parity(1__1) $p_2 = 0$	parity(1_01) $p_3 = 0$
$\text{parity}(p_1 p_2 p_3) = \text{parity}(000) = 0$		

Multi-Party Problem (Jake)

Before we talked about *rectangles*. Now: *cylinders*.

Definition (Cylinder)

A *cylinder in dimension i* is a subset S of the inputs $(\mathbb{B}^n)^k$ such that if $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k) \in S$, then for all $x_i' \in \mathbb{B}^n$, so is $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_k)$.

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Lemma (Generalize $\chi(f)$)

If every partition of $M(f)$ into m.c. cylinder intersections requires at least R of them, then $C(f) \geq \lceil log_2(R) \rceil$.

Multi-Party Discrepency Method (Max)

Definition (Multi-Party Discrepency)

Suppose

$$f : \underbrace{\mathbb{B}^n \times \ldots \times \mathbb{B}^n}_{k \text{ times}} \to \mathbb{B}$$

is a function. Then the *k-party discrepency of f* is defined as follows, where T ranges over all cylinder intersections of f.

$$\text{Disc}(f) = \frac{1}{(2^n)^k} \max_T \Big| \sum_{(x_1,\ldots,x_k) \in T} f(x_1, \ldots, x_k) \Big|$$

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Man this is really complicated. Could we lower-bound it statistically?

Multi-Party Discrepency Method (Max)

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Definition ((k, n)-Cube)

A (k, n)-cube is a set D of the form $D = \{a_1, a_1'\} \times \ldots \times \{a_k, a_k'\}$ where each $a_i, a_i' \in \mathbb{B}^n$. A point $\vec{d} \in D$ is a vector $(x_1, x_2, \ldots, x_k)$ s.t. each $x_i \in \{a_i, a_i'\}$.

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Definition ($\mathcal{E}$)

Let $f : (\mathbb{B}^n)^k \to \mathbb{B}$ be a function.

$$\mathcal{E}(f) = \mathop{\mathbb{E}}_{\substack{D \text{ is a} \\ (k,n)\text{-cube}}} \left[\prod_{\vec{d} \in D} f(\vec{d}) \right]$$

I.e., $\mathcal{E}(f) = \mathbb{E}[$*given an arbitrary cube D, what is the product of the image of f over all the points $\vec{d} \in D$?*$]$

Multi-Party Discrepency Method (Max)

Although somewhat scary-looking, this definition pays dividends immediately.

Lemma (k-Party Discrepency Bound)

If $f : (\mathbb{B}^n)^k \to \mathbb{B}$ is a function, then $Disc(f) \leq (\mathcal{E}(f))^{1/2^k}$.

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Proof sketch:

Proof.

Given any cylinder intersection and (n, k)-cube, what is the expectation / the image of the cube? What if we only consider points in the cylinder intersection? Derive a lower bound on $\mathcal{E}(f)$ like

$$\mathcal{E}(f) \geq E_{x_1,\ldots,x_k}[f(x_1,\ldots,x_k)(1 \text{ if } (x_1,\ldots,x_k) \in C \text{ else } 0)]^{2k}$$

given a cylinder intersection C. Argue from the def. of the k-party discrepency that this gives a natural lower-bound $\mathcal{E}(f) \geq \text{Disc}(f)^{2k}$. But this implies $\text{Disc}(f) \leq (\mathcal{E}(f))^{1/2^k}$, and we're done. $\square$

Non-Deterministic (Jake)

- Defined similarly to NP.

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- Consider a two party problem (we can generalize to multi-party from here). Each player is given their input along with some nondeterministic guess z of length m that may depend on the given inputs. $C(f) = m +$ communication, and $f(x, y) = 1$ iff $\exists z$ that makes the players output 1.

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- NP^{CC} is the class of nondeterministic functions f s.t. $C(f) = n^k$. coNP^{CC} is defined similarly, i.e., $g(x,y) = 1 - f(x,y)$ for $f \in \mathrm{NP}^{CC}$.

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- Claim: $\mathrm{NP}^{CC} \cap \mathrm{coNP}^{CC} = \mathrm{P}^{CC}$.
- For some f with non-deterministic $C(f) = k$ and $\overline{f}$ with non-deterministic $C(f) = \ell$.
- It can be shown that the deterministic $C(f) \leq 10k\ell$.

New Rules!

Figure: A coin.

- You're allowed to be wrong sometimes.
- You have a lava lamp or a coin or something.
- Instead of $C(f)$ the complexity of f, we discuss $R(f) = \mathbb{E}[C(f)]$ the *expected* complexity of f.

Two Models

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- **Public Coin:** Everyone can see the same random string ahead of time.

Randomized (Max)

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- **Public Coin:** Everyone can see the same random string ahead of time.
- **Private Coin:** Everyone can flip their own coin(s) in private.

Randomized (Max)

Example (=)

Consider the equality function $f(x, y) = 1$ if $x = y$ else 0.

- **Public Coin:** Alice sends the dot-product $\langle x, \text{coin} \rangle$ to Bob. Bob replies 1 if $\langle x, \text{coin} \rangle = \langle y, \text{coin} \rangle$ else 0. The chance of an erroneous "accept" is 1/2, so the protocol can be repeated three times (6 bits of communication) to get the desired result.
- **Private Coin:** Alice and Bob agree on a small set of possible random strings ahead of time. Alice sends her (privately random) choice from the set, using $\mathcal{O}(\log n)$ bits. Then the communication proceeds as if it were a public coin with that choice.

References I