Communication Complexity

Jake Kinsella and Max von Hippel

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If Alice knows x, and Bob knows y, how many bits of information must they communicate, in order for both Alice and Bob to know $f(x, y)$?

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Introduction (Max)

Consider a two-party communication problem, in which the participants

(a) Alice

and

(b) Bob

participate to compute a function:

$$f : \underbrace{\mathbb{B}^n}_{\text{Alice's input}} \times \underbrace{\mathbb{B}^n}_{\text{Bob's input}} \to \underbrace{\mathbb{B}}_{\text{global output}}$$

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The players can come up with a *protocol* $\Pi = (p_1, \ldots, p_t)$, namely, for some natural $t \in \mathbb{N}$, a sequence of t-many functions $p_i : \mathbb{B}^* \to \mathbb{B}^*$ such that the communication between the players looks like this ...

Introduction (Max)

Alice is given input x.

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Introduction (Max)

Alice is given input x.

Hello Bob. I can't reveal x, but $p_1(x)$ is $\mathtt{p1}$.

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Bob is given input y.

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Alice is given input x.

Hello Bob. I can't reveal x, but $p_1(x)$ is $\mathtt{p1}$.

Bob is given input y.

Thanks Alice. I can't reveal y, but $p_2(y, \mathtt{p1})$ is $\mathtt{p2}$.

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Pleasure doing business with you Bob. My final clue for you is that $p_{n-1}(x, \texttt{p1}, \ldots, \texttt{pn-2})$ is $\texttt{pn-1}$.

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Rad. Then $p_n(y, \texttt{p1}, \ldots, \texttt{pn-1})$ is $\texttt{pn}$. TTFN!

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- The functions p_i can be *anything* so long as they are well-defined. E.g., could solve the Halting Problem.
- After the final message, *both parties* must know $f(x, y)$.

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Definition (Communication Complexity)

Suppose Π is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the *communication complexity* of Π, denoted $C(\Pi)$, is t.

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Definition (Communication Complexity)

Suppose Π is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the *communication complexity* of Π, denoted $C(\Pi)$, is t.

Definition ($C(f)$)

The communication complexity of f, denoted $C(f)$, is the minimum communication complexity achieved by any protocol for f.

Parity (Jake)

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Example (Are the number of 1s in xy even (0), or odd (1)?)

$f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is precisely $(x, y) \mapsto \bigoplus xy$.

Parity (Jake)

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$f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is precisely $(x, y) \mapsto \bigoplus xy$.

Example protocol Π:

$\mathrm{P1} = \text{parity}(x)$.

Parity (Jake)

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$\text{P1} = \text{parity}(x)$.

$\text{P2} = \text{parity}(y) \bigoplus \text{P1}$

Parity (Jake)

Example (Are the number of 1s in xy even (0), or odd (1)?)

$f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is precisely $(x, y) \mapsto \bigoplus xy$.

Example protocol Π:

$\mathtt{P1} = \text{parity}(x)$.

$\mathtt{P2} = \text{parity}(y) \bigoplus \mathtt{P1}$

Now both Alice and Bob know $f(x, y) = \mathtt{P2}$. $C(f) \leq 2$ because $C(\Pi) = 2$ and Π implements f. But $C(f) \geq 2$ because f depends on x and y. Hence $C(f) = 2$.

Halting (Jake)

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Example (A_{TM})

$H : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is precisely $\langle M, x \rangle \mapsto 1$ if M halts on x else 0.

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$\mathtt{P1} = y$.

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P1 $= y$.

P2$=$ (M does/doesn't accept y).

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Example protocol Π:

P1 = y.

P2= (M does/doesn't accept y).

Both players have unlimited computation power. We are only interest in communication complexity.

Methods (Max)

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If we find a protocol Π, then we know $C(f)$ is at most $C(\Pi)$.

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If we find a protocol Π, then we know $C(f)$ is at most $C(\Pi)$.
What if we don't know any protocol Π?

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If we find a protocol Π, then we know $C(f)$ is at most $C(\Pi)$. What if we don't know any protocol Π?

- Could we upper-bound $C(f)$ without knowing Π?

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What if the only protocols we find seem really lousy?

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TL;DR: yup.

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We begin with a motivating observation.

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We begin with a motivating observation.

Lemma (Communication Equality is Image Equality)

If Alice and Bob exchange the same sequence of messages when Alice gets x *and Bob gets* y *as they do when Alice gets* x' *and Bob gets* y'*, then* $f(x, y) = f(x', y')$*.*

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Proof.

Π is deterministic and f is a function. $\square$

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Idea: an efficient protocol will efficiently group together inputs that go to the same output.

Fooling Set Method (Jake)

Idea: an efficient protocol will efficiently **group together inputs that go to the same output.**

Definition (Fooling Set)

If $f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is a function, a *fooling set* for f is a set $S \subseteq \mathbb{B}^n \times \mathbb{B}^n$ such that for some choice $b \in \mathbb{B}$ $f(S) = \{b\}$ but, for all distinct $(x, y), (x', y') \in S$, $(\neg b) \in f(\{x, x'\} \times \{y, y'\})$.

Basically, a fooling set is a group of inputs that go to the same output, but which is *brittle* to argument-swapping. In some sense these *brittle* sets lower-bound the difficulty in grouping like inputs.

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Basically, a fooling set is a group of inputs that go to the same output, but which is *brittle* to argument-swapping. In some sense these *brittle* sets lower-bound the difficulty in grouping like inputs.

Lemma (Fooling Set Method)

If f has a size-M fooling set, then $C(f) \geq log_2(M)$.

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Example (Set-Disjointness)

$\textsc{Disj} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is the function that maps (A, B) to 1 if $A \cap B = \varnothing$ else 0.

How many fooling sets does $\textsc{Disj}$ have?

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How many fooling sets does $\textsc{Disj}$ have? Notice A, B are disjoint iff $A \oplus B$ is $\mathbf{1}$.

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How many fooling sets does DISJ have? Notice A, B are disjoint iff $A \oplus B$ is $\mathbf{1}$. There are 2^n possible values A.

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How many fooling sets does $\textsc{Disj}$ have? Notice A, B are disjoint iff $A \oplus B$ is $\mathbf{1}$. There are 2^n possible values A. Hence 2^n values (A, B) s.t. $A \oplus B = \mathbf{1}$. None of these distinct $(A, B), (A', B')$ satisfy $A \oplus B = A \oplus B'$ or $A \oplus B = A' \oplus B$ else they wouldn't be distinct.

Fooling Set Method (Jake)

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$$\therefore C(\textsc{Disj}) \geq \log_2(2^n) = n$$

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NTS: If f has a size-M fooling set then $C(f) \geq \log_2(M)$.

Proof.

For a contradiction suppose a protocol Π exists for f s.t. $C(\Pi) < \log_2(M)$.

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For a contradiction suppose a protocol Π exists for f s.t. $C(\Pi) < \log_2(M)$. Then Π yields at most $2^{C(\Pi)} < 2^{\log_2(M)} = M$ distinct communication patterns.

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Then (x, y') must yield the same communication pattern as (x, y) as Bob cannot possibly tell the difference. The argument is symmetric for (x', y) and Alice. One of the two must yield a contradiction and we are done. $\square$

Tiling Method (Max)

Communication Complexity

Jake Kinsella and Max von Hippel

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With the *fooling set* method, we lower-bounded $C(f)$. Now we'll introduce a new method that both lower- and upper-bounds $C(f)$.

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Definition ($M(f)$)

The *matrix of* f, denoted $M(f)$, is the $2^n \times 2^n$ matrix whose (x, y)th entry is the value $f(x, y)$.

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Example ($M(\vee)$)

	00	01	10	11
00	00	01	10	11
01	01	01	11	11
10	10	11	10	11
11	11	11	11	11

- The green cells are Alice's possible inputs x.
- The blue cells are Bob's possible inputs y.
- The uncolored cells are the matrix $M(f)$.

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Definition (Combinatorial Rectangle)

A *combinatorial rectangle* in $M(f)$ is any submatrix of M. We say a rectangle $A \times B$ in $M(f)$ is *monochromatic* if for all x, x' in A and y, y' in B, $M_{x,y} = M_{x',y'}$.

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<u>Idea:</u> Each event in a protocol Π splits $M(f)$ into two or more combinatorial rectangles of still-possible values for $f(x, y)$.

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Intuition: Much like splitting a circuit C into "C where the first bit is 0" and "C where the first bit is 1".

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Let's see an example ...

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Example (Π = LeastSignificantBit, $f =<$)

- $f : \mathbb{B}^3 \times \mathbb{B}^3 \to \mathbb{B}$ is the function that maps (x, y) to 1 if $x < y$ else 0.
- Π = LeastSignificantBit is the naïve protocol where Alice and Bob read off their bits from right to left.

	000	001	010	011	100
000	0	1	1	1	1
001	0	0	1	1	1
010	0	0	0	1	1
011	0	0	0	0	1
100	0	0	0	0	0

Alice: "x = __0"

	000	001	010	011	100
000	0	1	1	1	1
010	0	0	0	1	1
100	0	0	0	0	0

Alice: "x = __1"

	000	001	010	011	100
001	0	0	1	1	1
011	0	0	0	0	1

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Now we get to the punchline.

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A *monochromatic tiling* of $M(f)$ is a partition of $M(f)$ into disjoint monochromatic rectangles.

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It's thinking time.

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Definition (Monochromatic Tiling)

A *monochromatic tiling* of $M(f)$ is a partition of $M(f)$ into disjoint monochromatic rectangles.

It's thinking time.

- Then the leaves of the tree induced by Π and rooted at $M(f)$ clearly form a monochromatic tiling of $M(f)$.

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- Then the leaves of the tree induced by Π and rooted at $M(f)$ clearly form a monochromatic tiling of $M(f)$.
- The number of leaves in a binary tree can be used to upper-bound its depth.

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- Then the leaves of the tree induced by Π and rooted at $M(f)$ clearly form a monochromatic tiling of $M(f)$.
- The number of leaves in a binary tree can be used to upper-bound its depth.
- The depth of the binary tree induced by Π is exactly $C(\Pi)$.

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Theorem (The Punchline)

Let $\chi(f)$ denote the minimum number of rectangles in any monochromatic tiling of $M(f)$.

$$log_2\chi(f) \leq C(f) \leq 16\left(log_2\chi(f)\right)^2$$

Tiling Method (Max)

NTS: $\log_2 \chi(f) \leq C(f)$.

Proof.

Assume $C(f)$.

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Assume $C(f)$. Then $\exists$ a protocol Π in which $\leq C(f)$ bits are communicated between the 2 participants.

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Assume $C(f)$. Then $\exists$ a protocol Π in which $\leq C(f)$ bits are communicated between the 2 participants. For simplicity suppose each bit is communicated individually.

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NTS: $\log_2 \chi(f) \leq C(f)$.

Proof.

Assume $C(f)$. Then $\exists$ a protocol Π in which $\leq C(f)$ bits are communicated between the 2 participants. For simplicity suppose each bit is communicated individually. Then Π induces a tree whose max depth is $C(f)$, whose leaves form a monochromatic partition of $M(f)$.

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Assume $C(f)$. Then $\exists$ a protocol Π in which $\leq C(f)$ bits are communicated between the 2 participants. For simplicity suppose each bit is communicated individually. Then Π induces a tree whose max depth is $C(f)$, whose leaves form a monochromatic partition of $M(f)$. Every m.c. partition / $M(f)$ requires $\geq \chi(f)$ rectangles, so the tree induced by Π has $\geq \chi(f)$ leaves. But it's a binary tree so its depth is at least $\log_2 \chi(f)$. $\square$

Tiling Method (Max)

NTS: $C(f) \leq 16\left(\log_2 \chi(f)\right)^2$. [1]

Proof.

- Let $M_1, \ldots, M_{\chi(f)}$ be a monochromatic partitioning of $M(f)$ known ahead of time to both Alice (on the "left") and Bob (on the "right"). Each rectangle M_i can alternatively be written $X_i \times Y_i$.
- Let G_L, G_R be graphs whose nodes are $\{1, \ldots, \chi(f)\}$. There is an edge $i \to j$ in G_L (G_R resp.) if M_i and M_j have a row (column resp.) in common.
- Let $\deg_L(u)$ (resp. $\deg_R(u)$) denote the degree of the node u in the graph G_L (resp. G_R.)
- Let x be Alice's input and y Bob's input.

Tiling Method (Max)

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Proof.

- We'll describe the protocol in "rounds". During the rounds, Alice keeps track of Y (a set containing y) and Bob keeps track of X (a set containing x), both of which are initially $\mathbb{B}^n$.
- Both sides know the graphs G_L, G_R and the rectangles M_i ahead of time.

Tiling Method (Max)

NTS: $C(f) \leq 16\left(\log_2 \chi(f)\right)^2$. [1]

Proof.

Each stage proceeds as follows.

1. Alice looks for a rectangle $M_i = X_i \times Y_i$ s.t. $x \in X_i$ and $\deg_L(i) \leq 3\chi(f)/4$.
   1. If she finds some such rectangle then she sends i to Bob.
      1. Bob replies to indicate if $y \in M_i$.
      2. If so then the protocol ends because $f(x, y)$ is the color of M_i.
      3. Otherwise $X := X \cap X_i$, and each rectangle $M_\alpha = X_\alpha \cap Y_\alpha$ is replaced with $(X_i \cap X_\alpha) \times Y_\alpha$.
   2. Otherwise she replies that she found no such rectangle. In this case Bob does what Alice just attempted, symmetrically, with a small caveat ...

Tiling Method (Max)

NTS: $C(f) \leq 16\left(\log_2 \chi(f)\right)^2$. [1]

Proof.

- If neither Alice nor Bob could find any M_i with low-enough degree, then they both know that every node i in $G := G_L \cap G_R$ for which $(x, y) \in M_i$ has degree $\geq (3\chi(f)/4)^2 = 9\chi(f)/16 > \chi(f)/2$.
- Let i, j both have degree $\geq \chi(f)/2$ in G. Then some node z is adjacent to i and j in G, by the Pigeonhole Principle. Hence $M_i \cap M_z = \varnothing$ and $M_j \cap M_z = \varnothing$. But the rectangles are monochromatic, hence, M_i and M_j are the same color. So Alice needs to find an M_i containing x and some $y \in Y$ whose degree in G is at least $\chi(f)/2$; and Bob's procedure is symmetric.

Tiling Method (Max)

NTS: $C(f) \leq 16\left(\log_2 \chi(f)\right)^2$. [1]

Proof.

- In the worst case for each stage, the first participant sends "nothing found" (1 bit), the second participant sends some i ($\leq \log_2(\chi(f))$ bits), and the first participant replies with some j ($\leq \log_2(\chi(f))$ bits). So in the worst case each round requires $\leq 2 + 2\log_2(\chi(f))$ bits.
- The protocol ends after at most n rounds where $(3\chi(f)/4)^n \approx 1$, i.e., after $\log_{(4/3)}(\chi(f))$ rounds.
- So total communication complexity is $\leq \log_{(4/3)}(\chi(f)) * (1 + 2\log_2(\chi(f)))$.

Tiling Method (Max)

NTS: $C(f) \leq 16\left(\log_2 \chi(f)\right)^2$. [1]

Proof.

For $\chi(f) \geq 2$:

$$\begin{aligned}
C(f) &\leq \log_{(4/3)}(\chi(f)) * (1 + 2\log_2(\chi(f))) \\
&= \frac{\log_2(\chi(f))}{\log_2(4/3)} * (1 + 2\log_2(\chi(f))) \\
&< 2.5 * \log_2(\chi(f)) * (1 + 2\log_2(\chi(f))) \\
&\leq 2.5 * \log_2(\chi(f)) * 3\log_2(\chi(f))) \\
&= 7.5\log_2^2(\chi(f)) \\
&\leq 16\log_2^2(\chi(f))
\end{aligned}$$

There might be a small arithmetic error somewhere, since the jump from 7.5 to 16 seems rather large. -Max.

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When we partition $M(f)$ into some number of rectangles, the sizes of the rectangles must add up to the size of $M(f)$.

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When we partition $M(f)$ into some number of rectangles, the sizes of the rectangles must add up to the size of $M(f)$.

Hence, if $\chi(f) \leq K$ for some integer K, then $M(f)$ must have a m.c. rectangle containing at least $2^n * 2^n/K$ entries.

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Hence, if $\chi(f) \leq K$ for some integer K, then $M(f)$ must have a m.c. rectangle containing at least $2^n * 2^n / K$ entries.

Proof.

Suppose $\chi(f) \leq K$ for some integer K.

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Proof.

Suppose $\chi(f) \leq K$ for some integer K. If $\chi(f) = K$ then $\exists$ a partioning of $M(f)$ into K m.c. rects, in which case at least 1 must have size $\geq |M(f)|/K$, i.e., $2^n * 2^n/K$.

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Suppose that $M(f)$ contains a monochromatic rectangle $A \times B$ having at least $2^n * 2^n / K$ entries.

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Suppose that $M(f)$ contains a monochromatic rectangle $A \times B$ having at least $2^n * 2^n/K$ entries. Since $A \times B$ is monochromatic, this implies that:

$$\sum_{a \in A, b \in B} (-1)^{M_{a,b}} = \begin{cases} -1 * |A \times B| & \text{if it's colored 1} \\ +1 * |A \times B| & \text{if it's colored 0} \end{cases}$$

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So if we wrap an absolute value above our sum, we get:

$$\Big| \sum_{a \in A, b \in B} (-1)^{M_{a,b}} \Big| = \text{ the size of the rectangle } A \times B$$

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So if we wrap an absolute value above our sum, we get:

$$|\sum_{a \in A, b \in B} (-1)^{M_{a,b}}| = \text{ the size of the rectangle } A \times B$$

But we already assumed that $A \times B$ has at least $2^n * 2^n/K$ entries, hence:

$$|\sum_{a \in A, b \in B} (-1)^{M_{a,b}}| =\geq 2^n * 2^n/K$$

2-Party Discrepency Method (Max)

Let's divide both size by $2^n * 2^n$, for fun and profit.

$$\frac{1}{2^n * 2^n} \left| \sum_{a \in A, b \in B} (-1)^{M_{a,b}} \right| \geq 1/K$$

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Mathematicians like to name things.

Definition (Discrepency)

The *discrepency* of a rectangle $A \times B$ of $M(f)$ is exactly the following.

$$\mathrm{Disc}(A \times B) = \frac{1}{2^n * 2^n} \Big| \sum_{a \in A, b \in B} (-1)^{M_{a,b}} \Big|$$

The *discrepency of* $M(f)$ is the max disc among its rectangles.

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Now that we've named this thing, let's re-write our inequality.

$$\text{Disc}(A \times B) \geq 1/K$$

2-Party Discrepency Method (Max)

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Certainly $\chi(f) \leq \chi(f)$, so supplanting $\chi(f)$ for K in the statement, we get:

Lemma (2-Party Discrepency Method)

$$\frac{1}{Disc(A \times B)} \leq \chi(f)$$

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