The 2-Party Discrepency Method

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1 Introduction

When we partition $M(f)$ into some number of rectangles, the sizes of the rectangles must add up to the size of $M(f)$. Hence, if $\chi(f) \leq K$ for some integer K, then $M(f)$ must have a monochromatic rectangle containing at least $2^n * 2^n / K$ entries.

Proof. Suppose that $\chi(f) \leq K$ for some integer K. If $\chi(f) = K$ then there exists a partioning of $M(f)$ into K monochromatic rectangles, in which case at least one of those rectangles must have size $\geq |M(f)|/K$, i.e., $2^n * 2^n/K$. On the other hand if $\chi(f) < K$ then $\chi(f) = K'$ for some $K' < K$ and then $M(f)$ can be partitioned into K' monochromatic rectangles, at least of which has size $\geq |M(f)|/K'$, which is strictly larger than $|M(f)|/K$. So in either case the result holds and we are done. □

Now, suppose that $M(f)$ contains a monochromatic rectangle $A \times B$ having at least $2^n * 2^n/K$ entries. Since $A \times B$ is monochromatic, this implies that:

$$\sum_{a \in A, b \in B} (-1)^{M_{a,b}} = \begin{cases} -1 * \text{ the size of the rectangle } A \times B & \text{if it's colored 1} \\ 1 * \text{ the size of the rectangle } A \times B & \text{if it's colored 0} \end{cases}$$

So if we wrap an absolute value above our sum, we get:

$$|\sum_{a \in A, b \in B} (-1)^{M_{a,b}}| = \text{ the size of the rectangle } A \times B$$

But we already assumed that $A \times B$ has at least $2^n * 2^n/K$ entries, hence:

$$|\sum_{a \in A, b \in B} (-1)^{M_{a,b}}| = \text{ the size of the rectangle } A \times B \geq 2^n * 2^n/K$$

Let's divide both size by $2^n * 2^n$, for fun and profit.

$$\frac{1}{2^n * 2^n} |\sum_{a \in A, b \in B} (-1)^{M_{a,b}}| \geq 1/K$$

We are mathematicians, and mathematicians like to name things. Let's do that.

Definition 1 (Discrepency)**.** *The* discrepency *of a rectangle $A \times B$ of $M(f)$ is exactly the following.*

$$\text{Disc}(A \times B) = \frac{1}{2^n * 2^n} |\sum_{a \in A, b \in B} (-1)^{M_{a,b}}|$$

The discrency of $M(f)$ *is the maximum discrepency among all its rectangles.*

Now that we've named this thing, let's re-write our inequality.

$$Disc(A \times B) \geq 1/K$$

Taking inverses:

$$\frac{1}{Disc(A \times B)} \leq K$$

Certainly $\chi(f) \leq \chi(f)$, so supplanting $\chi(f)$ for K in the statement, we get:

$$\frac{1}{Disc(A \times B)} \leq \chi(f) \tag{1}$$

This result generalizes as follows.

Lemma 1 (2-Party Discrepency Method)**.** *Suppose $f : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}$ is a function. Then Equation 1 holds.*