The Multi-Party Discrepency Method

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We begin by generalizing our discrepency definition to a multi-party setting. The general form is basically the same as the 2-party form, except we reason about intersections in cylinders, rather than rectangles in partitions.

Definition 1 (Multi-Party Discrepency)**.** *Suppose*

$$f : \underbrace{\mathbb{B}^n \times \ldots \times \mathbb{B}^n}_{k \text{ times}} \to \mathbb{B}$$

*is a function. Then the k-*party discrepency of f *is defined as follows, where* T *ranges over all cylinder intersections of* f.

$$\text{Disc}(f) = \frac{1}{(2^n)^k} \max_T \Big| \sum_{(x_1,\ldots,x_k)\in T} f(x_1,\ldots,x_k) \Big|$$

For even moderaly sized n and k, this definition becomes unreasonable to compute on-the-fly. But if we could lower-bound the discrepency, for example, using a statistical method, then we could get a slightly looser (but, cheaper) lower-bound on the communication complexity. First we need two useful but non-intuitive definitions.

Definition 2 ((k,n)-Cube)**.** *A* (k,n)*-cube is a set D of the form*

$$D = \{a_1, a_1'\} \times \ldots \times \{a_k, a_k'\}$$

where each $a_i, a_i' \in \mathbb{B}^n$*. To be clear, a point in d is a vector* $(x_1, x_2, \ldots, x_k)$ *where each* $x_i \in \{a_i, a_i'\}$*.*

Definition 3 ($\mathcal{E}$)**.** *Let* $f : (\mathbb{B}^n)^k \to \mathbb{B}$ *be a function. Then:*

$$\mathcal{E}(f) = \mathop{E}_{\substack{D \text{ is a} \\ (k,n)\text{-cube}}} \Big[\prod_{\vec{d}\in D} f(\vec{d}) \Big]$$

In other words, $\mathcal{E}(f)$ *denotes the expectation over the question,* given an arbitrary cube D, what is the product of the image of f over all the points $\vec{d} \in D$?

Although somewhat scary-looking, this definition pays dividends immediately.

Lemma 1 (k-Party Discrepency Bound)**.** *If* $f : (\mathbb{B}^n)^k \to \mathbb{B}$ *is a function, then* $\text{Disc}(f) \leq (\mathcal{E}(f))^{1/2^k}$.

Why do we care? Well, we could approximate $\mathcal{E}(f)$ statistically, by checking a ton of random cubes. And in this way we could get a decent idea of what an upper bound on the discrepency looks like. But then recall that the logarithm of the inverse of the discrepency lower-bounds the complexity. Hence, any upper-bound on discrepency naturally induces a lower-bound on complexity. So we've found a way to compute a lower-bound on the complexity without doing all the work of computing the discrepency, which is kind of cool. The proof is pretty complicated, but here's a proof sketch.

Proof. Given any arbitrary cylinder intersection and (n,k)-cube, what is the expectation of the image of points in the cube under f? What if we only consider points falling in the cylinder intersection? Derive a lower bound on $\mathcal{E}(f)$ which looks something like

$$\mathcal{E}(f) \geq E_{x_1,\ldots,x_k}[f(x_1,\ldots.,x_k)(1 \text{ if } (x_1,\ldots,x_k) \in C \text{ else } 0)]^{2k}$$

given a cylinder intersection C. Argue from the definition of the k-party discrepency that this gives a natural lower-bound $\mathcal{E}(f) \geq Disc(f)^{2k}$. But this implies $Disc(f) \leq (\mathcal{E}(f))^{1/2^k}$, and we're done. $\square$