Communication Complexity

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If Alice knows x, and Bob knows y, how many bits of information must they communicate, in order for both Alice and Bob to know $f(x, y)$?

(Max)

Consider a two-party communication problem, in which the participants

(a) Alice

and

(b) Bob

participate to compute a function:

$$f : \underbrace{\mathbb{B}^n}_{\text{Alice's input}} \times \underbrace{\mathbb{B}^n}_{\text{Bob's input}} \to \underbrace{\mathbb{B}}_{\text{global output}}$$

(Max)

The players can come up with a *protocol* $\Pi = (p_1, \ldots, p_t)$, namely, for some natural $t \in \mathbb{N}$, a sequence of t-many functions $p_i : \mathbb{B}^* \to \mathbb{B}^*$ such that the communication between the players looks like this ...

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Thanks Alice. I can't reveal y, but $p_2(y, \texttt{p1})$ is $\texttt{p2}$.

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Pleasure doing business with you Bob. My final clue for you is that $p_{n-1}(x, \texttt{p1}, \ldots, \texttt{pn-2})$ is $\texttt{pn-1}$.

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Rad. Then $p_n(y, \texttt{p1}, ..., \texttt{pn-1})$ is $\texttt{pn}$. TTFN!

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Suppose Π is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the *communication complexity* of Π, denoted $C(\Pi)$, is t.

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Definition ($C(f)$)

The communication complexity of f, denoted $C(f)$, is the minimum communication complexity achieved by any protocol for f.

Parity (Jake)

Example (Are the number of 1s in xy even (0), or odd (1)?)

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Now both Alice and Bob know $f(x, y) = \texttt{P2}$. $C(f) \leq 2$ because $C(\Pi) = 2$ and Π implements f. But $C(f) \geq 2$ because f depends on x and y. Hence $C(f) = 2$.

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Example (A_{TM})

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Both players have unlimited computation power. We are only interest in communication complexity.

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TL;DR: yup.

Fooling Set Method (Jake)

TODO

Tiling Method (Max)

TODO

2-Party Discrepency Method (Max)

TODO

Multi-Party Problem (Jake)

TODO

Multi-Party Discrepency Method (Max)

TODO

Non-Deterministic (Jake)

TODO

Randomized (Max)

TODO