

The Tiling Method

Max von Hippel

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1 Problem Statement

Consider a two-party communication problem, in which the participants



(a) Alice

and



(b) Bob

participate to compute a function:

$$f: \underbrace{\mathbb{B}^n}_{\text{Alice's input}} \times \underbrace{\mathbb{B}^n}_{\text{Bob's input}} \rightarrow \underbrace{\mathbb{B}}_{\text{global output}}$$

The players can come up with a *protocol* $\Pi = (p_1, \dots, p_t)$, namely, for some natural $t \in \mathbb{N}$, a sequence of t -many functions $p_i: \mathbb{B}^* \rightarrow \mathbb{B}^*$ such that the communication between the players looks like this:

Alice is given input x .

Hi Bob. I'm not divulging x , but, $p_1(x) = p_1$.

Bob is given input y .

Thanks Alice. I'm not divulging y , but, $p_2(y, p_1) = p_2$.

Thanks Bob. Don't tell anyone, but: $p_3(x, p_1, p_2) = p_3$.

Is that so? Well, $p_4(y, p_1, p_2, p_3) = p_4$.

Wicked. In that case, $p_5(x, p_1, p_2, p_3, p_4) = p_5$.

...yada yada yada...

$p_t(x, p_1, p_2, p_3, p_4, \dots, p_{(t-1)}) = p_t$, and TTFN!

Critically, the functions p_i can be *anything* so long as they are well-defined. For example, p_2 could be the function that asks if $\langle y, p_2 \rangle$ is a word in ATM .

Definition 1 (Communication Complexity). Suppose Π is a protocol for f in which at most t bits are communicated between Alice and Bob. Then the communication complexity of Π , denoted $C(\Pi)$, is t . The communication complexity of f , denoted $C(f)$, is the minimum communication complexity achieved by any protocol for f .

Given some such function f , it would be nice if we could automatically compute a reasonable lower bound on its communication complexity.

2 The Tiling Method

One way to do this is with the *tiling method*. We will give the method immediately, and in tandem, we will illustrate the method using the function $f(x,y) = x < y$ where x,y are integers in $\{0,1,2,3\}$, encoded in \mathbb{B} oolean.

Definition 2 ($M(f)$). The matrix of f , denoted $M(f)$, is the $2^n \times 2^n$ matrix whose (x,y) th entry is the value $f(x,y)$. An example is given in Table 1.

	000	001	010	011	100
	0	1	2	3	4
000 = 0	0	1	1	1	1
001 = 1	0	0	1	1	1
010 = 2	0	0	0	1	1
011 = 3	0	0	0	0	1
100 = 4	0	0	0	0	0

Table 1: The matrix $M(<)$ for inputs $x,y \in \{0,1,2,3\}$. Values of x are given in the rows, while values of y are given in the columns. False (i.e. 0) values are marked red for clarity.

Definition 3 (Combinatorial Rectangle). A combinatorial rectangle in $M(f)$ is any submatrix of M . We say a rectangle $A \times B$ in $M(f)$ is monochromatic if for all x,x' in A and y,y' in B , $M_{x,y} = M_{x',y'}$. Examples are given in Figure 2.

And now for the punchline!

Theorem 1 (The Tiling Method). Denote by $\chi(f)$ the minimum number of rectangles in any monochromatic tiling of $M(f)$. Then $\log_2 \chi(f) \leq C(f) \leq 16(\log_2 \chi(f))^2$.

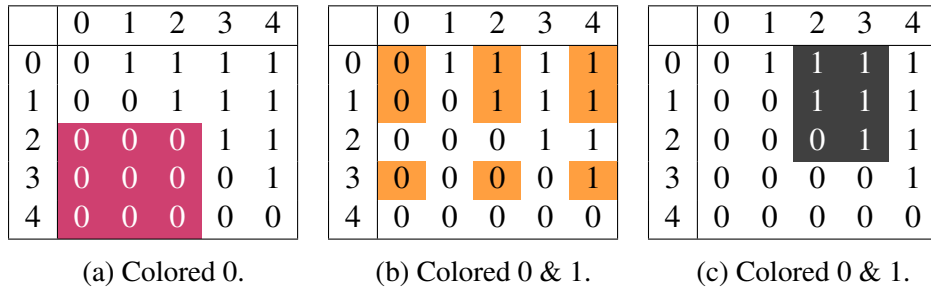


Figure 2: Some example rectangles of $M(<)$. The first rectangle, in **purple**, is monochromatically colored 0. The second rectangle, in **orange**, illustrates the flexibility of our rectangle definition, namely, that the rectangle does not actually need to be connected in the original matrix. Neither the **second** nor **third** rectangle is monochromatic.