Some open problems in analysis

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Abstract.
In this paper some open problems in analysis are formulated. These problems were formulated and discussed by the author at ICMAA6.

1. **Injectivity of the classical Radon transform**.
Consider the Radon transform:

$$Rf := \int_{\ell_{\alpha,p}} f ds, \tag{1}$$

where $\ell_{\alpha,p}$ is a straight line $\alpha \cdot x = p$ on the plane $x = \{x_1, x_2\}$, α is a unit vector, p is a real number, ds is the element of the arclength of the straight line.

Assume that

$$f \in L^1(\ell_{\alpha,p}) \tag{2}$$

for all p and α, that f is a continuous function, and that

$$|f(x)| \leq c(1 + |x|^m), \tag{3}$$

where $c = const > 0$, and $m \geq 0$ is a fixed number. Assume that

$$Rf = 0 \tag{4}$$

for all p and α.

Problem 1: *Does it follow from the assumptions (2)-(4) that $f = 0$?*

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There is a large literature on Radon transform (see, e.g., [3] and references therein). It is known (see, e.g, [1], [3]) that there are entire functions not vanishing identically, such that (2) and (4) hold.

The open problem is to understand what the weakest natural restriction on the growth of f at infinity is for the Radon transform to be injective. In other words, *under what weakest growth restriction at infinity do assumptions (2)-(4) imply $f = 0$?*

It is known (see [3]) that if $f \in L^1(\mathbb{R}^2, \frac{1}{1+|x|})$ and (4) holds, then $f = 0$, i.e., the Radon transform is injective on $L^1(\mathbb{R}^2, \frac{1}{1+|x|)})$.

2. **A uniqueness problem**.

Let L and M be elliptic, second order, selfadjoint, strictly positive Dirichlet operators in a bounded domain $D \subset R^n$, $n > 1$, with a smooth connected boundary S, and the coefficients of L and M be real-valued functions, so that all the functions below are real-valued. Let $a(x)$ and $b(x)$ be strictly positive functions, smooth in the closure of D. Let

$$Lu + a(x)v = 0 \text{ in } D, \quad -b(x)u + Mv = 0 \text{ in } D, \quad u = v = 0 \text{ on } S, \tag{5}$$

Problem 2: *Does* (5) *imply*

$$u = v = 0 \text{ in } D? \tag{6}$$

It is of no interest to give sufficient conditions for (6) to hold, such as, e.g., $|b - a|$ is small, or $L = M$, or some other conditions.

What is of interest is to answer the question as stated, without any additional assumptions, by either proving (6) or constructing a counterexample.

In the one-dimensional case the answer to the question (6) is yes (see [2]).

3. **A problem in operator theory**.

The question in two different forms is stated below as Problem 3.1 and Problem 3.2. These problems are closely related.

3.1. Let D be a bounded domain in R^3, D can be a box or a ball, $f \in L^2(D)$ be a function, $f \not\equiv 0$. Define

$$F(z) := \int_D f(x) \exp(iz \cdot x) dx, \quad z \in C^3.$$

The function $F(z)$ is an entire function of exponential type.

Let $L_j(z)$, $j = 1, 2$, be polynomials of degree not less than one, $\deg L_j(z) \geq 1$,

$$\mathcal{L}_j := \{z : z \in C^3, L_j(z) = 0\}$$

be the corresponding algebraic varieties.

Define Hilbert spaces $H_j := L^2(\mathcal{L}_j, dm_j)$, where $dm_j(z)$ are smooth, rapidly decaying, strictly positive measures on $\mathcal{L}_j$, such that any exponential $exp(iz \cdot x)$ with any $x \in R^3$ belongs to H_j. Define a linear operator T from H_1 into H_2 by the formula:

$$Th := \int_{\mathcal{L}_1} dm_1(u_1)h(u_1)F(u_1 + u_2) := g(u_2),$$

where $u_j \in \mathcal{L}_j$, $h \in H_1$, $g \in H_2$. We assume that the measures dm_j decay so rapidly that for any $h \in H_1$ the function $g = Th$ belongs to H_2, $Th \in H_2$. For example, this happens if the measures decay as $e^{-|z|^2}$.

Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *transversal*, which by definition means that there exist two points, one in $\mathcal{L}_1$ and one in $\mathcal{L}_2$, such that the union of the bases of the tangent spaces to $\mathcal{L}_1$ and to $\mathcal{L}_2$ at these points form a basis in C^3. The same setting is of interest in dimension $n > 3$ as well.

Problem 3.1: *Is it true that T is not a finite-rank operator?*

In other words, *is it true that the dimension of the range of T is infinite?*

Remark: The assumption that $f(x)$ is in $L^2(D)$ is important: if, for example, $f(x)$ is a delta-function, then the answer to the question of Problem 3.1 is no: the dimension of the range of T in this case is equal to 1 if the delta-function is supported at one point.

3.2. In the notations of Problem 3.1, choose points $p_m \in \mathcal{L}_2$, $m = 1, 2, \dots, M$, where M is an *arbitrary large* fixed integer. Consider the set $\mathcal{S}$ of M functions $F(z + p_m)$, $m = 1, 2, \dots, M$, where $z \in \mathcal{L}_1$, and $F(z)$ is defined above: it is the Fourier transform of a compactly supported $L^2(D)$ function, where D is a bounded domain in R^n, $n > 1$.

Problem 3.2: *Can one choose $p_m \in \mathcal{L}_2$ such that the above set $\mathcal{S}$ of M functions is linearly independent?*

In other words, can one choose $p_m \in \mathcal{L}_2$, $m = 1, 2, \dots, M$, such that the relation:

$$\sum_{m=1}^{M} c_m F(z + p_m) = 0 \quad \forall z \in \mathcal{L}_1 \tag{7}$$

implies $c_m = 0$ for all $m = 1, 2, \dots, M$? Here c_m are constants.

These questions arise in the study of Property C (see [4], p. 298).

4. **A problem related to the Pompeiu problem.**

Let $D \subset R^3$ be a bounded domain homeomorphic to a ball, with a real analytic boundary S. Let $u_j = u_j(x)$, $j = 1, 2, 3$, solve the problem:

$$\Delta u_j + k^2 u_j = 0 \quad \text{in} \quad D, \quad u_j|_S = 0, \tag{8}$$

where $k^2 > 0$ is a constant. Let $N = N_s$ be the unit normal to the surface S at the point $s \in S$, pointing out of D. Define the following vector-function:

$$u(x) = \sum_{j=1}^{3} u_j(x) e_j, \tag{9}$$

where $\{e_j\}_{j=1}^3$ is the standard Euclidean orthonormal basis of R^3. Let $[a, b]$ denote the cross product of two vectors a and b in R^3.

Assume that

$$u_N = [s, N_s] \quad \forall s \in S, \tag{10}$$

where $u = u(x)$ is defined in (9) and $u_j(x)$ solve problem (8).

Problem 4: *Do (8)-(10) imply that* $[s, N_s] = 0 \quad \forall s \in S$*?*

Conjecture: *Assumptions (8) and (10) imply*

$$[s, N_s] = 0 \quad \forall s \in S. \tag{11}$$

It is pointed out in [4], p. 416, that if (11) holds, then S is a sphere.

A proof of the above Conjecture implies a positive solution to the Pompeiu problem: see [4], Chapter 11, and [5], [6].

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