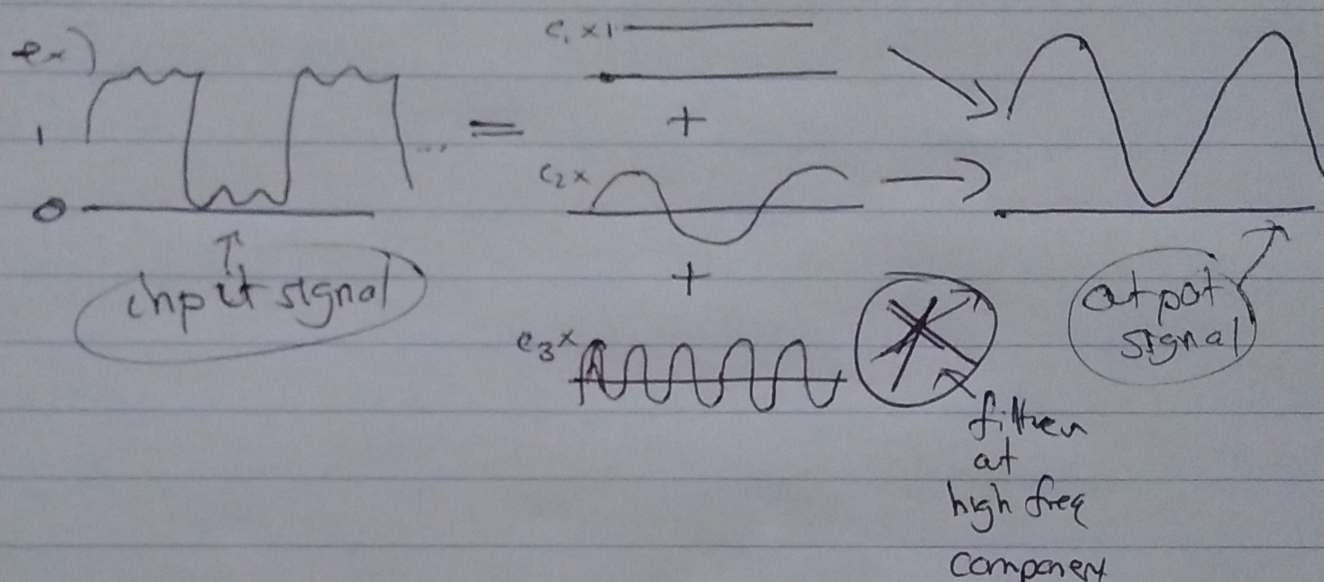
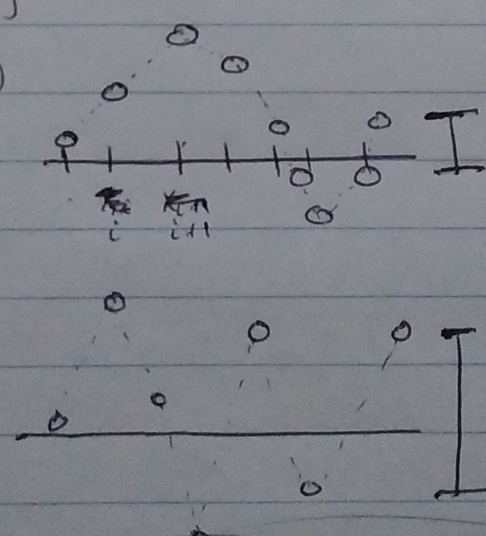
One of the main tools in conventional Digital Signal Processing (DSP) is to be able to decompose signals into components of varying frequencies and be able to filter certain components.

ex)



In standard DSP, frequency is related to how much a signal "varies" between adjacent samples

ex)



not much difference btwn adjacent samples = $|x_{i+1} - x_i|$ = small

large difference btwn adjacent samples = $|x_{i+1} - x_i|$ = large

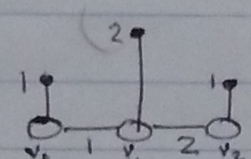
Thus freq ~ $\sum_{(i,i+1)} 1 \cdot (x_{i+1} - x_i)^2$ "total variation" of the signal

To gain mathematical & intuitive foundation for what frequency means on a graph we will extend this notion of the "total variation" of a signal being proportional to its freq.

Total Variation of signal x on Graph $G=(V,E)$ $= TV_G(x) \triangleq \underbrace{\sum_{e_{i,j} \in E} e_{i,j}(x_i - x_j)^2}$

"weighted sum of the squared difference of signal values btwn adjacent vertices"

ex)



$TV_G\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = \underset{e_{0,1}}{1}(2-1)^2 + \underset{e_{1,2}}{2}(1-2)^2 = 3$

<u>lemma:</u>

- $L \triangleq D - A$ { Laplacian of G
- $TV_G(x) \triangleq \sum_{e \in E} e_{i,j}(x_i - x_j)^2$

$\Rightarrow TV_G(x) = x^T L x$

In order to extend the idea of decomposing signals into constituent frequency components, we need an orthonormal basis

- In DSP this is the Fourier Basis $= \{\underbrace{e_{j\omega}}_{\text{complex exponentials}}\}_{i=0,\dots,N-1}$
  - The DFT is simply projecting our signal onto this basis.

Which basis should we use?

Proposition: Use the set of vectors which solves the following optim. problem

$$u_i = \underset{\substack{\bullet \|x\|=1 \\ \bullet x \perp \{u_0, \dots, u_{i-1}\}}}{\operatorname{argmin}} \; u^T L u \quad \left.\vphantom{\int}\right\} \text{min total var. over graph while orthog. to prev. vectors}$$

$$\Downarrow$$

Soln: $u_0, \; u_1, \; \dots, \; u_{N-1} \in \mathbb{R}^N$

$u_0^T L u_0 \leq u_1^T L u_1 \leq \dots \dots \leq u_{N-1}^T L u_{N-1}$

- we now have an orthonormal basis for $\mathbb{R}^N$ which has an increasing total variation $u_i^T L u_i$ w/ increasing i.

Amazing Fact:

- Let $(\lambda_0, v_0), \dots, (\lambda_{N-1}, v_{N-1})$ be the normalized ($\|v_i\|=1$) eigensystem of L, s.t. λ_i's are sorted: $0 \leq \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{N-1}$

- Then $u_i^T L u_i = \lambda_i$!!
$u_i = v_i$

Thus by producing eigensystem, and sorting eigenvalues in increasing order, we produce an orthonormal basis for $\mathbb{R}^n$ w/ the property that

$$\underbrace{\underset{\lambda_i}{\underset{\shortparallel}{TV_G(V_i)}} \leq \underset{\lambda_j}{\underset{\shortparallel}{TV_G(V_j)}}}_{}\ , \quad i \leq j$$

We thus interpret the eigenvalues as our frequencies

Doing so allows us to decompose graph signals onto a basis which preserves the classic DSP notions of frequency.

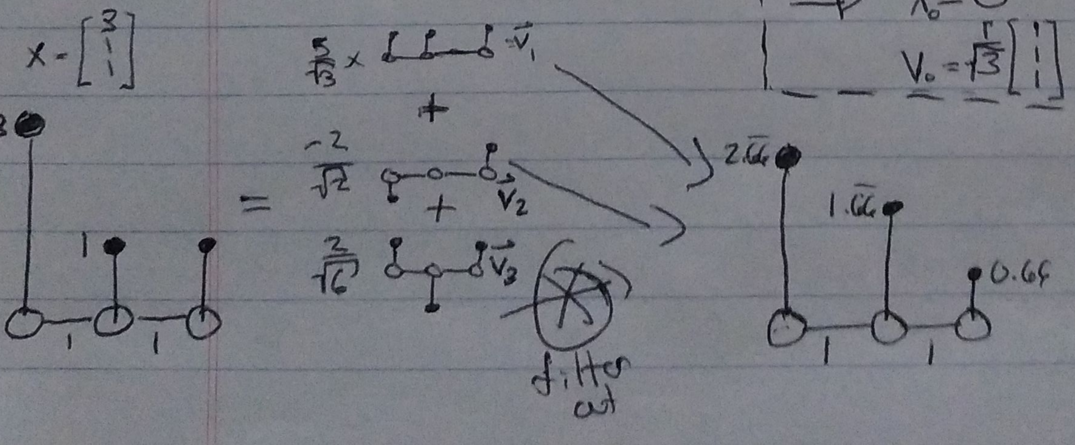
ex) $V_0 \overset{1}{—} V_1 \overset{1}{—} V_2 \iff L \triangleq D - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$\Rightarrow \lambda_0 = 0 \qquad \lambda_1 = 1 \qquad \lambda_2 = 3$

$V_0 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad V_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad V_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$x = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$



$X_{\text{filtered}} = \begin{bmatrix} 2.\overline{66} \\ 1.\overline{66} \\ 0.\overline{66} \end{bmatrix}$