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Dynamic pricing and inventory control for multiple products in a retail chain[☆]

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ABSTRACT

The biggest problems in the retail industry are related to price allocation to keep the product attractive to the market, without allowing arbitrage, together with adequate inventory control involving substitution and complementarity. The above objectives gains relevance if the uncertain behavior of the market is taken into account. Most of the existing literature addresses each of the above problems independently or as a combination of some of them and not as a whole that is related to each other. In this research, we present a model that allows setting optimal dynamic pricing and inventory policies in each period in which a planning horizon is divided. When assigning the price, the model avoids the presence of arbitrage, by allowing the use of fixed price policies (same price for the product in the stores) and/or variable price policies (different prices for the product in the stores). When making replenishments, it allows having an adequate quantity of each product, since it incorporates substitution and complementarity between them. The market demand in each store is represented by a probability distribution that is a function of the market by means of a seasonality parameter, price sensitivity, price and inventory, subject to a set of restrictions. We find that demand has uncertainty in the seasonality parameter and in the price sensitivity, due to variations outside the historical data with which it is calculated. To handle this uncertainty, we use a Robust Stochastic Optimization formulation, which we solve in an approximate way, using a scenario generation technique. We show computational experiments applied to a case with industrial data. The results show the advantages of using this methodology, allowing managers to make the right decisions and thus improve operating results.

1. Introduction

The retail sector is one of the most active and important in the world. In the United States, it accounts for $3.9 trillion of annual GDP and support 1 in 4 American jobs (National Retail Federation, 2022). For the most part, retailers market seasonal products, where dynamic pricing policies are often used. In this policy, an initial selling price is announced at the beginning of the season and the price is subsequently reduced or increased as the season progresses according to sales performance (Soysal & Krishnamurthi, 2012). For recent developments in the implementation of dynamic pricing policies, see Klein, Koch, Steinhardt, and Strauss (2020).

To set the price of each product, the retailer observes the behavior of sales as time progresses (day, week, fortnight) within the planning horizon (month, season, year) and uses this information to adjust prices (decrease or increases) in the future periods, all with the objective of maintaining an attractive price and thus increasing sales (Chatwin, 2000). This dynamic pricing policy falls under Revenue Management issues (Viglia & Abrate, 2019) and is also used by airlines, hotels and cargo operators in general. For other areas where this strategy is used, see Dutta and Mitra (2017) and Talluri and Van Ryzin (2004).

The problem of price allocation is particularly relevant for retailers, as they operate stores in different markets (regions, cities or countries) (Campbell, Datar, & Sandino, 2009). This situation forces retailers to set pricing policies based on the willingness to pay of the market served by each store (Koschate-Fischer, Stefan, & Hoyer, 2012) and, at the same time, avoid arbitrage. Arbitrage occurs when a third party takes advantage of the price difference between two stores of the same retailer and buys the product in the lower-priced store to sell it at a

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lower price in the catchment area of the higher-priced store (Chung, Rasmusen, et al., 2011).

Retailers also have to balance the quantities of products to be sold with the costs of having them available. This involves planning inventories (Azzi, Battini, Faccio, Persona, & Sgarbossa, 2014), where it is necessary to take into account the presence of substitute products (e.g., how many jackets of each brand to have), as well as complementary products (e.g., how much inventory to have of a pant that is complemented by a jacket) (McAuley, Pandey, & Leskovec, 2015). Finally, the retailer must set a price to sell the remaining units at the end of the season (Nakamura, 2008).

Today, many retailers use decision-making procedures based solely on intuition and previous history (Cohen & Perakis, 2020), which generates many stock-outs, among other problems. In addition, by not establishing coordination between stores, arbitrage occurs and, also, by not knowing the degree of correlation between products (substitution and complementarity) they end up having overstocks.

According to the above and the literature review presented in the following section, we did not find a model that maximizes the retailer's profit and incorporates all the concepts mentioned above in one formulation. Consequently, our objective is to develop and validate an optimization model that allows retailers to establish optimal dynamic pricing and inventory policies for the multiple products marketed by the retail chain in each period, that includes the presence of substitution and complementarity among its products and avoids the presence of arbitrage.

This research was motivated by a collaboration between academia and industry, and as a result the business problem presented, represents the reality of retail chain. On the other hand, the formulation and the algorithm used to address the problem under uncertainty, are also interesting from the perspective of optimization and statistics.

To identify the presence of substitution and complementarity, we assume that a multiple linear regression model (Gujarati & Porter, 1999) has been used, where the coefficients of the resulting regression model, tells us the degree of substitution (if negative) or complementarity (if positive), of a product with respect to another product taken as the dependent variable.

The main uncertainty source to be considered in our model is demand uncertainty. To represent demand, we use a probability distribution that adjusts in each period with a parameter of seasonality and another parameter that represents price sensitivity. However, those parameters are themselves the result of statistical estimates that are subject to noise and errors. Moreover, there are events that are not captured by the historical information (colder years or higher dollar values). Therefore, we find that we must strengthen the model. Since it is difficult to assign any specific distributional information to the estimation process of such parameters, we decided to implement Robust Optimization to model them so as to obtain policies that are protected from estimation errors.

The resulting robust stochastic optimization model with double protection could be large given, due to the number of SKU (stock keeping units) that are handled by the retailer. On the other hand, the model presents (as we will show) nonlinearities that increase complexity. Therefore, we address the problem using a method of successive generation of scenarios, from Bienstock and ÖZbay (2008), instead of trying to construct its robust counterpart. In their paper, Bienstock and ÖZbay (2008) describe how their methodology can find good results in problems related to inventory optimization.

In summary, these are the characteristics of our model:

- It optimizes inventory and dynamic pricing policies, thus increasing revenue and also reducing storage costs. This is especially relevant considering that most research tends to address only dynamic pricing policies and assumes the known inventories (Feng & Xiao, 2006).
- Avoids arbitrage between stores and, at the same time, sets the price according to the market served by each store, thus keeping the products attractive to consumers (Şen & Zhang, 2009). Most existing studies avoid arbitrage by setting the same price for the product in all stores. However, this deviates from reality, as the willingness to pay is different in each market (Koschate-Fischer et al., 2012). What is attractive to one market may not be attractive to another.
- It uses the coefficients of a multiple linear regression model to incorporate the degree of substitution and complementarity between products. Products are considered to be substitutes when a product can be purchased, in the absence of the one being sought (e.g. butter and margarine) and complementary are products that are consumed together (e.g. car and gasoline) (Moradi, 2021).
- It is able to adjust the price and inventory in each period, depending on the market served by the store to maximize profits, eliminating arbitrage and adjusting the inventory according to the degree of substitution and complementarity of its products. In this way, the model generates policies that seek to prevent the main causes of losses that occur in the retail trade (for example, selling a product at an unprofitable price in order to promote another, among others).

The article is structured as follows: In Section 2, we present the literature review. In Section 3, we present the demand assumptions, and the stochastic optimization model. In Section 4, we present the robust stochastic optimization model, along with the description of the demand function and the parameters to be protected. In Section 5, we present the results of the application of the model, which is carried out in a case study using retail chain data. In Section 6, we discuss the main managerial insights derived from our model. Finally, in Section 7, we present the main conclusions and possible areas for extension.

2. Literature review

Most research has included learning about demand behavior through a probability distribution, and the classic approaches to dynamic pricing policies consider a product (a product sold in one store), as in Dong, Simsek, and Topaloglu (2019), Şen and Zhang (2009), Aviv and Pazgal (2005) and Feng and Gallego (1995) among others. These models, which incorporate customer buying logic or learning from demand behavior, are more complex to solve, but return a more robust policy. That generally generates better expected revenue than a demand model without learning (Barbier, Anjos, Cirinei, & Savard, 2020). However, these models ignore the fact that buyers can switch to alternative classes when their requests are rejected, known as no-buy-up or buy-down (Talluri & Van Ryzin, 2004). In this paper we consider this switching of alternative classes by including the presence of substitution and complementarity between products in the model.

The consideration of a product sold in multiple stores, is addressed by Vinod (2021), Tsao and Sheen (2008), Chen, Ray, and Song (2006), Bernstein and Federgruen (2004) and Wang (2002) among others. The major problem in these lines of research lies in the need to avoid arbitrage, which, if not avoided, can lead to erroneous decisions with significant economic costs (Vinod, 2021). For this reason, two pricing policies are usually developed: fixed-price (in which the product has the same price in all stores) and variable-price (in which the price of the product varies from store to store) (Bernstein & Federgruen, 2004). Most papers on this topic, work with fixed price policies and ignore the fact that the willingness to pay in each market is different. Therefore, we work with a variable pricing policy, increasing profit, by taking into account of willingness to pay in each market and avoiding arbitrage, by controlling that the price of the same product in two different stores is lower than the cost of moving it between them.

The extension considering several products sold in one store has been addressed by Simchi-Levi, Sun, and Zhang (2022), Javanmard, Nazerzadeh, and Shao (2020), Saureé and Zeevi (2013), Adida and Perakis (2007), Bertsimas and De Boer (2005), and Kachani and Perakis (2002), among others. Perhaps the most relevant challenge in the existing literature is related to the fact that the price of a product is considered independently of other products (Kourentzes, Li, & Strauss, 2019) . This is because, in most research, the products are considered different and the target market is also different (Adida & Perakis, 2007); (Bertsimas & De Boer, 2005); (Biller, Chan, Simchi-Levi, & Swann, 2005)). However, in many cases, the price of each product cannot be controlled independently of other products, as indicated by Gallego and Van Ryzin (1997). In this sense, Cohen and Perakis (2020) addresses the presence of substitute and complementary products through cross-correlations, without including learning from demand behavior. We address the presence of substitution and complementarity between products through a multiple linear regression model, where the coefficients define the type of relationship between products and in addition, we incorporate learning in demand behavior, using a probability distribution.

Finally, the extension to multiple products sold in the chain retail, addressed in our research, has not been widely treated. However, some authors address the problem of setting dynamic pricing policies with known inventory; for example, Paschalidis and Liu (2002) they use Markov process for incorporate demand learning, and in Maglaras and Meissner (2006) this aspect is incorporated, modeling demand with a probability distribution. As for incorporating inventory uncertainty to jointly model inventory and pricing policies, Feng and Xiao (2006) treats fixed pricing policies as variables and ignores the fact that buyers may switch to alternative classes when their requests are rejected. In this sense, Kim and Bell (2011) employs a variable pricing policy and two types of substitution. In the first type, customers substitute an out-of-stock product for a similar one, and in the second a customer substitutes a lower-cost product for a similar higher-cost product.

In a similar vein, He, Wang, and Cheng (2013), investigates competition and evolution in multi-product supply chains in retail. In their review, they find that the daily strategy of low prices arises from the evolutionary behavior of its model, called the agent-based retail model, as the dominant pricing strategy in multi-product retail chains. However, they do not consider the presence of substitution and complementarity between products in the methodology. They do allow products to be priced differently in each store but do not include a restriction to prevent arbitrage-related losses. Recently, Javanmard et al. (2020) addressed the sale of multiple products in the retail chain by using a multinomial choice logit model to model the learning of demand behavior; this method also takes price sensitivity into account. It does not, however, optimize inventory, and it ignores potential market saturation, which could result in many units of surplus at the end of the planning horizon. On the other hand, they assume that they work with dissimilar products, which implies that the model does not contemplate substitution and complementarity in their products. This model also does not include a restriction to prevent arbitrage between its stores. Following this line, our research developed a fairly complete model that almost closely reflects the reality of retail operation, where we set optimal dynamic pricing (variables price) and inventory policies and take into account the presence of substitution and complementarity between products and at the same time, avoid the presence of arbitrage between stores.

As mentioned above, in our research we have uncertainty in the demand function. When we have uncertainty, the most widely used methodology is robust optimization (Sun, Wang, & Xue, 2021). Traditional robust optimization assumes the variation of a certain model parameter and the model parameter usually varies in intervals (Ziaei & Jabbarzadeh, 2021). In this paper, we assume some probabilistic behavior, but we establish robustness for some parameters of the demand distribution, which in turn affects the parameters of the probability distribution (Yang, Couillet, & McKay, 2015). This is in line with what is known as Robust Stochastic Optimization (Li, Lu, Li, Wang, & Zhu, 2022; Mahtab, Azeem, Ali, Paul, & Fathollahi-Fard, 2022), which is identified as a bridge between the robust and stochastic worlds (Tan, Ju, Reed, Rao, Peng, Li, & Pan, 2015). This line of research, is quite recent, with the earliest work dating back to 2007 with Chen, Sim, and Sun (2007).

In their paper, Chen et al. (2007) uses random variables, called variances, to generate forward and backward values to capture the distribution. This model is used in Xu, Huang, Qin, Cao, and Sun (2010) for solid waste management using a robust linear stochastic optimization problem. Baringo and Amaro (2017) formulate a problem using a robust stochastic optimization stochastic model with uncertainties in daily market prices and electric vehicle driving requirements. For more information on stochastic robust optimization, the reader is referred to Chen and Xione (2020), (Huang, Qu, Yang, & Liu, 2021) and Gabrel, Murat, and Thiele (2014). Research in robust stochastic optimization protects a single parameter with uncertainty (Akbari-Dibavar, Nojavan, Mohammadi-Ivatloo, & Zare, 2020; Caunhye & Alem, 2021; Chen, Sim, & Xiong, 2020). Our problem as mentioned above and will be appreciated in the formulation, it is necessary to protect two parameters, which makes our model quite complex, but at the same time allows managers to make decisions adjusted to reality.

In addition, Ferrer, Oyarzún, and Vera (2012) presents a simpler pricing model in which a distribution is assumed and robustness is applied to the model parameters, which are established via forecasting. The simple structure of their model achieves analytic solutions that meet the robustness requirements. The model we present in this paper is more complex as it considers various other characteristics of the problem, and the computations of the robust solutions are not straightforward. We have devised a solution scheme, based on scenario generations, which allows us to compute a good approximate solution to the problem. We illustrate that on a specific application using data from an actual retailer.

As can be seen, there are opportunities to consider a model that seeks optimal policies of dynamic prices (variable prices) and inventories, for a scenario of multiple products sold in the retail chain, which involves the complementarity and substitution of products, as well such as arbitration prevention. In addition, the incorporation of the uncertainty of demand and its effect on the price is a relevant characteristic to consider (double protection). The fact that two of the parameters of the demand function are subject to statistical estimation errors and noise is important and should be taken into account. Therefore, our model uses an approach based on robust stochastic optimization to address this problem and obtain price and inventory policies that are more tolerant to such estimation errors.

3. Dynamic pricing and inventory model

In this section, we present the details of the model, including the demand function. We consider the situation of a multi-store, multi-product retailer. These products can be grouped into families so that the products of each family represent substitution and complementarity. We use factor analysis (Yang & Trewn, 2004) to form the groups, which implies that the demand for the products of the same group is highly correlated and therefore there is little or no correlation from one group to another. Pricing and inventory decisions must be made on a finite planning horizon (i.e., week, month, season, or year).

The model presented below assumes that orders are placed at the beginning of each period and replenished immediately. The above does not escape from the actual operation of the retailer under study, given that by policy, suppliers in the case of third-party products and its distribution center, in the case of its own products, must supply the warehouses in less than 24 h. The number of units lost in each period, estimates on the retailer's historical data and too, allows the remaining units to be sold at a residual value at the end of the season.

Table 1
Sets.

Sets	Description
q	Product set: $\{1,\ldots,q,\ldots,Q\}$
m	Group set: $\{1,\ldots,m,\ldots,M\}$
l	Store set: $\{1,\ldots,l,\ldots,L\}$
t	Period set: $\{1,\ldots,t,\ldots,T\}$

Table 2
Parameters of the demand function.

Parameters	Description
μ_{qmlt}	Mean of the probability distribution of the demand for product q in group m in store l in period t
σ^2_{qmlt}	Variance of the probability distribution of the demand for product q in group m in store l in period t
γ_{qmlt}	Seasonality factor of the demand for product q in group m in store l in period t
ρ_{qmlt}	Scale factor of the demand for product q in group m in store l in period t
ϑ_{qmlt}	Factor related to the dispersion of the demand for product q in group m in store l in period t in the distribution
α_{qmlt}	Factor measuring price sensitivity of the demand for product q in group m in store l in period t
r_{qml}	Exponent of the negative binomial distribution of product q in group m in store l

Table 3
Parameters of the Retail model.

Parameters	Description
s_{qmlt}	Demand for product q of group m in store l in period t
c_{qmlt}	Cost per unit of product q in group m in store l in period t
K_{qmlt}	Cost of placing an order of product q in group m in store l in period t
dl_{qml}	Penalty for each unit of unmet demand for product q of group m in store l
dn_{qml}	Penalty for demand not satisfied in period t and to be satisfied in period $t+1$ for product q of group m in store l
w_{qml}	Residual value per unit of product q in group m in inventory in store l at the end of the season
$\bar{h}_{qml}$	Inventory cost for product q in group m, charged at the end of each period t in store l
IF_{qlt}	Storage capacity for product q in store l in period t
$p_{min_{qml}}$	Minimum price allowed for product q of group m in store l
$p_{max_{qml}}$	Maximum price allowed for product q in group m in store l
$Transp_{vw}$	Cost of transportation between stores v and w (may include import duties if the stores are in different countries)
η_{qmlt}	Factor that represents the percentage of lost orders for product q in group m, store l, period t
τ_{qml}	Fraction of lost demand for product q in group m, store l

3.1. Model

In this subsection, we define the sets and parameters of the demand function, as well as the parameters, variables and constraints of the retail model. There are Q products, which are divided into M homogeneous groups in terms of substitutability and complementarity between products. We consider L stores, and planning is done over T time periods. The sets, demand function, demand function parameters, retail model parameters, substitute product parameters, decision variables, and auxiliary variables are presented in Tables 1, 2, 3, 4, 5, and 6, respectively.

Demand Function:

To represent the demand function for the products, we use the classical model by Erenberg (1972), i.e., a negative binomial distribution that Subrahmanyan and Shoemaker (1996), approximates to a normal distribution. This approximation incorporates a seasonality factor γ_{qmlt} into the mean as well as another factor that represents the elasticity of demand with respect to price, α_{qmlt}. The mean and variance of the normal distribution that will be used for demand depends on these factors in Eqs. (1) and (2), respectively. The probability density of the demand for a specific price level $f_{qmlt}(s_{qmlt}|p_{qmlt})$ is presented in Fig. 1.

Table 4
Product substitution parameters.

Parameters	Description
$\beta_{qq'ml}$	Measures the complementarity or substitution of products q and q' of group m in store l. A negative value of β represents complementarity between q and q', and a plus sign indicates substitutes

Table 5
Decision variables.

Variables	Description
o_{qmlt}	Amount of product q of group m to be ordered in store l at the beginning of period t
p_{qmlt}	Price of product q, group m in store l during period t

Table 6
Auxiliary variables.

Variables	Description
I_{qmlt}	Inventory of product q, group m in store l at the start of period t
e_{qmlt}	Lost units of product q of group m in store l during period t, assumed to be a percentage of the real orders of the previous period
z_{qmlt}	Controls the variation of the protected factors of product q in group m in store l during period t
y_{qmlt}	Takes the value of 1 when there is an order of product q of group m in store l in period t and 0 otherwise

$$\mu_{qmlt} = \gamma_{qmlt} \times \rho_{qmlt} \times e^{-\alpha_{qmlt} p_{qmlt}} \tag{1}$$

$$\sigma^2_{qmlt} = \mu_{qmlt} \times \left(1 + \frac{\mu_{qmlt}}{\vartheta_{qmlt}}\right) \tag{2}$$

where ϑ_{qmlt} is an adjustment factor, and the periods of the seasonality factor γ_{qmlt} can be expressed in days, weeks, months or a time interval set by the decision maker (not necessarily similar). It can also be a period pre-established by the retailer.

Fig. 1 shows the units available at the beginning of the period $I_{qml(t-1)} + o_{qmlt}$. Depending on the number of available units, there is a probability $(1-\phi)$ of satisfying the demand and a probability ϕ of not satisfying the demand. Therefore, we define $E\left[g_{qmlt}(s_{qmlt}|o_{qmlt}, p_{qmlt})\right]$ as the expected demand and $E\left[h_{qmlt}(s_{qmlt}|o_{qmlt}, p_{qmlt})\right]$ as the expected lost demand; these are calculated in Eqs. (3) and (4), respectively.

$$E\left[g_{qmlt}(s_{qmlt}|o_{qmlt}, p_{qmlt})\right] = \int_{-\infty}^{I_{qml(t-1)}+o_{qmlt}} s_{qmlt} f_{qmlt}(s_{qmlt}|p_{qmlt}) ds \tag{3}$$

$$E\left[h_{qmlt}(s_{qmlt}|o_{qmlt}, p_{qmlt})\right] = \int_{I_{qml(t-1)}+o_{qmlt}}^{+\infty} s_{qmlt} f_{qmlt}(s_{qmlt}|p_{qmlt}) ds \tag{4}$$

We have defined the maximum demand that can be managed by the retailer, $Smax_{qmlt}$, as the cumulative demand up to a $1-\eta$ percentile of the demand distribution; the latter term is defined by the decision maker. This can be expressed as:

$$Smax_{qmlt} = \mu_{qmlt} + \phi_{1-\eta}\sigma_{qmlt} \tag{5}$$

where $\phi_{1-\eta}$ is the $1-\eta$ percentile of the standard normal distribution. It should be clarified that our model is capable of working with any other probability distribution that involves a seasonality factor and price sensitivity in its formulation.

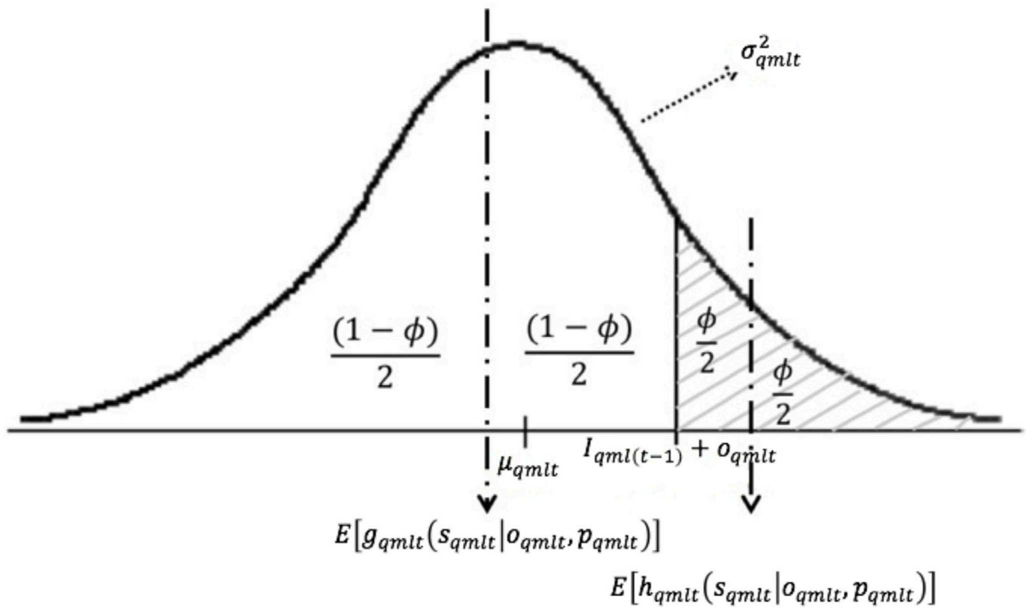

Fig. 1. Probability density of demand (By the authors).

3.2. Retail model (RM) formulation

The stochastic nonlinear optimization model is represented by:

$(RM)\ \max_{o,p} Ben(o,p)$

Subject to:

$$0 \leq I_{qml(t-1)} + o_{qmlt} \leq min_t \left\{ IF_{qml}, Smax_{qmlt} \right\} \quad \forall q,m,l,t \tag{6}$$

$$o_{qmlt} \geq \sum_{q'} \hat{\beta}_{qq'ml} o_{q'mlt} \quad \forall q,m,l,t \tag{7}$$

$$-Transp_{vw} \leq p_{qmvt} - p_{qmwt} \leq Transp_{vw} \quad \forall q,m,t,v,w,v \neq w \tag{8}$$

$$p_{min_{qml}} \leq p_{qmlt} \leq p_{max_{qml}} \qquad \forall q,m,l,t \tag{9}$$

$$\begin{aligned} o_{qmlt} \leq{} & I_{qmlt} - I_{qml(t-1)} + e_{qmlt} + \int_{-\infty}^{I_{qml(t-1)}+o_{qmlt}} s_{qmlt} f_{qmlt}(s_{qmlt}|p_{qmlt}) ds \\ & + \tau_{qml} \int_{I_{qml(t-1)}+o_{qmlt}}^{+\infty} s_{qmlt} f_{qmlt}(s_{qmlt}|p_{qmlt}) ds \\ & + (1-\tau_{qml}) \int_{I_{qml(t-2)}+o_{qml(t-1)}}^{+\infty} s_{qml(t-1)} f_{qml(t-1)}(s_{qml(t-1)}|p_{qml(t-1)}) ds \quad \forall q,m,l,t \end{aligned} \tag{10}$$

$$e_{qmlt} \geq \eta_{qmlt} o_{qml(t-1)} \quad \forall q,m,l,t,t > 1 \tag{11}$$

$$o_{qmlt} \leq M y_{qml(t-1)} \quad \forall q,m,l,t \tag{12}$$

$$p_{qmlt} \geq 0, o_{qmlt} \geq 0, I_{qmlt} \geq 0, e_{qmlt} \geq 0, \quad \forall q,m,l,t \tag{13}$$

where

$$\begin{aligned} Ben(o,p) ={} & \\ & \sum_{q=1}^{Q} \sum_{m=1}^{M} \sum_{l=1}^{L} \sum_{t=1}^{T-1} (\int_{-\infty}^{I_{qml(t-1)}+o_{qmlt}} s_{qmlt} f_{qmlt}(s_{qmlt}|p_{qmlt}) ds \\ & * p_{qmlt} - c_{qmlt} * o_{qmlt} - K_{qmlt} * o_{qmlt} \\ & - (I_{qmlt} - e_{qmlt}) * \bar{h}_{qmlt}) - \tau_{qml} * dl_{qml} \\ & * \int_{I_{qml(t-1)}+o_{qmlt}}^{+\infty} s_{qmlt} f_{qmlt}(s_{qmlt}|p_{qmlt}) ds \\ & - (1-\tau_{qml}) * dn_{qml} * \int_{I_{qml(t-1)}+o_{qmlt}}^{+\infty} s_{qmlt} f_{qmlt}(s_{qmlt}|p_{qmlt}) ds \\ & + \sum_{q=1}^{Q} \sum_{m=1}^{M} \sum_{l=1}^{L} (\int_{-\infty}^{I_{qml(T-1)}+o_{qmlT}} s_{qmlT} f_{qmlT}(s_{qmlT}|p_{qmlT}) ds \\ & * p_{qmlT} - c_{qmlT} * o_{qmlT} - K_{qmlT} * o_{qmlT} \\ & + (l_{qmlT} - e_{qmlT}) * w_{qml}) - \tau_{qml} * dl_{qml} \\ & * \int_{I_{qml(T-1)}+o_{qmlT}}^{+\infty} s_{qmlT} f_{qmlT}(s_{qmlT}|p_{qmlT}) ds \\ & - (1-\tau_{qml}) * dn_{qml} * \int_{I_{qml(T-1)}+o_{qmlT}}^{+\infty} s_{qmlT} f_{qmlT}(s_{qmlT}|p_{qmlT}) ds \end{aligned}$$

Here, the retailer's expected profit is maximized and the method considers the following: expected sales revenue, cost per unit ordered in a period, cost of storage applied to ending inventory less units lost in the period, cost of expected lost demand, the demand that was not satisfied in the previous period and, for the last period, the residual revenue of the remaining products.

Constraint (6) controls the inventory in each period, which must always be nonnegative and should not exceed the available physical space or the maximum allowed demand $Smax_{qmlt}$. Constraint (7) represents the presence of substitution and complementarity between products of the same group and expresses that the quantity to be ordered of one of the products, o_{qmlt}, is at least a linear combination of the orders for the other products within the same group, weighted by the β parameters. Constraint (8) prevents arbitrage by accounting for the price difference between the same product at two different stores and ensuring that this difference is less than the cost of moving them, $Transp_{vw}$. Constraint (9) establishes the upper and lower bounds for the price of the product. Constraint (10) controls the orders at the beginning of each period; the orders are less than or equal to the difference between the ending inventory of the period and the end of the previous period, plus the lost units, the expected demand, the lost demand that can be satisfied in the period and the lost demand that was not satisfied in the previous period. Constraint (11) defines the lost orders as a fixed percentage of the orders from the previous period, with the initial value set to zero. Constraint (12) activate the collection of replacement costs. where M is a very large value. Finally, constraint (13) represents the nature of the variables.

Meanwhile, constraint (10) operates as a soft restriction in which the auxiliary variable e_{qmlT} functions as a slack that stores the missing

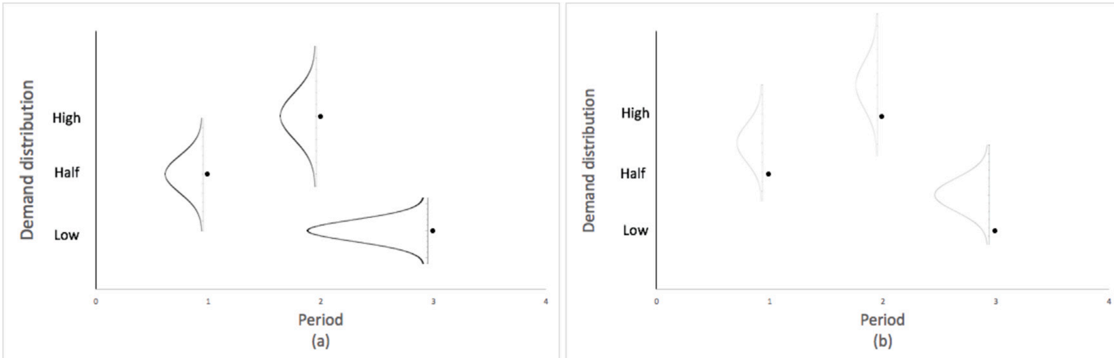

Fig. 2. Demand levels in the period (By the authors).

products; accordingly, the auxiliary variable does not allow I_{qmlt} to take negative values. Constraint (7) functions as a hard restriction in which $\beta_{qq'ml}$ represents the marginal rate of substitution. In other words, for each additional unit of $o_{q'mlt}$, $\beta_{qq'ml}$ units of o_{qmlt} are required. It is important to note that a negative $\beta_{qq'ml}$ value indicates that products q and q' are substitutes since the increase of one necessarily decreases the other. On the other hand, a positive $\beta_{qq'ml}$ value implies complementarity since increasing one necessarily increases the other.

With regards to restriction (7), the first step is to perform a factor analysis (Yang & Trewn, 2004) in which the products should be grouped according to their degree of correlation of demand (i.e., the products of the same group are highly correlated). Then, a multiple linear regression model is estimated, taking one of the products as the dependent variable and the rest of the products of the same group, as independent variables. For a more in-depth analysis of multiple linear regression and factor analysis, see Yang and Trewn (2004) and Gujarati and Porter (1999), respectively.

4. Model with robust protection of distribution parameters

As previously explained, demand is subject to uncertainty. This uncertainty is represented by means of a negative binomial probability distribution that is approximated to normal, which results in the stochastic optimization model presented in the previous section. Fig. 2a presents the probability distribution constructed with historical data of a product as a function of three periods and three levels of demand.

According to Fig. 2a and Eqs. (1) and (2), the higher the expected demand (represented by the black dots coinciding with the mean of the distribution), the larger the market (defined by the width of the Gaussian bell), so the model will raise the price to increase revenue. However, when demand falls, the market shrinks and the model will lower the price to maintain the attractiveness of the product in the market. This behavior is consistent with traditional revenue management theory. On the other hand, considering that inventories in one period are a function of the previous period, the model will adjust orders in each period to maintain adequate inventory, which is consistent with traditional inventory management theory, which requires adequate inventory to meet demand and avoid overstocking.

Following Fig. 2a, we could linearize the demand by working with the expected value in each period. However, due to a property of the normal distribution, we would only be satisfying 50% of the demand, and the rest would be lost. On the other hand, if we work with a confidence level of 95%, we will accurately estimate the actual demand 95% of the time, since the population mean would be within our confidence interval 95% of the time.

Fig. 2b shows a change in the behavior of demand in future periods (normal distribution curve). These changes are due to future fluctuations that were not considered when estimating demand with historical information (see Fig. 2a). For our problem, we have two important variations: The first, is due to temperatures lower or higher than those captured in the historical data (Ex.: in seasons colder or warmer than historical ones, demand for jackets increases or decreases). The second, is related to higher dollar values than historically seen.

Continuing with Fig. 2b, in the presence of either of the two variations expressed above, our theoretical expected demand defined by the black dots, will deviate from the market-defined real demand behavior (normal probability distribution) and therefore, we will make wrong decisions, since we will think that our market remains the same, when in fact it has changed when compared to Fig. 2a. In our model, we control for environmental variations with the seasonality factor (γ_{qmlt}) and dollar fluctuation with the price sensitivity (α_{qmlt}). Consequently, looking at Eqs. (1) and (2), it is necessary to protect the demand function from the risk and uncertainty of these parameters by robust optimization. To this end, we develop a robust stochastic optimization problem with double protection.

As mentioned above, we focus on the fluctuations of the parameters γ_{qmlt} and α_{qmlt}. To represent the fluctuation of the seasonality parameter of the γ_{qmlt} problem, we assume that γ_{qmlt} fluctuates around a nominal value of $\hat{\gamma}_{qmlt}$ with a variance of $\pm\nabla^1_{qmlt}$. Likewise, the mean demand sensitivity to price, α_{qmlt}, fluctuates around a nominal value $\hat{\alpha}_{qmlt}$ with variance equal to $\pm\nabla^2_{qmlt}$.

We now impose limits on the simultaneous variation of these two parameters; these limits are specific to products and time periods as per the robust approach developed by Bertsimas and De Boer (2005). To that end, we introduce the auxiliary variables z^1_{qmlt} and z^2_{qmlt}, and we write:

$$\gamma_{qmlt} = \hat{\gamma}_{qmlt} + \nabla^1_{qmlt} z^1_{qmlt} \tag{14}$$

$$\alpha_{qmlt} = \hat{\alpha}_{qmlt} + \nabla^2_{qmlt} z^2_{qmlt} \tag{15}$$

where the auxiliary variables take values in the interval of $[-1, 1]$ but their simultaneous variation must satisfy:

$$\sum_{t=1}^{T}\sum_{q=1}^{Q}\left|z^1_{qmlt}\right| \leq \Omega \ \forall m, l \tag{16}$$

$$\sum_{t=1}^{T}\sum_{q=1}^{Q}\left|z^2_{qml}\right| \leq \Omega \ \forall m, l \tag{17}$$

Ω is the "uncertainty budget" that controls the level of protection imposed on the simultaneous variation of the parameters γ_{qmlt} and α_{qmlt}.

$D(\Omega)$ denotes the set of all possible combinations of the parameters (γ, α) such that their variations are controlled by Relations (16) and (17). This is the uncertainty set, as it is called in robust optimization.

Next, let us use $X(\gamma, \alpha)$ to denote the set of all combinations of the decision variables (p, o) that satisfy the constraints of the problem. Notice that this set depends on the parameters (γ, α) of the probability distribution. We formulate the following robust version of the problem, which we call the retail robust problem:

$$\begin{array}{lll} \max & \theta & \\ s.a. & Ben(p, o, \gamma, \alpha) \geq \theta & \forall(\gamma, \alpha) \in D(\Omega) \\ & (p, o) \in X(\gamma, \alpha) & \forall(\gamma, \alpha) \in D(\Omega) \end{array}$$

where the dependence of the objective function Ben and the constraints on the parameters γ and α is explicit. In particular, constraint (10) is the one that depends on (γ, α), and variables e_{qmlT} take care of the slack needed to satisfy all scenarios.

The problem above has an infinite number of constraints and, due to the specific analytic form of the constraints, they cannot be easily transformed into a robust counterpart. Instead, we have approximated the solution to this problem through the successive generation of scenarios proposed by Bienstock and ÖZbay (2008). In their document, they propose an algorithm that adapts well to problems with hundreds of periods of time when the demand is uncertain (regardless of the type of distribution that it represents), as is the case in the model presented in this research. We now explain the specific implementation.

The idea of successive generation of scenarios is similar to Benders' decomposition principle in which a Master problem with a finite number of scenarios is constructed and is later complemented with other scenarios generated by a "satellite" problem, which, in this approach, is called the "adversarial" problem.

To illustrate this, suppose we are in iteration r of the algorithm and we have already generated a partial set of scenarios with the parameters $(\gamma^1, \alpha^1) \in \mathcal{D}(\Omega), \ldots, (\gamma^r, \alpha^r) \in \mathcal{D}(\Omega)$. The decision problem in this iteration is then:

Initialize D = $\emptyset$, L = $-\infty$ and U = $+\infty$

1. Master problem. Let p^r, o^r and θ^r be the solution to the problem:

$$\begin{array}{lll} \multicolumn{3}{l}{\max_{o,p}\theta} \\ s.t. & Ben(p, o, \gamma^k, \alpha^k) \geq \theta & k = 1, \ldots, r \\ & (p, o) \in X(\gamma^k, \alpha^k) & k = 1, \ldots, r \\ Set & L \leftarrow p^r, o^r, \theta^r & \end{array}$$

2. Adversarial problem. Let γ^r and α^r be the solution to the problem

$$\begin{array}{ll} \multicolumn{2}{l}{\min_{\gamma,\alpha} Ben(p^r, o^r, \gamma, \alpha)} \\ s.t. & (\gamma, \alpha) \in \mathcal{D}(\Omega) \\ & (p, o) \in X(\gamma, \alpha) \\ Set & U \leftarrow \gamma^r, \alpha^r, Ben^r \end{array}$$

3. Termination test. If $U - L$ is small enough then EXIT

4. Formulation update. Otherwise, add Set D and return to Step 1

This is an approximation that is justified on the grounds that the economic value of pricing, inventory and demand satisfaction is the most relevant component for the decision maker. Hence, feasibility is subordinate to scenarios that are good for the economic value of the problem. This problem is nonlinear and is solved approximately. Our proposal is a simplification, as it might miss some satisfactory scenarios, but we will later show that, from a managerial perspective of the problem, it delivers favorable results.

5. Results

This study arose from the retail industry's need to improve its methods of price allocation and inventory control in each store, given that these were done on the basis of the experience of the manager in charge of each store. However, this procedure results in excessive price reductions, excessive inventories in some stores and shortages in others, which causes additional transportation costs between stores to move the products.

Another problem is due to the lack of knowledge of the existence of substitution and complementarity between products, which caused sales to be lost because there were no substitutes and/or complementary products to the product sought by the customer and, lastly, a no less important problem was the presence of arbitrage, since each store manager programmed the discounts independently.

Consequently, the purpose of the following section is to analyze the results in a specific application to the retail sector. While for confidentiality reasons, we consider two unidentified products with correlated demand sold in two stores over a four-period planning horizon, the problem addresses the proposed methodology and can be replicated and scaled to handle more products and stores.

Table 7
Costs for the case study.

Description	q = 1	q = 2
Cost (c_{qml})	77,99	48,71
Holding cost ($\bar{h}_{qml}$)	7,80	4,87
Leftover value (w_{qml})	38,99	24,36
Unmet demand (d_{qml})	3,90	2,44

Table 8
Demand function parameters.

Parameter	q = 1, l = 1	q = 2, l = 1	q = 1, l = 2	q = 2, l = 2
Price sensitivity (α_{qmlt})	0,0086	0,0084	0,0094	0,0094
Mean control (ρ_{qmlt})	3.255,68	7.652,27	3.936,55	8.652,27
Variance control (ϑ_{qmlt})	12,76	13,37	17,37	18,37

Table 9
Seasonality factor (γ_{qmlt}).

Period (t)	q = 1, l = 1	q = 1, l = 2	q = 2, l = 1	q = 2, l = 2
1	1,0215	1,1539	0,8752	1,1539
2	0,9113	0,8260	0,8117	0,8710
3	0,7816	0,8102	0,8373	0,8946
4	0,8113	0,8105	0,8034	0,8375

The models developed in this section were implemented on a computer with an Intel Core processor with 2.8 GHz core i7 and 4 GB of RAM. The models were solved using the LSGRG solver (FrontlineSolver, 2021), which was implemented as an Excel add-in. This solver is capable of finding global solutions to difficult problems, both linear and, as is the case here, non-linear. For more information on the LSGRG solver and other solvers that can solve problems like those discussed in this document, see FrontlineSolver (2021).

In the following subsection, we describe the problem as well as the case study data. We then solve the robust problem. Finally, we study the impact of uncertainty on the optimality, feasibility and solution structure.

5.1. Description of the test instance

The case we present here is based on actual data from a retail operator in Latin America, which we do not identify for reasons of confidentiality. Table 7 presents a summary of the costs for each product, which is indicated by the index q.

The retailer's policy is to work with a minimum price $\left(p_{min_{qml}}\right)$ so as to guarantee a profit of 20% of the cost of the product. These minimum prices are \$97.49 and \$60.89 for products 1 and 2, respectively. On the other hand, the maximum price $\left(p_{max_{qml}}\right)$ is the one that makes the average of the demand function $\left(\mu_{qmlt}\right)$ equal to zero; it is computed in Eq. (18). The retailer also wants the price of the product in one period to be lower than the price in the previous period; this is expressed in relation (19).

$$pmax_{qmlt} = \left(ln\left(\gamma_{qmlt}\right) + ln\left(\rho_{qmlt}\right) - ln(0.0001)\right)/\alpha_{qml} \tag{18}$$

$$p_{qml(t-1)} \leq p_{qmlt} \qquad \forall q, m, l, \tag{19}$$

Table 8 presents the value of the implemented parameters $\left(\alpha_{qml}\right)$, $\left(\rho_{qmlt}\right)$ and $\left(\vartheta_{qmlt}\right)$. Table 9 presents the value of $\left(\gamma_{qmlt}\right)$, for each product q and store l.

The cost of moving a product between stores $\left(Transp_{vw}\right)$ is \$ 3,726.75, the correlation coefficient $\left(\hat{\beta}_{ml}\right)$ required for Constraint (11) is equal to 0.99, indicating that the products are complementary, and the lost products correspond to 2% of the orders of the previous period. Finally, the storage capacity, IF_{qml} for the case study, is present in Table 10 for each product q in each store l.

According to the retailer requirements, initially the costs of placing an order were not taken into account ($K_{qmlt} = 0$), nor was it taken into

Table 10
Storage capacity (IF_{qml}).

Period (t)	q = 1, l = 1	q = 1, l = 2	q = 2, l = 1	q = 2, l = 2
1	1,250	2,959	1,072	2,569
2	1,116	2,230	995	2,061
3	994	2,326	1,068	2,276
4	1,080	2,342	1,080	2,150

Table 11
Optimal order and pricing policy according to the retail model.

Description		Product 1 (q = 1)				Product 2 (q = 2)			
		t = 1	t = 2	t = 3	t = 4	t = 1	t = 2	t = 3	t = 4
Store 1 (l = 1)	Price	172.91	172.91	168.55	163.17	188.02	181.97	174.69	173.89
	Order	787.06	786.68	786.65	786.47	2,008.88	1,873.25	1,794.74	979.80
Store 2 (l = 2)	Price	171.85	171.85	167.61	161.99	189.90	183.40	175.81	174.87
	Order	787.23	745.15	744.94	786.52	1,884.22	1,891.92	1,212.02	1,183.54

Table 12
Inventory movement for store 1 (l = 1) under the retail model.

Description	Product 1 (q = 1)				Product 2 (q = 2)			
	t = 1	t = 2	t = 3	t = 4	t = 1	t = 2	t = 3	t = 4
Initial inventory	787.1	810.8	876.7	1,005.2	2,008.9	2,026.2	2,326.0	1,757.2
Final inventory	24.1	90.1	218.7	291.5	153.0	531.3	777.4	219.5
Expected sales	630.6	616.4	588.6	645.3	1,587.0	1,342.7	1,413.9	1,328.7
Unmet expected demand	132.3	88.5	53.7	52.6	268.9	112.1	97.3	173.2
Lost units	0.0	15.7	15.7	15.7	0.0	40.2	37.5	35.9

Table 13
Inventory movement for store 2 (l = 2) under the retail model.

Description	Product 1 (q = 1)				Product 2 (q = 2)			
	t = 1	t = 2	t = 3	t = 4	t = 1	t = 2	t = 3	t = 4
Initial inventory	787.2	823.0	880.8	932.0	1,884.2	2,061.0	1,828.6	1,428.6
Final inventory	77.9	135.9	145.5	185.6	169.1	616.6	245.1	4.4
Expected sales	627.8	613.7	658.1	674.4	1,519.9	1,328.6	1,406.1	1,193.5
Unmet expected demand	81.6	57.8	62.3	57.1	195.2	78.1	139.6	206.5
Lost units	0.0	15.7	14.9	14.9	0.0	37.7	37.8	24.2

account that a demand that is not satisfied in one period can be satisfied in the following period ($\tau_{qmlt} = 1$). However, later on, the results are compared with the case when the cost of placing an order is included ($K_{qmlt} > 0$) and how the solution is affected when there is lost demand in one period, and can be satisfied in the next ($0 < \tau_{qmlt} < 1$).

In the following section, we analyze the results of the solution to the problem: first using the nominal values of the parameters, and we subsequently analyze how variability affects the structure of the results by solving the robust problem.

5.2. Retail model results

First, we show the results of the retail model (RM). The results indicate a maximum benefit of 1,754,021 monetary units. Table 11 shows the optimal solution of price and orders for each product in each period. The inventory movements for each product and each period are presented in Tables 12 and 13 for stores 1 and 2, respectively.

As can be seen in Tables 11–13, the movements in period 1 of product 1 in the store 1 are as follows: In store 1 a price of 172,91 monetary units (MU) is assigned and 787,2 units (Table 11) are ordered. Based on the previous policy, the store has an initial inventory of 630,6 units, the expected sales are 81,6 units, there are no lost units, the expected lost sales are 21,68 units, and the ending inventory is 24,1 units (Table 12). The analysis for the other combinations of store, period and product, are done in a similar way.

The following sections present certain variations in the optimal value and in the structure of the solution; these are caused by uncertainty in the demand.

5.3. Retail Stochastic Robust Model (RSRM)

Before analyzing the RSRM problem, it is necessary to observe the historical behavior of the demand for the product or similar products sold in previous periods. In the data, we observe that γ_{qmlt} had a maximum variation of 15% of the nominal value and $\hat{a}_{qml}$ had a maximum variation of 20% of the nominal value. According to a statistical analysis, both variations can be represented with a uniform distribution. In addition, depending on the number of periods considered in the model, the uncertainty of the budget Ω takes values between 0 (nominal case) and 4 (worst case). In the next section we discuss the optimality and feasibility of the RSRM.

5.3.1. Optimality and feasibility

Fig. 3a shows the optimum value of the robust model as a function of the uncertainty budget Ω. In the figure, we observe : The value of 1,754,021 corresponds to the optimal value of RSRM without protection (Ω= 0) and is equal to the optimal value of RM. The value of 1,408,544 corresponds to the optimal value of RSRM for a conservative scenario (Ω= 3) and the value of 1,366,673 to the optimal value of RSRM for full protection (Ω= 4). These labels are discussed later.

As predicted, Fig. 3a shows that the expected profit decreases as the variability of the protected parameter increases. Initially we also observe a reduction in the expected benefit of up to values of 0.5 of the uncertainty budget. This is due to a property of the function used to represent the price sensitivity in the demand function (Eq. (1)), where we observe a negative slope at the beginning, which then increases to an asymptotic value. On the other hand, as the protection parameter increases, the expected benefit decreases, reaching its minimum value when the problem is fully protected.

It is imperative to study the solutions in terms of the feasibility of their implementation. To do this, we generated several scenarios using a Monte-Carlo simulation and observed the behavior of the solutions for different values of Ω, in each of the variations of γ_{qmlt} and α_{qml}. The simulation was run for 800 scenarios that were generated based on a uniform distribution with the previously-mentioned variations (Fig. 3b).

Fig. 3b shows the feasibility of the implementation of RSRM as a function of the uncertainty budget Ω. In the figure, three values are also observed: The value of 26.53% corresponds to the probability that the scenario will satisfy the demand distribution when the RSRM is not protected (Ω= 0). This probability is 73.46% for the conservative scenario (Ω= 3) and 100% when RSRM is fully protected (Ω= 4). An important point is to study the solutions in terms of the feasibility of their implementation. To that end, it should be noted that 100% feasibility is achieved when the problem is fully protected, but that could be very expensive for the system. On the other hand, for lower protection values, it is possible that the solution might not always be feasible.

Fig. 3b is more interesting when viewed in combination with Fig. 3a; this enables us to analyze the acceptable values of Ω in terms of their capacity to provide robust solutions. As one can observe, the RSRM without protection achieves the greatest benefit, but it has a low probability of being feasible. Conversely, the RSRM with full protection has 100% feasibility but achieves the least benefit. After analyzing both figures and considering the decision maker's risk aversion, the company could decide on a conservative scenario with Ω equal to 3,

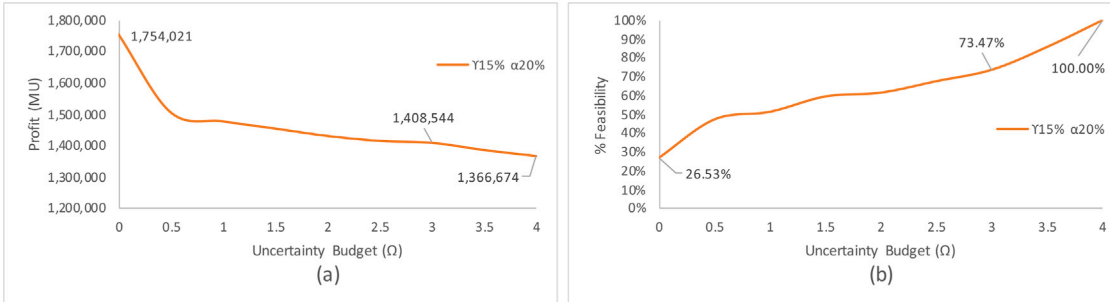

Fig. 3. Optimality and feasibility (By the authors).

Table 14
Optimal pricing and ordering policies, RSRM.

Description		Product 1 (q = 1)				Product 2 (q = 2)			
		t = 1	t = 2	t = 3	t = 4	t = 1	t = 2	t = 3	t = 4
Store 1 ($l = 1$)	Price	140.28	140.28	140.28	140.28	162.5	160.23	157.31	157.31
	Order	840.11	797.95	797.93	750.27	2,098.41	1,538.29	1,403.18	1,125.52
Store 1 ($l = 2$)	Price	155.33	155.33	155.33	155.33	179.16	174.80	168.78	162.31
	Order	668.95	632.52	695.03	823.59	1,631.11	1,463.99	1,480.74	1,228.77

Table 15
Inventory flow for store 1 (l = 1), RSRM.

Description	Product 1 (q = 1)				Product 2 (q = 2)			
	t = 1	t = 2	t = 3	t = 4	t = 1	t = 2	t = 3	t = 4
Initial inventory	840.1	979.95	832.2	854.5	2,098.4	1,538.3	1,403.2	1,125.5
Final inventory	0.0	34.3	104.2	145.5	137.89	260.8	251.3	0.0
Expected sales	680.3	630.4	626.6	620.6	1,666.5	1,231.7	1,234.4	1,125.5
Expected lost sales	159.8	116.5	85.5	72.4	294.0	141.7	147.6	234.8
Lost units	0.0	16.8	15.9	15.9	0.0	41.9	30.8	28.1

which would result in a benefit of $1,408,544. With this decision, the implementation has a 73.46% chance of feasibility. If it were to become infeasible, the retailer would only lose 3.05% of profit as compared with the RSRM with full protection (i.e., the difference between $ 1,408,544 and $1,366,673). In the next section, we analyze the results of implementing this policy.

5.3.2. Solution structure

Perhaps one of the most interesting analyses is to observe the structure of the solution in terms of robustness, taking into account that the objective is to obtain a stable price, despite demand uncertainty.

The conservative analysis presented in the previous section, with an uncertainty budget of Ω= 3 and a variation of 15% in γ_{qmlt} and from the 20% in α_{qmlt}, results in a benefit of $ 1,408,544, which is lower than the benefit obtained without protection, $ 1,754,021. Optimal pricing and ordering polices in each period of the conservative scenario are presented in Table 14, while Tables 15 and 16 present the movement of inventories for stores 1 and 2, respectively.

The interpretation of Tables 14–16 for RSRM is analogous to Tables 11–13 of RM. As the tables show, the values of the variables change as the protection level of the problem increases in RM. In a conservative scenario, RSRM suggests lowering prices by about 14.19% and 6.52% on average for products 1 and 2, respectively. The Orders have a reduction of 3.40% and 7.17%, respectively. The inventory has a reduction of 7.13% and 23.77%, respectively. The lost units are reduced by 4.63% and 10.92% for products 1 and 2, respectively.

While the expected demand in RSRM is reduced by 2.03% and 8.40% for products 1 and 2, respectively, and the expected loss is

Table 16
Inventory flow for store 2 (l = 2), RSRM.

Description	q = 1				q = 2			
	t = 1	t = 2	t = 3	t = 4	t = 1	t = 2	t = 3	t = 4
Initial inventory	668.9	642.5	733.7	891.4	1,631.1	1464.0	1,647.1	1,470.4
Final inventory	9.9	38.7	67.8	0.0	0.0	166.4	241.6	1,570.4
Expected sales	556.3	517.7	580.8	741.6	1,372.9	1,142.9	1,255.9	1,228.4
Expected lost sales	102.7	72.7	72.4	135.9	258.1	122.1	120.4	212.4
Lost units	0.0	13.4	12.6	13.9	0.0	32.6	29.3	29.6

increased with respect to RM, it should be noted that in practice, it is highly unlikely that RM will actually occur.

Another interesting point is the 4.63% reduction of lost units since these are included in the budget for the year, which implies that any savings goes directly to net profit. On the other hand, by having a balanced inventory that includes substitution and product complementarity, we estimated that there could be an additional increase of 2.3% in profit.

In summary, with the savings generated by not having excess inventory and the reduction of lost units, RSRM with a moderate scenario and a feasibility probability of 73.46% will satisfy the demand distribution and generate an increase in profits by at least 6.93%. To this increase in profit, we must add savings from inventory management since the model did not allow us to include this value. The main managerial insights are presented below.

5.4. Recover lost demand (τ < 1)

After conducting the study, it was necessary to observe the behavior of the model when part of the demand lost in one period is recovered in the next (τ < 1). Fig. 4A shows the profit as a function of τ < 1, it shows an increase in profit when the percentage of lost demand is reduced, being this behavior consistent with the operation, since part of the demand that was previously considered lost can now be recovered in the following period.

Fig. 4B, shows how inventory costs increase as the expected demand loss τ = 0 decreases. This makes sense since satisfying part of the lost demand in the following period implies that we must increase inventories and this entails a higher cost of storing them.

5.5. Cost of placing an order

Recalling that the cost of placing an order includes all administrative costs incurred at the time of placing the order and verification against delivery. In accordance with the above, in our research, the need arose to see how this cost affects the retail operation. Accordingly, a study was conducted in which the cost was varied as a proportion of the product cost (Fig. 5). The above converts our nonlinear stochastic

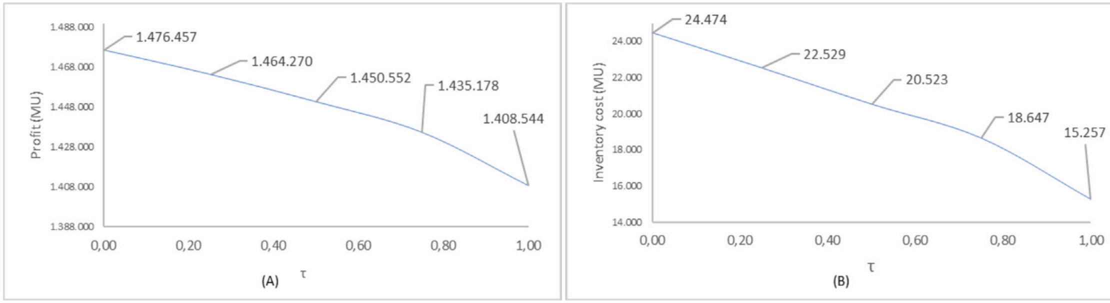

Fig. 4. Profit and inventory cost as a function of τ (By the authors).

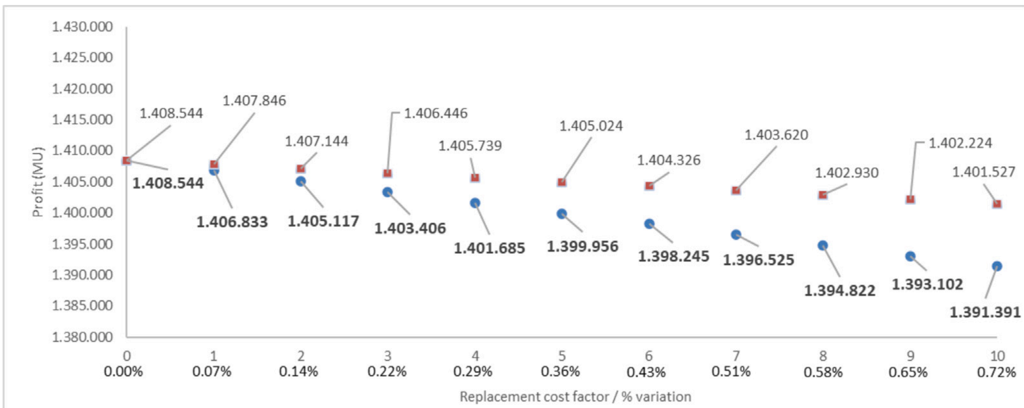

Fig. 5. Replacement cost (By the authors).

robust optimization problem, into a nonlinear integer problem, as a consequence the solver requires more execution time to find a solution, going from 0.35 s on average to 4.25 s.

Fig. 5 shows the profit as a function of: first, the replacement cost as a fraction of the unit cost of each product (first row on the horizontal axis). Second, the probability of the difference between the solution offered by the solver for the entire nonlinear problem and the value of that solution, when the replacement cost is not taken into account (second row on the horizontal axis). In Fig. 5, the bold labels, represent the values delivered by the solver and the light labels, are the benefits without the replacement cost.

According to Fig. 5, it can be observed that when there is no replacement cost, the difference between this and the value found by the solver for $K = 1$ is 0.00%. This percentage increases to 0.72%, when the replacement cost is 10 times the product cost. With the above, we can conclude that the contribution of the replacement cost in our problem is marginal, if we take into account that the retailer buys volumes of product and that when prorating the replacement cost to the units, it will be much lower than the product cost.

6. Managerial insights

In this section we present some of the most important managerial insights that follow from our work.

We have presented an optimization model that can support the pricing and ordering decisions in a retailer, taking into account the various characteristics of the operation, in particular, the consideration of substitution and complementarity between products that has an effect to be considered. The model allows setting prices in each period, for each store, and controls the inventory to meet market demand.

The policies produced by our model allow to keep products attractive to customers and at the same time minimizes the cost of keeping them available. In addition, it avoids overstocking and reduces loss of demand, since orders are controlled at the beginning of each period. Setting separate prices in each store would lead one to believe that a market could take advantage of this price differential. This economic weakness is avoided since the price difference between two stores is less than the cost of moving between them.

Demand is not known with precision, and therefore, we model its uncertainty by means of a probability distribution, whose parameters must be estimated based on historical information. Seasonality and price sensitivity factors are introduced to model demand and our development shows that it is important to take this into account. However, the exact effect fluctuates due to many factors, including changes in consumer behavior and financial and economic fluctuation.

Despite the above, our robust optimization formulation can obtain good solutions considering a protection to the seasonality and price sensitivity parameters of the demand–price model. Managers can evaluate the effect of variability in the estimation of these parameters and obtain solutions with different performance. One of the main insights from this is that there is no need to be fully conservative: Good and efficient solutions can be obtained with reasonable degrees of protection in the model. Managers can then test different variability protection scenarios and evaluate the solutions proposed by the model to make final decisions.

Handling unmet demand is a major problem in retail. Our model implements the possibility of postponing orders or missing orders altogether, or any combination in between. Again, managers could use the model to observe the effect of changing the proportion of unfilled orders that are deferred to future periods and use those results to define a good policy.

The model can also be used to analyze the effects of fixed order cost. In our particular situation, we conclude that the effect is rather marginal, but the model can be used to analyze a different situation and, again, define a threshold value for which the fixed cost has a significant effect on the ordering policy. Finally, our model allows us to sell the remaining units at the end of the scheduling horizon at reasonable values, without incurring the large discounts that sometimes lead to losses.

7. Conclusions

In this research, we develop a robust stochastic optimization model for seasonal products sales in the retail sector, which establishes optimal inventory policies and dynamic pricing. The model allocates prices in each period into which the planning horizon is divided. In order to keep the product attractive to customers in each store, it incorporates learning from demand behavior and at the same time avoids the presence of arbitrage.

It also allows to adjust the inventory in each period according to the market covered by the various stores. It allows the replenishment of alternative products when customers cannot find what they are looking for (substitute products) and encourages the sale of other products (complementary products). Accordingly, it minimizes the number of units left over at the end of the planning horizon.

Demand is represented by a continuous probability distribution, with two adjustment parameters: one for seasonality and one for price sensitivity. Since seasonality is influenced by weather variations and price sensitivity is affected by dollar variations, it was necessary to perform a double protection to find solutions immune to these two variations. With the above, solutions adjusted to the retailer's reality were found.

Based on the experimental results, it was found that it was not necessary to fully protect the model to satisfy demand efficiently. However, in the case when a moderate protection is used, it was necessary to lower prices, which resulted in lower inventory and reduced demand satisfaction. At the same time, an attractive price of the products was maintained, since their price is adjusted to the market's willingness to pay, and even an adequate inventory including substitute and complementary products was maintained, since it adjusts the orders in each period.

Subsequently, a lost demand analysis was performed, where it was considered that a portion can be sold in the next period. We found that this leads to an increase in profit since this recovered demand increases revenues in the next period. We also analyzed how the cost of placing an order affects profit marginally due to the large volumes of products sold by retailers. Finally, the results in the case under study allow to concluded that retailers can increase their profits by at least 6.93% by using this formulation.

This research has been carried out in response to a particular situation in the retail sector. The developed tool can compute robust solutions while using tactical decision design to simultaneously assess the sensitivity of the model to uncertainty in specific parameters. Therefore, several extensions are possible, including the addition of a function that protects against uncertainty in revenues and costs. Another interesting extension would be to use the model with perishable products in which a term representing the deterioration or obsolescence of the product over time can be considered. Finally, a heuristic could be developed to find solutions in reasonable computational times, in the case of larger problems.

The model has the potential to be used in other industrial sectors such as airlines, hotels, car rentals, communications, tour operators, casinos, cruises, general cargo, theaters, sporting events, manufacturing and any other sector that simultaneously analyzes prices, orders, inventory and other factors such as quality control (Rios G., 2011). Risk aversion could also be incorporated (Ferrer et al., 2012).

CRediT authorship contribution statement

John H. Rios: Methodology, Software, Writing – original draft, Validation, Formal analysis, Investigation, Data curation, Visualization. **Jorge R. Vera:** Conceptualization, Writing – review & editing, Resources, Supervision, Project administration, Funding acquisition.

Data availability

The data that has been used is confidential.

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