GDA MAP ESTIMATES

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Deriving the MAP estimates $\{\phi, \mu_0, \mu_1, \Sigma\}$ of Gaussian Discriminant Analysis.

Proof.

Given

$$
\begin{aligned}
p(y|x, \phi, \mu_0, \mu_1, \Sigma) &= \prod_{i=1}^{m} p(x^{(i)}, y^{(i)}|\phi, \mu_0, \mu_1\Sigma) \\
if \quad l(\phi, \mu_0, \mu_1\Sigma) &= \log p(y|x, \phi, \mu_0, \mu_1, \Sigma) \\
l(\phi, \mu_0, \mu_1\Sigma) &= \log \prod_{i=1}^{m} p(x^{(i)}, y^{(i)}|\phi, \mu_0, \mu_1\Sigma) \\
&= \log \prod_{i=1}^{m} p(x^{(i)}|\mu_0, \mu_1\Sigma)p(y^{(i)}|\phi) \\
p(y^{(i)}|\phi) &= \phi^{y}(1-\phi)^{1-y}
\end{aligned} \tag{0.1}
$$

$$
\begin{aligned}
l(\phi, \mu_0, \mu_1\Sigma) &= \sum_{i=1}^{m} \left(\{y^{(i)} = 1\} \log \left\{\mathcal{N}(x^{(i)}; \mu_1, \Sigma)p(y^i = 1)\right\} + 1\{y^{(i)} = 0\} \log \left\{\mathcal{N}(x^{(i)}; \mu_0, \Sigma)p(y^i = 0)\right\}\right) \\
&= \sum_{i=1}^{m} 1\{y^{(i)} = 1\} \log \left\{\frac{1}{((2\pi)^n|\Sigma|)^{1/2}} . \exp\left\{-\frac{1}{2}(x^{(i)} - \mu_1)^T\Sigma^{-1}(x - \mu_1)\right\}\right\} + y^{(i)} \log \phi \quad + \\
& 1\{y^{(i)} = 0\} \log \left\{\frac{1}{((2\pi)^n|\Sigma|)^{1/2}} . \exp\left\{-\frac{1}{2}(x^{(i)} - \mu_0)^T\Sigma^{-1}(x^{(i)} - \mu_0)\right\}\right\} + (1 - y^{(i)}) \log(1 - \phi)
\end{aligned}
$$

if $-\frac{1}{2}\log(2\pi)^n|\Sigma| = K$

$$
\begin{aligned}
l(\phi, \mu_0, \mu_1\Sigma) = \sum_{i=1}^{m} & 1\{y^{(i)} = 1\} \left\{K - \frac{1}{2}(x^{(i)} - \mu_1)^T\Sigma^{-1}(x^{(i)} - \mu_1) + y^{(i)} \log \phi\right\} \\
+ \qquad & 1\{y^{(i)} = 0\} \left\{K - \frac{1}{2}(x^{(i)} - \mu_0)^T\Sigma^{-1}(x^{(i)} - \mu_0) + (1 - y^{(i)}) \log(1 - \phi)\right\}
\end{aligned} \tag{0.2}
$$

Now taking the partial derivatives for the MAP estimates:

First, the derivative with respect to ϕ Working out $\frac{\delta l}{\delta \phi}$

$$
\begin{aligned}
\frac{\delta l}{\delta \phi} &= \sum_{i=1}^{m} \left\{ 1\{y^{(i)}=1\} \left\{ 0+0+\frac{y^{(i)}}{\phi} \right\} + 1\{y^{(i)}=0\} \left\{ 0+0-\frac{1-y^{(i)}}{1-\phi} \right\} \right\} \\
&= \sum_{i=1}^{m} \left\{ 1\{y^{(i)}=1\} \frac{y^{(i)}}{\phi} + 1\{y^{(i)}=0\} \frac{y^{(i)}-1}{1-\phi} \right\} \\
\text{if} \frac{\delta l}{\delta \phi} &= 0 \\
0 &= \frac{1}{\phi(1-\phi)} \sum_{i=1}^{m} 1\{y^{(i)}=1\} y^{(i)}(1-\phi) + 1\{y^{(i)}=0\} \phi (y^{(i)}-1) \\
0 &= \frac{1}{\phi(1-\phi)} \sum_{i=1}^{m} 1\{y^{(i)}=1\}(1-\phi) - 1\{y^{(i)}=0\}(\phi) \\
0 &= \sum_{i=1}^{m} 1\{y^{(i)}=1\} - \sum_{i=1}^{m} 1\{y^{(i)}=1\}\phi + \{y^{(i)}=0\}\phi \\
0 &= \sum_{i=1}^{m} 1\{y^{(i)}=1\} - \sum_{i=1}^{m} \phi \\
0 &= \sum_{i=1}^{m} 1\{y^{(i)}=1\} - m\phi \\
\therefore \phi_{MAP} &= \frac{\sum_{i=1}^{m} 1\{y^{(i)}=1\}}{m}
\end{aligned}
\tag{0.3}
$$

Deriving μ_1

Taking the partial derivative of $\frac{\delta l}{\delta \mu_1}$ of Eq. 0.2 and Since $1\{y^{(i)}=0\}$ doesn't depend on μ_1, $\frac{\delta l}{\delta \mu_1}$ is

$$
\begin{aligned}
\frac{\delta l}{\delta \mu_1} &= \sum_{i=1}^{m} 1\{y^{(i)}=1\} \frac{\delta}{\delta \mu_1} \left\{ K - \frac{1}{2}(x^{(i)}-\mu_1)^T \Sigma^{-1} (x-\mu_1) + y^{(i)} \log \phi \right\} + 0 \\
&= \sum_{i=1}^{m} 1\{y^{(i)}=1\} \left\{ 0 + -\frac{1}{2} \left(0 - 2\Sigma^{-1} x^{(i)} + 2\Sigma^{-1}\mu_1 + 0 \right) \right\} \\
&= \sum_{i=1}^{m} 1\{y^{(i)}=1\} \left(\Sigma^{-1} x - \Sigma^{-1} \mu_1 \right) \\
&= \Sigma^{-1} \sum_{i=1}^{m} 1\{y^{(i)}=1\} \left(x^{(i)} - \mu_1 \right)
\end{aligned}
$$

$$
\begin{aligned}
\text{Setting } \frac{\delta l}{\delta \mu_1} &= 0 \\
=> 0 &= \sum_{i=1}^{m} 1\{y^{(i)}=1\} x^{(i)} - \sum_{i=1}^{m} 1\{y^{(i)}=1\} \mu_1 \\
&= \sum_{i=1}^{m} 1\{y^{(i)}=1\} x^{(i)} - \mu_1 \sum_{i=1}^{m} 1\{y^{(i)}=1\} \\
\therefore \mu_{1MAP} &= \frac{\sum_{i=1}^{m} 1\{y^{(i)}=1\} x^{(i)}}{\sum_{i=1}^{m} 1\{y^{(i)}=1\}}
\end{aligned}
\tag{0.4}
$$

Following the same procedure as above,μ_0 is:

$$\mu_{0MAP} = \frac{\sum_{i=1}^{m} 1\{y^{(i)} = 0\}x^{(i)}}{\sum_{1=1}^{m} 1\{y^{(i)} = 0\}} \tag{0.5}$$

Deriving the Σ

Rewriting Eq. 0.2 fully capturing all terms of Σ

$$\begin{aligned} l(\phi, \mu_1, \mu_0, \Sigma) = -\frac{mn}{2}\log(2\pi) + -\frac{m}{2}\log|\Sigma| + \sum_{i=1}^{m} 1\{y^{(i)} = 1\}\left\{-\frac{1}{2}(x^{(i)} - \mu_1)^T\Sigma^{-1}(x^{(i)} - \mu_1) + y^{(i)}\log\phi\right\} \\ + \qquad \{y^{(i)} = 0\}\left\{-\frac{1}{2}(x^{(i)} - \mu_0)^T\Sigma^{-1}(x^{(i)} - \mu_0) + (1 - y^{(i)})\log(1-\phi)\right\} \end{aligned}$$

Taking the partial derivative of l w.r.t Σ, $\quad \frac{\delta l}{\delta \Sigma}$ of Eq. 0.2

$$\begin{aligned} \frac{\delta l}{\delta \Sigma} &= -\frac{m}{2}.\frac{\delta}{\delta \Sigma}\log|\Sigma| - \frac{1}{2}\sum_{1=1}^{m} 1\{y^{(i)} = 1\}\left\{\frac{\delta}{\delta \Sigma}(x^{(i)} - \mu_1)^T\Sigma^{-1}(x^{(i)} - \mu_1)\right\} \\ + &\qquad 1\{y^{(i)} = 0\}\left\{\frac{\delta}{\delta \Sigma}(x^{(i)} - \mu_0)^T\Sigma^{-1}(x^{(i)} - \mu_0)\right\} \end{aligned} \tag{0.6}$$

Since Σ is linear, the derivative of the determinant is computed as

$$-\frac{m}{2}.\frac{\delta}{\delta \Sigma}\log|\Sigma| = -\frac{m|\Sigma|.(\Sigma^{-1})^T}{2|\Sigma|} = -\frac{m(\Sigma^{-1})^T}{2} \tag{0.7}$$

Computing the derivative of part dependent on μ_1

$$\begin{aligned} \frac{\delta}{\Sigma}(x^{(i)} - \mu_1)^T\Sigma^{-1}(x^{(i)} - \mu_1) &= -\Sigma^{-T}\{(x^{(i)} - \mu_1)^T(x^{(i)} - \mu_1)\}^T\Sigma^{-T} \\ &= -\Sigma^{-T}(x^{(i)} - \mu_1)(x^{(i)} - \mu_1)^T\Sigma^{-T} \end{aligned} \tag{0.8}$$

Following similar procedure as Eq. 0.8, the derivative of the part dependent on μ_0 is

$$\frac{\delta}{\Sigma}(x^{(i)} - \mu_0)^T\Sigma^{-1}(x^{(i)} - \mu_0) = -\Sigma^{-T}(x^{(i)} - \mu_0)(x^{(i)} - \mu_0)^T\Sigma^{-T} \tag{0.9}$$

Substituting the results of Eqs.0.7, 0.8, 0.9 into Eq. 0.6, we get

$$\begin{aligned} \frac{\delta l}{\delta \Sigma} &= -\frac{m(\Sigma^{-1})^T}{2} + \frac{1}{2}\sum_{1=1}^{m} 1\{y^{(i)} = 1\}\Sigma^{-T}(x^{(i)} - \mu_1)(x^{(i)} - \mu_1)^T\Sigma^{-T} \\ + &\qquad \{y^{(i)} = 0\}\Sigma^{-T}(x^{(i)} - \mu_0)(x^{(i)} - \mu_0)^T\Sigma^{-T} \\ &= \frac{1}{2\Sigma^T}\left(-m + \frac{1}{\Sigma^T}\sum_{1}^{m}\{y^{(i)} = 1\}(x^{(i)} - \mu_1)(x^{(i)} - \mu_1)^T + \{y^{(i)} = 0\}(x^{(i)} - \mu_0)(x^{(i)} - \mu_0)^T\right) \end{aligned} \tag{0.10}$$

Setting Eq. 0.10 to zero, and since Σ is symmetric, $\Sigma^T = \Sigma$, $\frac{\delta l}{\delta \Sigma}$ becomes:

$$\begin{aligned}
0 &= -m\Sigma^T + \sum_1^m \{y^{(i)} = 1\}(x^{(i)} - \mu_1)(x^{(i)} - \mu_1)^T + \{y^{(i)} = 0\}(x^{(i)} - \mu_0)(x^{(i)} - \mu_0)^T \\
m\Sigma &= \sum_1^m \{y^{(i)} = 1\}(x^{(i)} - \mu_1)(x^{(i)} - \mu_1)^T + \{y^{(i)} = 0\}(x^{(i)} - \mu_0)(x^{(i)} - \mu_0)^T \\
\therefore \Sigma_{MAP} &= \frac{1}{m}\sum_1^m \{y^{(i)} = 1\}(x^{(i)} - \mu_1)(x^{(i)} - \mu_1)^T + \{y^{(i)} = 0\}(x^{(i)} - \mu_0)(x^{(i)} - \mu_0)^T \\
&= \frac{1}{m}\sum_1^m (x^{(i)} - \mu_y)(x^{(i)} - \mu_y)^T
\end{aligned} \tag{0.11}$$

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