Discrete Differential Geometry

Maxwell Thum

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Introduction

This is an interactive blueprint to help with the formalisation of several definitions and results from *discrete differential geometry*, using Keenan Crane's textbook as a general template.

The actual Lean code can be found at (`https://github.com/maxwell-thum/DDG_Lean3`).

This blueprint is adapted from the blueprint of Thomas F. Bloom and Bhavik Mehta's Unit Fractions project (`https://github.com/b-mehta/unit-fractions`), which was itself based on the blueprint created by Patrick Massot for the Sphere Eversion project (`https://github.com/leanprover-community/sphere-eversion`).

This blueprint uses Patrick Massot's leanblueprint plugin (`https://github.com/PatrickMassot/leanblueprint`) for plasTeX (`http://plastex.github.io/plastex/`).

Chapter 1

Combinatorial Surfaces

1.1 Abstract simplicial complexes

Definition 1.1. An *abstract simplicial complex* $\mathcal{K}$ is a set of non-empty[1] finite sets called *simplices* such that for all $\sigma \in \mathcal{K}$, if $\sigma' \subseteq \sigma$ and $\sigma' \neq \varnothing$, then $\sigma' \in \mathcal{K}$.

Abstract simplicial complexes are the combinatorial or topological analogs of *geometric simplicial complexes*, which we will see shortly. Abstract simplicial complexes capture the connectivity of geometric simplicial complexes without their geometry.

Definition 1.2. The *degree* or *dimension* of an (abstract) simplex with $k+1$ vertices is k.

Definition 1.3. An (abstract) simplex of degree k is called an *abstract k-simplex*. The set of all k-simplices is denoted F_k.

For instance, 0-simplices are points or vertices, 1-simplices are line segments or edges, 2-simplices are triangles or faces, and 3-simplices are tetrahedra.

A nonempty subset of an abstract simplex is called a *face*. By definition of an abstract simplicial complex, all of the faces of a simplex are themselves simplices. A proper subset of a simplex is called a *proper face*.

Definition 1.4. Given two simplicial complexes $\mathcal{K}$ and $\mathcal{K}'$, we say that $\mathcal{K}'$ is a *subcomplex* of $\mathcal{K}$ if $\mathcal{K}' \subseteq \mathcal{K}$, that is, every simplex in $\mathcal{K}'$ is a simplex in $\mathcal{K}$ as well.

Definition 1.5. If the set of degrees of the simplices of an abstract simplicial complex has a maximum k, then that abstract simplicial complex is said to be an *abstract simplicial k-complex.*

Definition 1.6. Let $\mathcal{K}$ be an abstract simplicial k-complex. If every simplex is a face of some k-simplex, then we say $\mathcal{K}$ is *pure* and call it a *pure simplicial k-complex.*

Definition 1.7. The *star* $\mathrm{St}(\sigma)$ of a simplex σ in an abstract simplicial complex $\mathcal{K}$ is the set of all simplices in $\mathcal{K}$ having σ as a face.

The star $\mathrm{St}(S)$ of a set S of simplices in $\mathcal{K}$ (i.e., a subset of $\mathcal{K}$) is the set of all simplices in $\mathcal{K}$ having some simplex in S as a face. Equivalently, $\mathrm{St}(S) = \bigcup_{\sigma \in S} \mathrm{St}(\sigma)$.

Definition 1.8. The *closure* $\mathrm{Cl}(S)$ of a set of simplices $S \subseteq \mathcal{K}$ is the smallest (by inclusion) simplicial complex containing S.

[1] Some authors allow simplices to be empty.

Lemma 1.9.

$$\mathrm{Cl}(S) = \bigcup_{\sigma \in S} \mathcal{P}(\sigma) \quad \varnothing.$$

Definition 1.10. The *link* $\mathrm{Lk}(\sigma)$ of a simplex $\sigma \in \mathcal{K}$ is the set of all simplices $\tau \in \mathcal{K}$ such that σ and τ are disjoint and their union is a simplex. In set-builder notation,

$$\mathrm{Lk}(\sigma) := \{\tau \in \mathcal{K} \mid \tau \cap \sigma = \varnothing, \tau \cup \sigma \in \mathcal{K}\}.$$

Lemma 1.11. For any set of simplices $S \subseteq \mathcal{K}$,

$$\mathrm{Lk}(S) = \mathrm{Cl}(\mathrm{St}(S)) \quad \mathrm{St}(\mathrm{Cl}(S)).$$

1.1.1 Oriented ASCs

Definition 1.12. An *orientation* on a simplex is a choice of a permutation of its vertices. This permutation and every permutation related to it by even permutations is called *positively oriented*. Every permutation related to it by odd permutations is called *negatively oriented*.

An *oriented abstract simplicial complex* is an abstract simplicial complex with an orientation assigned to each of its simplices.

1.1.2 The halfedge mesh

1.2 Simplicial complexes

Definition 1.13. A *(geometric) simplicial complex* is...

Definition 1.14. The *underlying abstract simplicial complex* of a geometric simplicial complex is the abstract simplicial complex whose abstract simplices are simply the sets of vertices of each geometric simplex.

Definition 1.15. The *geometric realization* of an abstract simplicial complex $\mathcal{K}$ is a geometric simplicial complex whose underlying abstract simplicial complex is $\mathcal{K}$ itself. In particular, we define it as...

Chapter 2

Discrete Exterior Calculus

Throughout this chapter, we let $\mathcal{K}$ be an oriented abstract simplicial complex.

2.1 Discrete Differential Forms

Definition 2.1. A *(real-valued) discrete (differential) k-form* on an oriented abstract simplicial complex $\mathcal{K}$ is a map $\alpha\colon F_k \to \mathbb{R}$. The set of all discrete k-forms on $\mathcal{K}$ is denoted Ω^k.

2.1.1 Discretization

Recall the notions of a *manifold* and *differential (k-)forms*. From here on, let M be an n-dimensional triangulable topological manifold. Let $t\colon |\mathcal{K}| \to M$ be a triangulation of M.

Definition 2.2. The *discretization* on $\mathcal{K}$ of a differential k-form α on M is a discrete differential k-form defined by integrating α over the image in M of each (oriented) k-simplex in $\mathcal{K}$. In other words, for all $\sigma \in F_k$,

$$\hat{\alpha}_\sigma := \int_{t(|\sigma|)} \alpha.$$

We denote this map by $\int_k$.

2.2 Discrete Exterior Calculus

2.2.1 The discrete exterior derivative

Definition 2.3. The *discrete exterior derivative* on k-forms is a map $\hat{\mathrm{d}}_k\colon \hat{\Omega}^k \to \hat{\Omega}^{k+1}$ such that $\hat{\mathrm{d}}_k \circ \int_k = \int_{k+1} \circ \mathrm{d}_k$.

We are not yet able to prove this in Lean, but

Theorem 2.4. The discrete exterior derivative is unique and given by

$$(\mathrm{d}_k\alpha)(\sigma) = \sum_{\tau \in F_k \cap \mathcal{P}(\sigma)} \operatorname{sgn}(\tau, \sigma)\alpha(\tau)$$

for all $\alpha \in \Omega^k$ and $\sigma \in F_{k+1}$.