Discrete Differential Geometry

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Introduction

This is an interactive blueprint to help with the formalisation of several definitions and results from *discrete differential geometry*, using Keenan Crane's textbook as a general template.

The actual Lean code can be found at (`https://github.com/maxwell-thum/DDG_Lean3`).

This blueprint is adapted from the blueprint of Thomas F. Bloom and Bhavik Mehta's Unit Fractions project (`https://github.com/b-mehta/unit-fractions`), which was itself based on the blueprint created by Patrick Massot for the Sphere Eversion project (`https://github.com/leanprover-community/sphere-eversion`).

This blueprint uses Patrick Massot's leanblueprint plugin (`https://github.com/PatrickMassot/leanblueprint`) for plasTeX (`http://plastex.github.io/plastex/`).

Chapter 1

Combinatorial Surfaces

1.1 Abstract simplicial complexes

Definition 1.1. An *abstract simplicial complex* $\mathcal{K}$ is a set of non-empty[1] finite sets called *simplices* such that for all $\sigma \in \mathcal{K}$, if $\sigma' \subseteq \sigma$ and $\sigma' \neq \varnothing$, then $\sigma' \in \mathcal{K}$.

Abstract simplicial complexes are the combinatorial or topological analogs of *geometric simplicial complexes*, which we will see shortly. Abstract simplicial complexes capture the connectivity of geometric simplicial complexes without their geometry.

Definition 1.2. The *degree* or *dimension* of an (abstract) simplex with $k+1$ vertices is k.

Definition 1.3. An (abstract) simplex of degree k is called an *abstract k-simplex.*

For instance, 0-simplices are points or vertices, 1-simplices are line segments or edges, 2-simplices are triangles or faces, and 3-simplices are tetrahedra.

Definition 1.4. A nonempty subset of an abstract simplex is called a *face.*

By definition of an abstract simplicial complex, all of the faces of a simplex are themselves simplices. A proper subset of a simplex is called a *proper face.*

Definition 1.5. Given two simplicial complexes $\mathcal{K}$ and $\mathcal{K}'$, we say that $\mathcal{K}'$ is a *subcomplex* of $\mathcal{K}$ if $\mathcal{K}' \subseteq \mathcal{K}$, that is, every simplex in $\mathcal{K}'$ is a simplex in $\mathcal{K}$ as well.

Definition 1.6. If the set of degrees of the simplices of an abstract simplicial complex has a maximum k, then that abstract simplicial complex is said to be an *abstract simplicial k-complex.*

Definition 1.7. Let $\mathcal{K}$ be an abstract simplicial k-complex. If every simplex is a face of some k-simplex, then we say $\mathcal{K}$ is *pure* and call it a *pure simplicial k-complex.*

Definition 1.8. The *star* $\mathrm{St}(\sigma)$ of a simplex σ in an abstract simplicial complex $\mathcal{K}$ is the set of all simplices in $\mathcal{K}$ having σ as a face.

The star $\mathrm{St}(S)$ of a set S of simplices in $\mathcal{K}$ (i.e., a subset of $\mathcal{K}$) is the set of all simplices in $\mathcal{K}$ having some simplex in S as a face. Equivalently, $\mathrm{St}(S) = \bigcup_{\sigma \in S} \mathrm{St}(\sigma)$.

[1]Some authors allow simplices to be empty.

Definition 1.9. The *closure* $\mathrm{Cl}(S)$ of a set of simplices $S \subseteq \mathcal{K}$ is the smallest (by inclusion) simplicial complex containing S.

Lemma 1.10.

$$\mathrm{Cl}(S) = \bigcup_{\sigma \in S} \mathcal{P}(\sigma) \quad \varnothing.$$

Definition 1.11. The *link* $\mathrm{Lk}(\sigma)$ of a simplex $\sigma \in \mathcal{K}$ is the set of all simplices $\tau \in \mathcal{K}$ such that σ and τ are disjoint and their union is a simplex. In set-builder notation,

$$\mathrm{Lk}(\sigma) := \{\tau \in \mathcal{K} \mid \tau \cap \sigma = \varnothing, \tau \cup \sigma \in \mathcal{K}\}.$$

Lemma 1.12. For any set of simplices $S \subseteq \mathcal{K}$,

$$\mathrm{Lk}(S) = \mathrm{Cl}(\mathrm{St}(S)) \quad \mathrm{St}(\mathrm{Cl}(S)).$$

1.1.1 The halfedge mesh

1.2 Simplicial complexes

Chapter 2

Discrete Exterior Calculus