Introduction and Motivation

The Feynman-Kac formula is a generalization of the Kolmogorov Backward Equation, both of which provide a connetion between stochastic processes and PDEs. The motivation for this DRP is from my Market Making project, where the authors use the Feynman-Kac formula to solve the HJB equation for the optimal control problem. The Feynman-Kac formula states that, under sufficient technical conditions, if a PDE is of the form

$$\frac{\partial v}{\partial t} = Av - qv \tag{1}$$

and subject to the condition

$$v(0,x) = f(x) \tag{2}$$

where A is interpreted as the infinitesimal generator applied to the function $x \to v(t,x)$ then

$$v(t,x) = E^x[exp(-\int_0^t q(X_s)ds)f(X_t)] \tag{3}$$

Under sufficient technical conditions, the converse also holds. Notice how this can be seen as an extension of Kolmogorov's Backward Equation, which states that, again under sufficient technical conditions, if a PDE is of the form

$$\frac{\partial u}{\partial t} = Au \tag{4}$$

and subject to the condition

$$u(0,x) = f(x) \tag{5}$$

where A is interpreted as the infinitesimal generator applied to the function $x \to u(t,x)$ then

$$u(t,x) = E^x[f(X_t)] \tag{6}$$

Under sufficient technical conditions, the converse also holds. In general, the Feynman-Kac formula has applications in stochastic optimal control, mathematical finance, physics, and stochastic models.

Important Proofs, Definitions, and Results

Ito vs Stratanovich Interpretation

As a result of the quadratic variation of brownian motion, there are actually an infininte number of possible different stochastic calculi that can be defined. In general, for equations of the form

$$dX = \alpha_t dt + noise \tag{7}$$

two interpretations are used for the noise term:

$$dX = \alpha_t dt + \sigma dB_t \tag{8}$$

$$dX = \alpha_t dt + \sigma \circ dB_t \tag{9}$$

The first term is called the Ito interpretation, and the second term is called the Stratanovich interpretation. This notation is just a shorthand for process of the form

$$X = \int_0^t \alpha_t dt + \int_0^t \sigma_t dB_t \tag{10}$$

$$X = \int_0^t \alpha_t dt + \int_0^t \sigma_t \circ dB_t \tag{11}$$

where the first equation is an Ito process, and the second could be called a Stratanovich process. The Ito process can be reached through a process most analgous to taking left riemann sums, and the Stratanovich process can be reached through a process most analgous to taking midpoint reimann sums. It is important to note that for σ_t that is smooth in time the interprations agree. We are going to focus our work on the Ito interpretation.

Markov Property

Intuitively, the Markov Property states that the future behavior of a process given what has happened up to time t is the same as the behavior obtained when starting the process at X_t. The precise statement is this:

$$E^x[f(X_{t+h})|\mathscr{F}_t](\omega) = E^{X_t(\omega)}[f(X_h)] \tag{12}$$

The proof is not too difficult but is out of the scope of this work.

Generator and Dynkin's Formula

The generator of a stochastic process X_t is given by

$$Af(x) = \lim_{t\to 0} \frac{E^x[f(X_t) - f(x)]}{t} \tag{13}$$

Consider the case of 2-dimensional stochastic process (X,t) where $dX = \sigma_t dB_t$ (so we are working in the Ito interpretation). Then in this case the generator Av has an intuitive interpretation as the expected change of a value function, and

$$Av = \frac{\partial v}{\partial t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 v}{\partial x^2} \tag{14}$$

Then Dynkin's Formula is almost a stochastic, weaker analogue to the fundamental theorem of calculus:

$$E^x[f(X_t)] = f(x) + E^x[\int_0^t Af(X_s)ds] \tag{15}$$

Proof of Feynman-Kac

For my work in applied areas, I am not as concerned with uniqueness, so I will only present one half of the proof of Feynman-Kac. Let $Y_t = f(X_t)$ and $Z_t = exp(-\int_0^t q(X_s)ds)$ where X_t is a 1-dimensional Ito diffusion. Then $dY_t = (\frac{\partial f}{\partial x}u + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2})dt + \frac{\partial f}{\partial x}\sigma dB_t$. But we see that $dZ_t = -Z_t q(X_t)dt$, and so $d(Y_t Z_t) = Y_t dZ_t + Z_t dY_t$, as $dZdY = 0$. Looking at

$$\frac{1}{r}(E^x[v(t,X_r)] - v(t,x)) = \frac{1}{r}[E^x[E^{X_r}[Z_t f(X_t) - E^x[Z_t f(X_t)]]]] \tag{16}$$

$$= \frac{1}{r}E^x[E^x[f(X_{t+r}exp(-\int_0^t q(X_{s+r}ds))|F_r)] - E^x[Z_t f(X_t)|F_r]] \tag{17}$$

$$= \frac{1}{r}E^x[Z_{t+r}exp(\int_0^r q(X_s)ds)f(X_{t+r} - Z_t f(X_t))] \tag{18}$$

$$= \frac{1}{r}E^x[f(X_{t+r}Z_{t+r} - f(X_t)Z_t)] + \frac{1}{r}E^x[f(X_{t+r}Z_{t+r})exp(\int_0^r q(X_s)ds) - 1] \tag{19}$$

which tends to $\frac{\partial v}{\partial t} + qv$ as $r \to 0$.

Selected Applications

Black Scholes

Suppose we want to value some derivative on an underyling. Let the price of the underlying be denoted by

$$dX = u_t X dt + \sigma_t X dW \tag{20}$$

To arrive at the price of our derivative, we can change measure (assuming we can hedge costlessly in the underlying) and we have

$$dX = rX_t dt + \sigma_t X \widetilde{dW} \tag{21}$$

Our Feynman-Kac formula above says equivalenty that, if

$$-\frac{\partial v}{\partial t} = Av - qv \tag{22}$$

and subject to the condition

$$v(T,x) = f(x) \tag{23}$$

where A is interpreted as the infinitesimal generator applied to the function $x \to v(t,x)$ then

$$v(t,x) = E^x[exp(-\int_t^T q(X_s)ds)f(X_t)] \tag{24}$$

Letting $q = r$ and changing measures, we have

$$\begin{aligned} v(t,x) &= \widetilde{E}^x[exp(-\int_t^T rds)f(X_t)] && (25)\\ &= \widetilde{E}^x[exp(-r(T-t)f(X_t))] && (26)\end{aligned}$$

Under the risk-neutral probability measure we see that the generator of our process

$$Av = rX_t \frac{\partial v}{\partial x} + \frac{1}{2}\sigma_t^2 X_t^2 \frac{\partial^2 v}{\partial x^2} \tag{27}$$

Substituting into the Feynman-Kac formula, we arrive at the Black-Scholes equation

$$0 = \frac{\partial v}{\partial t} + rX_t \frac{\partial v}{\partial x} + \frac{1}{2}\sigma_t^2 X_t^2 \frac{\partial^2 v}{\partial x^2} - rv \tag{28}$$

subject to

$$v(T,x) = f(x) \tag{29}$$

where f is the payoff curve of our derivative at expiry.

Market Making

It's well known in the traditional Market-Making literature that the Market Maker's problem reduces to solving a Hamilton Jacobi Bellman Equation of similar form to:

$$\begin{aligned} (\frac{\partial}{\partial t} + L)v + \max_{\delta^+} Ae^{-k\delta^+}[v(t,s,q-1,x+(s+\delta^+)) - v(t,s,q,x)] \\ + \max_{\delta^-} Ae^{-k\delta^-}[v(t,s,q+1,x-(s-\delta^+)) - v(t,s,q,x)] = 0, && (30)\\ u(T,s,q,x) = \phi(s,q,x) && (31)\end{aligned}$$

Consider the case of linear utility where

$$\phi(s,q,x) = x + qs \tag{32}$$

We have the correspodning value function

$$u(t,s,q,x) = \max_{(\delta^+,\delta^-)} E_{t,s,q,x}[X(T) + Q(T)S(T)] \tag{33}$$

Applying the ansatz

$$u(t,s,q,x) = x + \theta_0(t,s) + q\theta_1(t,s) \tag{34}$$

and using our first-order maxima condition from calculus, we are left with

$$\delta_*^{\pm} = \frac{1}{k} \pm (\theta_1 - s) \tag{35}$$

$$r_* = \theta_1 \tag{36}$$

Letting

$$f^{\pm}(\delta^{\pm}) = Ae^{-k\delta^{\pm}}[\mp s + \delta^{\pm} - \mp\theta_1] \tag{37}$$

it is seen that our ansatz satisfies

$$(\frac{\partial}{\partial t} + L)(\theta_0 + q\theta_1) + \frac{2A}{ek}\cosh[k(\theta_1 - s)] = 0, \tag{38}$$

$$\theta_0(T,s) = 0 \tag{39}$$

$$\theta_1(T,s) = s \tag{40}$$

Through application of the Feynman-Kac formula to these equations, it can be seen that

$$\theta_1(t,s) = E_{t,s}[S(T)] \tag{41}$$

$$\theta_0(t,s) = \frac{2A}{ek}E_{t,s}[\int_t^T \cosh[k(\theta_1(\zeta, S(\zeta)) - S(\zeta)]d\zeta] \tag{42}$$

solving the Market-Making optimal control problem.

Citations

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