Introduction and Motivation

The Feynman-Kac formula is a generalization of the Kolmogorov Backward Equation, both of which provide a connetion between stochastic processes and PDEs. The motivation for this DRP is from my Market Making project, where the authors use the Feynman-Kac formula to solve the HJB equation for the optimal control problem. The Feynman-Kac formula states that, under sufficient technical conditions, if a PDE is of the form

$$\frac{\partial v}{\partial t} = Av - qv \tag{1}$$

and subject to the condition

$$v(0,x) = f(x) \tag{2}$$

where A is interpreted as the infinitesimal generator applied to the function $x \to v(t,x)$ then

$$v(t,x) = E^x[exp(-\int_0^t q(X_s)ds)f(X_t)] \tag{3}$$

Under sufficient technical conditions, the converse also holds. Notice how this can be seen as an extension of Kolmogorov's Backward Equation, which states that, again under sufficient technical conditions, if a PDE is of the form

$$\frac{\partial u}{\partial t} = Au \tag{4}$$

and subject to the condition

$$u(0,x) = f(x) \tag{5}$$

where A is interpreted as the infinitesimal generator applied to the function $x \to u(t,x)$ then

$$u(t,x) = E^x[f(X_t)] \tag{6}$$

Under sufficient technical conditions, the converse also holds. In general, the Feynman-Kac formula has applications in stochastic optimal control, mathematical finance, physics, and stochastic models.

Important Proofs, Definitions, and Results

Ito vs Stratanovich Interpretation

Ito Process

Markov Property

Kolmogorov's Backward Equation

Feynman-Kac Formula

Selected Applications

Black Scholes

Suppose we want to value some derivative on an underyling. Let the price of the underlying be denoted by

$$dX = u_t X dt + \sigma_t X dW \tag{7}$$

To arrive at the price of our derivative, we can change measure (assuming we can hedge costlessly in the underlying) and we have

$$dX = rX_t dt + \sigma_t X \widetilde{dW} \tag{8}$$

Our Feynman-Kac formula above says equivalenty that, if

$$-\frac{\partial v}{\partial t} = Av - qv \tag{9}$$

and subject to the condition

$$v(T,x) = f(x) \tag{10}$$

where A is interpreted as the infinitesimal generator applied to the function $x \to v(t,x)$ then

$$v(t,x) = E^x[exp(-\int_t^T q(X_s)ds)f(X_t)] \tag{11}$$

Letting $q = r$ and changing measures, we have

$$v(t,x) = \widetilde{E}^x[exp(-\int_t^T rds)f(X_t)] \tag{12}$$

$$= \widetilde{E}^x[exp(-r(T-t)f(X_t))] \tag{13}$$

Under the risk-neutral probability measure we see that the generator of our process

$$Av = rX_t\frac{\partial v}{\partial x} + \frac{1}{2}\sigma_t^2 X_t^2 \frac{\partial^2 v}{\partial x^2} \tag{14}$$

Substituting into the Feynman-Kac formula, we arrive at the Black-Scholes equation

$$0 = \frac{\partial v}{\partial t} + rX_t\frac{\partial v}{\partial x} + \frac{1}{2}\sigma_t^2 X_t^2 \frac{\partial^2 v}{\partial x^2} - rv \tag{15}$$

subject to

$$v(T,x) = f(x) \tag{16}$$

where f is the payoff curve of our derivative at expiry.

Market Making