1 Model Dynamics Summary

Summarizing Buy Low Sell High, we model

$$d\lambda^+ = \beta(\theta - \lambda^+)dt + nd\overline{M_t^+} + vd\overline{M_t^-} \tag{1}$$

$$d\lambda^- = \beta(\theta - \lambda^-)dt + vd\overline{M_t^+} + nd\overline{M_t^-} \tag{2}$$

$$d\alpha_t = -\zeta\alpha_t dt + \sigma_\alpha dB_t + \varepsilon^+ d\overline{M_t^+} + \varepsilon^- d\overline{M_t^-} \tag{3}$$

$$dk^+ = \beta_k(\theta_k - k^+)dt + n_k d\overline{M_t^+} + v_k d\overline{M_t^-} \tag{4}$$

$$dk^- = \beta_k(\theta_k - k^-)dt + v_k d\overline{M_t^+} + n_k d\overline{M_t^-} \tag{5}$$

$$dX_t = (S_t + \delta_{t-}^+)dN_t^+ - (S_t - \delta_{t-}^-)dN_t^- \tag{6}$$

$$dq_t = dN_t^- - dN_t^+ \tag{7}$$

2 Hamilton-Jacobi-Bellman Derivation

Market making is a stochastic optimal control problem where we aim to maximize a functional of the form

$$V(x,t) = \max_{\delta_a \delta_b} \mathbb{E}[\int_t^T L(t,x,\delta_a,\delta_b)dt + V(T,\cdot)] \tag{8}$$

. Many market-making papers have no running-cost, but the paper we are considering has a value function of the form

$$\Phi(t,X_t,S_t,q_t,\alpha_t,\boldsymbol{\lambda}_t,\boldsymbol{k}_t) = \sup_{\delta_u^+,\delta_u^- t\le u\le T\in\mathscr{A}} \mathbb{E}[X_T + q_T S_T - \phi\int_t^T q_s^2 ds|\mathscr{F}_t] \tag{9}$$

Since market making has optimal substructure, we can write

$$\Phi(t,X_t,S_t,q_t,\alpha_t,\boldsymbol{\lambda}_t,\boldsymbol{k}_t) = \sup_{\delta_u^+,\delta_u^- t\le u\le T\in\mathscr{A}} \mathbb{E}[\int_t^{t+\Delta t} -\phi q_s^2 ds + \Phi(t+\Delta t, X_{t+\Delta t}, S_{t+\Delta t}, q_{t+\Delta t}, \alpha_{t+\Delta t}, \boldsymbol{\lambda}_{t+\Delta t}, \boldsymbol{k}_{t+\Delta t})] \tag{10}$$

Subtracting Φ from both sides, dividing by Δt, and taking the limit as Δt goes to zero / computing the stochastic derivative / using Ito's Lemma and then doing tedious algebraic manipulations, we are left with the Hamilton-Jacobi-Bellman PIDE presented in Buy Low Sell High.

2.1 Resources

1. Das, Sanjiv R. "The surprise element: Jumps in interest rates." Journal of Econometrics, vol. 106, no. 1, 2002, pp. 27–65, https://doi.org/10.1016/s0304-4076(01)00085-9.

2. Rasmussen, Jakob Gulddahl. "Lecture Notes: Temporal Point Processes and the Conditional Intensity Function." arXiv.Org, 1 June 2018, arxiv.org/abs/1806.00221.

3. Laub, Patrick J., et al. "Hawkes Processes." arXiv.Org, 10 July 2015, arxiv.org/abs/1507.02822.

4. Aït-Sahalia, Yacine, et al. "How and When Are High-Frequency Stock Returns Predictable?" NBER, 22 Aug. 2022, www.nber.org/papers/w30366.

5. Cartea, Álvaro, et al. "Buy Low Sell High: A High Frequency Trading Perspective." SSRN, 26 Nov. 2011, `deliverypdf.ssrn.com/delivery.php?ID=701020068111102104094119094073115106050051026007034010 0amp;EXT=pdf&INDEX=TRUE`.

6. Sophie Laruelle - Events.Chairefdd.Org, `events.chairefdd.org/wp-content/uploads/2013/06/CAHIER_MICRO_1.pdf`. Accessed 18 Oct. 2023.