1 Intuition Behind Alpha Dynamical Model

If as a trader we had all of the information in the entire world, then perhaps stock market movements wouldn't look random and we could instead model them as determinstic processes. However we don't have all of the information, and so we model price information as a combination of deterministic and stochastic processes. One possible stock price model is $dX = \alpha_t dt + \sigma dB_t$, where B_t is a Weiner process, or Brownian motion. In this model, α_t captures the trend in stock price at a given time that we are able to predict. We expect α to have certain characteristics:

1. We can predict both upwards and downwards movements
2. These movements generally don't last for a long time when they are present (mean reversion)
3. Cannot necessarily predict how α will change at a given time, can only predict certain tendencies like mean reversion
4. Buy orders should be tied to positive alpha, and sell orders to negative alpha

So then one possible model for the dynamics of alpha is

$$d\alpha_t = k(\theta - \alpha_t)dt + \sigma_\alpha dB_{\alpha_t} + \varepsilon d\bar{M}^+ - \varepsilon d\bar{M}^- \tag{1}$$

, where epsilon captures the effects that buy and sell market orders have on the mid-price, and k is the rate of α mean reversion, and θ is the long term mean reversion level for α during the trading period. As an interesting note, jump diffusion models related to this one are often used in the electricty markets to model spikes in electricty prices, though they may have additional seasonal elements incorporated.

1.1 Ornstein-Uhlenbeck Jump Process

Examine (1). This is a mean-reverting Ornstein-Uhlenbeck process with long term mean 0 and jumps when influential market orders arrive. Das (1), gives a discrete-time MLE estimator for a related process, where

$$dr = k(\theta - r)dt + vdz + Jd\pi(h) \tag{2}$$

, where J plays a role similar to ε except is normally distributed with mean μ and variance γ^2 and $d\pi(h)$ a role similar to $d\overline{M_t^{\pm}}$. Das finds the transition probabilities to be

$$\begin{aligned} f[r(s)|r(t)] = q\exp(&\frac{-(r(s)-r(t)-k(\theta-r(t))\Delta t-\mu)^2}{2(v^2\Delta t+\gamma^2)})\frac{1}{\sqrt{2\pi(v^2\Delta t+\gamma^2)}} \\ &+(1-q)\exp(\frac{-(r(s)-r(t)-k(\theta-r(t))\Delta t)^2}{2(v^2\Delta t)})\frac{1}{\sqrt{2\pi(v^2\Delta t)}} \end{aligned} \tag{3}$$

Modifying Das' work, I treat ε similar to J, except ε is a constant and does not follow a distribution. Additionally, it is key to note that unlike in the interest rate case, we know exactly when the market orders arrived in general, and conveniently the form of our transition probabilties changes. The transitiion probabilties for our α process can be given as $f[\alpha(t)|\alpha(t+\Delta t)]$, where s is sufficiently close to t such that only one or no orders arrive between s and t are now given by

$$\begin{aligned} f[\alpha(t+\Delta t)|\alpha(t)] = \mathbb{1}M^+_{[t,t+\Delta t)}exp(&\frac{-(\alpha(s)-\alpha(t)-k(\theta-\alpha(t))\Delta t-\varepsilon)^2}{2\sigma_\alpha^2\Delta t})\frac{1}{\sqrt{2\pi\sigma_\alpha^2\Delta t}} \\ &+\mathbb{1}M^-_{[t,t+\Delta t)}exp(\frac{-(\alpha(s)-\alpha(t)-k(\theta-\alpha(t))\Delta t+\varepsilon)^2}{2\sigma_\alpha^2\Delta t})\frac{1}{\sqrt{2\pi\sigma_\alpha^2\Delta t}} \\ &+\mathbb{1}M^0_{[t,t+\Delta t)}exp(\frac{-(\alpha(s)-\alpha(t)-k(\theta-\alpha(t))\Delta t)^2}{2\sigma_\alpha^2\Delta t})\frac{1}{\sqrt{2\pi\sigma_\alpha^2\Delta t}} \end{aligned} \tag{4}$$

Then the log-likelihood for our model is given by $\log(\prod_{t_i} f[\alpha(t_{i+1})|\alpha(t_i)])$. Notice that in the interest rate case since we don't know exactly when the jumps occurred, this log-likelihood must be maximized numerically. However since we when trades arrive and thus when we expect our jumps to occur, we are left with a log-likelihood that can be maximized by taking the gradient and solving algebraically.

2 Resources

1. Rasmussen, Jakob Gulddahl. "Lecture Notes: Temporal Point Processes and the Conditional Intensity Function." arXiv.Org, 1 June 2018, arxiv.org/abs/1806.00221.

2. Laub, Patrick J., et al. "Hawkes Processes." arXiv.Org, 10 July 2015, arxiv.org/abs/1507.02822.