Graphical Abstract

Comparing Relative Reachable Sets about Nearly Circular Orbits

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Highlights

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- Research highlight 1
- Research highlight 2

Comparing Relative Reachable Sets about Nearly Circular Orbits

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Abstract

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1. Introduction

We will examine four different methods for calculating the relative reachable sets: one single impulse ΔV_0 at t_0, one constant impulse over the time period $[t_0, t_1)$ normalized such that $||\Delta V_1|| = ||\Delta V_0||$, and an energy optimal and fuel optimal control approach.

2. Relative Reachable Set Calculations

2.1. Fixed-Time Single Impulse

The geometry of the reachable set that we obtain from analysis of the single impulse problem can be found through analysis of the associated state-transition matrix. While in general the state-transition matrix associated with linear systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x_0} \tag{1}$$

is given by

$$\mathbf{\Phi}(t, \tau) \equiv \mathbf{U}(t)\mathbf{U}^{-1}(\tau) \tag{2}$$

where $\mathbf{U}(t)$ satisfies

$$\dot{\mathbf{U}}(t) = \mathbf{A}(t)\mathbf{U}(t), \quad \mathbf{U}(t_0) = \mathbf{I} \tag{3}$$

in this case it is well known that state transition matrix is given analytically as

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)} \tag{4}$$

The position-velocity reachable set associated with this formulation of the problem is a 6-dimensional ellipsoid with semi-axes $\mathbf{h}_i$ given by $\mathbf{h}_i = s_i\mathbf{u}_i$, where $s_i, \mathbf{u}_i$ are the singular values and their associated left singular vectors associated with $\mathbf{\Phi}(t, t_0)$ and $\mathbf{A}$ is the Clohessy-Wiltshire Equations. The position reachable-set is ultimately given by the projection of this 6-dimensional ellipsoid onto the first three coordinates. The closed form of $\mathbf{\Phi}(t, t_0)$ is given by (appendix entry or reference).

2.2. Fixed-Time Constant Thrust

The geometry of the reachable set associated with the constant thrust formulation of the problem can be derived in a process similar to the one used in (2.1) (use
ref and
label for references of section numbers eqns numbers and figure numbers). Suppose that constant thrust normalized appropriately is applied during $[t_0, t_1)$, and subsequently no thrust is applied from $[t_1, T]$. Then the state $\mathbf{x}_t$ during $[t_1, T]$ is determined by $\mathbf{x}_t = \mathbf{\Phi}(t_1, t)\mathbf{x}_{t1}$ with $\mathbf{\Phi}$ as given in (4), for $t_1 \leq t \leq T$. During $[t_0, t_1)$, the dynamics are determined by $\mathbf{x}_t = \mathbf{\Gamma}(t, t_0)$, where $\mathbf{\Gamma}$ is the input transition matrix as found in (appendix entry or reference). The reachable set is then again a 6 dimensional ellipsoid with axes given by the singular value decomposition of $\mathbf{\Gamma}(t_1, t_0)\mathbf{\Phi}(T, t_1)$. Similarly, the position reachable set is given by the projection of this 6-dimensional ellipsoid onto the first three coordinates.

2.2.1. RIC Frame

2.2.2. ECI Frame

What we have considered so far is based on a RIC frame constant thrust burn (So the acceleration is in a constant direction in the rotating RIC frame). For completeness, it is also worth considering thrusts that are fixed in direction in the inertial ECI frame. The Ankersen thesis describes the ITM for this case as well. Comparing this with the RIC frame fixed reachable set is slightly interesting/low effort.

2.3. Energy Limited

A brief summary of the literature surrounding energy reachable sets goes here, along with a summary of the relevant calculations/quadratic form. You will then emphasize the novelty of looking at the geometry of the position reachable set described below using a generalized eigenvalue framework.

2.3.1. Position Reachable Set

The reachable set that we obtain from analysis of the energy optimal control problem is in terms of a six-dimensional quadratic form in position and velocity. The eigenvalue decomposition of this quadratic form describes the geometry of the reachable set as an ellipsoid in $\mathbb{R}^6$. While this may be useful in itself, the position reachable set may also be necessary in some instances. The reachable set in position only consists of the positions which can be reached with any final velocity. In order to characterize the geometry of this set, a constrained optimization problem arises

$$\max_{\mathbf{x}^T\mathbf{N}\mathbf{x}=E} \mathbf{x}^T \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \max_{\mathbf{x}^T\mathbf{N}\mathbf{x}=E} \mathbf{r}^T\mathbf{r} \tag{5}$$

Using the method of lagrange multipliers, this leads to a generalized eigenvalue problem that must be solved

$$\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \lambda\mathbf{N}\mathbf{x} \tag{6}$$

Given that both of these matrices are positive semi-definite, the resulting three nonzero eigenvalues will have orthogonal associated eigenvectors. The projection of these eigenvectors onto the first three coordinates give the axes of the position reachable set ellipsoid. The semi-axes of the position ellipsoid are

$$\mathbf{v}_i^{\mathbf{r}} = \sqrt{\frac{E}{\mathbf{v}_i^T\mathbf{N}\mathbf{v}_i}} \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{v}_i \tag{7}$$

2.4. Thrust Limited

[**?**] equation 21 gives a method to parameterize the relative reachable set. I have some Mathematica code I can share for doing this too. You will be able to vary 1 or 2 parameters to find points along the reachable set (in-plane or full 3d position). You don't control where you find them along the reachable set though, so when you find one, you will then compare it to one of the ellipsoidal sets by mapping back that direction with the ellipsoidal set energy reachable set method

2.5. Fuel Limited

This one is a reach goal. Likely, a linear programming approach will be the best way to go about it (where you pick a direction and solve a maximization problem for how far you can get along that direction)

3. Comparison Methodology

3.1. Comparison Between an Ellipsoidal Set and an Arbitrary Set

Given a point on an arbitrary reachable set, find how much further or less far in that direction the ellipsoidal set can go

3.2. Comparison Between Ellipsoidal Sets

Show how this previous approach extends into solving that generalized eigenvalue problem that tells you the minimum and maximum ratios/direction

4. Analysis

4.1. Comparing the Impulsive and Constant Thrust Reachable Sets

Show comparisons, and add a note about when the ITM*STM is singular and show where it is singular (that implicit plot you had of the set where det()=0 for various values of t1,t2)

4.2. Comparing Low Thrust Reachable Sets

Compare the Energy limited and Thrust limited Reachable Sets, and potentially add in the (fuel limited and thrust limited) set against the thrust limited set for various levels of fuel constraint lower than the fuel used by blasting at full limited thrust the whole time

t2 determines impulsive set t1, t2 determines const thrust reachable set. For a given value of t1, t2 come up with the ratios of the reachable sets in a reasonable way. If the ratio is γ we want to look at the stationary points of gamma and also the arg stationary. As we vary t1, t2 what do the values of γ look like and what do the vectors $\hat{r}_f$ look like. Make a plot of t_2 fixed, what do the γ look like as a function of t_1. What happens with t_1 fixed an vary t_2 look at the γ_i. (Fixed value of thrust). Color the γ depending on whether they are greater or less than 1. Values of t_2.

Appendix A. Sample Appendix Section

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