UNIVERSITÀ DEGLI STUDI DI PADOVA

A.Y. 2024-2025: Quantum Information and Computing

EVALUATED HOMEWORK

Exercise 1: Temporal evolution

Let's consider a qubit with basis states $\{|0\rangle, |1\rangle\}$ and the operator defined by

$$\hat{H} = \hbar[\Delta\omega\hat{\sigma}_z + \gamma\hat{\sigma}_x] \tag{1}$$

with $\Delta\omega$ and γ real parameters with the unit of frequency.

1. Show that the operator $\hat{H}$ has the properties to be considered the Hamiltonian of the system.
2. Evaluate the eigenvalues $E_\pm$ and the eigenvectors of the Hamiltonian $\hat{H}$. Plot the energy gap $E_+ - E_-$ in function of $\Delta\omega$ and γ.
3. Evaluate $|\psi_+(t)\rangle$ and $|\psi_-(t)\rangle$, defined as the temporal evolution of the states $|+i\rangle$ and $|-i\rangle$, where
$$|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\,, \quad |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\,.$$
4. Determine the Bloch's vectors $\vec{r}_\pm(t)$ of the states $|\psi_+(t)\rangle$ and $|\psi_-(t)\rangle$. Evaluate the scalar product $\vec{r}_+(t)\cdot\vec{r}_-(t)$ in function of time and give an interpretation of the result.
5. Determine the trajectory on the Bloch' sphere of the vectors $\vec{r}_\pm(t)$ in function of time and evaluate at which times the states $|\psi_+(t)\rangle$ and $|\psi_-(t)\rangle$ coincide with the input states $|\pm i\rangle$.

Exercise 2

Consider a two-level system, A, coupled with an ancillary system B. The system B is a *qutrit*, namely a three-level system spanned by $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$. The system A is prepared in the state

$$|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A\,,$$

while the ancillary system B is initially prepared into $|0\rangle_B$.

The A and B systems interact by the unitary operator $\hat{\mathcal{U}}$ whose action is defined by:

$$\begin{aligned} \hat{\mathcal{U}}|0\rangle_A|0\rangle_B &= \frac{1}{\sqrt{5}}(\sqrt{2}|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B + \sqrt{2}|0\rangle_A|2\rangle_B) \\ \hat{\mathcal{U}}|1\rangle_A|0\rangle_B &= \frac{1}{2}(|1\rangle_A|0\rangle_B - \sqrt{2}|0\rangle_A|1\rangle_B + |1\rangle_A|1\rangle_B) \end{aligned} \tag{2}$$

1. Evaluate the final joint state after the interaction.
2. Calculate the resulting state on the subsystem A.
3. Calculate the generalized measurement operators acting on A when the system B is measured in the "computational" basis $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$.
4. Evaluate the post-measurement state when the output $|2\rangle_B$ is obtained.

5. Calculate the corresponding POVM elements. What is their rank?

6. Calculate the output probabilities of the previous POVM when the input state is the mixed state

$$\rho = \frac{1}{5}|0\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1| \tag{3}$$

Exercise 3

Consider arbitrary "spin" observables $\hat{A} = a_x\hat{\sigma}_x + a_z\hat{\sigma}_z$ and $\hat{B} = b_x\hat{\sigma}_x + b_z\hat{\sigma}_z$ with $a_x^2 + a_z^2 = b_x^2 + b_z^2 = 1$. The operators $\hat{\sigma}_{x,z}$ are the Pauli matrices. Let's also consider the two-qubit nonmaximally entangled state

$$|\Phi(\theta)\rangle_{AB} = \cos\frac{\theta}{2}|0\rangle_A|0\rangle_B + \sin\frac{\theta}{2}|1\rangle_A|1\rangle_B$$

1. Demonstrate that the eigenvalues of $\hat{A}$ and $\hat{B}$ are ± 1.

2. Evaluate the action of the possible combinations $\hat{\sigma}_i \otimes \hat{\sigma}_j$ with $i, j = x$ or z on the state $|\Phi(\theta)\rangle_{AB}$, i.e.

$$\hat{\sigma}_i \otimes \hat{\sigma}_j|\Phi(\theta)\rangle_{AB}\,, \qquad \forall i, j = x, z \tag{4}$$

3. Use the above result to calculate following expectation value

$$\langle \hat{A} \otimes \hat{B}\rangle_\theta = \langle\Phi(\theta)|\hat{A} \otimes \hat{B}|\Phi(\theta)\rangle \tag{5}$$

4. Consider the four operators $\hat{A}_k = \vec{a}_k \cdot \vec{\sigma}$, $\hat{B}_k = \vec{b}_k \cdot \vec{\sigma}$ with $k = 0, 1$. Define $\vec{a}_0$, $\vec{a}_1$, $\vec{b}_0$ and $\vec{b}_1$ as the vectors in the (x, z) plane that allow to maximally violate the CHSH inequality with the maximally entangled state $|\Phi(\frac{\pi}{2})\rangle_{AB}$. Determine for which value of θ it is possible to violate the CHSH inequality with the observables defined by $\vec{a}_0$, $\vec{a}_1$, $\vec{b}_0$ and $\vec{b}_1$.

5 (extra) Determine if it is possible to find other observables (θ dependent) that allow to increase the violation of the CHSH inequality for arbitrary values of θ.