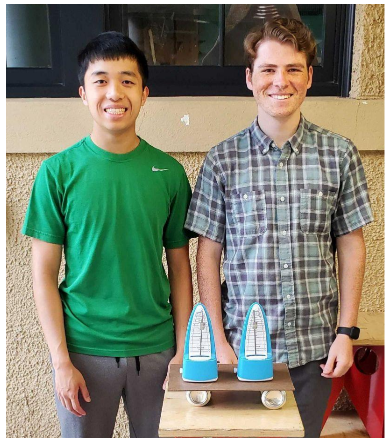
Metronome Madness

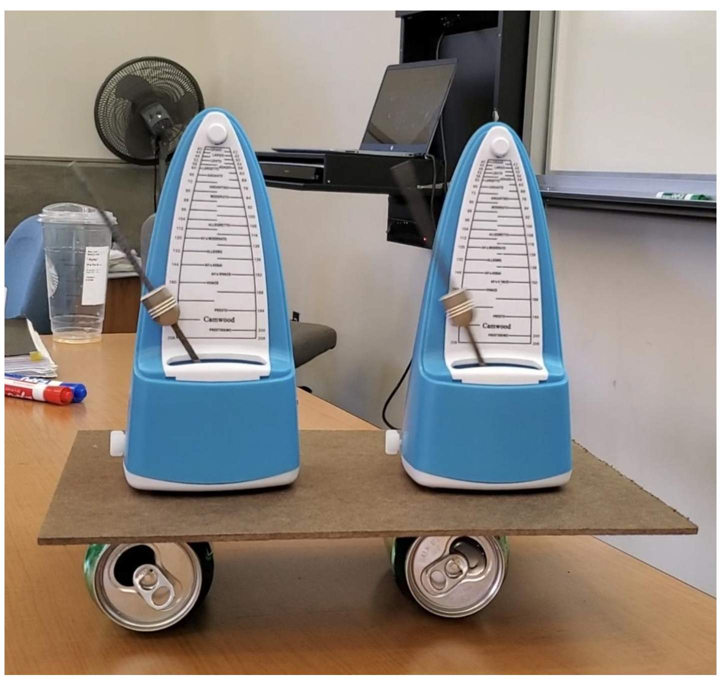
Max Ahlquist and Ryan Nguyen



Question:

Ryan is dutifully practicing the violin while using a metronome, but Max next door is blasting Chicken Fried by Zac Brown Band, making Ryan unable to hear his metronome. To increase the metronome's volume, he whips out his second metronome (always pack extra socks and metronomes) and places both metronomes on a thin rectangular plate parallel to the ground. The plate (his lucky piece of duron which he carries around everywhere) is then placed on two empty cylindrical LaCroix Cans (taken from a d.school event he did not attend) whose respective cylindrical axes are parallel to each other, parallel to the metronomes' revolute axes, and perpendicular to the local vertical. Due to rolling, the plate translates horizontally in a direction perpendicular to the cylinders' axes. Unfortunately, Ryan is unable to start the metronomes at the exact same time so they are not synchronized. However, having seen a cool Youtube video (A, B), he knows that they will eventually synchronize, but how long will that take?

(Special Thanks to David Levinson for his analysis of the metronome escapement Mechanism. The analysis' success hinged on his provided help!)

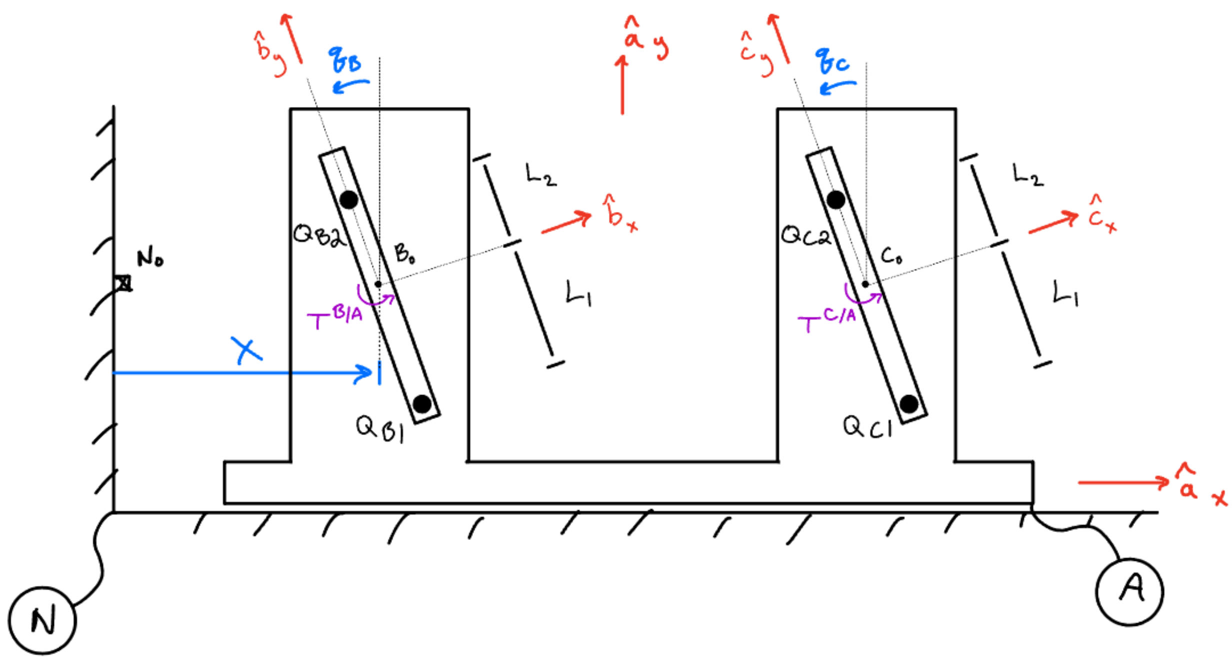


Model:

The figure below shows a rigid body A consisting of two metronome chassis welded on top of the rectangular plate positioned on the left and right sides, respectively. Rigid body A translates horizontally left and right on Earth (regarded as a Newtonian Frame N). Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are fixed in A with $\hat{\mathbf{a}}_x$ horizontally-right, $\hat{\mathbf{a}}_y$ vertically upward, and $\hat{\mathbf{a}}_z = \hat{\mathbf{a}}_x \times \hat{\mathbf{a}}_y$ (also horizontal).

A rigid pendulum-like metronome arm B, modeled as a massless thin rod, is supported by the left metronome's chassis at pivot point Bo. B has a simple rotation in A about $\hat{\mathbf{a}}_z$. Particles QB1 and QB2 are fixed on distal ends of B. Right-handed orthogonal unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are fixed in B. $\hat{\mathbf{b}}_y$ is directed along B (from QB1 to Bo to QB2), and $\hat{\mathbf{b}}_z = \hat{\mathbf{a}}_z$. B's orientation in A is determined by setting $\hat{\mathbf{b}}_i = \hat{\mathbf{a}}_i$ (i = x, y, z) and subjecting B to a right-handed rotation in A characterized by $q_B\, \hat{\mathbf{a}}_z$.

Metronome arm C is similar to B, being supported by the right metronome chassis at pivot point Co. Fixed in C are right-handed orthogonal unit vectors $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$ and particles QC1 and QC2. C's orientation in A is determined by setting $\hat{\mathbf{c}}_i = \hat{\mathbf{a}}_i$ (i = x, y, z) and subjecting C to a right-handed rotation in A characterized by $q_C\, \hat{\mathbf{a}}_z$.



Assumptions:

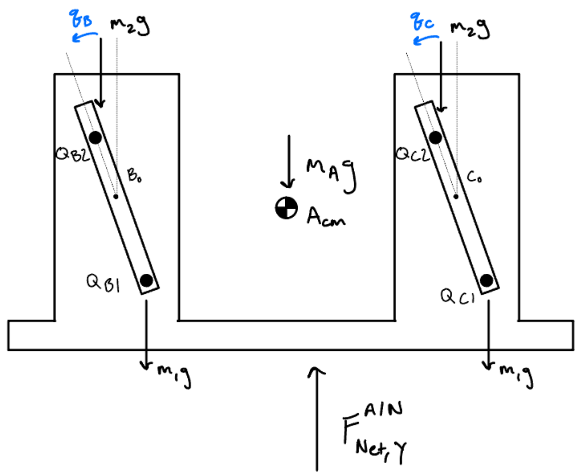
- Assume the empty LaCroix cans are sufficiently light so that we can omit them from the model resulting in A sliding frictionlessly on N.[1]

[1] While this assumption is perfectly reasonable if the soda cans are massless and possess zero inertia properties, this isn't completely realistic. See the optional problem at the bottom to explore this concept further.

Identifiers:

Description	Symbol	Type	(Initial) Value
Earth's gravitational acceleration	g	Constant	$9.8\ m/s^2$
Mass of QB1, QC1	m_1	Constant	0.2 kg
Mass of QB2, QC2	m_2	Constant	0.05 kg
Mass of A	m_A	Constant	0.3 kg
$-\hat{\mathbf{b}}_y$ measure of the position vector from Bo to QB1 ($-\hat{\mathbf{c}}_y$ measure of position vector from Co to QC1.)	L1	Constant	0.05 m
$\hat{\mathbf{b}}_y$ measure of the position vector from Bo to QB2. ($\hat{\mathbf{c}}_y$ measure of position vector from Co to QC2.)	L2	Constant	0.08 m
Van Der Pol Oscillator Factor	k	Constant	0.001 N*m*s
Desired amplitude of metronome for Van Der Pol Oscillator	q_o	Constant	30^o
$\hat{\mathbf{a}}_x$ measure of position vector from No (a point fixed in N) to Bo.	x	Variable	0 m
Angle from $\hat{\mathbf{a}}_y$ to $\hat{\mathbf{b}}_y$ with a $+\hat{\mathbf{a}}_z$ sense	q_B	Variable	35^o (initial value)
Angle from $\hat{\mathbf{a}}_y$ to $\hat{\mathbf{c}}_y$ with a $+\hat{\mathbf{a}}_z$ sense	q_C	Variable	12^o (initial value)
Torque on B from A in the $\hat{\mathbf{a}}_z$ direction (from torsional spring/escapement [Van Der Pol torque equation])	$T^{B/A}$	Variable	$-\ k \cdot ((\frac{2q_B}{q_o})^2 - 1) \cdot \dot{q}_B$
Torque on C from A in the $\hat{\mathbf{a}}_z$ direction (from torsional spring/escapement [Van Der Pol torque equation])	$T^{C/A}$	Variable	$-\ k \cdot ((\frac{2q_C}{q_o})^2 - 1) \cdot \dot{q}_C$

1. Draw a Free Body Diagram of the System consisting of A, B, C, QB1, QB2, QC1, and QC2. Circle the following forces and torques which show up in this Free Body Diagram.



Note: $F^{A/N}_{Net,y}$ is not applied at the same point on A for all time, but rather moves in such a way that the sum of the moments acting on the system about any point equals 0.

$F^{A/N}_{Net,y}$ *	$F^{A/B}_{x}$	$F^{A/B}_{y}$	$F^{A/C}_{x}$	$F^{A/C}_{y}$
m^1g	m^2g	m^Ag	$T^{B/A}$	$T^{C/A}$

*$F^{N/A}_{Net,y}$ is the sum of all forces from N on A in the $\hat{\mathbf{a}}_y$ direction.

2. For each of the following forces/torques, determine if they contribute to work **on the system**. Explain.

   a. $F^{QB1}_{gravity}$ Force due to gravity on QB1 — Yes/No
   b. $F^{N/A}_{Net,y}$ Net Force on A from N in the $\hat{\mathbf{a}}_y$ direction — Yes/No
   c. $F^{A/B}_{x}$ Force between A and B in the $\hat{\mathbf{a}}_x$ direction — Yes/No
   d. $T^{B/A}$ Torque between A and B in the $\hat{\mathbf{a}}_z$ direction — Yes/Yes

$P = F \cdot v,\ P = T \cdot \omega,\ W = \int Pdt$ If P = 0, W = 0

a. $F = -m_1g*\hat{\mathbf{a}}_y$, $v^{QB1} = \dot{x}*\hat{\mathbf{a}}_x + L1*\dot{q}_B*\hat{\mathbf{b}}_x$: $P = -m_1{}^*g*L1*\sin q_B*\dot{q}_B$ => W is not 0
b. $F = \text{something}*\hat{\mathbf{a}}_y$, $v^{Ao} = \dot{x}*\hat{\mathbf{a}}_x$: P = 0 => W = 0
c. $F^{A/B}_{x} = -F^{B/A}_{x}$, forces are equal and opposite, points have the same velocity => P = 0, W

= 0. (Alternative explanation: if we wrote out a formula for system work, $F_x^{A/B}$ would not appear in the proposed formula because of the canceling out.)

d. $T^{B/A} = - T^{A/B}$, $\omega^B = \dot{q}_B * \hat{\mathbf{a}}_z$, $\omega^A = 0$ =>Because the angular velocity of A and B are not the same, P = is non zero, W is not 0

3. Before solving, determine which of the following forces and torques will appear in Kane's equations of motion.

$F_{Net,y}^{N/A}$	$F_x^{A/B}$	$F_y^{A/B}$	$F_x^{A/C}$	$F_y^{A/C}$
m^1g	m^2g	m^Ag	$T^{B/A}$	$T^{C/A}$

4. Determine Kane's equations of motions for the system in the form shown below.

Integrated Torques*

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_1L_1^2 + m_2L_2^2 & 0 & (m_1L_1 - m_2L_2)\cos(q_B) \\ 0 & m_1L_1^2 + m_2L_2^2 & (m_1L_1 - m_2L_2)\cos(q_C) \\ (m_1L_1 - m_2L_2)\cos(q_B) & (m_1L_1 - m_2L_2)\cos(q_C) & m_A + 2m_1 + 2m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_B \\ \ddot{q}_C \\ \ddot{X} \end{bmatrix} +$$

$$\begin{bmatrix} m_1gL_1\sin(q_B) - m_2gL_2\sin(q_B) - T^{B/A} \\ m_1gL_1\sin(q_C) - m_2gL_2\sin(q_C) - T^{C/A} \\ m_2L_2\sin(q_B)\dot{q}_B^2 + m_2L_2\sin(q_C)\dot{q}_C^2 - m_1L_1\sin(q_B)\dot{q}_B^2 - m_1L_1\sin(q_C)\dot{q}_C^2 \end{bmatrix}$$

Separated Torques*

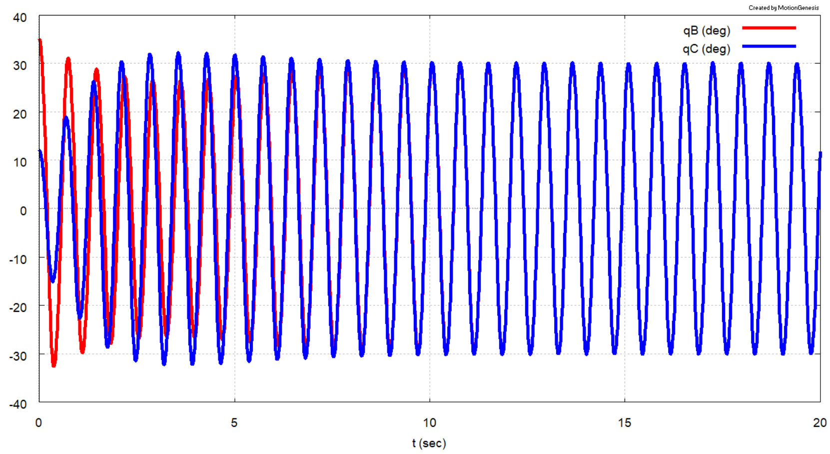
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_1L_1^2 + m_2L_2^2 & 0 & (m_1L_1 - m_2L_2)\cos(q_B) \\ 0 & m_1L_1^2 + m_2L_2^2 & (m_1L_1 - m_2L_2)\cos(q_C) \\ (m_1L_1 - m_2L_2)\cos(q_B) & (m_1L_1 - m_2L_2)\cos(q_C) & m_A + 2m_1 + 2m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_B \\ \ddot{q}_C \\ \ddot{X} \end{bmatrix}$$

$$\begin{bmatrix} m_1gL_1\sin(q_B) - m_2gL_2\sin(q_B) \\ m_1gL_1\sin(q_C) - m_2gL_2\sin(q_C) \\ m_2L_2\sin(q_B)\dot{q}_B^2 + m_2L_2\sin(q_C)\dot{q}_C^2 - m_1L_1\sin(q_B)\dot{q}_B^2 - m_1L_1\sin(q_C)\dot{q}_C^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{B/A} \\ T^{C/A} \end{bmatrix}$$

5. Simulate this system's motion for 20 seconds with the initial conditions above, starting from rest. Use a time step of 0.01s. Defining qDif as $q_B - q_C$, determine the value t_{sync} of time t that |qDif| decreases to be consistently less than 0.01 deg.

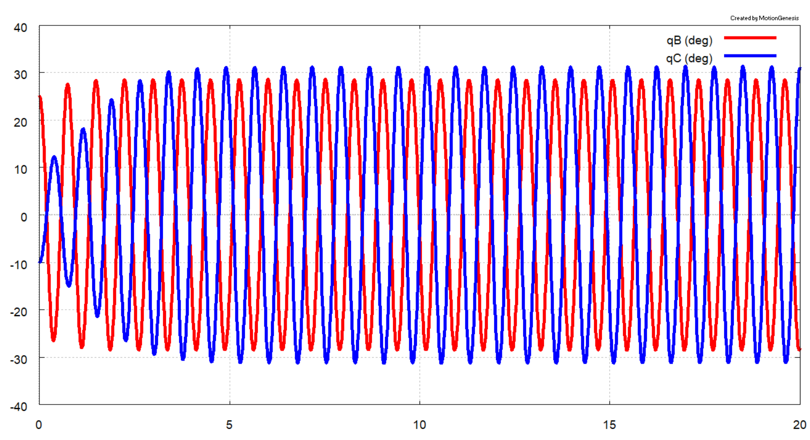
Result: (two decimal places)

t_{sync} = 15.76 sec



6. Denoting the change in the system's gravitational potential energy as deltaV (PE - PE_{intial}) and the work done by the torsional spring/escapement mechanism as $W_{metronomes}$ the simulation results show KePlusPEMinusWork = K + deltaV − $W_{metronomes}$ is ~0 (to within 10^{-5}). **True/False**
   - Optional: If you are having trouble with this energy check, consider plotting deltaV (the system's change in potential energy) and Wmetronomes separately first before combining in KePlusPEMinusWork.

7. Play around with the initial conditions of qB and qC. Can you get a simulation where the two metronomes 180o out of phase? Note: Van Der Pol oscillators drive the metronomes to oscillate between ±qo (±30o). In light of this, experiment with initial conditions between ±50o. Write down your initial conditions where the metronomes do not start perfectly 180o out of phase and submit a plot.
   Result: (This is one of many possible results)
   q_B = 25 deg
   q_C = -10 deg



- **Optional:** The instructor team (Max and Ryan) added a 3rd identical metronome D with the same mass and length values as B and C. Guess whether or not it is possible for two metronomes to synchronize while the third is 180° out of phase. In view of your answer, is it always possible to synchronize an odd number of metronomes? Is it always possible to synchronize an even number of metronomes?

Not Possible: with three metronomes, it is not possible for the system to reach a stable equilibrium with only two synchronized and the third 180° out of phase. The uneven distribution of mass in this configuration will move the plate left and right, causing the metronome 180° out of phase to synchronize with the other two metronomes. It is always possible to synchronize an odd number of metronomes, but with an even number of metronomes it is sometimes possible to have a set of metronomes act 180 degrees out of phase.

- **Optional:** As seen in [1], solve for the equations of motion where the plate rolls on (massive) cylinders rather than sliding on a frictionless surface. What terms are added to the equations as a result of this?

From Kane's equations, the only new added term will be in the generalized effective forces with respect to the generalized coordinate $\dot{x}$. The generalized moment of effective forces and generalized effective forces from the contribution of two cans is

$$\ddot{x}\left(\frac{I_{can}}{2r^2} + \frac{m_{can}}{2}\right)$$

Note: this term can be combined into the mass matrix, increasing the effective mass of the system

$$m_{eff} = m_A + 2m_1 + 2m_2 + \frac{I_{can}}{2r^2} + \frac{m_{can}}{2}$$