Math 451 Test #2

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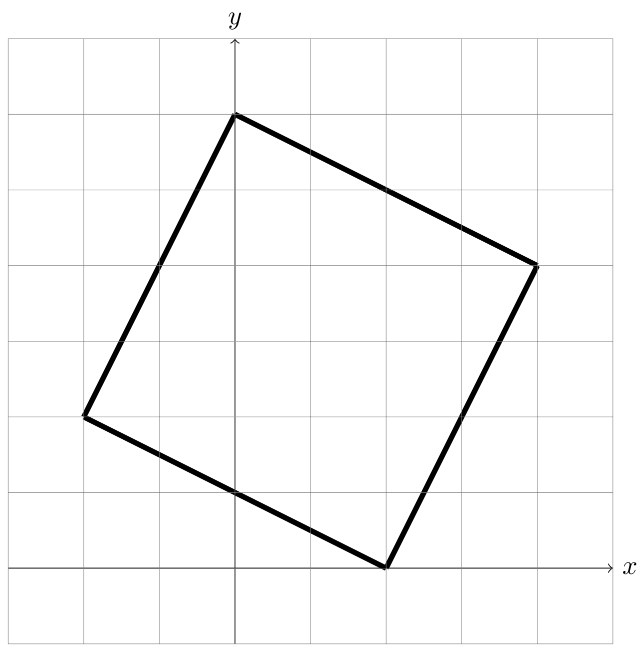
October 26, 2018

Problem 1 (26 points)

Randomly generate an ordered pair (x, y) inside the quadrilateral with vertices (-2, 2), (2, 0), (4, 4), and (0, 6). Let X be the random variable denoting the x-coordinate of the random ordered pair.

(a) (6 points) Find the probability density function $f_X(x)$ of X.

We can use the *tikz* package to plot our quadrilateral:



Here we notice that the probability of generating an ordered pair(x, y) inside the quadrilateral is proportional to the height within the quadrilateral at a given x-coordinate. We also note that we have to break up our support of X into three parts: $-2 \leq x < 0$, $0 \leq x < 2$, and $2 \leq x \leq 4$. Using these supports we see that:

$$f_X(x) = \begin{cases} k\left((2x+6) - \left(-\frac{1}{2}x+1\right)\right), & \text{if } -2 \leq x < 0; \\ k\left(\left(-\frac{1}{2}x+6\right) - \left(-\frac{1}{2}x+1\right)\right), & \text{if } 0 \leq x < 2; \\ k\left(\left(-\frac{1}{2}x+6\right) - (2x-4)\right), & \text{if } 2 \leq x \leq 4; \\ 0, & \text{otherwise.} \end{cases}$$

We can simplify this to get

$$f_X(x) = \begin{cases} k\left(\frac{5}{2}x+5\right), & \text{if } -2 \leq x < 0; \\ 5k, & \text{if } 0 \leq x < 2; \\ k\left(-\frac{5}{2}x+10\right), & \text{if } 2 \leq x \leq 4; \\ 0, & \text{otherwise.} \end{cases}$$

Now we use the fact that

$$1 = \int_{-\infty}^{\infty} f_X(x)dx$$

to find our proportionality constant, k:

$$\begin{gathered}
1 = \int_{-2}^{0} k\left(\frac{5}{2}x + 5\right) dx + \int_{0}^{2} 5k dx + \int_{2}^{4} k\left(-\frac{5}{2}x + 10\right) dx, \\
1 = k\left(\int_{-2}^{0} \frac{5}{2}x + 5dx + \int_{0}^{2} 5dx + \int_{2}^{4} -\frac{5}{2}x + 10dx\right), \\
1 = k\left(\left(\frac{5}{4}x^2 + 5x\right)\Big|_{-2}^{0} + (5x)\Big|_{0}^{2} + \left(-\frac{5}{4}x^2 + 10x\right)\Big|_{2}^{4}\right), \\
1 = k\left((-5+10) + (10) + (-20+40-(-5+20))\right) \\
k = \frac{1}{20}.
\end{gathered}$$

Note that this value of k makes sense because it is the inverse of the area of our quadrilateral. We substitute k back into our p.d.f. $f_X(x)$ to find

$$f_X(x) = \begin{cases} \frac{1}{20}\left(\frac{5}{2}x + 5\right), & \text{if } -2 \le x < 0; \\ \frac{5}{20}, & \text{if } 0 \le x < 2; \\ \frac{1}{20}\left(-\frac{5}{2}x + 10\right), & \text{if } 2 \le x \le 4; \\ 0, & \text{otherwise,} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{8}x + \frac{1}{4}, & \text{if } -2 \le x < 0; \\ \frac{1}{4}, & \text{if } 0 \le x < 2; \\ -\frac{1}{8}x + \frac{1}{2}, & \text{if } 2 \le x \le 4; \\ 0, & \text{otherwise.} \end{cases}$$

(b) (6 points) Find and *graph* the cumulative distribution function $F_X(x)$ of *X*.

Recall that we define the c.d.f. $F_X(x)$ to be the probability that $X \le x$ for some $x \in \mathbb{R}$. Since $X \sim f_X(x)$, we can express $F_X(x)$ as the integral

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt.$$

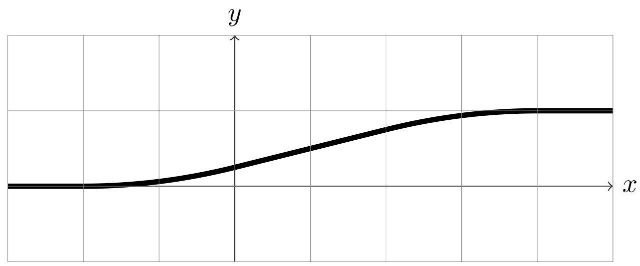
Since our $f_X(x)$ is a piecewise function this becomes:

$$F_X(x) = \begin{cases} 0, & \text{if } x < -2; \\ \int_{-2}^{x} \frac{1}{8}t + \frac{1}{4}dt, & \text{if } -2 \le x < 0; \\ \int_{-2}^{0} \frac{1}{8}t + \frac{1}{4}dt + \int_{0}^{x} \frac{1}{4}dt, & \text{if } 0 \le x < 2; \\ \int_{-2}^{0} \frac{1}{8}t + \frac{1}{4}dt + \int_{0}^{2} \frac{1}{4}dt + \int_{2}^{x} -\frac{1}{8}t + \frac{1}{2}dt, & \text{if } 2 \le x \le 4; \\ 1, & \text{if } 4 < x. \end{cases}$$

We can simplify this significantly to get:

$$F_X(x) = \begin{cases} 0, & \text{if } x < -2; \\ \frac{1}{16}(x+2)^2, & \text{if } -2 \leq x < 0; \\ \frac{1}{4}x + \frac{1}{4}, & \text{if } 0 \leq x < 2; \\ -\frac{1}{16}x(x-8), & \text{if } 2 \leq x \leq 4; \\ 1, & \text{if } 4 < x. \end{cases}$$

We can plot this using the *tikz* package:



(c) (3 points) Find *E(X)*.

From the definition of the expected value for a continuous random variable we know that

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

For our $f_X(x)$ this becomes

$$\begin{aligned} E[X] &= \int_{-2}^{0} \frac{1}{8}x^2 + \frac{1}{4}x dx + \int_{0}^{2} \frac{1}{4}x dx + \int_{2}^{4} \frac{1}{2}x - \frac{1}{8}x^3 dx, \\ &= \left(\frac{1}{24}x^3 + \frac{1}{8}x^2\right)\Big|_{-2}^{0} + \left(\frac{1}{8}x^2\right)\Big|_{0}^{2} + \left(\frac{1}{4}x^2 - \frac{1}{24}x^3\right)\Big|_{2}^{4}, \\ &= \frac{8}{24} - \frac{4}{8} + \frac{4}{8} + \frac{16}{4} - \frac{64}{24} - \frac{4}{4} = \frac{8}{24}, \\ E[X] &= 1 \end{aligned}$$

(d) (3 points) Find the 60th percentile of *X*. That is, find the cutoff *c* such that $\mathbf{Pr}\{X \leq c\} =$ 0.6.

We search for and c such that

$$0.6 = F_X(x).$$

Note that $F_X(0) = \frac{1}{4} < 0.6$ and $F_X(2) = \frac{3}{4} > 0.6$, so we search for c between 0 and 2. I.e. we have

$$0.6 = \frac{1}{4}c + \frac{1}{4},$$

$$c = 1.4.$$

Thus our 60th percentile of X is 1.4.

(e) (8 points) Let $Z = X^2$. Find the probability density function $f_Z(z)$ of Z. Clearly specify the domain of $f_Z(z)$.

Recall that our p.m.f. $f_X(x)$ is:

$$f_X(x) = \begin{cases} \frac{1}{8}x + \frac{1}{4}, & \text{if } -2 \le x < 0; \\ \frac{1}{4}, & \text{if } 0 \le x < 2; \\ -\frac{1}{8}x + \frac{1}{2}, & \text{if } 2 \le x \le 4; \\ 0, & \text{otherwise.} \end{cases}$$

Here we have that the support of X is $S_X = \{(-2,0),(0,2),(2,4)\}$. We let $A_1 = (-2,0)$, $A_2 = (0,2)$, and $A_3 = (2,4)$ represent our support of X. Note that A_1 and A_2 overlap for $Z = X^2$. We'll keep this in mind for later. We know from Theorem 1 in Class 14 that

$$f_Z(z) = f_X(g^{-1}(z)) \left| \frac{dx}{dz} \right|,$$

where $X = g^{-1}(Z) = \pm\sqrt{Z}$ and $\frac{dx}{dy} = \pm\frac{1}{2}\frac{1}{\sqrt{z}}$. We now compute $f_Z(z)$ on $A_1, A_2,$ and A_3. For A_1 we have $g^{-1}(z) = -\sqrt{z}$ and $\frac{dx}{dy} = -\frac{1}{2}\frac{1}{\sqrt{z}}$. We then have:

$$\begin{aligned} f_Z(z) &= f_X(-\sqrt{z}) \left| -\frac{1}{2}\frac{1}{\sqrt{z}} \right|, \\ &= \left(\frac{1}{8}(-\sqrt{z}) + \frac{1}{4}\right)\left(\frac{1}{2}\frac{1}{\sqrt{z}}\right), \\ &= \frac{1}{16}\left(\frac{2\sqrt{z}}{z} - 1\right). \end{aligned}$$

For A_2 we have $g^{-1}(z) = \sqrt{z}$ and $\frac{dx}{dy} = \frac{1}{2}\frac{1}{\sqrt{z}}$. We then have:

$$\begin{aligned} f_Z(z) &= f_X(\sqrt{z}) \left| \frac{1}{2}\frac{1}{\sqrt{z}} \right|, \\ &= \frac{1}{4}\left(\frac{1}{2}\frac{1}{\sqrt{z}}\right), \\ &= \frac{\sqrt{z}}{8z}. \end{aligned}$$

For A_3 we have $g^{-1}(z) = \sqrt{z}$ and $\frac{dx}{dy} = \frac{1}{2}\frac{1}{\sqrt{z}}$. We then have:

$$\begin{aligned} f_Z(z) &= f_X(\sqrt{z}) \left| \frac{1}{2}\frac{1}{\sqrt{z}} \right|, \\ &= \left(-\frac{1}{8}\sqrt{z} + \frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{\sqrt{z}}\right), \\ &= \frac{1}{16}\left(\frac{4\sqrt{z}}{z} - 1\right). \end{aligned}$$

Now we note that $f_Z(z)$ adds for A_1 and A_2 and that A_3 stands alone. We then have:

$$f_Z(z) = \begin{cases} \frac{1}{16}\left(\frac{2\sqrt{z}}{z} - 1\right) + \frac{\sqrt{z}}{8z}, & \text{if } 0 \le z \le 4; \\ \frac{1}{16}\left(\frac{4\sqrt{z}}{z} - 1\right), & \text{if } 4 < z < 16; \\ 0, & \text{otherwise.} \end{cases}$$

We simplify this to get

$$f_Z(z) = \begin{cases} \frac{1}{16}\left(\frac{4\sqrt{z}}{z} - 1\right), & \text{if } 0 \le z \le 4; \\ \frac{1}{16}\left(\frac{4\sqrt{z}}{z} - 1\right), & \text{if } 4 < z < 16; \\ 0, & \text{otherwise.} \end{cases}$$

This is equivalent to just writing:

$$f_Z(z) = \begin{cases} \frac{1}{16}\left(\frac{4\sqrt{z}}{z} - 1\right), & \text{if } 0 \le z \le 16; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 2 (8 points)

Alice tosses a coin that comes up heads with probability p_a, and Bob tosses a coin that comes up heads with probability p_b. They toss their coins alternately and independently until someone gets the first head to be the winner of a brand new car. Suppose Alice starts tossing first. Let T be the number of tosses taken in this contest.

(a) (3 points) Does T have the memoryless property? Explain.

No. Recall from class that the memoryless property is

$$Pr\{T > k + l | T > k\} = Pr\{T > l\},$$

which we can rewrite as:

$$\frac{Pr\{T > k + l\}}{Pr\{T > k\}} = Pr\{T > l\},$$

where k, l are some positive integers. If T has the memoryless property, then the property should hold for all $k, l \in \mathbb{Z}^+$. Consider the case where $k = 1$ and $l = 2$. On the left hand side of the equation we have:

$$\begin{aligned} &\frac{Pr\{T > 1 + 2\}}{Pr\{T > 1\}}, \\ &= \frac{1 - Pr\{T \le 3\}}{1 - Pr\{T \le 1\}}, \\ &= \frac{1 - (Pr\{T = 1\} + Pr\{T = 2\} + Pr\{T = 3\})}{1 - Pr\{T = 1\}}, \\ &= \frac{1 - (p_a + p_b(1 - p_a) + p_a(1 - p_a)(1 - p_b))}{1 - p_a}, \\ &= \frac{1 - (2p_a + p_b - 2p_ap_b - p_a^2 + p_a^2p_b)}{1 - p_a}, \\ &= \frac{1 - 2p_a - p_b + 2p_ap_b + p_a^2 - p_a^2p_b}{1 - p_a}. \end{aligned}$$

On the right hand of the equation we have

$$\begin{aligned} &Pr\{T > 2\}, \\ &= 1 - Pr\{t \le 2\}, \\ &= 1 - (p_a + p_b(1 - p_a)), \\ &= 1 - p_a - p_b + p_ap_b. \end{aligned}$$

Since the two sides of our memoryless property are not equivalent in at least one case, we can say that T does not have the memoryless property.

(b) (5 points) Find $E(T)$.

To answer this we need to define the p.m.f. $f_T(t)$ of T. I do this by considering the first few coin tosses and then generalize a formula from that. Here are the cases I consider and their corresponding probabilities:

$$Pr(T=1) = p_a,$$

$$Pr(T=2) = p_b(1-p_a),$$
$$Pr(T=3) = p_a(1-p_a)(1-p_b),$$
$$Pr(T=4) = p_b(1-p_a)^2(1-p_b),$$
$$Pr(T=5) = p_a(1-p_a)^2(1-p_b)^2,$$

We could keep doing this forever, but that would not be an effective use of time. Instead, we construct the following p.m.f.:

$$f_T(t) = \begin{cases} p_a, & \text{if } t=1; \\ p_b(1-p_a)^{t-1}(1-p_b)^{t-2}, & \text{if } t=2,4,6,\dots; \\ p_a(1-p_a)^{t-2}(1-p_b)^{t-2}, & \text{if } t=3,5,7,\dots; \\ 0, & \text{otherwise.} \end{cases}$$

To calculate the expected value we take

$$E[T] = \sum_{t=1}^{\infty} t f_T(t),$$
$$= p_a + \sum_{t=2,4,\dots} t p_b(1-p_a)^{t-1}(1-p_b)^{t-2} + \sum_{t=3,5,\dots} t p_a(1-p_a)^{t-2}(1-p_b)^{t-2}.$$

This is the expected value. The rest of the work from here on is to simplify this expression. First, let $P = (1-p_a)(1-p_b)$. This yields:

$$E[T] = p_a + p_b(1-p_a)\sum_{t=2,4,\dots} tP^{t-2} + p_a \sum_{t=3,5,\dots} tP^{t-2}.$$

Now let's simplify this in parts. We begin with

$$p_b(1-p_a)\sum_{t=2,4,\dots} tP^{t-2} = p_b(1-p_a)\left(2P^0 + 4P^2 + 6P^4 + \dots\right),$$

$$= 2p_b(1-p_a)\sum_{t=0}^{\infty}(t+1)P^{2t},$$
$$= \frac{2p_b(1-p_a)}{(P^2-1)^2}.$$

We now simplify

$$p_a \sum_{t=3,5,\dots} tP^{t-2} = p_a(3P + 5P^2 + 7P^5 + \dots),$$

$$= p_a P \sum_{t=0}^{\infty}(2t+3)P^{2t},$$

$$= p_a P\left(\frac{3-P^2}{(P^2-1)^2}\right).$$

We can put all these components together to get

$$E[T] = p_a + \frac{2p_b(1-p_a)}{(P^2-1)^2} + p_a P\left(\frac{3-P^2}{(P^2-1)^2}\right),$$

$$= p_a + \frac{1}{(P^2-1)^2}(2p_b - 2p_a p_b + 3p_a P - p_a P^3).$$

I get the impression this should be much simpler than what I've found, but alas what more can I do?

Problem 3 (8 points)

Alice and her n friends [$n \geq 1$, so there are totally $(n+1)$ people] decide the following rule for sharing a cash prize totaling K dollars: Each person tosses a p-coin, $0 \leq p \leq 1$. The K dollars will be equally shared by those who toss a head. If no one tosses a head, then no one will receive any prize. Let W be the prize received by Alice. Find the probability mass function $f_W(w)$ of W and calculate $E(W)$.

We note that we can think of this as a set of *n+1* **Bernoulli** trials, with each trial having a probability of success of p. In this case, if we let Y represent the number of heads Alice and her friends toss then we can say that Y has the binomial distribution:

$$f_Y(y) = \binom{n+1}{y} p^y (1-p)^{n+1-y}$$

This tells us the probability of getting y heads in $n+1$ trials. Since Alice's payoff is the K dollars divided by the number of heads that are tossed (or 0 if no heads are tossed), we can say that $\frac{K}{y}$ models her prize for $1 \leq y \leq n+1$. To find the $Pr\{W = w\}$ we can use:

$$w = \frac{K}{y},$$

$$y = \frac{K}{w},$$

$$Pr\{W = w\} = f_W(w) = f_Y\left(\frac{K}{w}\right) = \binom{n+1}{\frac{K}{w}} p^{\frac{K}{w}} (1-p)^{n+1-\frac{K}{w}}$$

Note that this only makes sense for certain discrete values of w. Namely, w must be one of $K, \frac{K}{2}, \frac{K}{3}, \ldots, \frac{K}{n+1}$ to be used in this equation. We can express $f_W(w)$ more formally as:

$$f_W(w) = \begin{cases} (1-p), & \text{if } w = 0; \\ \binom{n+1}{\frac{K}{w}} p^{\frac{K}{w}} (1-p)^{n+1-\frac{K}{w}}, & \text{if } w = K, \frac{K}{2}, \frac{K}{3}, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

To calculate $E[W]$ we take

$$E[W] = \sum_{w=0}^{\infty} w f_W(w),$$

$$E[W] = K\binom{n+1}{1} p(1-p)^n + \frac{K}{2}\binom{n+1}{2} p^2 (1-p)^{n-1} + \cdots + \frac{K}{n+1}\binom{n+1}{n+1} p^{n+1}$$

I'm not exactly sure how to simplify this to something nicer.

Problem 4 (8 points)

A deck consists of $m \geq 2$ cards of which exactly two are Joker cards. You randomly select one card at a time, without replacement, from a well shuffled deck until both Joker cards are selected. Let J be the number of cards drawn. Find the probability mass function $f_J(j)$ of J and calculate $E(J)$.

Here we can think of our sample space as being divided into two types of cards: We have two Joker-type cards and $m-2$ non-Joker-type cards. To find the p.m.f. $f_J(j) = Pr\{J = j\}$ we can break $Pr\{J = j\}$ into the intersection of two events: event A and event B. Event A is the event that exactly one Joker is drawn out of the first $j-1$ draws from the deck and event B is the event that a Joker is drawn on the j^{th} draw. We have:

$$Pr\{J = j\} = Pr\{A \cap B\} = Pr\{A\}Pr\{B|A\}.$$

To get $Pr\{A\}$ we choose one Joker from our two jokers and then choose $j-2$ non-jokers out of our $m-2$ non-joker cards. We then divide by the total number of ways of choosing $j-1$ cards out of m cards. That is,

$$\begin{aligned} Pr\{A\} &= \frac{\binom{2}{1}\binom{m-2}{j-1-1}}{\binom{m}{j-1}}, \\ &= 2\left(\frac{(m-2)!}{m!}\frac{(j-1)!}{(j-2)!}\frac{(m-j+1)!}{(m-j)!}\right), \\ &= 2\left(\frac{(j-1)(m-j+1)}{m(m-1)}\right). \end{aligned}$$

To get $Pr\{B|A\}$ we follow a similar process, but account for the fact that one joker and $j-2$ non-joker cards have already been picked. Doing so we see that:

$$\begin{aligned} Pr\{B|A\} &= \frac{\binom{1}{1}\binom{m-2-(j-1)}{0}}{\binom{m-(j-1)}{1}}, \\ &= \frac{1}{m-j+1}. \end{aligned}$$

We can multiply to find $f_J(j)$:

$$\begin{aligned} f_J(j) &= 2\left(\frac{(j-1)(m-j+1)}{m(m-1)}\right)\left(\frac{1}{m-j+1}\right), \\ &= \frac{2(j-1)}{m(m-1)}. \end{aligned}$$

We note that this makes sense for when $j = 2, 3, \ldots, m$. When j is not one of those integers we require that $f_J(j) = 0$. To write this more fomally, we have:

$$f_J(j) = \begin{cases} \frac{2(j-1)}{m(m-1)}, & \text{if } j = 2, 3, \ldots, m; \\ 0, & \text{otherwise.} \end{cases}$$

To calculate $E[J]$ we take

$$E[J] = \sum_{j=2}^{m} j f_J(j) = \sum_{j=2}^{m} j \frac{2(j-1)}{m(m-1)}.$$

We can simplify this:

$$
\begin{aligned}
E[J] &= \frac{2}{m(m-1)}\left[\sum_{j=2}^{m} j^2 + \sum_{j=2}^{m} j\right], \\
&= \frac{2}{m(m-1)}\left[\sum_{j=1}^{m} j^2 + \sum_{j=1}^{m} j\right], \\
&= \frac{2}{m(m-1)}\left[\frac{m(m+1)(2m+1)}{6} - \frac{m(m+1)}{2}\right], \\
&= \frac{2(m+1)}{m-1}\left[\frac{2m+1}{6} - \frac{1}{2}\right], \\
&= \frac{2(m+1)}{m-1}\left[\frac{2(m-1)}{6}\right], \\
&= \frac{2(m+1)}{3}.
\end{aligned}
$$

Thus we've found that $E[J] = \frac{2(m+1)}{3}$. This marks the end of my test!