Math 451 HW #19

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Question 1 (Textbook: 4.10)

The random pair (X, Y) has the distribution:

	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	1/12	1/6	1/12
$Y = 3$	1/6	0	1/6
$Y = 4$	0	1/3	0

(a) Show that X and Y are dependent.

From our two way table we can construct the following marginal pmfs:

$$f_X(x) = \begin{cases} \frac{1}{4}, & \text{if } \ x = 1; \\ \frac{1}{2}, & \text{if } \ x = 2; \\ \frac{1}{4}, & \text{if } \ x = 3; \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{3}, & \text{if } \ y = 2; \\ \frac{1}{3}, & \text{if } \ y = 3; \\ \frac{1}{3}, & \text{if } \ y = 4; \\ 0, & \text{otherwise.} \end{cases}$$

For X and Y to be independent we require $f_{X,Y}(x, y) = f_X(x)f_Y(y)$, for all x, y. Consider the case where $X = 1, Y = 4$. We then have

$$f_{X,Y}(x, y) = 0,$$

and

$$f_X(x)f_Y(y) = \frac{1}{4}\frac{1}{4} = \frac{1}{16} \neq 0.$$

Thus X and Y must be dependent.

(b) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

Consider the following table:

	$U = 1$	$U = 2$	$U = 3$
$V = 2$	1/12	1/6	1/12
$V = 3$	1/12	1/6	1/12
$V = 4$	1/12	1/6	1/12

Question 2 (Textbook: 4.14)

Suppose X and Y are independent random variables that are distributed by $Normal(0, 1^2)$.

(a) Find $Pr\{X^2 + Y^2 < 1\}$.

Note that here we take the double integral:

$$\int\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dxdy$$

over $\{X^2 + Y^2 < 1\}$ to get the $Pr\{X^2 + Y^2 < 1\}$. To integrate this we make the following substitutions:

$$x = rcos(\theta),$$

$$y = rsin(\theta).$$

With these substitutions our integral becomes:

$$\int_0^{2\pi} \int_0^1 \frac{1}{2\pi} e^{-\frac{1}{2}r^2} rdrd\theta,$$

$$= \int_0^1 re^{-\frac{1}{2}r^2} dr,$$

$$= -e^{-\frac{1}{2}r^2}\Big|_0^1,$$

$$= 1 - \frac{1}{\sqrt{e}}.$$

Thus we have found

$$Pr\{X^2 + Y^2 < 1\} = 1 - \frac{1}{\sqrt{e}}.$$

(b) Find $Pr\{X^2 < 1\}$, after verifying that X^2 is distributed by χ_1^2.

Note that we have shown in class that X^2 is distributed by χ_1^2, so we do not repeat our work again here. To find $Pr\{X^2 < 1\}$ we follow a similar process as in part (a) by taking the following integral:

$$Pr\{X^2 < 1\} = \int_{\{X^2<1\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx,$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

We can ask Wolfram Alpha (or whatever numerical integration program we wish to use) to compute this integral for us. Doing so we see that:

$$Pr\{X^2 < 1\} \approx 0.341.$$