Math 451 HW #16

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Question 1.

Let $f_X(x|\alpha, \beta)$ be the p.d.f. of the $Gamma(\alpha, \beta)$ distribution.

(a) Sketch $f_X(x|\alpha = 2, \beta = 1)$, $f_X(x|\alpha = 2, \beta = 3)$, and $f_X(x|\alpha = 2, \beta = 5)$ on the same plot.

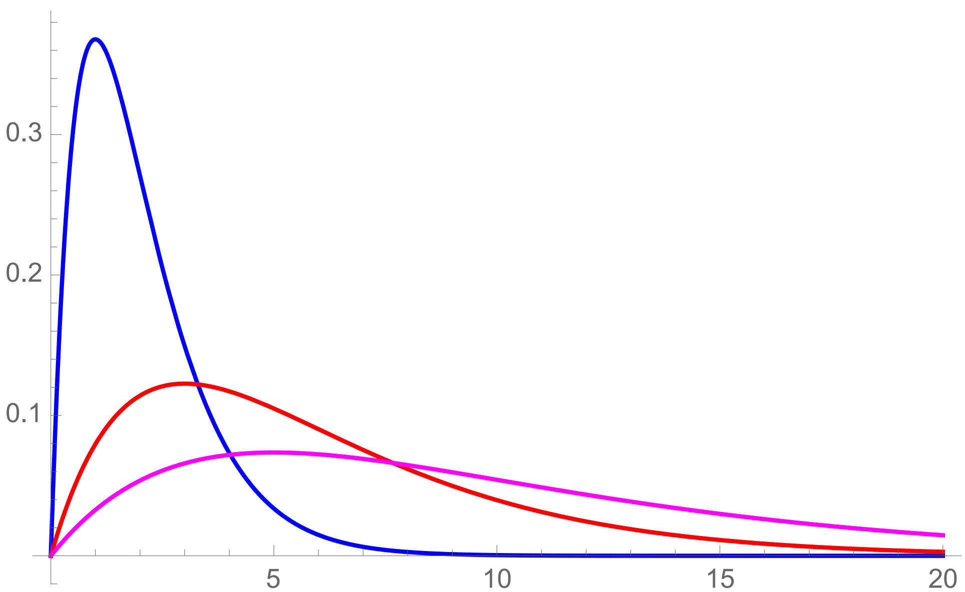
Recall that

$$f_X(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \;\; 0 < x < \infty,$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$

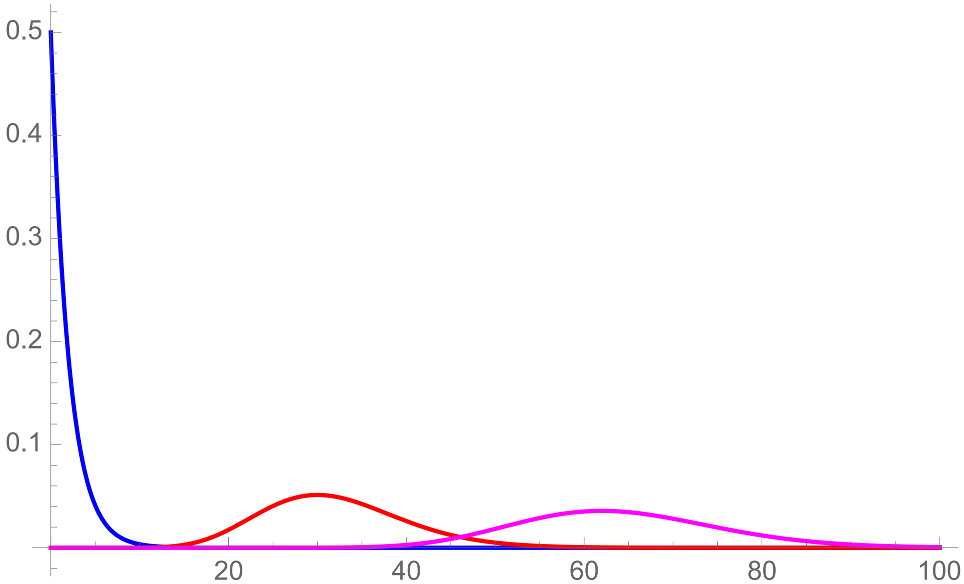
We can plot $f_X(x|\alpha = 2, \beta = 1)$, $f_X(x|\alpha = 2, \beta = 3)$, and $f_X(x|\alpha = 2, \beta = 5)$ on the same plot using mathematica:



Here note that blue represents $f_X(x|\alpha = 2, \beta = 1)$, red represents $f_X(x|\alpha = 2, \beta = 3)$, and magenta represents $f_X(x|\alpha = 2, \beta = 5)$.

(b) Sketch $f_X(x|\alpha = 1, \beta = 2)$, $f_X(x|\alpha = 16, \beta = 2)$, and $f_X(x|\alpha = 32, \beta = 2)$ on the same plot.

Here we also use mathematica to plot our probability density functions:



Here note that blue represents $f_X(x|\alpha = 1, \beta = 2)$, red represents $f_X(x|\alpha = 16, \beta = 2)$, and magenta represents $f_X(x|\alpha = 32, \beta = 2)$.

Question 2.

Let X be a continuous random variable. The *median* ζ of X satisfies

$$Pr\{X \leq \zeta\} = 0.5 \text{ and } Pr\{X \geq \zeta\} = 0.5.$$

Now let X be an exponential random variable with parameter $\beta > 0$.

(a) Find the median ζ of X in terms of β.

We seek a constant ζ such that $Pr\{X \leq \zeta\} = 0.5$. Since X is an exponential random variable we know that it has the pdf:

$$f_X(x) = \frac{1}{\beta}e^{-x/\beta}, \;\; 0 < x < \infty.$$

We can integrate this to get the cdf

$$F_X(x) = Pr\{X \leq x\} = \int_0^x \frac{1}{\beta}e^{-t/\beta}dt, \;\; 0 < x < \infty.$$

Note that both the pdf and cdf of X are both zero if x is non-positive. We can use the cdf with $F_X(\zeta) = 0.5$ to calculate the median:

$$\begin{gathered} F_X(\zeta) = 0.5 = \int_0^\zeta \frac{1}{\beta}e^{-t/\beta}dt, \;\; 0 < \zeta < \infty, \\ 0.5 = -e^{-t/\beta}\Big|_0^\zeta, \\ 0.5 = -e^{-\zeta/\beta} + 1, \\ 0.5 = e^{-\zeta/\beta}, \\ \zeta = -\beta ln(0.5) \end{gathered}$$

Thus we've found the median of X to be $\zeta = -\beta ln(0.5)$.

(b) Show that the mean of X exceeds the median of X.

Note that the mean of X is simply the expected value of X. We calculate this as

$$E[X] = \int_0^\infty x \left(\frac{1}{\beta}e^{-x/\beta}\right) dx,$$

$$= \frac{1}{\beta}\int_0^\infty xe^{-x/\beta}dx.$$

To evaluate this we use integration by parts with $u = x$ and $dv = e^{-x/\beta}dx$:

$$= \frac{1}{\beta}\left(-\beta xe^{-x/\beta}\Big|_0^\infty - \int_0^\infty -\beta e^{-x/\beta}dx\right),$$

$$= \frac{1}{\beta}\left(0 - \left(\beta^2 e^{-x/\beta}\Big|_0^\infty\right)\right),$$

$$= \frac{1}{\beta}\left(\beta^2\right),$$

$$E[X] = \beta.$$

We now show that our mean is larger than our median:

Because $1 > -ln(0.5)$ and $\beta > 0$ we know that $\beta > -\beta ln(0.5)$. Thus the mean exceeds the median for an exponential random variable.