Math 490 HW #11

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Question 1.

a. Show the work of finding σ_ψ discussed in Example 1 of this class.

Recall that we are trying to use the Importance Sampling method with Monte Carlo random simulations to get an estimate for

$$I = \int_0^{2\pi} e^{cos(x)} dx.$$

Also recall that we rewrote this integral as

$$I = \int_0^{2\pi} \frac{\psi(x)}{g(x)} dx,$$

where $\psi(x)$ was given by $g(x)e^{cos(x)}$. We used the central limit theorem to achieve the Monte Carlo Importance Sampling estimator:

$$I \approx Z_N^{IS} = \frac{\psi(Y_1) + \psi(Y_2) + \cdots + \psi(Y_N)}{N},$$

where $Y_1, Y_2, \ldots, Y_N$ are independently and identically distributed over the p.d.f. $g(\cdot)$. Remember that

$$g(y) = \begin{cases} \frac{2}{3\pi}, & \text{if } 0 \le y < \frac{\pi}{2}; \\ \frac{1}{3\pi}, & \text{if } \frac{\pi}{2} \le y < \frac{3\pi}{2}; \\ \frac{2}{3\pi}, & \text{if } \frac{3\pi}{2} \le y < 2\pi; \\ 0, & \text{otherwise.} \end{cases}$$

We then have that

$$E[Z_N^{IS}] = E[\psi(Y_1)] = I,$$
$$\sigma_\psi^2 = Var[\psi(Y)] = E[\psi(Y)^2] - E[\psi(Y)]^2,$$
$$\sigma_\psi^2 = E\left[\frac{e^{2cos(Y)}}{g(Y)^2}\right] - I^2.$$

We can compute $E\left[\frac{e^{2cos(Y)}}{g(Y)^2}\right]$

$$= \int_0^{2\pi} (\psi(Y))^2 g(y) dy,$$
$$= \int_0^{2\pi} \frac{1}{g(y)} e^{2cos(y)} dy,$$

$$= \int_0^{\frac{\pi}{2}} \frac{3\pi}{2} e^{2cos(y)} dy + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3\pi e^{2cos(y)} dy + \int_{\frac{3\pi}{2}}^{2\pi} \frac{3\pi}{2} e^{2cos(y)} dy \approx 72.5612$$

That is, σ_ψ^2 is given by

$$\sigma_\psi^2 \approx 72.5612 - (7.9549)^2 \approx 9.2803.$$

Thus we have

$$\sigma_\psi \approx 3.0464.$$

b. Show the work of finding the c.d.f. $G(\cdot)$ and the inverse $G^{-1}(\cdot)$ discussed in Example 1 of this class.

We find the c.d.f. $G(\cdot)$ by integrating our p.d.f. $g(\cdot)$ from $-\infty$ to x. Although our p.d.f. $g(\cdot)$ is discontinuous, each component is linear, which makes integration rather trivial. To be explicit, however, we should note the following:

$$\int_0^{\frac{\pi}{2}} \frac{2}{3\pi} dt = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{3\pi} dt = \int_{\frac{3\pi}{2}}^{2\pi} \frac{2}{3\pi} dt = \frac{1}{3}.$$

We can use these results to calculate $G(\cdot)$:

$$G(x) = \begin{cases} 0, & \text{if } x < 0; \\ \int_0^x \frac{2}{3\pi} dt, & \text{if } 0 \leq x < \frac{\pi}{2}; \\ \frac{1}{3} + \int_{\frac{\pi}{2}}^x \frac{1}{3\pi} dt, & \text{if } \frac{\pi}{2} \leq x < \frac{3\pi}{2}; \\ \frac{2}{3} + \int_{\frac{3\pi}{2}}^x \frac{2}{3\pi} dt, & \text{if } \frac{3\pi}{2} \leq x < 2\pi; \\ 1, & \text{if } x > 2\pi. \end{cases}$$

Evaluating the remaining integrals, we see that

$$G(x) = \begin{cases} 0, & \text{if } x < 0; \\ \frac{2}{3\pi} x, & \text{if } 0 \leq x < \frac{\pi}{2}; \\ \frac{1}{3} + \frac{1}{3\pi}(x - \frac{\pi}{2}), & \text{if } \frac{\pi}{2} \leq x < \frac{3\pi}{2}; \\ \frac{2}{3} + \frac{2}{3\pi}(x - \pi), & \text{if } \frac{3\pi}{2} \leq x < 2\pi; \\ 1, & \text{if } x > 2\pi. \end{cases}$$

Thus we have found $G(\cdot)$. We now seek $G^{-1}(\cdot)$, which we can do by setting $y = G(x)$ and solving for y in each section. Doing this we see that $G^{-1}(\cdot)$ is given by:

$$G^{-1}(y) = \begin{cases} \frac{3\pi}{2} y, & \text{if } 0 \leq y < \frac{1}{3}; \\ \frac{\pi}{2} + 3\pi(y - \frac{1}{3}), & \text{if } \frac{1}{3} \leq y < \frac{2}{3}; \\ \pi + \frac{3}{2\pi}(y - \frac{2}{3}), & \text{if } \frac{2}{3} \leq y \leq 1; \end{cases}$$

Question 2.

a. If sample size $N = 100$, what is the mean and the standard deviation of the importance sampling estimator Z_{100}^{IS} discussed in Example 1 of this class?

As we know, our mean does not change with our sample size. Thus our mean, $E\left[Z_{100}^{IS}\right]$, is given by

$$E\left[Z_{100}^{IS}\right] = E\left[Z_1^{IS}\right] = I \approx 7.9549.$$

We also know that our standard deviation decreases by a factor of the square root of our sample size. Thus when $N = 100$ our standard deviation will be a tenth of the standard deviation of σ_ψ,

$$\sigma_{100} = SD\left[Z_{100}^{IS}\right] = \frac{1}{10}\sigma_\psi \approx 0.30464.$$

b. With sample size $N = 100$, use R to simulate 1024 importance sampling estimates of I. Make a histogram of those 1024 importance sampling estimates, and report the mean and the standard deviation of those 1024 estimates

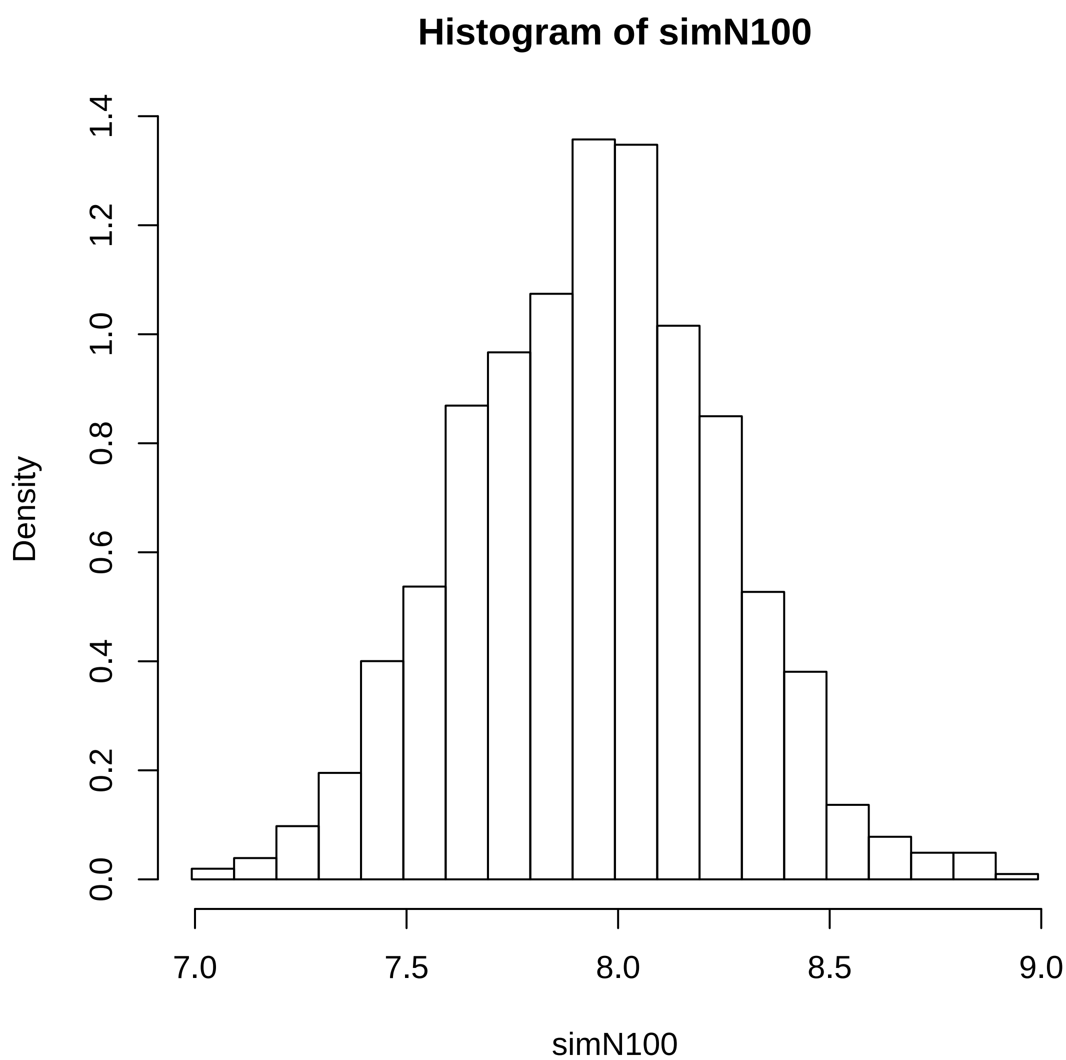
```
gpdf <- function(x) {  # define p.d.f. g(.) #
    ifelse((x >= 0 & x < (pi/2)), 2/(3*pi),
      ifelse((x >= (pi/2) & x < (3*pi/2)), 1/(3*pi),
        ifelse((x >= (3*pi/2) & x <= (2*pi)), 2/(3*pi), 0
        )
      )
    )
}

Ginv <- function(y) {  # define c.d.f. inverse G^(-1)(.) #
    ifelse(y < 0, 0,
      ifelse((y >= 0 & y < 1/3), 3*pi*y/2,
        ifelse((y >= 1/3 & y < 2/3), 3*pi*y-pi/2,
          ifelse((y >= 2/3 & y < 1), 3*pi*y/2 + pi/2, 2*pi
          )
        )
      )
    )
}

cdfInv <- function(n) {  # CDF inversion sampling #
    U <- runif(n);
    Ginv(U);
}

clt <- function(sam, rep){
    obs <- NULL;
    for (i in 1:rep){
        y <- cdfInv(sam);
        psi <- exp(cos(y)) / gpdf(y);
        psibar <- mean(psi);
        obs <- c(obs, psibar);
    }
    obs;
}
```

```
simN100 = clt(100, 1024)
hist(simN100, breaks=seq(min(simN100), max(simN100)+0.1, 0.1), prob=T)
```



The mean of our simulation is

```
[1] 7.944717
```

The standard deviation is

```
[1] 0.3079704
```

These are not too far off from our calculations.

c. If sample size $N = 100$, how likely is the importance sampling estimator Z_{100}^{IS} to yield an estimate of I within error ± 0.01? Use R to simulate this probability with 10000 runs.

We can run the following R code to calculate this probability:

```r
simN100 = clt(100, 10000)
length(simN100[abs(7.9549 - simN100) <= 0.01]) / length(simN100)
```

```
## [1] 0.0246
```

Thus we see that we have about a 2% chance of estimating I within ± 0.01. This is not great.

d. How large should N be such that the importance sampling estimator Z_{100}^{IC} would yield an estimate of I within error ± 0.01 with probability 95%?

Recall from previous homeworks that N is given by

$$N = \left(\frac{1.96\sigma}{\epsilon}\right)^2,$$

$$N \approx \left(\frac{1.96(3.0464)}{0.01}\right)^2 \approx 356,522.$$