Math 451 HW #15

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Question 1.

Get a *POWERBALL* ticket and read the prize tiers information on the back of the ticket. Let *X* be the prize (in dollars) of a single play.

(a) Use a table to represent the p.m.f. of *X*

After looking up the rules for the *POWERBALL* lottery game I found that the game works as follows: pick 5 numbers out of 69 for the "white" balls and pick 1 number out of 26 for the "red" ball. The hosts of the game pick 5 white balls and 1 red ball to be winning. Depending on how the numbers you picked match up you can win one of the following prizes: \$4, \$7, \$100, \$50,000, \$1,000,000, and the Jackpot.

To win \$4 you need to pick the winning red ball and either 1 or 0 of the winning white balls. The probability that this will occur is:

$$Pr\{X = \$4\} = \frac{\binom{5}{0}\binom{64}{5}}{\binom{69}{5}} * \frac{\binom{1}{1}\binom{25}{0}}{\binom{26}{1}} + \frac{\binom{5}{1}\binom{64}{4}}{\binom{69}{5}} * \frac{\binom{1}{1}\binom{25}{0}}{\binom{26}{1}} \approx 3.697\text{e}{-2}.$$

To win \$7 you need to pick 2 of the winning white balls and the winning red ball, or exactly 3 of the winning white balls. The probability that this will occur is:

$$Pr\{X = \$7\} = \frac{\binom{5}{2}\binom{64}{3}}{\binom{69}{5}} * \frac{\binom{1}{1}\binom{25}{0}}{\binom{26}{1}} + \frac{\binom{5}{3}\binom{64}{2}}{\binom{69}{5}} * \frac{\binom{1}{0}\binom{25}{1}}{\binom{26}{1}} \approx 3.151\text{e}{-3}.$$

To win \$100 you need to pick 3 of the winning white balls and the winning red ball, or exactly 4 of the winning white balls. The probability that this will occur is:

$$Pr\{X = \$100\} = \frac{\binom{5}{3}\binom{64}{2}}{\binom{69}{5}} * \frac{\binom{1}{1}\binom{25}{0}}{\binom{26}{1}} + \frac{\binom{5}{4}\binom{64}{1}}{\binom{69}{5}} * \frac{\binom{1}{0}\binom{25}{1}}{\binom{26}{1}} \approx 9.637\text{e}{-5}.$$

To win \$50,000 you need to pick 4 of the winning white balls and the winning red ball. The probability that this will occur is:

$$Pr\{X = \$50,000\} = \frac{\binom{5}{4}\binom{64}{1}}{\binom{69}{5}} * \frac{\binom{1}{1}\binom{25}{0}}{\binom{26}{1}} \approx 1.095\text{e}{-6}.$$

To win \$1,000,000 you need to pick all 5 of the winning white balls. The probability that this will occur is:

$$Pr\{X = \$1,000,000\} = \frac{\binom{5}{5}\binom{64}{0}}{\binom{69}{5}} * \frac{\binom{1}{0}\binom{25}{1}}{\binom{26}{1}} \approx 8.556\text{e}{-8}.$$

To win the jackpot you need to pick all 5 of the winning white balls and the winning red ball. The probability that this will occur is:

$$Pr\{X = Jackpot\} = \frac{\binom{5}{5}\binom{64}{0}}{\binom{69}{5}} * \frac{\binom{1}{1}\binom{25}{0}}{\binom{26}{1}} \approx 3.422\text{e}{-9}.$$

Finally, you can also not win any prize. This happens when you pick 0 of the winning white and red balls, 1 of the winning white balls, or 2 of the winning white balls. The probability that this will occur is:

$$Pr\{X=\$0\} = \frac{\binom{5}{0}\binom{64}{5}}{\binom{69}{5}} * \frac{\binom{1}{0}\binom{25}{1}}{\binom{26}{1}} + \frac{\binom{5}{1}\binom{64}{4}}{\binom{69}{5}} * \frac{\binom{1}{0}\binom{25}{1}}{\binom{26}{1}} + \frac{\binom{5}{2}\binom{64}{3}}{\binom{69}{5}} * \frac{\binom{1}{0}\binom{25}{1}}{\binom{26}{1}} \approx 9.598\text{e}{-1}.$$

We can put all of these probabilities in a table to get our p.m.f.:

$X = x$	x = 0	x = 4	x = 7	x = 100	x = 50,000	x = 1,000,000	x = Jackpot
$f_X(x)$	9.598e−1	3.697e−2	3.151e−3	9.637e−5	1.095e−6	8.556e−8	3.422e−9

(b) How much prize should be awarded for the "jackpot" in order for the game to be "fair," i.e. so that $E[X] = \$2$? [The cost of a single ticket is \$2.]

We calculate the expected value and set it equal to \$2 to solve for the size of the *Jackpot*:

$$\begin{aligned} E[X] = \$2 &= 0(9.598\text{e}{-1}) + 4(3.697\text{e}{-2}) + 7(3.151\text{e}{-3}) + 100(9.637\text{e}{-5}) \\ &\quad + 50,000(1.095\text{e}{-6}) + 1,000,000(8.556\text{e}{-8}) + Jackpot(3.422\text{e}{-9}), \end{aligned}$$

$$Jackpot = \frac{2 - (4(3.697\text{e}{-2}) + 7(3.151\text{e}{-3}) + 100(9.637\text{e}{-5}) + 50,000(1.095\text{e}{-6}) + 1,000,000(8.556\text{e}{-8}))}{3.422\text{e}{-9}},$$

$$Jackpot \approx \$490,974,868.$$

This is quite large. According to the official powerball website the current Jackpot is estimated at \$40 dollars, which is much smaller than the required Jackpot size for the game to be fair. Because of this I don't think that I will be buying a powerball ticket anytime soon.

Question 2.

Suppose we independently generate random real numbers $W_1, W_2, \ldots, W_n$ from the interval (0, 1). Find the smallest n that yields a probability greater than 0.95 of generating at least one random number exceeding 0.98.

(a) Solve for n using the binomial method.

Let Y be the number of observed random numbers greater than 0.98. Because the random numbers are uniformly distributed over (0,1) we can say that $Y \sim Binomial(n, 0.02)$. We want n such that $Pr\{Y \geq 1\} > 0.95$. We can rewrite this as

$$1 - Pr\{Y = 0\} > 0.95,$$

$$1 - \left(\binom{n}{0}(0.02)^0(0.98)^n\right) > 0.95,$$

$$0.05 > (0.98)^n,$$

$$\frac{ln(0.05)}{ln(0.98)} < n,$$

$$n = 149$$

(b) Solve for n using the Poisson approximation method.

Let X be the number of observed random numbers greater than 0.98. We can then approximate X using the Poisson distribution, $Poisson(\lambda \approx 0.02n)$. Like before, we search for $Pr\{X \geq 1\} > 0.95$, which we can rewrite as:

$$1 - Pr\{X = 0\} > 0.95,$$
$$Pr\{X = 0\} < 0.05.$$

Using the Poisson approximation we write this as

$$\frac{(0.02)^0}{0!} e^{-0.02n} < 0.05,$$
$$-0.02n < ln(0.05),$$
$$n > \frac{ln(0.05)}{-0.02},$$
$$n = 150.$$

This is close to the true value we got with our binomial method, so I'm content with the Poisson method as an approximation.