Math 451 HW #11

Maxwell Levin

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Question 1.

Find the p.d.f. of Y if $Y = e^X$ and X has the following p.d.f.

$$f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2}, \text{ for } 0 < x < \infty,$$

where σ^2 is a positive constant.

We know from class that

$$f_Y(y) = f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big|.$$

Note that in our case, $Y = g(x) = e^X$, so $X = g^{-1}(y) = ln(y)$ and $\frac{dx}{dy} = \frac{1}{y}$. We can then use this to get

$$f_Y(y) = f_X(ln(y))\Big|\frac{1}{y}\Big|,$$

$$f_Y(y) = \frac{1}{\sigma^2} ln(y) exp\left(-\frac{1}{2}\left(\frac{ln(y)}{\sigma}\right)^2\right)\frac{1}{y}.$$

Note that our range for y is $1 < y < \infty$, so this becomes

$$f_Y(y) = \begin{cases} \frac{1}{\sigma^2} ln(y) exp\left(-\frac{1}{2}\left(\frac{ln(y)}{\sigma}\right)^2\right)\frac{1}{y} & \text{if } 1 < y < \infty; \\ 0 & \text{otherwise.} \end{cases}$$

Question 2.

Suppose X has the geometric p.d.f. $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \ldots$. Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its probability mass function.

We know that $Y = \frac{X}{X+1}$ with $X \sim f_X(x)$, where $x = 0, 1, 2, \ldots$. To create a probability mass function for Y, we need to figure out what $Pr\{Y = y\}$ is. We do this by performing a substitution for Y and simplifying:

$$f_Y(y) = Pr\{Y = y\},$$

$$= Pr\left\{\frac{X}{X+1} = y\right\},$$

$$= Pr\left\{X = \frac{y}{1-y}\right\}$$

We use the definition of $f_X(x)$ to get

$$f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}.$$

Note that this makes sense when y obeys the relation $Y = \frac{X}{X+1}$, so $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$. Thus we have

$$f_X(x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}} & \text{if } y = 0, \frac{1}{2}, \frac{2}{3}, \ldots; \\ 0 & \text{otherwise.} \end{cases}$$

Question 3.

Let X have a p.d.f. $f_X(x) = \frac{2}{9}(x+1)$, $-1 \le x \le 2$. Find the p.d.f. of $Y = X^2$. Note that Theorem 2.1.8 is not directly applicable in this problem.

We follow the same process as we did in Question 1. First, we note that $Y = g(x) = X^2$, $X = \pm\sqrt{Y}$, and $\frac{dx}{dy} = \pm\frac{1}{2\sqrt{y}}$. Unlike Question 1, our $g(x)$ is not one-to-one on our interval from (-1, 2). We have to break up our support into two partitions for x: we let $A_1 = (-1, 0)$ and $A_2 = (0, 2)$ so that $g(x)$ is one-to-one on A_1 and A_2. We now compute $f_Y(y)$ on A_1 and A_2:

$$A_1 : f_Y(y) = f_X(-\sqrt{y})\left| -\frac{1}{2\sqrt{y}}\right|,$$

$$f_Y(y) = \frac{1}{9\sqrt{y}}(1 - \sqrt{y}).$$

$$A_2 : f_Y(y) = f_X(\sqrt{y})\left|\frac{1}{2\sqrt{y}}\right|,$$

$$f_Y(y) = \frac{1}{9\sqrt{y}}(1 + \sqrt{y}).$$

We now note that our overlap in y occurs when $x \in (-1, 1)$ and that the overlap is $y \in (0, 1)$ due to the fact that $Y = X^2$. Taking this into account we have:

$$f_Y(y) = \begin{cases} \frac{1}{9\sqrt{y}} & \text{if } 0 < y < 1; \\ \frac{1}{9\sqrt{y}}(1 + \sqrt{y}) & \text{if } 1 < y < 4; \\ 0 & \text{otherwise.} \end{cases}$$