Math 451 HW #20

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Question 1.

Let X and Y be the life spans (in hours) of two electronic devices, and their joint p.d.f. is given below.

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x-2y}, & \text{if } 0 < x < y < \infty; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c .

We can use the fact that

$$\int_{all X,Y} f_{X,Y}(x,y)dxdy = 1$$

to calculate c:

$$\begin{aligned}
\int_{0<x<y<\infty} ce^{-x-2y}dxdy &= \int_0^\infty \int_0^y ce^{-x}e^{-2y}dxdy, \\
&= \int_0^\infty ce^{-2y}\left(-e^{-x}\right)\Big|_0^y dy, \\
&= c\int_0^\infty \left(e^{-2y} - e^{-3y}\right)dy, \\
&= c\left(-\frac{1}{2}e^{-2y} + \frac{1}{3}e^{-3y}\right)\Big|_0^\infty, \\
&= \left(\frac{1}{2} - \frac{1}{3}\right)c.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{1}{6}c &= 1, \\
c &= 6.
\end{aligned}$$

(b) Find $\mathbf{Pr\{X + Y \leq 4\}}$.

We can find this by taking the integral

$$\int 6e^{-x-2y}dxdy$$

over the set of (x, y) such that $x + y \leq 4$. We can encode this condition into our bounds of integration as follows:

$$\begin{aligned}
&\int_0^2 \int_x^{4-x} 6e^{-x-2y}dydx, \\
&= \int_0^2 6e^{-x}\left(-\frac{1}{2}e^{-2y}\right)\Big|_x^{4-x} dx,
\end{aligned}$$

$$= \int_0^2 6e^{-x} \left(-\frac{1}{2}e^{-8+2x} - \left(-\frac{1}{2}e^{-2x} \right) \right) dx,$$

$$= 3\int_0^2 e^{-3x} - e^{x-8} dx,$$

$$= 3\left(-\frac{1}{3}e^{-3x} - e^{x-8} \right) \Big|_0^2,$$

$$= (-e^{-6} - 3e^{-6}) - (-e^{0} - 3e^{-8}),$$

$$= 1 + \frac{3}{e^8} - \frac{4}{e^6} \approx 0.991$$

Thus the probability that $Pr\{X + Y \leq 4\}$ is about 0.991.

(c) Find the marginal p.d.f. of X, and calculate E(X) and Var(X).

We can find the marginal p.d.f. of X by integrating out y:

$$f_X(x) = \int_x^\infty 6e^{-x}e^{-2y} dy,$$

$$= 6e^{-x}\left(-\frac{1}{2}e^{-2y} \right) \Big|_x^\infty,$$

$$= 3e^{-3x}.$$

Note that this only makes sense when $x > 0$. If $x < 0$ then the p.d.f. is 0.

We calculate the expected value by

$$E[X] = \int_{-\infty}^\infty x f_X(x) dx = \int_0^\infty 3xe^{-3x} dx,$$

$$= \left(3x\left(-\frac{1}{3}e^{-3x} \right) \right) \Big|_0^\infty - \int_0^\infty \left(-\frac{1}{3}e^{-3x}(3) \right) dx,$$

$$= 0 + \left(-\frac{1}{3}e^{-3x} \right) \Big|_0^\infty,$$

$$= \frac{1}{3}.$$

To calculate the variance we use the fact that

$$Var[X] = E[X^2] - E[X]^2.$$

We calculate $E[X^2]$ by

$$E[X^2] = \int_{-\infty}^\infty x^2 f_X(x) dx = \int_0^\infty 3x^2 e^{-3x} dx,$$

$$= \left(-x^2 e^{-3x} \right) \Big|_0^\infty - \int_0^\infty (-2xe^{-3x}) dx,$$

$$= \int_0^\infty (2xe^{-3x}) dx,$$

$$= \left(-\frac{2}{3}e^{-3x} \right) \Big|_0^\infty - \int_0^\infty \left(-\frac{2}{3}e^{-3x} \right) dx$$

$$= \frac{2}{3}\int_0^\infty e^{-3x}dx,$$

$$= -\frac{2}{9}e^{-3x}\Big|_0^\infty,$$

$$= \frac{2}{9}.$$

Thus the variance is:

$$Var[X] = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

(d) Find the marginal p.d.f. of Y, and calculate E(Y) and Var(Y).

We can find the marginal p.d.f. of Y by integrating out x:

$$f_Y(y) = \int_0^y 6e^{-2y}e^{-x}dx,$$

$$= 6e^{-2y}\left(-e^{-x}\right)\Big|_0^y,$$

$$= 6e^{-2y}\left(1 - e^{-y}\right).$$

Note that this only makes sense when $y > 0$. If $y < 0$ then the p.d.f. is 0.

We calculate the expected value by

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_0^\infty 6e^{-2y}\left(1 - e^{-y}\right)ydy.$$

We could integrate this manually by using integration by parts as we did in part (c), but I am running short on time so I will use Wolfram alpha to evaluate this integral instead. By integrating we see that

$$E[Y] = \frac{5}{6}.$$

To calculate the variance we take:

$$Var[Y] = E[Y^2] - E[Y]^2.$$

To get $E[Y^2]$ we need to evaluate the integral

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_0^\infty 6e^{-2y}\left(1 - e^{-y}\right)y^2dy.$$

To save time, we also use Wolfram Alpha as opposed to length integration by parts by hand. Doing so we see that

$$E[Y^2] = \frac{19}{18}.$$

Thus the variance is given by

$$Var[Y] = \frac{19}{18} - \left(\frac{5}{6}\right)^2 = \frac{13}{36}.$$

(e) Are X and Y independent? Justify your answer.

No. Recall that two random variables that are jointly distributed are called independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

For us we have

$$f_{X,Y}(x,y) = 6e^{-x-2y},$$

and

$$\begin{aligned} f_X(x)f_Y(y) &= (3e^{-3x})(6e^{-2y}(1-e^{-y})), \\ &= 18e^{-3x-2y} - 18e^{-3x-3y}, \end{aligned}$$

where $0 < x < y < \infty$. Since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ we can say that X and Y are not independent.

(f) Find $\mathbf{Cov(X,Y)}$.

We know from class notes 28 that we can calculate the covariance by

$$Cov(X,Y) = E[XY] - \mu_x\mu_y.$$

We calculate $E[XY]$ below:

$$E[XY] = \int_0^\infty \int_0^y 6xye^{-x-2y}dxdy.$$

This is another case where we could solve this using integration by parts, but due to my own time constraints I will evaluate the integral in Wolfram Alpha. Doing so we see that

$$E[XY] = \frac{7}{18}.$$

Thus the covariance is given by

$$\begin{aligned} Cov(X,Y) &= \frac{7}{18} - \left(\frac{1}{3}\right)\left(\frac{5}{6}\right), \\ Cov(X,Y) &= \frac{1}{9}. \end{aligned}$$

Since the covariance is greater than 0 we can say that X and Y are positively correlated.

(g) Find $\rho(\mathbf{X,Y})$

Recall from class notes 28 that we can calculate the correlation coefficient $\rho(X,Y)$ by

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x\sigma_y}.$$

We can use our result from (f) to get

$$\rho(X,Y) = \frac{1/9}{\sqrt{1/9}\sqrt{13/36}} = \frac{2}{\sqrt{13}}.$$

(h) Find the conditional p.d.f. of Y given $\mathbf{X = x, \forall x > 0}$

We know from class (Class notes 27) that

$$f_{Y|X}(y|X = x) = Pr\{Y = y|X = x\} = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

For us this becomes

$$f_{Y|X}(y|X = x) = \frac{6e^{-x-2y}}{3e^{-3x}} = 2e^{2x-2y}.$$

Note that this only makes sense when $0 < x < y < \infty$. Otherwise the conditional probability density function of Y given X is 0.