Math 451 HW #23

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November 19, 2018

Question 1.

There are n sets of twins, attending a party. Each twin wears and identical hat to his or her twin sibling, and there are n different kinds of hats. Each person hands their hat to a hat clerk. When the part ends, the hat clerk returns the hats completely at random. Let X be the number of people with their own hats back. Find E[X] and Var[X].

We can calculate $E[X]$ using a sum of indicators $X_1, X_2, \ldots, X_{2n}$, where

$$X_i = \begin{cases} 1, & \text{if the ith person gets their hat back;} \\ 0, & \text{otherwise.} \end{cases}$$

We caculate the expected value by:

$$\begin{aligned} E[X] &= E[X_1 + X_2 + \cdots + X_{2n}], \\ &= \frac{2}{2n}(1) + \frac{2}{2n}(1) + \cdots + \frac{2}{2n}(1), \\ E[X] &= 2. \end{aligned}$$

We now calculate the variance of X by

$$Var[X] = \sum_{i=1}^{2n} Var(X_i) + \sum_{i=1}^{2n} \sum_{j=1, i \neq j}^{2n} Cov(X_i, X_j).$$

Where we can calculate $Var(X_i)$ by:

$$\begin{aligned} Var(X_i) &= E[X_i^2] - E[X_i]^2, \\ &= \left(\frac{1}{n}(1)^2 + \left(1 - \frac{1}{n}\right)(0)^2\right) - \left(\frac{1}{n}(1) + \left(1 - \frac{1}{n}\right)(0)\right)^2, \\ Var[X_i] &= \frac{1}{n} - \frac{1}{n^2}. \end{aligned}$$

We now calculate the covariance in two parts to account for the possibility of two partygoers being twins. If persons i and j are twins then $Cov(X_i, X_j)$ is given by:

$$\begin{aligned} Cov(X_i, X_j) &= E[X_i X_j] - E[X_i]E[X_j], \\ &= \frac{2}{2n}\frac{1}{2n-1} - \frac{1}{n}\frac{1}{n}, \\ &= \frac{1}{n}\left(\frac{1}{2n-1} - \frac{1}{n}\right). \end{aligned}$$

Note that there are $2n$ twins total, so there are $2n$ twin terms in the covariance double sum. Since there are $(2n)^2$ terms total there are $(2n)^2 - 2n - 2n = 2n(2n-2)$ non-twin terms in the covariance double sum, each of which looks like:

$$Cov(X_i, X_j) = \frac{2}{2n}\frac{2}{2n-1} - \frac{1}{n}\frac{1}{n},$$

$$= \frac{1}{n}\left(\frac{2}{2n-1} - \frac{1}{n}\right).$$

We now calculate the variance by

$$\begin{aligned}
Var(X) = 2n\left(\frac{1}{n} - \frac{1}{n^2}\right) + 2n\left(\frac{1}{n}\left(\frac{1}{2n-1} - \frac{1}{n}\right)\right) + 2n(2n-2)\left(\frac{1}{n}\left(\frac{2}{2n-1} - \frac{1}{n}\right)\right), \\
= 2\left(1 - \frac{1}{n}\right) + 2\left(\frac{1}{2n-1} - \frac{1}{n}\right) + 2(2n-2)\left(\frac{2}{2n-1} - \frac{1}{n}\right), \\
2\left(1 - \frac{1}{n} + \frac{1}{2n-1} - \frac{1}{n} + \frac{4n}{2n-1} - 2 - \frac{4}{2n-1} + \frac{2}{n}\right), \\
2\left(\frac{4n-3}{2n-1} - 1\right), \\
= 4\left(\frac{n-1}{2n-1}\right).
\end{aligned}$$