Math 490 HW #21

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Question 1.

Let π be the probability distribution of the random variable denoing the minimum value obtained from rolling two fair tentrahedra. Use the transition matrix:

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

a. Find the probability distribution π.

We could do this by hand, but it would be rather tedious. Instead we ask our computer to run the following code in Python:

```
unique_sum = {}
for d1 in range(1, 5):
  for d2 in range(1, 5):
    new_sum = min(d1, d2)
    if new_sum in unique_sum:
      unique_sum[new_sum] += 1
    else:
      unique_sum[new_sum] = 1

for key in unique_sum:
  print(key, ": ", (unique_sum[key] / 16))
```

```
1 :  0.4375
2 :  0.3125
3 :  0.1875
4 :  0.0625
```

Thus we see that

$$\pi = \left[\frac{7}{16}, \frac{5}{16}, \frac{3}{16}, \frac{1}{16}\right].$$

b. Verify that Q is the transition matrix of a regular Markov Chain.

We can verify that **Q** is a regular transition matrix by raising it various powers of n until we see that all of its entries are strictly positive. We can do this in R by running the following code:

```
Q = matrix(c(1/2, 1/2, 0, 0,
             1/2, 0, 1/2, 0,
             0, 1/2, 0, 1/2,
             0, 0, 1/2, 1/2),
           nrow=4, ncol=4, byrow=TRUE)
```

```r
Q %^% 2
```

```
     [,1] [,2] [,3] [,4]
[1,] 0.50 0.25 0.25 0.00
[2,] 0.25 0.50 0.00 0.25
[3,] 0.25 0.00 0.50 0.25
[4,] 0.00 0.25 0.25 0.50
```

We see that there are still some zeros so we try a higher number:

```r
Q %^% 3
```

```
      [,1]  [,2]  [,3]  [,4]
[1,] 0.375 0.375 0.125 0.125
[2,] 0.375 0.125 0.375 0.125
[3,] 0.125 0.375 0.125 0.375
[4,] 0.125 0.125 0.375 0.375
```

We see that **Q** has all positive entries when n = 3. Thus we have shown that **Q** is a regular transition matrix.

c. Run the Metropolis-Hastings Algorithm to simulate the Markov Chain with π as the stationary distribution. The chain may start with state 1. Make a relative frequency table of the 1024 values of the simulated chain.

We run the following code in R:

```r
stationary = c(7/16, 5/16, 3/16, 1/16)

initial <- c(1, 0, 0, 0);

metro <- function(step, initial, Q) {
    x <- sample(length(initial), 1, prob=initial);
          # chain starts with the initial state

    for (j in 1:step) {
        q.x <- Q[x, ];
        y <- sample(length(q.x), 1, prob=q.x);
        u <- runif(1);
        if ( u <= stationary[y]*Q[y,x]/(stationary[x]*Q[x,y]) ) {
            x <- y;
        }
    }
    x;
}

# In R Console:
simul <- replicate(1024, metro(500, initial, Q));
table(simul) / length(simul);
```

```
simul
         1          2          3          4 
0.43164062 0.31640625 0.18554688 0.06640625 
```

This seems pretty similar to the probability distribution π that we generated in part (a)!

d. Find the transition matrix $\mathbf{P} = [p_{x,y}]$ of the Markov Chain constructed under the Metropolis-Hastings Algorithm.

We run the following code in R to find the transition matrix:

```
P <- matrix(c(rep(0, 16)), nrow = 4, ncol=4, byrow = T);

for (x in 1:4) {
    for (y in (1:4)[-x]) {
        if (Q[x, y] > 0) {
            P[x, y] = Q[x, y] * min(
            1, (stationary[y] * Q[y,x]) / (stationary[x] * Q[x,y]));
        }
    }
}

for (x in 1:4) {
    P[x, x] = 1 - sum(P[x,]);
}

P
```

```
          [,1]      [,2]      [,3]      [,4]
[1,] 0.6428571 0.3571429 0.0000000 0.0000000
[2,] 0.5000000 0.2000000 0.3000000 0.0000000
[3,] 0.0000000 0.5000000 0.3333333 0.1666667
[4,] 0.0000000 0.0000000 0.5000000 0.5000000
```

e. Use R to verify that π is the stationary distribution of the Markov Chain constructed under the Metropolis-Hastings Algorithm.

We run the following commands in R to see what the Metropolis-Hastings Algorithm thinks our stationary distribution is:

```
trP <- t(P);  # transpose matrix P
eigenSys = eigen(trP)
fractions(eigenSys$vectors[ ,1] / sum( eigenSys$vectors[ ,1] ))
```

```
[1] 7/16 5/16 3/16 1/16
```

We see that this is exactly what we got for π.