Math 451 HW #10

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Question 1.

A continuous random variable X has the following probability density function:

$$f_X(x) = \begin{cases} k\left(2+x^3\right), & \text{if } 0 \le x \le 4; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of k.

Since $f_X(x)$ is a probability distribution function, we require

$$1 = \int_{-\infty}^{\infty} f_X(x)dx.$$

Since our $f_X(x)$ is 0 except in the range $0 \le x \le 4$, this integral becomes

$$1 = \int_0^4 k(2+x^3)dx.$$

We can now solve for k by integrating:

$$1 = \left(2kx + \frac{kx^4}{4}\right)\Big|_0^4,$$

$$1 = 8k + 64k \implies k = \frac{1}{72}.$$

We can plug our value of k back into our p.d.f. $f_X(x)$ to get

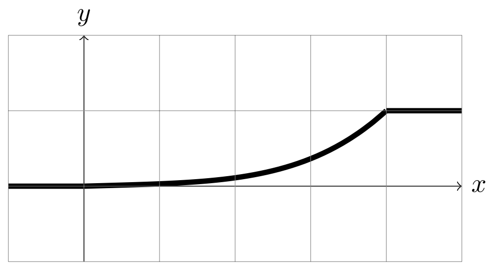
$$f_X(x) = \begin{cases} \frac{1}{36} + \frac{1}{72}x^3, & \text{if } 0 \le x \le 4; \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find the cumulative distribution function (c.d.f.) $F_X(x)$ of X. Also sketch $F_X(x)$.

We define the c.d.f. $F_X(x)$ to be the probability that $X \le x$ for some $x \in \mathbb{R}$. Since $X \sim f_X(x)$, we can express $F_X(x)$ as the integral

$$\int_{-\infty}^{x} f_X(t)dt = \begin{cases} 0 & \text{if } x < 0; \\ \int_0^x \frac{1}{36} + \frac{1}{72}t^3 dt, & \text{if } 0 \le x \le 4; \\ 1, & \text{if } x > 4. \end{cases}$$

My LaTeX skills have improved, and I can now plot functions using the *tikz* package. We sketch the c.d.f. $F_X(x)$ below



(c) Find $Pr\{X < 2|X > 1\}$.

We note that $Pr\{X < 2|X > 1\}$ is equivalent to

$$\frac{Pr\{x > 1 \cap x < 2\}}{Pr\{x > 1\}}.$$

We can further rewrite this as:
$$\frac{Pr\{x < 2\} - Pr\{x < 1\}}{1 - Pr\{x < 1\}}.$$

We now calculate $Pr\{x < 2\}$ and $Pr\{x < 1\}$ by using our c.d.f. $F_X(x)$:

$$Pr\{x < 2\} = F_X(2) = \frac{1}{36}(2) + \frac{1}{288}(2)^4 = \frac{1}{9},$$

$$Pr\{x < 1\} = F_X(1) = \frac{1}{36}(1) + \frac{1}{288}(1)^4 = \frac{1}{32}.$$

We then have
$$\frac{Pr\{x < 2\} - Pr\{x < 1\}}{1 - Pr\{x < 1\}} = \frac{\frac{1}{9} - \frac{1}{32}}{1 - \frac{1}{32}} = \frac{23}{279}.$$

(d) Find the 25th percentile of X. [Note: If x is the 25th percentile of a continuous random variable X, then x satisfies $Pr\{X \le x\} = 0.25$.] You may need to find the 25th percentile numerically.

To calculate the 25th percentile of X we need to solve the equation

$$0.25 = \frac{1}{36}x + \frac{1}{288}x^4$$

for X. We do this numerically by running the following python code that I've adapted from the Computational Physics course I took last Fall:

```
from math import fabs


def F_X(x):
  """Our function F_X(x) where we subtract 0.25 so that one side is 0,
  as required by our root-finding algorithm"""
  return x/36 + (x**4)/288 - 0.25


# We search for a root in the interval [a, b]
a = 0
```

```python
b = 4

# Stops the algorithm when the accuracy reaches six decimal places
error = 1000
delta = 1e-6

# The main loop of our algorithm
while error > delta:
    # Start x' in the middle of the interval
    xp = (a + b)/2

    if ( F_X(xp) * F_X(a) > 0):  # Increment our lower bound
        a = xp
    else:  # Decrement the upper bound
        b = xp

    # Calculate the error as the absolute difference here
    error = fabs(F_X(b) - F_X(a))

# Print the final value
print((a + b)/2)
```

```
2.667843818664551
```

Note that we have to be careful with the interval we set with this code; if there are multiple roots within the set interval, I'm not sure which one it will find. We expect our root to be somewhere between 0 and 4, so this is what we set here.

My code tells us that the 25th percentile of X is when $x \approx 2.6678$.

(e) Let $Y = \lfloor X \rfloor$, where $Y = \lfloor \cdot \rfloor$ denotes the floor function. Find the probability mass function (p.m.f.) of Y. [Note that the floor function discretizes a continuous random variable.]

We construct the following table

$Y = y$	$y = 0$	$y = 1$	$y = 2$	$y = 3$	otherwise
$f_Y(y)$	$\int_0^1 \frac{1}{36} + \frac{1}{72}x^3 dx$	$\int_1^2 \frac{1}{36} + \frac{1}{72}x^3 dx$	$\int_2^3 \frac{1}{36} + \frac{1}{72}x^3 dx$	$\int_3^4 \frac{1}{36} + \frac{1}{72}x^3 dx$	0

We evaluate these integrals to get

$Y = y$	$y = 0$	$y = 1$	$y = 2$	$y = 3$	otherwise
$f_Y(y)$	9/288	23/288	73/288	183/288	0

(f) Also find the cumulative distribution function (c.d.f.) of Y defined in part (e).

The c.d.f. of $F_Y(y)$ is defined as $Pr\{Y \leq y\}$. To calculate this, we can simply use our previous table. We do this in the following manner:

$Y = y$	$y < 0$	$y = 0$	$y = 1$	$y = 2$	$y \geq 3$
$F_Y(y)$	0	$\frac{9}{288}$	$\frac{9+23}{288}$	$\frac{9+23+73}{288}$	$\frac{9+23+73+183}{288}$

Thus our cumulative distribution function $F_Y(y)$ is:

$Y = y$	$y < 0$	$y = 0$	$y = 1$	$y = 2$	$y \geq 3$
$F_Y(y)$	0	9/288	32/288	105/288	1