Math 451 HW #4

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Question 1.

In the game of dominoes, each piece is marked with three numbers. The pieces are symmetrical so that the pairs in the tuple are not ordered (so, for example, $(2,2,6) = (6,2,2)$, but $(2,6,2) \neq (2,2,6)$). How many different pieces can be formed using the numbers $1, 2, \ldots, n$?

In the case where the dominoes are simply ordered pairs we have just two cases: a pair (i.e. $(1,1)$), or two distinct numbers (i.e. $(1,2)$). We see that there are n distinct ways to create a pair, and that there are $n(n-1)$ ways to create an ordered pair. Since our pairs are not ordered, we divide this by two. Thus the number of distinct ways to create a 2-domino piece from n numbers is

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

In the case where our dominos are constructed from three numbers we have more cases to consider. The first case is that all three numbers are the same (i.e. $(1,1,1)$). There are simply n distinct ways to do this. Our second case is that we have a pair of numbers like $(1,1,2)$ or $(2,1,1)$ (Note that these are considered to be the same piece.) The number of ways we can do this is $n(n-1)$, but to prevent overcounting we divide by 2. Our third case is that we have a set of numbers like $(1,2,1)$. There are simply $n(n-1)$ ways to do this. Our last case is that we have three distinct numbers (i.e. $(1,2,3)$). In this case we have $n(n-1)(n-2)$ ways to do this, but we divide this by 2 to prevent overcounting due to symmetry (i.e. $(1,2,3)$ is the same as $(3,2,1)$, but not $(1,3,2)$). The total number of ways to create a 3-domino piece from n numbers is

$$n + \frac{n(n-1)}{2} + n(n-1) + \frac{n(n-1)(n-2)}{2},$$

$$\frac{1}{2}n(n^2+1).$$

Question 2.

If n balls are placed at random into n cells, find the probability that exactly one cell remains empty and also find the probability that exactly two cells remain empty. We may assume that $n \geq 3$.

We first note that the number of ways to distribute n distinct balls into n distinct cells is n^n.

We first consider the case where exactly one cell is to remain empty. There are $\binom{n}{1}$ ways to pick one cell to remain empty. Since we now have $n-1$ cells and n balls, for exactly one cell to remain empty we will have to fill one cell with two balls and the rest with exactly one each. We note that there are $\binom{n}{2}$ ways to pick two balls to group together. We now treat this grouped ball as a single ball, and distribute the $n-1$ balls into the $n-1$ cells. There are $(n-1)!$ ways to do this. Thus the total number of ways to distribute n balls into n cells with exactly one empty cell is

$$\binom{n}{1}\binom{n}{2}(n-1)!.$$

Thus the probability that this will occur is given by

$$\frac{\binom{n}{1}\binom{n}{2}(n-1)!}{n^n} = \frac{\binom{n}{2}n!}{n^n}.$$

We now consider the case where exactly two cells are to remain empty. Now we have two main cases. The first case is that we have one cell with 3 balls, 2 cells with 0 balls, and $n-3$ cells with 1 ball each. We can think about this in a similar way to the last problem, but with some slight modifications. Instead of picking one cell to remain empty, we now pick 2. There are $\binom{n}{2}$ ways to do this. Now we pick 3 balls to group together instead of two balls. There are $\binom{n}{3}$ ways to do this. We now put that group set of balls into our remaining pile of balls and distribute our new set of $n-2$ balls into $n-2$ cells. There are $(n-2)!$ ways to do this. Thus there are

$$\binom{n}{2}\binom{n}{3}(n-2)!$$

ways of doing this.

Our second possible case is that we have 2 cells with 2 balls each instead of just one cell with 3 balls. In this case we have $\binom{n}{2}$ ways to pick the first group of 2 balls and $\binom{n-2}{2}$ ways to pick the second group of balls. We still have $\binom{n}{2}$ ways to pick the empty cells and $(n-2)!$ ways to distribute the groups of balls into $n-1$ cells. The number of ways of doing this is

$$\binom{n}{2}\binom{n-2}{2}\binom{n}{2}(n-2)!.$$

Thus the probability of distributing n balls into n cells with exactly two empty cells is

$$\frac{\binom{n}{2}\binom{n}{3}(n-2)! + \binom{n}{2}\binom{n-2}{2}\binom{n}{2}(n-2)!}{n^n},$$

$$= \frac{n!\left(\binom{n}{3} + \binom{n}{2}\binom{n-2}{2}\right)}{2n^n}.$$