Math 451 HW #3

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September 12, 2018

Question 1.

If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$, can A and B be disjoint?

We assume that $P(\cdot)$ is a probability set function on some set $\mathbb{S}$ and that A and B are subsets of $\mathbb{S}$. Since we are given $P(B^c) = \frac{1}{4}$ we can say that

$$P(B^c \cup (B^c)^c) = P(\mathbb{S}) = 1,$$

$$P(B^c \cup B) = 1.$$

Since B and B^c are disjoint by definition this reduces to:

$$P(B^c) + P(B) = 1,$$

$$P(B) = 1 - P(B^c) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Suppose now that A and B are disjoint. Since $P(\cdot)$ is a probability set function we know that

$$P(A \cup B) = P(A) + P(B).$$

That is,

$$P(A \cup B) = \frac{1}{3} + \frac{3}{4} = \frac{13}{12} > 1.$$

This is impossible! The largest probability permitted is 1. Thus A and B are not disjoint.

Question 2.

Let $\mathbb{N}$. denote the set of natural numbers. For each natural number, n, let $\mathbb{N}_n = \{1, 2, \ldots, n\}$. If A is a subset of $\mathbb{N}$ with its cardinality denoted by $|A|$, and the limit

$$D(A) = \lim_{n\to\infty} \frac{|A \cap \mathbb{N}_n|}{n}$$

exists, then $D(A)$ is called the density of A. Is $D(\cdot$ a probability set function?

To check if $D(\cdot)$ is a probability set function, we need to check for non-negativity, normalization, and countable additivity. We first check non-negativity. Since the cardinality of any set is always greater than or equal to zero, we see that our negativity condition is satisfied. We now check our normality condition. Consider

$$D(\mathbb{N}) = \lim_{n\to\infty} \frac{|\mathbb{N} \cap \mathbb{N}_n|}{n} = 1.$$

Thus $D(\cdot)$ satisfies the normality condition. We now check the countable additivity property. Let $A_1, A_2, \ldots$ be a set of mutually exclusive events in $\mathbb{N}$. Consider

$$D(\cup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \frac{|\cup_{i=1}^{\infty} A_i \cap \mathbb{N}_n|}{n},$$

$$
\begin{aligned}
&= \frac{|\cup_{i=1}^{\infty} A_i \cap \mathbb{N}|}{|\mathbb{N}|}, \\
&= \frac{|\cup_{i=1}^{\infty} A_i|}{|\mathbb{N}|}, \\
&= \frac{|A_i| + |A_2| + \dots|}{|\mathbb{N}|}, \\
&= \sum_{i=1}^{\infty} \frac{|A_i|}{|\mathbb{N}|}, \\
&= \sum_{i=1}^{\infty} D(A_i).
\end{aligned}
$$

Thus we see that $D(\cdot)$ satisfies the countable additivity condition. Since $D(\cdot)$ is non-negative, normalized, and satisfies the countable additivity condition, we can say that $D(\cdot)$ is a probability set function on $\mathcal{P}(\mathbb{N})$.