Math 451 HW #9

Maxwell Levin

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Question 1.

Toss a 0.6-coin four times. Let W be the random variable denoting the difference between the number of heads and the number of tails.

(a) There are $2^4 = 16$ sample points in the sample space. List all sample points in the event where $\{W = -2\}$.

Note that $\{W = -2\}$ occurs only when there are three tails and one head, assuming that W represents the number of heads minus the number of tails. We note that there are four ways of tossing one head and three tails: (H, T, T, T), (T, H, T, T), (T, T, H, T), (T, T, T, H).

(b) Find $Pr\{-2 \leq W \leq 2\}$.

Note that W is only less than -2 when $W = -4$ and that W is only larger than 2 when $W = 4$. In other words, $\{-2 \leq W \leq 2\}$ excludes the cases when only heads or only tails are tossed. Thus we can calculate $Pr\{-2 \leq W \leq 2\}$ by taking

$$Pr\{-2 \leq W \leq 2\} = 1 - Pr\{W = -4 \cup W = 4\} = 1 - \left(Pr\{W = -4\} + Pr\{W = 4\}\right),$$

$$Pr\{-2 \leq W \leq 2\} = 1 - \left((0.4)^4 + (0.6)^4\right) = 0.8448.$$

Note that we substitute 0.4 in for the probability of tossing tails and 0.6 in for the probability for tossing heads because we have a 0.6-coin.

(c) Find the p.m.f. $f_W(w)$ of W.

We do this by creating a table for each case.

$W = w$	-4	-2	0	2	4
$f_W(w)$	$1(0.4^4)$	$4(0.4^3 \cdot 0.6)$	$6(0.4^2 \cdot 0.6^2)$	$4(0.4 \cdot 0.6^3)$	$1(0.6^4)$

We simplify this to get:

$W = w$	-4	-2	0	2	4
$f_W(w)$	0.0256	0.1536	0.3456	0.3456	0.1296

Note that we also manually define $f_W(w) = 0$ when $W = w$ is not one of $\{-4, -2, 0, 2, 4\}$.

Question 2.

Randomly generate an ordered pair (x,y) inside the triangular region with vertices (0,0), (4,0), and (4,3). Let X and Y be the random variables denoting the x-coordinate and the y-coordinate respectively.

(a) Find the p.d.f. $f_X(x)$ of X.

My LaTeX skills are not yet strong enough to make a plot of our triangle, but we should still note from our setup that the probability density function $f_X(x)$ is proportional to the height of the triangle. We should also note that when x is outside of the triangle then the probability of it being picked is 0. We can write

$$f_X(x) = \begin{cases} k\left(\frac{3}{4}x + 3\right), & \text{if } 0 \leq x \leq 4; \\ 0, & \text{otherwise}, \end{cases}$$

where k is some proportionality constant relating the height of our triangle to the probability of our x-coordinate. Because $f_X(x)$ should be a probability density function, we know that the integral of $f_X(x)$ from negative infinity to positive infinity should be 1. Also note that since $f_X(x)$ is 0 everywhere except the interval from 0 to 4, this integral becomes:

$$\int_0^4 k\left(\frac{-3}{4}x + 3\right) dx = 1.$$

We integrate this to get

$$\frac{-3k}{8}x^2 + 3kx\Big|_0^4 = 6k.$$

So $k = 1$ makes $f_X(x)$ a p.d.f. Thus we have

$$f_X(x) = \begin{cases} \frac{1}{2} - \frac{1}{8}x, & \text{if } 0 \leq x \leq 4; \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find $Pr\{1 \leq X \leq 3\}$.

We can find $Pr\{1 \leq X \leq 3\}$ by taking the integral of $f_X(x)$ from 1 to 3:

$$\int_1^3 \left(\frac{1}{2} - \frac{1}{8}x\right) dx = \frac{1}{2}x - \frac{1}{4}x^2\Big|_1^3 = \frac{1}{2}.$$

Thus we have a 50% chance of picking a point (x,y) where the x-coordinate is between 1 and 3.

(c) Find the p.d.f. $f_Y(y)$ of Y.

Similar to part (a), we note that the probability density function $f_Y(y)$ is proportional to a measure of the width of the triangle: namely the distance between the y-axis and the diagonal of triangle. Solving for x in the equation $y = \frac{-3}{4}x + 3$ yields $x = \frac{-4}{3}(y - 3)$. We can write:

$$f_Y(y) = \begin{cases} k\left(\frac{-4}{3}y + 4\right), & \text{if } 0 \leq y \leq 3; \\ 0, & \text{otherwise.} \end{cases}$$

We again take the integral of $f_Y(y)$ from negative to positive infinity, noting that only the region from 0 to 3 is applicable to get

$$\int_0^3 \left(4k - \frac{4k}{3}y\right) dy = 1,$$

$$4ky - \frac{2k}{3}y^2\Big|_0^3 = 12k - 6k \implies k = \frac{1}{6}.$$

Thus we have

$$f_Y(y) = \begin{cases} \frac{2}{3} - \frac{2}{9}y, & \text{if } 0 \leq y \leq 3; \\ 0, & \text{otherwise.} \end{cases}$$