Math 451 HW #26

Maxwell Levin

December 9, 2018

Question 1

Let X and Y be two independent uniform distributions on the interval (0,2). Let $\mathbf{U = X + 2Y}$, and let $\mathbf{V = 2X - Y}$.

(a) Find the joint pdf $\mathbf{f_{U,V}(u,v)}$ of U and V. (Be sure to write the support)

We first note that:

$$X = U - 2Y,$$
$$Y = 2X - V.$$

We now find X and Y in terms of just U and V:

$$V = 2(U - 2Y) - Y,$$
$$V = 2U - 5Y,$$
$$Y = \frac{2U - V}{5},$$
$$U = X + 2(2X - V),$$
$$U = 5X - 2V,$$
$$X = \frac{U + 2V}{5}.$$

We now compute the Jacobian J of our system by:

$$J = det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix},$$

$$= det \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} = -\frac{1}{25} - \frac{4}{25} = -\frac{1}{5}.$$

We also note that since X and Y are independent $f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{4}$. Thus we know that $f_{U,V}(u,v)$ is given by

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{U+2V}{5}, \frac{2U-V}{5}\right) \left|-\frac{1}{5}\right|,$$
$$f_{U,V}(u,v) = \frac{1}{20}.$$

Note that this only makes sense when

$$0 < \frac{U+2V}{5}, \frac{2U-V}{5} < 2,$$

i.e.

$$0 < U + 2V < 10,$$
$$0 < 2U - V < 10.$$

(b) Find the marginal pdf $\mathbf{f_U(u)}$ of $\boldsymbol{U}$.

We can find this by integrating out V from the joint pdf. Note that our support for the joint pdf is the area bounded by the four lines:

$$U + 2V = 0 : V = -\frac{1}{2}U,$$
$$U + 2V = 10 : V = 5 - \frac{1}{2}U,$$
$$2U - V = 0 : V = 2U,$$
$$2U - V = 10 : V = 2U - 10.$$

Thus we have

$$f_U(u) = \begin{cases} 0, & \text{if } u \notin (0,6); \\ \int_{-\frac{1}{2}u}^{2u} \frac{1}{20} dv, & \text{if } u \in (0,2); \\ \int_{-\frac{1}{2}u}^{5-\frac{1}{2}u} \frac{1}{20} dv, & \text{if } u \in (2,4); \\ \int_{2u-10}^{5-\frac{1}{2}u} \frac{1}{20} dv, & \text{if } u \in (4,6). \end{cases}$$

$$f_U(u) = \begin{cases} 0, & \text{if } u \notin (0,6); \\ \frac{1}{8}u, & \text{if } u \in (0,2); \\ \frac{1}{4}, & \text{if } u \in (2,4); \\ \frac{3}{4} - \frac{1}{8}u, & \text{if } u \in (4,6). \end{cases}$$

(c) Find the marginal pdf $\mathbf{f_V(v)}$ of $\boldsymbol{V}$.

We now look at the area bounded by the four lines:

$$V = -\frac{1}{2}U : U = -2V,$$
$$V = 5 - \frac{1}{2}U : U = 10 - 2V,$$
$$V = 2U : U = \frac{1}{2}V,$$
$$V = 2U - 10 : U = \frac{1}{2}V + 5.$$

Note this is the same area as before, but we've solved for U instead of V so that we can look at horizontal distances for our bounds of integration. I would make a plot for this, but it's near the end of the semester and my time is short. We compute $f_V(v)$ by

$$f_U(u) = \begin{cases} 0, & \text{if } v \notin (-2,4); \\ \int_{-2v}^{\frac{1}{2}v+5} \frac{1}{20} du, & \text{if } v \in (-2,0); \\ \int_{\frac{1}{2}v}^{\frac{1}{2}v+5} \frac{1}{20} du, & \text{if } v \in (0,2); \\ \int_{\frac{1}{2}v}^{10-2v} \frac{1}{20} du, & \text{if } v \in (2,4). \end{cases}$$

$$f_U(u) = \begin{cases} 0, & \text{if } v \notin (-2,4); \\ \frac{1}{8}v + \frac{1}{4}, & \text{if } v \in (-2,0); \\ \frac{1}{4}, & \text{if } v \in (0,2); \\ \frac{1}{2} - \frac{1}{8}v, & \text{if } v \in (2,4). \end{cases}$$

(d) Find $\mathbf{Cov(U, V)}$.

Note that this is equivalent to:

$$\begin{aligned}
Cov(X+2Y, 2X-Y) &= Cov(X, 2X) + Cov(X, -Y) + Cov(2Y, 2X) + Cov(2Y, -Y), \\
&= 2Var[X] + 0 + 0 - 2Var[Y], \\
&= 2Var[X] - 2Var[X], \\
&= 0.
\end{aligned}$$

Thus the covariance is 0.

(e) Are U and V independent? Justify your answer.

No. They would be independent if

$$f_{U,V}(u, v) = f_U(u) f_V(v)$$

for all u and v in our support. However, consider the case where $2 < u < 4$ and $0 < v < 2$. We then have

$$\begin{aligned}
f_{U,V}(u, v) &= \frac{1}{20}, \\
f_U(u) f_V(v) &= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \neq \frac{1}{20}.
\end{aligned}$$

Thus they are not independent.

Question 2

Alice and Bob go fishing. On a typical fishing trip, the time X in hours until Alice catches her first fish can be modelled by an exponential distribution with mean $\beta_{\mathbf{A}}$ hours/fish, and the time Y in hours until Bob catches his first fish can be modelled by an exponential distribution with mean $\beta_{\mathbf{B}}$ hours/fish. Assume their fishing times X and Y are independent. Define Alice's waiting time W as follows:

$$W = \begin{cases} 0, & \text{if } X > Y; \\ Y - X, & \text{if } X < Y. \end{cases}$$

Find the cdf of W.

Note that

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) = \frac{1}{\beta_A} e^{-x/\beta_A} \frac{1}{\beta_B} e^{-y/\beta_B}, \; 0 < x, y < \infty$$

because X and Y are independent. Also recall from Test #3 that we found

$$Pr\{X < Y\} = \frac{\beta_A}{\beta_A + \beta_B}.$$

Thus we have

$$Pr\{X > Y\} = 1 - Pr\{X < Y\} = \frac{\beta_B}{\beta_A + \beta_B}.$$

Note that this probability is equivalent to $Pr\{W = 0\}$. We now compute $Pr\{W > 0\}$ by

$$Pr\{W > 0\} = Pr\{Y < X + w\},$$

$$= \int_0^\infty \int_x^{x+w} f_{X,Y}(x,y)dydx,$$

$$= \int_0^\infty \frac{1}{\beta_A} e^{-x/\beta_A} \int_x^{x+w} \frac{1}{\beta_B} e^{-y/\beta_B} dydx,$$

$$= \int_0^\infty \frac{1}{\beta_A} e^{-x/\beta_A} \left(-e^{-y/\beta_B} \Big|_x^{x+w} \right) dx,$$

$$== \int_0^\infty \frac{1}{\beta_A} e^{-x(1/\beta_A + 1/\beta_B)} \left(1 - e^{-w/\beta_B} \right) dx,$$

$$= \left(-\frac{1}{\beta_A(1/\beta_A + 1/\beta_B)} e^{-x(1/\beta_A+1/\beta_B)} \Big|_0^\infty \right) (1 - e^{-w/\beta_B}),$$

$$= \frac{\beta_B}{\beta_A + \beta_B} (1 - e^{-w/\beta_B}).$$

Note that our cdf of W is given by

$$F_W(w) = \begin{cases} 0, & \text{if } w < 0; \\ Pr\{W = 0\}, & \text{if } w = 0; \\ Pr\{W = 0\} + Pr\{W > 0\}, & \text{if } w > 0. \end{cases}$$

$$F_W(w) = \begin{cases} 0, & \text{if } w < 0; \\ \frac{\beta_B}{\beta_A+\beta_B}, & \text{if } w = 0; \\ \frac{\beta_B}{\beta_A+\beta_B} + \frac{\beta_B}{\beta_A+\beta_B}(1 - e^{-w/\beta_B}), & \text{if } w > 0. \end{cases}$$