Math 451 HW #13

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Question 1.

Ley Y be a geometric random variable with p.m.f.

$$Pr\{Y = y\} = f_Y(y) = p(1-p)^{y-1},\ y = 1, 2, 3, \dots$$

We have show that $E(Y) = \mu_Y = \frac{1}{p}$ and that $Var(Y) = \sigma_Y^2 = \frac{1-p}{p^2}$.

(a) Find the conditional probability $Pr\{Y$ is divisible by 4 $|Y$ is even$\}$.

We know that

$$Pr\{Y \text{ is divisible by } 4\ |Y \text{ is even}\} = \frac{Pr\{Y \text{ is divisible by } 4 \cap Y \text{ is even}\}}{Pr\{Y \text{ is even}\}},$$

and furthermore that

$$\frac{Pr\{Y \text{ is divisible by } 4 \cap Y \text{ is even}\}}{Pr\{Y \text{ is even}\}} = \frac{Pr\{Y \text{ is divisible by } 4\}}{Pr\{Y \text{ is even}\}}.$$

We now compute $Pr\{Y$ is divisible by $4\}$:

$$\begin{aligned} Pr\{Y \text{ is divisible by } 4\} &= Pr\{Y = 4\} + Pr\{Y = 8\} + Pr\{Y = 12\} + \dots, \\ &= p(1-p)^{4-1} + p(1-p)^{8-1} + p(1-p)^{12-1} + \dots, \\ &= p(1-p)^3(1 + (1-p)^4 + (1-p)^8 + \dots), \\ &= p(1-p)^3\left(\frac{1}{1-(1-p)^4}\right). \end{aligned}$$

And now we compute $Pr\{Y$ is even$\}$:

$$\begin{aligned} Pr\{Y \text{ is even}\} &= Pr\{Y = 2\} + Pr\{Y = 4\} + Pr\{Y = 6\} + \dots, \\ &= p(1-p)^{2-1} + p(1-p)^{4-1} + p(1-p)^{6-1} + \dots, \\ &= p(1-p)(1 + (1-p)^2 + (1-p)^4 + \dots), \\ &= p(1-p)\left(\frac{1}{1-(1-p)^2}\right). \end{aligned}$$

We can now use these to compute

$$\begin{aligned} \frac{Pr\{Y \text{ is divisible by } 4\}}{Pr\{Y \text{ is even}\}} &= \frac{p(1-p)^3\left(\frac{1}{1-(1-p)^4}\right)}{p(1-p)\left(\frac{1}{1-(1-p)^2}\right)}, \\ &= \frac{(1-p)^2(1-(1-p)^2)}{1-(1-p)^4}. \end{aligned}$$

(b) Compute and express $E(Y(Y-1)(Y-2))$ in terms of p.

We have

$$E(Y(Y-1)(Y-2)) = \sum_{y=3}^{\infty} y(y-1)(y-2)p(1-p)^{y-1} = p(1-p)^2 \sum_{y=3}^{\infty} y(y-1)(y-2)(1-p)^{y-3}.$$

Our knowledge of geometric series yields the following information:

$$\frac{1}{1-r} = \sum_{y=0}^{\infty} r^y,$$

$$\frac{d^3}{dr^3}\left[\frac{1}{1-r}\right] = \frac{d^3}{dr^3}\left[\sum_{y=0}^{\infty} r^y\right],$$

$$\frac{6}{(1-r)^4} = \sum_{y=3}^{\infty} y(y-1)(y-2)r^{y-3}.$$

We notice that our ratio r is $(1-p)$ and so we have:

$$\frac{6}{(1-(1-p))^4} = \sum_{y=3}^{\infty} y(y-1)(y-2)(1-p)^{y-3},$$

$$\frac{6}{p^4} = \sum_{y=3}^{\infty} y(y-1)(y-2)(1-p)^{y-3}.$$

We then have

$$p(1-p)^2 \sum_{y=3}^{\infty} y(y-1)(y-2)(1-p)^{y-3} = p(1-p)^2 \frac{6}{p^4},$$

$$= \frac{6(1-p)^2}{p^3}.$$

Thus $E(Y(Y-1)(Y-2)) = \frac{6(1-p)^2}{p^3}$.

(c) Use part (b) to compute the skewness measure $\gamma_1 = E[\frac{(Y-\mu_Y)^3}{\sigma_Y^3}]$ of Y.

Before we begin simplifying, we recall from class that $E(Y) = \mu_Y = \frac{1}{p}$, $E(Y^2) = \frac{2-p}{p^2}$, and $Var(Y) = \sigma_Y^2 = \frac{1-p}{p^2}$.

We now use the linearity property to rewrite γ_1 as:

$$\gamma_1 = \frac{1}{\sigma_Y^3} E[(Y-\mu_Y)^3],$$

$$= \frac{1}{\sigma_Y^3} E[Y^3 - 3\mu_Y Y^2 + 3\mu_Y^2 Y - \mu_Y^3],$$

$$= \frac{1}{\sigma_Y^3} \left(E[Y^3] - 3\mu_Y E[Y^2] + 3\mu_Y^3 - \mu_Y^3\right),$$

$$= \frac{1}{\sigma_Y^3} \left(E[Y(Y-1)(Y-2) + 3Y^2 - 2Y] - 3\mu_Y E[Y^2] + 2\mu_Y^3\right),$$

$$= \frac{1}{\sigma_Y^3} \left(E[Y(Y-1)(Y-2)] + 3E[Y^2] - 2\mu_Y - 3\mu_Y E[Y^2] + 2\mu_Y^3\right),.$$

We can now substitute our results from class and part (b) and simplify:

$$= \left(\frac{p^2}{1-p}\right)^{\frac{3}{2}} \left(\frac{6(1-p)^2}{p^3} + 3\left(1-\frac{1}{p}\right)\left(\frac{2-p}{p^2}\right) - \frac{2}{p} + \frac{2}{p^3}\right),$$

$$= \left(\frac{1}{1-p}\right)^{\frac{3}{2}} \left(\frac{6(1-p)^2 + (3p-1)(2-p) + 2 - 2p^2}{p^3}\right),$$

$$= \left(\frac{1}{1-p}\right)^{\frac{3}{2}} \left(p^2 - 5p + 6\right),$$

$$\gamma_1 = \frac{(p-2)(p-3)}{(1-p)^{\frac{3}{2}}}$$

We've found our skewness measure γ_1.

(d) Compute and express $E[\frac{1}{Y}]$ in terms of *p*. [Note: $E[\frac{1}{Y}] \neq \frac{1}{E(Y)}$]

Our knowledge of geometric series yields the following information:

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots,$$

$$\int \frac{1}{1-r} dr = \int 1 + r + r^2 + r^3 dr,$$

$$-log(1-r) = r + \frac{1}{2}r^2 + \frac{1}{3}r^3 + \frac{1}{4}r^4 \ldots$$

$$-log(1-r) = r\left(1 + \frac{1}{2}r + \frac{1}{3}r^2 + \frac{1}{4}r^3 \ldots\right).$$

$$\frac{-log(1-r)}{r} = 1 + \frac{1}{2}r + \frac{1}{3}r^2 + \frac{1}{4}r^3 \ldots.$$

Now that we've established this background we note that:

$$E\left[\frac{1}{Y}\right] = \sum_{y=1}^{\infty} \frac{1}{y} p(1-p)^{y-1},$$

$$= p\left(1 + \frac{1}{2}(1-p) + \frac{1}{3}(1-p)^2 + \frac{1}{4}(1-p)^3 + \ldots\right).$$

We note that our ratio here is $(1-p)$ so this becomes

$$E\left[\frac{1}{Y}\right] = p\left(\frac{-log(1-(1-p))}{1-p}\right),$$

$$E\left[\frac{1}{Y}\right] = -\frac{plog(p)}{1-p}.$$