Math 451 HW #25

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Question 1

Roll two fair dice until the pair "(6,6)" is observed. Let Y denote the sum of all values that have been rolled. Find E[Y] and Var[Y].

We can express Y as

$$Y = Y_1 + Y_2 + \cdots + Y_{x-1} + Y_x,$$

where X is the number of rolls until (6,6) is observed and Y_i is the sum of a single two-dice roll. We know that $X \sim Geometric(p = 1/36)$ and that $E[X] = 1/p = 36$ and $Var[X] = (1-p)/p^2 = 1260$.

We can find $E[Y]$ by

$$E[Y] = E[E[Y|X]] = \sum_{all x} E[Y|X] f_X(x).$$

Note that $E[Y|X]$ is given by

$$E[Y|X] = 12 + (x-1)\sum_{y=2}^{11} y f_Y(y),$$

and that

$Y = y$	$Y=2$	$Y=3$	$Y=4$	$Y=5$	$Y=6$	$Y=7$	$Y=8$	$Y=9$	$Y=10$	$Y=11$
$Pr\{Y=y\}$	1/35	2/35	3/35	4/35	5/35	6/35	5/35	4/35	3/35	2/35

We can use this table to compute $E[Y|X]$. Doing so we get $E[Y|X] = (x-1)\left(\frac{240}{35}\right) + 12$. Thus we can calculate $E[Y]$ by

$$\begin{aligned}
E[Y] &= \sum_{x=1}^{\infty}\left(\left(\frac{240}{35}\right)(x-1) + 12\right)\left(\frac{1}{36}\right)\left(\frac{35}{36}\right)^{x-1}, \\
&= \frac{240}{35}\sum_{x=1}^{\infty} x\frac{1}{36}\left(\frac{35}{36}\right)^{x-1} + \left(12 - \frac{240}{25}\right)\sum_{x=1}^{\infty}\frac{1}{36}\left(\frac{35}{36}\right)^{x-1}, \\
&= \frac{240}{35}E[X] + \left(12 - \frac{240}{25}\right)(1), \\
&= 252.
\end{aligned}$$

We now compute the variance by

$$\begin{aligned}
Var[Y] &= Var[E[Y|X]] + E[Var[Y|X]], \\
&= Var\left[\frac{240}{35}(X-1) + 12\right] + E[Var[Y_1 + Y_2 + \cdots + Y_{x-1} + 12|X]], \\
&= \left(\frac{240}{35}\right)^2 Var[X] + E[Var[Y_1|X] + Var[Y_2|X] + \cdots + Var[Y_{x-1}|X] + 0], \\
&= \left(\frac{240}{35}\right)^2 (1260) + E[(X-1)Var[Y_1|X]],
\end{aligned}$$

$$
\begin{aligned}
&= \frac{36 \cdot 240^2}{35} + E[(X-1)(E[Y_1^2|X] - E[Y_1|X]^2)],\\
&= \frac{36 \cdot 240^2}{35} + E\left[(X-1)\left(\frac{1830}{35} - \left(\frac{240}{35}\right)^2\right)\right],\\
&= \frac{36 \cdot 240^2}{35} + (E[X]-1)\left(\frac{1830}{35} - \left(\frac{240}{35}\right)^2\right),\\
&= \frac{36 \cdot 240^2}{35} + (35)\left(\frac{1830}{35} - \left(\frac{240}{35}\right)^2\right),\\
&= 59430.
\end{aligned}
$$

Question 2

An urn contains 6 identical balls numbered 1 to 6. You randomly choose one ball at a time from the urn without replacement until the ball numbered 3 is selected. Let W denote the sum of the values on the selected balls. Find E[W] and Var[W].

I ran out of time to solve this problem for today, but I'll figure it out over the weekend so that I will be prepared if this comes up on the final exam.

Question 3

Suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a uniform(0, 1) random variable.

(a) Find E[Y] and Var[Y].

Note that $E[Y|X] = np = nX$ and $Var[Y|X] = np(1-p) = nX(1-X)$. We can find $E[Y]$ by

$$
\begin{aligned}
E[Y] &= E[E[Y|X]],\\
&= E[np] = E[nX],\\
&= \int_0^1 nxf_X(x)dx,\\
&= n\int_0^1 xdx,\\
&= \frac{1}{2}n.
\end{aligned}
$$

we now calculate variance by

$$
\begin{aligned}
Var[Y] &= E[Var[Y|X]] + Var[E[Y|X]],\\
&= E[nX(1-X)] + Var[nX],\\
&= nE[X] - nE[X^2] + n^2Var[X],\\
&= \frac{1}{2}n - n\int_0^1 x^2dx + n^2\int_0^1 x^2dx - n^2(E[X])^2,\\
&= \frac{1}{2}n - \frac{1}{3}n + \frac{1}{3}n^2 - \frac{1}{4}n^2,\\
&= \frac{1}{6}n + \frac{1}{12}n^2,\\
&= \frac{1}{12}n(n+2).
\end{aligned}
$$

(b) Find the joint distribution of *X* and *Y*.

Recall that

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|X=x),$$
$$= \binom{n}{y} x^y (1-x)^{n-y}.$$

(c) Find the marginal distribution of *Y*.

We can find $f_Y(y)$ by:

$$f_Y(y) = \int_0^1 f_{Y|X}(y|X=x)dx,$$
$$= \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} dx,$$
$$= \int_0^1 \frac{n!}{y!(n-y)!} x^y (1-x)^{n-y} dx,$$
$$\frac{1}{n+1} \int_0^1 \frac{(n+1)!}{y!(n-y)!} x^y (1-x)^{n-y} dx.$$

Recall that

$$1 = \int_0^1 \frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)} x^{s-1} (1-x)^{t-1} dx.$$

Here we set $s = y+1$ to get $\Gamma(s) = y!$, $t = n-y+1$ to get $\Gamma(t) = (n-y)!$, and $s+t = n+2$ to get $\Gamma(s+t) = (n+1)!$. Doing so we see that

$$\frac{1}{n+1} \int_0^1 \frac{(n+1)!}{y!(n-y)!} x^y (1-x)^{n-y} dx = \frac{1}{n+1} \int_0^1 \frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)} x^{s-1} (1-x)^{t-1} dx,$$

I.E. that

$$f_Y(y) = \frac{1}{n+1}.$$