Math 451 HW #21

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Question 1.

You roll three fair dice. Let X be the minimum and let Y be the median.

(a) Use a two-way table to show the joint pmf of X and Y.

We construct the following table:

	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$	$X=6$
$Y=1$	$16/6^3$	0	0	0	0	0
$Y=2$	$27/6^3$	$13/6^3$	0	0	0	0
$Y=3$	$21/6^3$	$21/6^3$	$10/6^3$	0	0	0
$Y=4$	$15/6^3$	$15/6^3$	$15/6^3$	$7/6^3$	0	0
$Y=5$	$9/6^3$	$9/6^3$	$9/6^3$	$9/6^3$	$4/6^3$	0
$Y=6$	$3/6^3$	$3/6^3$	$3/6^3$	$3/6^3$	$3/6^3$	$1/6^3$

(b) Find $\mathbf{Pr\{X+Y \leq 4\}}$.

To find this probability we can take $\sum f_{X,Y}(x,y)$ over all (x,y) such that $x+y \leq 4$. Doing so we see that

$$Pr\{X+Y \leq 4\} = \frac{16}{6^3} + \frac{27}{6^3} + \frac{13}{6^3} + \frac{21}{6^3} = \frac{77}{216} \approx 0.356.$$

(c) Find the marginal pmf of X, $\mathbf{E[X]}$, and $\mathbf{Var[X]}$.

We can calculate the marginal pmf of X by taking the sum of all the values in the columns for $X=1, X=2, \ldots, X=6$. Doing so we see that

$$f_X(x) = \begin{cases} 91/6^3, & \text{if } x = 1; \\ 61/6^3, & \text{if } x = 2; \\ 37/6^3, & \text{if } x = 3; \\ 19/6^3, & \text{if } x = 4; \\ 7/6^3, & \text{if } x = 5; \\ 1/6^3, & \text{if } x = 6; \\ 0, & \text{otherwise.} \end{cases}$$

We can calculate $E[X]$ by taking the sum $\sum_1^6 x f_X(x)$. Doing so we see that

$$E[X] = 1\left(\frac{91}{6^3}\right) + 2\left(\frac{61}{6^3}\right) + 3\left(\frac{37}{6^3}\right) + 4\left(\frac{19}{6^3}\right) + 5\left(\frac{7}{6^3}\right) + 6\left(\frac{1}{6^3}\right),$$

$$E[X] = \frac{441}{216} = \frac{49}{24} \approx 2.042.$$

We now calculate $Var[X] = E[X^2] + E[X]^2$ by first taking

$$E[X^2] = 1^2\left(\frac{91}{6^3}\right) + 2^2\left(\frac{61}{6^3}\right) + 3^2\left(\frac{37}{6^3}\right) + 4^3\left(\frac{19}{6^3}\right) + 5^3\left(\frac{7}{6^3}\right) + 6^3\left(\frac{1}{6^3}\right),$$

$$E[X^2] = \frac{1183}{216} \approx 5.477.$$

We can now calculate $Var[X]$ by

$$Var[X] = \frac{1183}{216} - \left(\frac{49}{24}\right)^2 \approx 1.308.$$

(d) Find the marginal pmf of Y, E[Y], and Var[Y].

We can calculate the marginal pmf of Y by taking the sum of all the values in the rows for $Y = 1, Y = 2, \ldots, Y = 6$. Doing so we see that

$$f_Y(y) = \begin{cases} 16/6^3, & \text{if } y = 1; \\ 40/6^3, & \text{if } y = 2; \\ 52/6^3, & \text{if } y = 3; \\ 52/6^3, & \text{if } y = 4; \\ 40/6^3, & \text{if } y = 5; \\ 16/6^3, & \text{if } y = 6; \\ 0, & \text{otherwise.} \end{cases}$$

We can calculate $E[Y]$ by taking the sum $\sum_1^6 y f_Y(y)$. Doing so we see that

$$E[Y] = 1\left(\frac{16}{6^3}\right) + 2\left(\frac{40}{6^3}\right) + 3\left(\frac{52}{6^3}\right) + 4\left(\frac{52}{6^3}\right) + 5\left(\frac{40}{6^3}\right) + 6\left(\frac{16}{6^3}\right),$$

$$E[Y] = \frac{756}{216} = \frac{7}{2} = 3.5.$$

We now calculate $E[Y^2]$ by first taking

$$E[Y^2] = 1^2\left(\frac{16}{6^3}\right) + 2^2\left(\frac{40}{6^3}\right) + 3^2\left(\frac{52}{6^3}\right) + 4^3\left(\frac{52}{6^3}\right) + 5^3\left(\frac{40}{6^3}\right) + 6^3\left(\frac{16}{6^3}\right),$$

$$E[Y^2] = \frac{3052}{216} = \frac{763}{54} \approx 5.477.$$

We can now calculate $Var[Y]$ by

$$Var[Y] = E[Y^2] - E[Y]^2 = \frac{763}{54} - 3.5^2 \approx 1.879.$$

(e) Find the conditional pmf of Y given X = x, x = 1, 2, 3, 4, 5, 6.

Note that the conditional pmf of Y given $X = x$ is given by the following expression

$$f_{y|X}(y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

So then we would have

$$f_{y|X}(y|X=1) = \frac{f_{X,Y}(1,y)}{f_X(1)},$$
$$f_{y|X}(y|X=2) = \frac{f_{X,Y}(2,y)}{f_X(2)},$$
$$\vdots$$
$$f_{y|X}(y|X=6) = \frac{f_{X,Y}(6,y)}{f_X(6)},$$

as our pmf's of Y given X. This would take up a lot of space to write out by hand, so what we can instead do is write up our results in a matrix, where each column represents the distribution of Y given $X = x$ for $x = 1, 2, \ldots, 6$. We do this below:

	$f_{y\|X}(y\|X=1)$	$f_{y\|X}(y\|X=2)$	$f_{y\|X}(y\|X=3)$	$f_{y\|X}(y\|X=4)$	$f_{y\|X}(y\|X=5)$	$f_{y\|X}(y\|X=6)$
$Y=1$	16/91	0	0	0	0	0
$Y=2$	27/91	13/61	0	0	0	0
$Y=3$	21/91	21/61	10/37	0	0	0
$Y=4$	15/91	15/61	15/37	7/19	0	0
$Y=5$	9/91	9/61	9/37	9/19	4/7	0
$Y=6$	3/91	3/61	3/37	3/19	3/7	1

(f) Are X and Y independent? Justify your answer.

No. There are plenty of cases where $f_{x,Y}(x,y) \neq f_X(x)f_Y(y)$. Consider one such example where $X = 2, Y = 1$. In this case we have

$$f_{X,Y}(2,1) = 0,$$
$$f_X(2)f_Y(1) = \left(\frac{61}{6^3}\right)\left(\frac{16}{6^3}\right),$$

which are clearly not equal. Thus x and Y are not independent.

(g) Find the correlation coefficient, $\rho(\mathbf{X}, \mathbf{Y})$.

Recall that we calculate $\rho(X,Y)$ by

$$\rho(x,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y},$$

and that

$$Cov(X,Y) = E[XY] - \mu_x \mu_y.$$

To calculate $E[XY]$ we can calculate

$$E[XY] = \sum_{x=1}^{6}\sum_{y=1}^{6} xy f_{X,Y}(x,y),$$

which we can calculate by using our table in part (a). Doing so we find that

$$E[XY] = \frac{1756}{216} = \frac{439}{54} \approx 8.130,$$
$$Cov(X,Y) = \frac{439}{54} - \left(\frac{49}{24}\right)\left(\frac{7}{2}\right) \approx 0.984,$$
$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var[X]Var[Y]}} \approx 0.627.$$