Math 451 HW #17

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Question 1.

Show that the normal probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \ , \quad -\infty < x < \infty,$$

has points of inflection at $x = \mu - \sigma$ and $x = \mu + \sigma$.

To find the points of inflection we need to set the second derivative of $f(x)$ equal to zero and solve for x. I first do some prep work:

$$\frac{d}{dx}\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] = -\frac{1}{\sigma^2}(x-\mu),$$

$$\begin{aligned}
\frac{d}{dx}[f(x)] &= \frac{1}{\sigma\sqrt{2\pi}}\left(-\frac{1}{\sigma^2}(x-\mu)\right) exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \\
&= -\frac{1}{\sigma^3\sqrt{2\pi}}(x-\mu) exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \\
&= -\frac{x}{\sigma^3\sqrt{2\pi}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) + \frac{\mu}{\sigma^3\sqrt{2\pi}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).
\end{aligned}$$

Now that we have the first derivative of $f(x)$, we can take the second derivative of $f(x)$ by using the product rule on the first derivative:

$$\begin{aligned}
\frac{d^2}{dx^2}[f(x)] &= -\frac{1}{\sigma^3\sqrt{2\pi}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) + \frac{x(x-\mu)}{\sigma^5\sqrt{2\pi}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) - \frac{\mu(x-\mu)}{\sigma^5\sqrt{2\pi}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \\
&= \frac{1}{\sigma^5\sqrt{2\pi}}\left[-\sigma^2 + x(x-\mu) - \mu(x-\mu)\right] exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \\
&= \frac{1}{\sigma^5\sqrt{2\pi}}\left[x^2 - 2\mu x + \mu^2 - \sigma^2\right] exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).
\end{aligned}$$

We now set this equal to zero and solve for x to get the points of inflection:

$$\frac{d^2}{dx^2}[f(x)] = 0 = \frac{1}{\sigma^5\sqrt{2\pi}}\left[x^2 - 2\mu x + \mu^2 - \sigma^2\right] exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right),$$

$$\begin{aligned}
0 &= x^2 - 2\mu x + \mu^2 - \sigma^2, \\
\sigma^2 &= (x-\mu)^2, \\
\pm\sigma &= x - \mu, \\
x &= \mu \pm \sigma.
\end{aligned}$$

Thus we've shown that $f(x)$ has points of inflection at $x = \mu - \sigma$ and $x = \mu + \sigma$.

Question 2.

If X is normally distributed with mean μ and standard deviation σ, it can be numerically shown that

$$Pr\{\mu - 1\sigma \leq X \leq \mu + 1\sigma\} \approx 0.680 \quad (1)$$
$$Pr\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} \approx 0.950 \quad (2)$$
$$Pr\{\mu - 3\sigma \leq X \leq \mu + 3\sigma\} \approx 0.997 \quad (3)$$

(a) Compute the corresponding probabilities in (1), (2), and (3) for a continuous uniform distibution over the interval [0, 1].

Recall that a continuous random variable X that is uniformly distributed has the pdf

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } \ a \leq x \leq b; \\ 0 & \text{otherwise.} \end{cases}$$

To find the probability that $Pr\{\alpha \leq X \leq \beta\}$, where we assume that $\alpha, \beta \in [a, b]$ we can take

$$Pr\{\alpha \leq X \leq \beta\} = \int_\alpha^\beta f(x)dx = \int_\alpha^\beta \frac{1}{b-a}dx = \frac{1}{b-a}(\beta - \alpha).$$

For equation (1) this becomes:

$$Pr\{\mu - 1\sigma \leq X \leq \mu + 1\sigma\} = \frac{1}{b-a}(\mu + 1\sigma - (\mu - 1\sigma)) = \frac{2}{b-a}\sigma.$$

We can further simplify this using the fact that $\sigma = \sqrt{\frac{(b-a)^2}{12}} = \frac{\sqrt{3}(b-a)}{6}$ for a uniform distribution (from Classnotes 22):

$$Pr\{\mu - 1\sigma \leq X \leq \mu + 1\sigma\} = \frac{2}{b-a}\frac{(b-a)\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \approx 0.577 \quad (1).$$

We can do this for (2) as well:

$$Pr\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} = \frac{1}{b-a}(\mu + 2\sigma - (\mu - 2\sigma)) = \frac{4}{b-a}\sigma,$$

$$= \frac{4}{b-a}\frac{(b-a)\sqrt{3}}{6} = \frac{2\sqrt{3}}{3} \approx 1.155.$$

Note that this probability is greater than 1, which is curious. I believe this implies that at least one of our assumptions about $\alpha, \beta \in [a, b]$ is incorrect. To account for this I believe we can take the minimum of 1 and whatever number we calculate using our process above. Thus for (2) we have:

$$Pr\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} = 1 \quad (2).$$

Keeping this in mind while we calculate (3) we get:

$$Pr\{\mu - 3\sigma \leq X \leq \mu + 3\sigma\} = \frac{1}{b-a}(\mu + 3\sigma - (\mu - 3\sigma)) = \frac{6}{b-a}\sigma,$$

$$= \frac{6}{b-a}\frac{(b-a)\sqrt{3}}{6} = \sqrt{3} \approx 1.732 \implies 1 \quad (3).$$

(b) Compute the corresponding probabilities in (1), (2), and (3) for an exponential distribution with parameter β.

We know that if X is a continuous random variable that is exponentially distributed has the pdf:

$$f(x) = \frac{1}{\beta}e^{-\frac{1}{\beta}x}, \quad 0 < x < \infty.$$

We can obtain the cdf of X by integrating:

$$F(x) = \int_0^x \frac{1}{\beta}e^{-\frac{1}{\beta}t}dt = -e^{-\frac{1}{\beta}t}\Big|_0^x = 1 - e^{-\frac{1}{\beta}x}.$$

Note that we only use this if x is larger than 0. Otherwise, our cdf would be 0.

We know from homework #15 that $E[X] = \mu = \beta$ for an exponential distribution, and we know from our classnotes that $Var[X] = \beta^2$ so $\sigma = \beta$. Note that as in part (a), we can get the probability that $\{\alpha \leq X \leq \beta\}$ by integrating our pdf from α to β. We do this in a sligtly smarter way this time by noticing that in equation (1), $\alpha = \mu - 1\sigma = \beta - \beta = 0$ and that the α for both (2) and (3) will be negative. Thus we can simply use our cdf $F(x)$ with the $\mu + 1\sigma, \mu + 2\sigma, \mu + 3\sigma$ as our x to obtain the probabilities we search for. This is:

$$Pr\{\mu - 1\sigma \leq X \leq \mu + 1\sigma\} = F(\mu + 1\sigma) = F(2\beta) = 1 - e^{-\frac{1}{\beta}(2\beta)} = 1 - \frac{1}{e^2} \approx 0.865. \quad (1)$$

$$Pr\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} = F(\mu + 2\sigma) = F(3\beta) = 1 - e^{-\frac{1}{\beta}(3\beta)} = 1 - \frac{1}{e^3} \approx 0.950. \quad (2)$$

$$Pr\{\mu - 3\sigma \leq X \leq \mu + 3\sigma\} = F(\mu + 3\sigma) = F(4\beta) = 1 - e^{-\frac{1}{\beta}(4\beta)} = 1 - \frac{1}{e^4} \approx 0.982. \quad (3)$$