Math 451 HW #6

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Question 1.

Verify the following identities for $n \geq 2$.

a) $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$**.**

Recall that for some $x, y \in \mathbb{R}$ we have

$$\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = (x+y)^n.$$

Consider $x = -1, y = 1$. In this case we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1)^{n-k} = (-1+1)^n,$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

b) $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$**.**

Take

$$\frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}y^{n-k}.$$

and

$$\frac{d}{dx}(x+y)^n = n(x+y)^{n-1}.$$

Because we know that

$$\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = (x+y)^n,$$

we also know that the derivatives of each side with respect to x are also equal, i.e.

$$\sum_{k=1}^{n} k\binom{n}{k}x^{k-1}y^{n-k} = n(x+y)^{n-1}.$$

Consider $x = 1, y = 1$. We then have

$$\sum_{k=1}^{n} k\binom{n}{k}(1)^{k-1}(1)^{n-k} = n(1+1)^{n-1},$$

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1},$$

c) $\sum_{k=1}^{n}(-1)^{k+1}k\binom{n}{k} = 0$**.**

Recall from (b) that

$$\sum_{k=1}^{n} k\binom{n}{k}x^{k-1}y^{n-k} = n(x+y)^{n-1}.$$

Consider $x=-1, y=1$. We then have

$$\sum_{k=1}^{n} k\binom{n}{k}(-1)^{k-1}(1)^{n-k} = n(-1+1)^{n-1},$$

$$\sum_{k=1}^{n} k\binom{n}{k}(-1)^{k-1} = 0.$$

Since $(-1)^{k-1} = (-1)^{k+1}$ we then see our identity holds:

$$\sum_{k=1}^{n} k\binom{n}{k}(-1)^{k+1} = 0.$$

Question 2.

For $n \geq 1$, guess a formula for the following sum, and prove your guess.

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Consider $(x+1)^{2n}$. We know from Theorem 1 on class notes 5 that

$$(x+1)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}x^k.$$

We also know that

$$((x+1)^n)^2 = (x+1)^{2n},$$

and thus that

$$\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)^2 = \sum_{k=0}^{2n}\binom{2n}{k}x^k.$$

We can expand this to see that

$$\left(\binom{n}{0}x^0 + \binom{n}{1}x^1 + \cdots + \binom{n}{n}x^n\right)^2 = \binom{2n}{0}x^0 + \binom{2n}{1}x^1 + \cdots + \binom{2n}{n}x^2 + \cdots + \binom{2n}{2n}x^{2n}.$$

Note that because the two sides are equal it must be true that the x^n terms on both sides are equal to one another. Thus we have

$$\left(\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}\right)x^n = \binom{2n}{n}x^n.$$

We can rearrange this to get:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$