R Notebook

```r
library(phaseR)
library(deSolve)
library(mosaic)
```

Problem #1

Show that the following systems are equivalent:

$$
\begin{aligned}
\dot{x} &= x(1-x^2-y^2)-y \qquad & \dot{r} &= r(1-r^2) \\
\dot{y} &= y(1-x^2-y^2)+x \qquad & \dot{\theta} &= 1
\end{aligned}
$$

$$
\begin{aligned}
r^2 &= x^2+y^2 \quad \rightarrow \quad r\dot{r} = x\dot{x}+y\dot{y} \\
r\dot{r} &= x[x(1-x^2-y^2)-y]+y[y(1-x^2-y^2)+x] \\
&= x^2(1-x^2-y^2)-yx+y^2(1-x^2-y^2)+yx \\
&= (x^2+y^2)(1-x^2-y^2) \\
&= (x^2+y^2)(1-(x^2+y^2)) \\
&= r^2(1-r^2) \\
\therefore \dot{r} &= r(1-r^2)
\end{aligned}
$$

$$
\begin{aligned}
\tan(\theta) &= \frac{y}{x} \quad \rightarrow \quad \sec^2(\theta)\dot{\theta} = \frac{x\dot{y}-y\dot{x}}{x^2} \\
\dot{\theta} &= \frac{x(y(1-x^2-y^2)+x)-y(x(1-x^2-y^2)-y))\cos^2(\theta)}{x^2} \\
&= \frac{1}{r^2}\left[xy(1-(x^2+y^2))+x^2-xy(1-(x^2+y^2))+y^2\right] \\
&= \frac{x^2+y^2}{r^2} \\
\therefore \dot{\theta} &= 1
\end{aligned}
$$

Problem #2

Convert the following system to cartesian:

$$
\begin{aligned}
\dot{r} &= r(1-r^2)+\mu r\cos(\theta) \\
\dot{\theta} &= 1
\end{aligned}
$$

$$
\begin{aligned}
x &= r\cos(\theta) \quad \rightarrow \quad \dot{x} = \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta) \\
y &= r\sin(\theta) \quad \rightarrow \quad \dot{y} = \dot{r}\sin(\theta) - r\dot{\theta}\sin(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= \Big[r(1-r^2) + \mu x\Big]\cos(\theta) - r\sin(\theta) \\
&= x(1-r^2) + \mu x\cos(\theta) - y \\
&= \boxed{x(1-x^2-y^2) + \frac{\mu x^2}{x^2+y^2} - y} \\
\dot{y} &= \Big[r(1-r^2) + \mu x\Big]\sin(\theta) - r\cos(\theta) \\
&= y(1-r^2) + \mu x\sin(\theta) - x \\
&= \boxed{y(1-x^2-y^2) + \frac{\mu xy}{x^2+y^2} - x}
\end{aligned}
$$

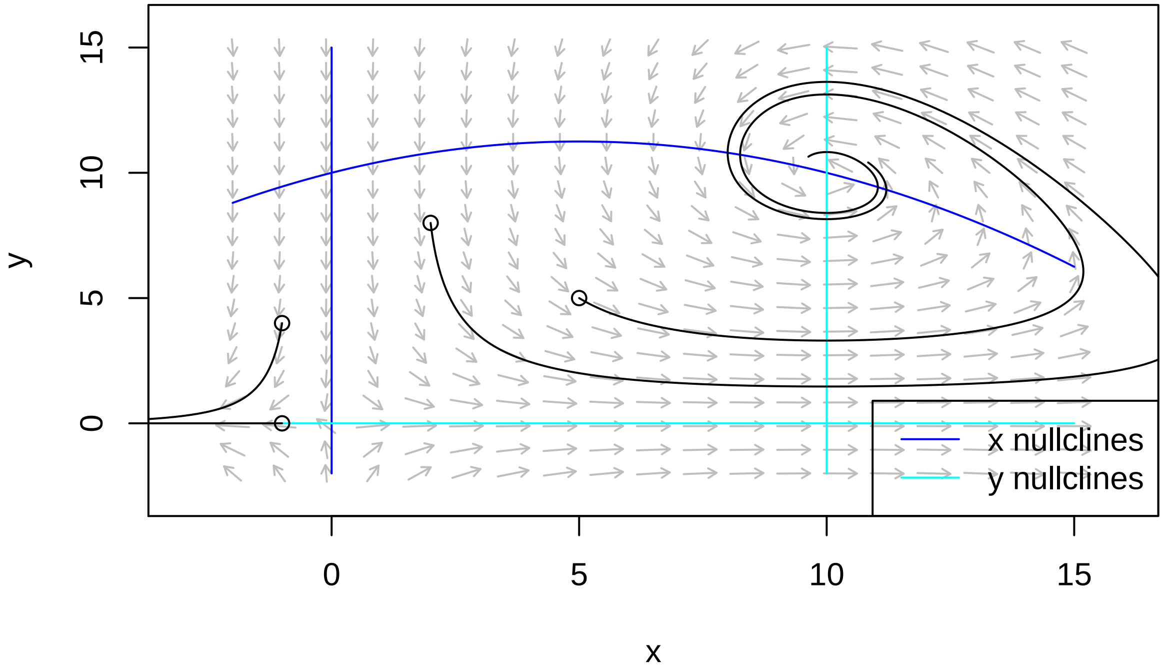
Problem #3

System A

Algebraically solving for equilibrium gives $y = 0, x = 10$ as y nullcines and $x = 0, y = \frac{-1}{20}x^2 - \frac{1}{2}x + 10$ nullclines. Intersections of nullclines occur at $(0,0)$ and $(10,10)$.

```
a <- function(t,state,parameters){
  with(as.list(c(state,parameters)),{
    x <- state[1];y <- state[2]
    dx <- x*(1-x/20)-(x*y)/(x+10)
    dy <- 3*y*(x/(x+10)-1/2)
    list(c(dx,dy))
  })}
a_flowField  <- flowField(a,xlim = c(-2,15), ylim = c(-2,15),
                          parameters = c(),points = 19,add = FALSE)
a_nullclines <- nullclines(a,xlim = c(-2,15),ylim = c(-2,15),
                           parameters = c(a=0), points = 500)
state <- matrix(c(-1,0,5,5,2,8,-1,4),4, 2, byrow = TRUE)
a_trajectory <- trajectory(a,y0 = state,tlim = c(0, 20),parameters = c(a=0),add=TRUE)
```

```
## Note: col has been reset as required
```



System B

Algebraically solving for equilibrium gives $y = 0, x = 5, x = -15$ as y nullclines and $x = 0, y = -\frac{3}{2}\left(\frac{1}{40}x^2 + \frac{25}{40}x - 15\right)$ x nullclines. Intersections of nullclines occur at $(0, 0)$ and $(-15, 0)$.

$$3x(1 - \frac{x}{40}) = \frac{2xy}{x + 15}$$

$$3x(1 - \frac{x}{40})(x + 15) = 2xy$$

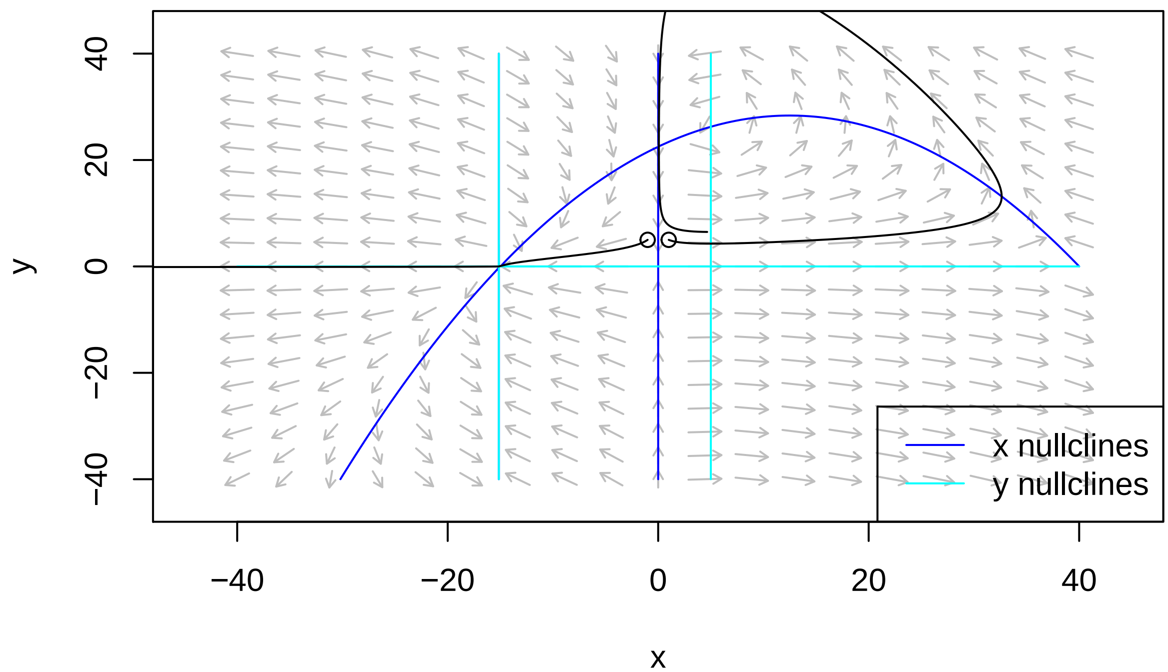
$$3x(x + 15 - \frac{x^2}{40} - \frac{15x}{40}) = 2xy$$

$$y = \boxed{-\frac{3}{2}\left(\frac{1}{40}x^2 + \frac{25}{40}x - 15\right)}$$

```
b <- function(t,state,parameters){
  with(as.list(c(state,parameters)),{
    x <- state[1];y <- state[2]
    dx <- 3*x*(1-x/40)-(2*x*y)/(x+15)
    dy <- y*(2*x/(x+15)-1/2)
    list(c(dx,dy))
  })}
b_flowField  <- flowField(b,xlim = c(-40,40), ylim = c(-40,40),
                          parameters = c(),points = 19,add = FALSE)
b_nullclines <- nullclines(b,xlim = c(-40,40),ylim = c(-40,40),
                           parameters = c(a=0), points = 500)
```

```r
state <- matrix(c(-1,5,1,5),2, 2, byrow = TRUE)
b_trajectory <- trajectory(b,y0 = state,tlim = c(0, 10),parameters = c(a=0),add=TRUE)
```

```
## Note: col has been reset as required
```



System C

Algebraically solving for equilibrium gives $y = 0, x = 30$ as y nullcines and $x = 0, y = -(\frac{1}{20}x - 20)$ nullclines. Intersections of nullclines occur at $(0, 0)$ and $(30, 21.5)$.

$$
\begin{aligned}
x(1 - \frac{x}{20}) &= \frac{xy}{x+20} \\
x(1 - \frac{x}{20})(x+20) &= xy \\
x(x + 20 - \frac{x^2}{20} - x) &= xy \\
y &= \boxed{-(\frac{1}{20}x - 20)}
\end{aligned}
$$

Unfortunately, I was not able to get the phaseR plots to work for this system. I kept getting an infinite loop error.

```r
# c <- function(t,state,parameters){
#   with(as.list(c(state,parameters)),{
#     x <- state[1];y <- state[2]
#     dx <- x*(1-x/20)-(x*y)/(x+20)
```

```
#     dy <- 3*y*(x/(x+10)-3/4)
#     list(c(dx,dy))
#   })}
# c_flowField  <- flowField(c,xlim = c(-1,1), ylim = c(-1,1),
#                           parameters = c(),points = 19,add = FALSE)
# c_nullclines <- nullclines(c,xlim = c(-1,1),ylim = c(-1,1),
#                            parameters = c(a=0), points = 500)
# state <- matrix(c(0,1,1,0),2, 2, byrow = TRUE)
# c_trajectory <- trajectory(c,y0 = state,tlim = c(0, 10),parameters = c(a=0),add=TRUE)
```

Problem #4

(A)

The function f should be assumed to have a negatve derivative because the chemical in S is an inhibitor and therefore an increase in concentration will lead to decreased response.

(B)

v-nullclines:

$$V(f(S) - r) = 0V = 0, f(S) = r$$

s-nullclines

$$\frac{pV}{W+V} - qS = 0\frac{pV}{W+V} = qSS = \frac{p}{q}\left(\frac{V}{V+Q}\right)$$

Equilibrium point:

$$S = \frac{p}{q}\left(\frac{V}{V+Q}\right) \quad \text{with} \quad S = r \quad \text{gives} V = \frac{\frac{qr}{p}W}{1 - \frac{qr}{p}}$$

Linearization:

$$\begin{aligned}
\dot{V} &= V[f(S) - r] & \dot{S} &= \frac{pV}{W+V} - qS \\
\dot{V}_V &= f(S) - r & \dot{S}_V &= \frac{pW}{(W+V)^2} \\
\dot{V}_S &= Vf'(S) & \dot{S}_S &= -q \\
J &= \begin{pmatrix} \dot{V}_V & \dot{V}_S \\ \dot{S}_V & \dot{S}_S \end{pmatrix} \\
J &= \begin{pmatrix} f(S) - r & \frac{pW}{(W+V)^2} \\ Vf'(S) & -q \end{pmatrix}
\end{aligned}$$

After plugging in the equilibrium points...

$$J = \begin{pmatrix} 0 & -\gamma \\ \alpha & -q \end{pmatrix}$$

(C)

The determinant is positive, the trace is negative. This means it is some form of sink, the type of which is determined by the relative values of γ, α and q.

Problem #5

$$\frac{dN}{d\tau} = rN - cNP - \rho EN \frac{dP}{d\tau} = bNP - mP - \sigma EP$$

Non-dimensionalization without substitutions:

$$\frac{dx}{dt} = \frac{x}{rN^{*2}}(r - c\frac{y}{P^*} - \rho E)\frac{dy}{dt} = \frac{y}{rN^{*2}}(b\frac{c}{N^*} - m - \sigma E)$$

Non-dimensionalized with substitutions $t = r\tau$, $x = \frac{N}{N^*}$, $y = \frac{P}{P^*}$ where $N^* = \frac{b}{r}$ and $P^* = \frac{c}{r}$:

$$\frac{dx}{dt} = \frac{xr}{b^2}(r - yr - \rho E)\frac{dy}{dt} = \frac{yr}{c^2}(xr - m - \sigma E)$$

with steady states:

$$x = 0, y = 0, r = 0 y = 1 - \frac{\rho E}{r}, \quad x = \frac{m + \sigma E}{r}$$