Biomath HW03

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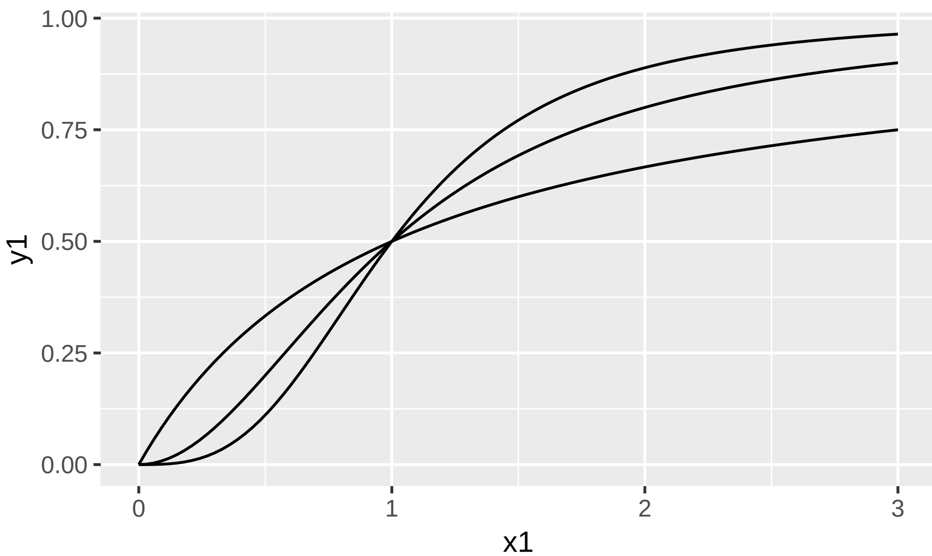
Problems 1,2,3

```
x1 <- seq(0,3,length.out = 101)
y1 <- x1/(1+x1)
y2 <- (x1^2)/(1+x1^2)
y3 <- (x1^3)/(1+x1^3)
ggplot(data=data.frame(y1=y1,y2=y2,y3=y3,x1=x1)) +
  geom_line(aes(x=x1,y=y1)) +
  geom_line(aes(x=x1,y=y2)) +
  geom_line(aes(x=x1,y=y3))
```



As n increases for the function $f(x) = \frac{x^n}{1+x^n}$ where n is a positive integer, the function has the same values for $f(0) = 0, f(x) = 1$ as $x \to \infty$ with differing behavior in between. More specifically, $f(x)$ is lesser near $x = 0$ and greater near $x = 1, 2, 3...$ as n increases. Intuitively, as n increases the $+1$ term in the denominator becomes more or less "impactful" to the resulting ratio depending on whether $x > 1$ or $x < 1$; as x increases with $x > 1, n$ large, the resulting value approaches 1 faster since the x^n value departs from 1 significantly faster.

Problem 4

Given $\frac{dx}{dt} = \Phi(t) - \lambda x$ and $y = f(x)$, ##a)

$$\begin{aligned} &\text{Known:} \\ x &= f^{-1}(y) \\ \frac{dy}{dx} &= f'(f^{-1}(y)) \end{aligned} \qquad \begin{aligned} \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} \\ &= f'(f^{-1}(y)) * \Big[\Phi(t) - \lambda x\Big] \\ &= f'(f^{-1}(y))\Big[\Phi(t) - \lambda f^{-1}(y)\Big] \end{aligned}$$

b)

Considering the particular case $y = f(x) = \frac{x}{k+x}$:

i)

Finding $x = f^{-1}(y)$:

$$\begin{aligned} y &= \frac{x}{k+x} \\ x &= yk + yx \\ yk &= x - yx = x(1-y) \\ x &= \frac{yk}{1-y} \end{aligned}$$

ii)

Finding $f'(x)$:

$$f'(x) = \frac{d}{dx}\frac{x}{k+x} = \frac{k}{(k+x)^2}$$

iii)

Finding $f'(f^{-1}(y))$:

$$\begin{aligned} f'(f^{-1}(y)) &= \frac{k}{\Big(k + f^{-1}(y)\Big)^2} = \frac{k}{k^2 + \frac{2yk^2}{(1-y)} + \frac{y^2k^2}{(1-y)^2}} \\ &= \frac{k(1-y)^2}{k^2(1-y)^2 + 2yk^2(1-y) + y^2k^2} \\ &= \frac{k(1-y)^2}{k^2 - 2k^2y + k^2y^2 + 2yk^2 - 2y^2k^2 + y^2k^2} = \frac{(1-y)^2}{k} \end{aligned}$$

iv)

Finding $\frac{dy}{dt}$:

$$\begin{aligned} \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} &= f'(f^{-1}(y))\Big[\Phi(t) - \lambda f^{-1}(y)\Big] \\ &= \frac{(1-y)^2}{k}\Big[\Phi(t) - \lambda\frac{yk}{(1-y)}\Big] \\ &= (1-y)\Big[\Phi(t)\frac{(1-y)}{k} - \lambda y\Big] \\ &= (1-y)\Big[\frac{\Phi(t)}{k} - \Big(\lambda + \frac{\Phi(t)}{k}\Big)y\Big] \end{aligned}$$

v)

Solving $\frac{dy}{dt}$ with $\Phi \equiv 0$:

$$\begin{aligned}
\frac{dy}{dt} &= (1-y)\Big[\frac{\Phi(t)}{k} - \Big(\lambda + \frac{\Phi(t)}{k}\Big)y\Big] \\
&= \lambda y(y-1)
\end{aligned}$$

Using separation of variables:

$$\begin{aligned}
\int \frac{1}{y(y-1)} dy &= \int \lambda dt \\
\ln\Big(\frac{1}{y} - 1\Big) &= \lambda t + C \\
\frac{1}{y} - 1 &= Ce^{\lambda t} \\
y &= \frac{1}{Ce^{\lambda t} + 1}
\end{aligned}$$

c)

Derive $\frac{dy}{dt}$ with $f(x) = \frac{x^h}{k^h + x^h}$:

i)

Finding $x = f^{-1}(y) = ...$

$$\begin{aligned}
y &= \frac{x^h}{k^h + x^h} \\
x^h &= yk^h + yx^h \\
yk^h &= x^h - yx^h = x^h(1-y) \\
x &= \sqrt[h]{\frac{yk^h}{1-y}}
\end{aligned}$$

ii)

Finding $f'(x)$ using quotient rule:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\frac{x^h}{k^h + x^h} \\
&= \frac{hx^{h-1}(k^h + x^h) - hx^{h-1}x^h}{(k^h + x^h)^2} \\
&= \frac{hx^{h-1}k^h + hx^{h-1}x^h - hx^{h-1}x^h}{k^{2h} + 2x^h k^h + x^{2h}} \\
f'(x) &= \frac{hk^h x^{h-1}}{k^{2h} + 2x^h k^h + x^{2h}}
\end{aligned}$$

iii)

Finding $f'(f^{-1}(y))$

$$\begin{aligned}
f'(f^{-1}(y)) &= \frac{hk^h \sqrt[h]{\frac{yk^h}{1-y}}^{h-1}}{k^{2h} + 2\sqrt[h]{\frac{yk^h}{1-y}}^h k^h + \sqrt[h]{\frac{yk^h}{1-y}}^{2h}} \\
&= \frac{hk^h \left(\frac{yk^h}{1-y}\right)^{\frac{h-1}{h}}}{k^{2h} + 2\left(\frac{yk^h}{1-y}\right)k^h + \left(\frac{yk^h}{1-y}\right)^2} \\
&= \frac{hk^{2h-1}y^{\frac{h-1}{h}}(1-y)^{-\frac{h-1}{h}}}{k^{2h} + 2yk^{2h}(1-y)^{-1} + y^2k^{2h}(1-y)^{-2}} \\
&= \frac{hk^{-1}y^{\frac{h-1}{h}}(1-y)^{\frac{h+1}{h}}}{(1-y)^2 + 2y(1-y) + y^2} \\
&= \frac{hk^{-1}y^{\frac{h-1}{h}}(1-y)^{\frac{h+1}{h}}}{1 - 2y + y^2 + 2y - 2y^2 + y^2} \\
f'(f^{-1}(y)) &= hk^{-1}y^{\frac{h-1}{h}}(1-y)^{\frac{h+1}{h}}
\end{aligned}$$

iv)

Finding $\frac{dy}{dt}$:

$$\begin{aligned}
\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} &= f'(f^{-1}(y))\Big[\Phi(t) - \lambda f^{-1}(y)\Big] \\
&= hk^{-1}y^{\frac{h-1}{h}}(1-y)^{\frac{h+1}{h}}\left[\Phi(t) - \lambda\sqrt[h]{\frac{yk^h}{1-y}}\right] \\
&= hk^{-1}y^{\frac{h-1}{h}}(1-y)^{\frac{h+1}{h}}\left[\Phi(t) - \lambda y^{\frac{1}{h}}k(1-y)^{-\frac{1}{h}}\right] \\
&= hk^{-1}y^{\frac{h-1}{h}}(1-y)^{\frac{h+1}{h}}\Phi(t) - \lambda hy(1-y) \\
&= hk^{-1}y^{\frac{h-1}{h}}(1-y)^{\frac{h+1}{h}}\Phi(t) + \lambda hy^2 - \lambda hy
\end{aligned}$$

Problem 5

Write the rate equations governed by the given system.

$$\begin{aligned}
\frac{dS}{dt} &= -SEk_1 + Ck_2 \\
\frac{dC}{dt} &= SEk_1 - Ck_2 - Ck_3 = -\frac{dS}{dt} - Ck_3 \\
\frac{dP}{dt} &= Ck_3 \\
\frac{dE}{dt} &= Ck_3 - SEk_1 + Ck_2 = \frac{dP}{dt} + \frac{dS}{dt}
\end{aligned}$$

Problem 6

Write the rate equations governed by the given system.

$$\begin{aligned}
\frac{dX}{dt} &= k_1 - Xk_3 + Yk_2 \\
\frac{dY}{dt} &= Xk_3 - Yk_2 + Zk_4 - Yk_5 - 3Yk_6 &&= Xk_3 - Y(k_2 + k_5 + k_6) + Zk_4 \\
\frac{dZ}{dt} &= Yk_5 - Xk_4 + \frac{1}{3}Yk_6 - k_7 &&= Y(k_5 + \frac{1}{3}k_6) - Zk_4 - k_7
\end{aligned}$$

Problem 7

State the dimensions of α, β, γ where α, β, γ positive and $\beta > 2\gamma\lambda$ in the following equation:

$$\begin{aligned}
\frac{dx}{dt} &= \alpha y + \beta\frac{x^2}{\gamma^2 + x^2} - \lambda x \\
\left[\frac{dx}{dt}\right] &= \left[\alpha y\right] = \left[\beta\frac{x^2}{\gamma^2 + x^2}\right] = \left[\lambda x\right] = \frac{\text{concentration}}{\text{time}} \\
\left[x\right] &= \text{concentration}, \left[y\right] = \text{concentration} \\
\left[\alpha\right] &= \frac{\text{concentration}}{\text{time}} * \frac{1}{\text{concentration}} = \frac{1}{\text{time}} \\
\left[\gamma\right] &= \text{concentration} \\
\left[\beta\right] &= \frac{\text{concentration}}{\text{time}},\ \text{becuase } \left[\gamma^2 + x^2\right] = \text{concentration}^2, \left[\frac{x^2}{\gamma^2 + x^2}\right] = 1 \\
\left[\lambda\right] &= \frac{1}{\text{time}}
\end{aligned}$$

Problem 8

Determine all dimensions of variables in equations (1)-(4) in the model of inflammation paper:

$$\begin{aligned}
\left[M\right] &= \text{population (alveolar macrophage)} = \text{cells} & \left[\frac{dM}{dt}\right] &= \frac{\text{population}}{\text{time}} = \frac{\text{cells}}{\text{minute}} \\
\left[C\right] &= \text{concentration (inflammatory cytokines)} = \frac{pg}{mL} & \left[\frac{dC}{dt}\right] &= \frac{\text{concentration}}{\text{time}} = \frac{pg}{mL * \text{minute}} \\
\left[A\right] &= \text{concentration (anti-inflammatory cytokines)} = \frac{pg}{mL} & \left[\frac{dA}{dt}\right] &= \frac{\text{concentration}}{\text{time}} = \frac{pg}{mL * \text{minute}} \\
\left[B\right] &= \text{population (generic pathogen)} = \text{cells} & \left[\frac{dB}{dt}\right] &= \frac{\text{population}}{\text{time}} = \frac{\text{cells}}{\text{minute}}
\end{aligned}$$

Table 2 Parameter definitions for the chronic inflammation model

Parameter	Units	Definition
s	$\frac{\text{cells}}{\text{minute}}$	Source term for the alveolar macrophage population
r	$\frac{\text{cells}}{\frac{\text{pg}}{\text{mL}}\text{minute}}$	Recruitment rate for macrophages in response to inflammatory cytokine
m_d	$\frac{1}{\text{minute}}$	Natural rate macrophages leave due to death or migration
a	$\frac{\text{cells}}{\text{minute}}$	Auto-induction rate of inflammatory cytokine production
p_c	$\frac{\frac{\text{pg}}{\text{mL}}}{\text{cells}^2\cdot\text{minute}}$	Bacterial induced inflammatory cytokine production rate
c_d	$\frac{1}{\text{minute}}$	Rate of degradation for the inflammatory cytokine
p_a	$\frac{1}{\text{cells}\cdot\text{minute}}$	Production of anti-inflammatory cytokine
a_d	$\frac{1}{\text{minute}}$	Rate of degradation for the anti-inflammatory cytokine
k_1	$\frac{1}{\text{cells}}$	Saturation constant for inflammatory cytokine induced macrophage recruitment
k_2	$\frac{1}{\frac{\text{pg}}{\text{mL}}}$	Saturation constant for production of cytokine through auto induction
k_3	$\frac{1}{\frac{\text{pg}}{\text{mL}}}$	Saturation constant for the inhibition of inflammatory cytokine
g	$\frac{1}{\text{minute}}$	Bacterial growth rate
b_d	$\frac{1}{\text{cells}\cdot\text{minute}}$	Bacterial death rate due to macrophages

Figure 1: See table 2 here for the particulars of the parameters

Problem 9

Nondimensionalize $\frac{dN}{dt} = rN^2(1 - \frac{N}{K})$ with $x = \frac{N}{K}$, $\tau = rKt$:

$$\frac{dX}{d\tau} = \frac{dX}{dN}\frac{dN}{dt}\frac{dt}{d\tau}$$

$$\frac{dX}{dN} = \frac{1}{K}, \quad \frac{dN}{dt} = rN^2\Big(1 - \frac{N}{K}\Big), \quad \frac{dt}{d\tau} = \frac{1}{rK}$$

$$\frac{dX}{d\tau} = \frac{1}{K} * \frac{1}{rK} * rN^2\Big(1 - \frac{N}{K}\Big)$$

$$= \frac{N^2}{K^2}\Big(1 - \frac{N}{K}\Big)$$

$$\frac{dX}{d\tau} = X^2\big(1 - X\big)$$

Equilibria values are those values of X that satisfy $\frac{dX}{d\tau} = 0$:

$$\frac{dX}{d\tau} = X^2\big(1 - X\big)X = 0, 1$$

We examine the behavior of the second derivative $\frac{d^2X}{d\tau^2}$ at $X = 0, 1$ to determine stability of equilibira values:
* For $X = 0$, $\frac{d^2X}{d\tau^2} = 0$ and this gives us no information.
* For $X = 1$, $\frac{d^2X}{d\tau^2} < 0$ and we know this is a stable equilibrium.