MATH 463 Topics in Biomathematics
Homework 8: Due Wednesday April 15 at Noon

The largest knight at King Arthur's round table was Sir Cumference. He acquired his size from eating too much pi.

Exercises:

1. Show that the system

$$\dot{x} = x(1 - x^2 - y^2) - y,$$
$$\dot{y} = y(1 - x^2 - y^2) + x,$$

in polar coordinates is

$$\dot{r} = r(1 - r^2),$$
$$\dot{\theta} = 1.$$

2. Convert the system given in polar coordinates into Cartesian (xy) coordinates.

$$\dot{r} = r(1 - r^2) + \mu r\cos(\theta),$$
$$\dot{\theta} = 1.$$

3. Each of the following systems is a model for predator-prey interactions where x represents the prey population and y the predator population. For each system use a combination of analytic (*e.g.* equilibrium and linearization) and numerical methods (*e.g.* phase portraits in R) to determine the dynamics of each model.

   (a) $\dot{x} = x\left(1 - \frac{x}{20}\right) - \frac{xy}{x+10}$, $\dot{y} = 3y\left(\frac{x}{x+10} - \frac{1}{2}\right)$
   (b) $\dot{x} = 3x\left(1 - \frac{x}{40}\right) - \frac{2xy}{x+15}$, $\dot{y} = y\left(\frac{2x}{x+15} - \frac{1}{2}\right)$
   (c) $\dot{x} = x\left(1 - \frac{x}{20}\right) - \frac{xy}{x+20}$, $\dot{y} = 3y\left(\frac{x}{x+10} - \frac{3}{4}\right)$

4. This problem is concerned wth a theory of liver regeneration due to Bard. Normally, the rate of cell division in the liver is very low, but if up to two thirds of the liver of a rat is removed, then the liver grows back to its original size in about a week. Bard discusses to theories. One theory, based on the assumed existence of a growth stimulator, predicts that the liver volume V will overshoot its normal value before finally settling down to a steady state. Such an overshot has not been observed. Here we will shoe something of how an alternative inhibitor model can account for the facts. Bard assumes that liver cells are prevented from dividing by an inhibitor of short half-life. The inhibitor is synthesized by the liver at a rate proportional to its size and is secreted into the blood, where its concentration is the same as in the liver. Let $V(t)$ be the volume of the liver and $S(t)$ the concentration of the inhibitor. Bard postulates the equations

$$\dot{V} = V[f(S) - r], \tag{1}$$
$$\dot{S} = \frac{pV}{W+V} - qS, \tag{2}$$

with W (blood volume), r, p, q constants.

(a) Why should the function f be assumed to have a negative derivative?

(b) Show that the system (1)-(2) has a unique positive equilibrium, and that the linearization of the system has the form

$$\dot{u} = -\gamma v, \tag{3}$$
$$\dot{v} = \alpha u - qv, \tag{4}$$

with positive constants γ, α.

(c) What can you conclude about the qualitative behavior of the system (1)-(2) near its equilibrium point.

5. One can modify the Lotka-Volterra system to model fishing. With fishing, we have

$$\frac{dN}{d\tau} = rN - cNP - \rho EN, \tag{5}$$
$$\frac{dP}{d\tau} = bNP - mP - \sigma EP, \tag{6}$$

where E represents the fishing effort and ρ and σ are the catchability coefficients for the prey and predator, respectively.

(a) Nondimensionalize the system by setting $t = r\tau$, $x = \frac{N}{N^*}$ and $y = \frac{P}{P^*}$.

(b) Choose values of N^*, P^* that eliminate as many parameters as possible. **Hint:**[*]

(c) Find the biologically realistic steady-states.

[*] You should be able to reduce the system to have the form $x' = (1-l)x(1-y)$, $y' = ky(x-1)$.