MATH 463 Topics in Biomathematics
Homework 4: Due Friday February 28 at Noon

Exercises:

1. To study the dynamics of an infectious disease which has prevailed for a long time it is necessary to modify the basic SIR model so as to take death rates of the cohorts into account. Doing so leads to the system

$$\frac{dS}{dt} = \kappa N(t) - \alpha I(t)S(t) - \sigma S(t), \tag{1}$$
$$\frac{dI}{dt} = \alpha I(t)S(t) - \beta I(t) - \sigma I(t), \tag{2}$$
$$\frac{dR}{dt} = \beta I(t) - \sigma R(t), \tag{3}$$
$$\frac{dN}{dt} = (\kappa - \sigma)N(t). \tag{4}$$

   Interpret the meaning of each term in these equations. Draw an appropriate compartment diagram corresponding to the model equations.

2. Consider equations (1)-(4) in the article *General Model of Inflammation* posted on D2L. Derive the dimensionless version of the model shown in equations (5)-(8) of the paper.

3. For each of the given one-dimensional dynamical systems, determine the steady-states and their stability properties. You may use geometric, analytical, or computational methods.

   (a) $\dot{x} = 4x^2 - 16$
   (b) $\dot{x} = 1 - x^{14}$
   (c) $\dot{x} = x - x^3$
   (d) $\dot{x} = e^{-x}\sin(x)$
   (e) $\dot{x} = 1 + \frac{1}{2}\cos(x)$
   (f) $\dot{x} = 1 - 2$

4. Consider the model chemical reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 2X, \tag{5}$$

   in which one molecule of X combines with one molecule of A to form two molecules of X. This means that the chemical X stimulates its own production. a process called *autocatalysis*. This positive feedback process leads to a chain reaction, which eventually is limited by a "back reaction" in which $2X$ returns to $A + X$.

As we have seen, according to the law of mass action of chemical kinetics, if there is an enormous supply of molecule A, then the rate equation for the autocatalytic reaction is

$$\frac{dx}{dt} = k_1 a x - k_{-1} x^2,$$

where k_1 and k_{-1} are positive constants.

(a) Find all of the steady-state values for this equation.

(b) Determine if or when the steady-state values are stable.

5. Recall that in class we went through the steps to analyze the bifurcation behavior and obtain a bifurcation diagram for the supercritical pitchfork bifurcation of $\dot{x} = \alpha x - x^3$. Instead, follow the same steps to analyze the bifurcation behavior and obtain the bifurcation diagram for the subcritical pitchfork bifurcation of $\dot{x} = \alpha x + x^3$.