MATH 463 Topics in Biomathematics
Homework 3: Due Friday February 21 at Noon

Exercises:

1. Sketch a graph of the function $f(x) = \frac{x}{1+x}$ for $x \geq 0$.

2. Sketch a graph of the function $f(x) = \frac{x^2}{1+x^2}$ for $x \geq 0$.

3. For the function $f(x) = \frac{x^n}{1+x^n}$, where n is a positive integer, how does increasing the value of n change the graph of f?

4. Let x denote the concentration of a certain drug in the blood plasma, described by the following differential equation:
$$\frac{dx}{dt} = \phi(t) - \lambda x,$$
where $\phi(t)$ is the rate of drug administration and λ is the **clearance rate constant**. The **efficacy** of the drug, denoted as y, is given by some function f, so $y = f(x)$. Assume that this function has derivative f' and inverse f^{-1}.

   (a) Show that
$$\frac{dy}{dt} = f'(f^{-1}(y))(\phi(t) - \lambda f^{-1}(y)).$$

   (b) Consider the particular case
$$y = f(x) = \frac{x}{\kappa + x},$$
   where κ is a positive constant. Show that
$$\frac{dy}{dt} = (y-1)\left(\frac{\phi(t)}{\kappa} - \left(\lambda + \frac{\phi(t)}{\kappa}\right) y\right),$$
   and solve the equation in the case where $\phi(t) \equiv 0$, given that $y = y_0 > 0$ at time $t = 0$.

   (c) Derive a differential equation for y if the efficacy is given by the following function:
$$f(x) = \frac{x^h}{\kappa^h + x^h},$$
   where h is a positive constant.

5. Write the four rate equations for the concentrations of chemicals C, E, P, S governed by the reactions
$$S + E \underset{k_2}{\overset{k_1}{\rightleftarrows}} C, \tag{1}$$
$$C \xrightarrow{k_3} P + E. \tag{2}$$

6. Write the rate equations for the concentrations of chemicals X, Y, Z governed by the reactions

$$\text{(source)} \xrightarrow{k_1} X, \tag{3}$$

$$Y \underset{k_3}{\overset{k_2}{\rightleftarrows}} X, \tag{4}$$

$$Z \underset{k_5}{\overset{k_4}{\rightleftarrows}} Y, \tag{5}$$

$$3Y \xrightarrow{k_6} Z, \tag{6}$$

$$Z \xrightarrow{k_7} \text{(waste)}. \tag{7}$$

7. Let x denote the concentration of the messenger RNA (mRNA) of a certain gene. Production of this mRNA species is stimulated by a transcription factor Y, the concentration of which is y. Furthermore, mRNA production is also stimulated by the mRNA molecule itself (autocatalytic feedback). The average lifetime of a molecule of this mRNA species is $\frac{1}{\lambda}$. The following kinetics are proposed:

$$\frac{dx}{dt} = \alpha y + \beta \frac{x^2}{\gamma^2 + x^2} - \lambda x, \tag{8}$$

where α, β, and γ are all positive parameters, with $\beta > 2\gamma\lambda$.

   (a) State the dimensions of α, β, and γ.

8. Consider equations (1)-(4) in the article *General Model of Inflammation* posted on D2L. Use dimensional analysis to determine the unit dimensions of all of the parameters contained in the equations. Note that the units for the variables M, C, A, and B are listed in Table 1 of the paper. Of course, t has units of time.

9. Consider the differential equation

$$\frac{dN}{dt} = rN^2\left(1 - \frac{N}{K}\right),$$

that modifies the logistic population growth model. Nondimensionalize this equation similar to how we did in class with the logistic growth model but now setting $x = \frac{N}{K}$ and $\tau = rKt$. Also, determine the equilibria values and their stabilities for this equation.