MATH 463 Topics in Biomathematics
Homework 1: Due Friday February 7 at Noon

Exercises:

1. In this exercise, we derive a solution to the discrete-time model $N_{t+1} = \rho N_t$.

   (a) Observe that if the sequence N_t satisfies $N_{t+1} = \rho N_t$, then for any positive integer t' we must have

   $$\begin{aligned} N_{t'} &= \rho N_{t'-1} \\ &= \rho(\rho N_{t'-2}) \\ &= \cdots . \end{aligned}$$

   Fill in the "dots" and assume an initial population N_0 in order to derive the solution $N_t = \rho^t N_0$.

2. For the sequence $\{N_t\}$, where $N_t = \rho^t N_0$ with t a nonnegative integer and N_0 a positive number, show that if $0 < \rho < 1$, then $N_t \to 0$ as $t \to \infty$. On the other hand, show that if $\rho > 1$, then $N_t \to \infty$ as $t \to \infty$.

3. Consider again the exponential growth model derived in class

   $$\frac{dN}{dt} = rN, \ N(0) = N_0.$$

   Explain the pros and cons of this as a model for population growth. Give reasons why this model may not be appropriate as a model for all populations.

4. Use the method of separation to derive the solution $N(t) = N_0 e^{rt}$ for the initial value problem

   $$\frac{dN}{dt} = rN, \ N(0) = N_0.$$

5. In this exercise we consider an alternative to the exponential growth model. Specifically we consider the so-called logistic growth model. In particular, consider the problem

   $$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N, \ N(0) = N_0,$$

   and answer the following questions:

   (a) Is the population $N(t)$ always increasing? If not, when is the population increasing and when is it decreasing?

   (b) For what values of N is the rate of change of the population zero?

6. In this exercise, we derive the logistic growth model of the previous problem in a manner analogous to the way in which we derived the exponential growth model in class. In this case, we will assume that the per-capita growth rate r depends on a time-dependent concentration of a limited resource which we denote by $C(t)$. That is, we suppose that

$$\frac{\frac{dN}{dt}}{N} = r(C(t)).$$

The next steps lead to a derivation for an expression for $r(C(t))$.

   (a) Following an approach similar to as in class, expand $r(C)$ as a Taylor series

   $$r(C) = a + bC + \cdots.$$

   Now argue why to first order one would expect that $r(C) \approx bC$.

   (b) Use the previous step to derive the equation

   $$\frac{dN}{dt} = bC(t)N(t).$$

   Take note that this is one equation in two unknowns, such an equation can not be solved. Thus, now we will proceed by relating $C(t)$ and $N(t)$.

   (c) We expect that as the population increases, the concentration of the resource C should decrease. Explain how this statement is encapsulated by the expression

   $$\frac{dC}{dt} = -\alpha \frac{dN}{dt}.$$

   What is the meaning of the positive constant α?

   (d) Using the previous equation, explain why $C + \alpha N$ must be constant. Call this constant k.

   (e) Put together the last three steps to derive the equation

   $$\frac{dN}{dt} = b\left(k - \alpha N(t)\right) N(t).$$

   (f) Show that the previous equation can be rewritten as

   $$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right) N.$$

   (g) What is the meaning of the terms r and K. Explain when and why the logistic growth model may be a more appropriate mathematical model for the growth of certain populations.

7. Use the method of separation of variables to solve the problem

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right) N.$$

Hint: You will have to use partial fractions to compute the integral $\int \frac{dN}{(1-\frac{N}{K})N}$. Show that the solution to this problem can be written as

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)e^{-rt}}.$$

What does the graph of such a function look like? You may assume that $N_0, K, r > 0$.

8. Let $r = 0.5$ and $K = 10$, use R to compute a numerical solution to

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N, \; N(0) = 4,$$

from $t = 0$ to $t = 10$. Plot the result. Explain the observed behavior. What happens if you change the initial condition to 15?