Prob & Stat HW8

Question 1

X, Y independent random variables with X,Y ~ $geom(p)$: $p_X(k) = p_Y(y) = p(1-p)^{k-1}, 0 < p < 1$

If $W = X + Y$, show

$$p_W(w) = (w-1)p^2(1-p)^{w-2}, w \geq 1$$

Hint: $p_W(w) = \sum_{\text{all } k} p_X(k)p_y(w-k)$

$$\begin{aligned} p_W(w) &= \sum_{\text{all } k} p_X(k)p_y(w-k) \\ &= \sum_{\text{all } k} p(1-p)^{1-k}p(1-p)^{w-k-1} \\ &= \sum_{\text{all } k} p^2(1-p)^{w-2} \\ &= \boxed{(w-1)p^2(1-p)^{w-2}} \end{aligned}$$

Question 2

Use $\Gamma(r) = (r-1)\Gamma(r-1)$ to prove $\Gamma(n) = (n-1)!$ if n is a positive integer. Hint: prove by induction.

$\Gamma(1) = 1$ and $0! = 1$, which verifies the equation for $n = 1$. Assuming $\Gamma(k) = (k-1)!$, $\Gamma(k+1) = k\Gamma(k) = k(k-1)! = k!$.

Question 3

Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\begin{aligned} 1 &= E(Z^2) \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty z^2 e^{\frac{-z^2}{2}} dz \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{(2u)e^{-u}}{\sqrt{2}\sqrt{u}} du, u = \frac{z^2}{2} \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty u^{(\frac{3}{2}-1)} e^{-u} du \\ \sqrt{\pi} &= 2\Gamma(\frac{3}{2}) \\ \sqrt{\pi} &= 2 * \frac{1}{2}\Gamma(\frac{1}{2}) \\ &\boxed{\sqrt{\pi} = \Gamma(\frac{1}{2})} \end{aligned}$$

Question 4

```r
pnorm(1.33,0,1)-pnorm(-0.44,0,1)
```

```
## [1] 0.5782723
```

```r
pnorm(0.94,0,1)
```

```
## [1] 0.8263912
```

```r
1-pnorm(-1.48,0,1)
```

```
## [1] 0.9305634
```

```r
pnorm(-4.32,0,1)
```

```
## [1] 7.80146e-06
```

Question 5

Part (a)

```r
pnorm(2.07,0,1)-pnorm(0,0,1)
```

```
## [1] 0.4807738
```

Part (b)

```r
pnorm(-0.11,0,1)-pnorm(-0.64,0,1)
```

```
## [1] 0.1951184
```

Part (c)

```r
1-pnorm(-1.06,0,1)
```

```
## [1] 0.8554277
```

Part (d)

```r
pnorm(-2.33,0,1)
```

```
## [1] 0.009903076
```

Part (e)

```r
1-pnorm(4.61,0,1)
```

```
## [1] 2.013345e-06
```

Question 6

```r
qnorm(.75,0,1)-qnorm(.25,0,1)
```

```
## [1] 1.34898
```

Question 7

4.4.2

X is a random variable determined by the geometric distribution: $p_X(k) = (1-p)^{k-1}p$. Then, the number of tries it takes, on average, to achieve a success is the expected value of the geometric distribution: $\frac{1}{p}$.

```
p <- 0.10
1/p
```

```
## [1] 10
```

4.5.2

The probability that the seventh missile fired will be the fourth successful strike is equal to the probability of three successes within six fires, multiplied by the probability that the seventh is a success:

$$p_F(s) = \binom{s+f-1}{s-1} p^s (1-p)^f$$

where s is the number of successes, f is the number of failures and p is the probability of a success. Now,

$$\begin{aligned} P(7th = 4th \text{ strike}) &= \binom{3+3-1}{3-1} p^3 (1-p)^3 * p \\ &= \binom{5}{2} p^3 (1-p)^3 * p \\ &= \binom{5}{2} (.1)^4 (.9)^3 \end{aligned}$$

```
choose(5,2)*(.1^4)*(.9^3)
```

```
## [1] 0.000729
```

4.5.6

Here, we must achieve r successes and k failures. These can be organized in $\binom{k+r-1}{k}$ ways because the last trial must be a success. Therefore,

$$p_X(k) = \binom{k+r-1}{k} p^r (1-p)^k$$

4.2.2

```
sum(dbinom(1:10,size=10,prob=905/289411))
```

```
## [1] 0.03083403
```

4.2.10

```
observed <- c(109,65,22,3,1)/200
pois <- dpois(0:4,lambda=sum(c(0,1,2,3,4)*c(109,65,22,3,1))/200)
observed
```

```
## [1] 0.545 0.325 0.110 0.015 0.005
```

```r
pois
```

```
## [1] 0.543350869 0.331444030 0.101090429 0.020555054 0.003134646
```

```r
observed/pois
```

```
## [1] 1.0030351 0.9805577 1.0881347 0.7297475 1.5950766
```

The observed data is very closely represented by the pois distribution with lambda set to the average number of deaths.

4.2.22

$$p_X(2) = \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{\lambda e^{-\lambda}}{1!} = p_X(1)\lambda e^{-\lambda} = 2e^{-\lambda} \quad , \quad \lambda = 2$$

```r
dpois(4,lambda=2)
```

```
## [1] 0.09022352
```

4.2.26

(a)

Yes. This is a reasonable assumption because they are individual events measured by their frequency within a fixed time interval. ###(b)

```r
dpois(4,lambda=2.5)
```

```
## [1] 0.1336019
```

(c)

$$\int_0^{1/4} 2.5e^{-2.5y}dy = -e^{-2.5/4} + e^0$$

```r
-exp(-2.5/4)+exp(0)
```

```
## [1] 0.4647386
```

4.3.16

By the central limit theorem, the sum of faces showing should be approximately evenly distributed. The expected value of this distribution should be 350, given that it is symmetrical with $100 \leq y \leq 600$.

Continuity Correction:

$$P(a \leq X \leq b) = \int_{a-0.5}^{b+0.5} f(y)dy$$

where $f(y) = 1/500$, $100 \leq y \leq 600$.

4.3.18

Expected Value:

```r
EV <- sum(1:100*dpois(1:100,lambda=3))
EV
```

```
## [1] 3
```

Variance:

```r
Var2 <- sum((1:100)*(1:100)*dpois(1:100,lambda=3))-EV^2
Var2
```

```
## [1] 3
```

```r
Var2S <- Var2/4
Var <- sqrt(Var2)
```