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MXB101 Problem Solving Task 3 (10%)

Question 1a.

Show that $b = \frac{1}{a-1}$.

Solution:

We are given that $X \sim U(1, a)$ and $W = \sqrt{X}$. The cumulative distribution function (CDF) of W is $F_W(w) = b(w^2 - 1)$.

To find b, recognise CDFs range from 0 to 1 over the interval. Since $X \in [1, a]$, then

$$W = \sqrt{X} = [1, \sqrt{a}]$$

Hence, the maximum of CDF is at $w = \sqrt{a}$

$$F_W(\sqrt{a}) = b((\sqrt{a})^2 - 1)$$
$$= b(a - 1)$$

Since the CDF ranges from 0 to 1, the maximum of the CDF is $F_W(\sqrt{a}) = 1$

$$1 = b(a - 1)$$
$$\therefore b = \frac{1}{a - 1}$$

Hence $b = \frac{1}{a-1}$.

Question 1b.

The group choose the **maximum allowable region area**, a, such that the **average region area** is equal to 10 km^2. What is the average region side length, $\mathbb{E}(W)$?

Solution:

Since the region area X is uniformly distributed such that $X \sim U(1, a)$, the expectation for a uniform random variable is:

$$X \sim U(a, b) \rightarrow \mathbb{E}(X) = \frac{a+b}{2}$$

Hence given the average region area:

$$\mathbb{E}(X) = \frac{1+a}{2}$$
$$10 = \frac{1+a}{2}$$
$$\therefore a = 19$$

Given $W = \sqrt{X}, \mathbb{E}(W) = \mathbb{E}(\sqrt{X})$:

$$\mathbb{E}(W) = \int_1^a \sqrt{x} \cdot f_X(x) \mathrm{d}x$$
$$= \int_1^a \sqrt{x} \cdot \frac{1}{a-1} \mathrm{d}x$$
$$= \int_1^{19} \sqrt{x} \cdot \frac{1}{18} \mathrm{d}x$$
$$= \frac{1}{18} \left[\frac{2}{3} x^{\frac{3}{2}} \right]_1^{19}$$
$$= \frac{1}{18} \left(\frac{2}{3} \right) \left[19^{\frac{3}{2}} - 1^{\frac{3}{2}} \right]$$
$$= \frac{1}{27} \left(19\sqrt{19} - 1 \right)$$
$$\approx 3.0303$$

Hence, $\mathbb{E}(W) \approx 3.0303$km

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Question 1c.

The monthly monitoring cost comprises a base rate of $700 + $80 per km^2.

Question 1ci.

Write an expression for the monitoring cost, C, in terms of region area, X.

Solution:

$$C = 700 + 80X$$

Question 1cii.

Find the average monitoring cost.

Solution:

$$\begin{aligned}
\mathbb{E}(C) &= 700 + 80\mathbb{E}(X) \\
&= 700 + 80 \cdot 10 \\
&= 1500
\end{aligned}$$

Hence, the average monitoring cost is $1500.

Question 1ciii.

Find the variance of the monitoring cost.

Solution:

$$\begin{aligned}
Var(C) &= Var(700 + 80X) \\
&= 80^2 \cdot Var(X)
\end{aligned}$$

Recall the variance of a uniformly distributed random variable

$$\begin{aligned}
Var(X) &= \frac{(a-1)^2}{12} \\
&= \frac{(19-1)^2}{12} \\
&= \frac{18^2}{12} \\
&= 27
\end{aligned}$$

Hence:

$$\begin{aligned}
Var(C) &= 80^2 \cdot 27 \\
\therefore Var(C) &= 172800
\end{aligned}$$

Therefore, the variance of the monitoring cost is $172,800

Question 2a.
Use the conditional probability rule to argue that the probability of a randomly chosen wallaby having a joey in their pouch is 0.4836.
Solution:
Let F denote females where $\Pr(F) = 0.62$, hence $\Pr(M) = 1 - \Pr(F) = 0.38$ due to sexes being mutually exclusive. Let $J|F$ denote joeys in a female pouch where $\Pr(J|F) = 0.78$. Since males cannot carry joeys, $\Pr(J|M) = 0$.

By law of total probability, $p = \Pr(J)$ is given by:

$$p = \Pr(J) = \Pr(J|F)\Pr(F) + \Pr(J|M)\Pr(M)$$
$$= 0.78 \times 0.62 + 0 \times 0.38$$
$$= 0.4836$$

Hence, the probability of a randomly chosen wallaby having a joey in their pouch is 0.4836.

Question 2b.
On average, how many wallabies does a team in the field have to check before finding six that have a joey?
Solution:
The number of trials to get 6 joeys ($j = 6$) can be modelled by a negative-binomial distribution with mean:

$$\mathbb{E}(N) = \frac{j}{p}$$
$$= \frac{6}{0.4836}$$
$$\approx 12.41$$

On average, to find 6 joeys, 13 wallabies need to be checked (rounding up)

Question 2c.
What is the probability that the team checks fewer than three wallabies before finding the first with a joey?
Solution:
The number of trials N to the first success can be modelled $X{\sim}\text{Geom}(p)$

$$\Pr(N < 3) = \Pr(N = 1) + \Pr(N = 2)$$
$$= p + (1 - p)p$$
$$= p\big(1 + (1 - p)\big)$$
$$= 0.4836(1 + 1 - 0.4836)$$
$$\therefore \Pr(N < 3) \approx 0.7338$$

Hence, the probability that the team checks fewer than three wallabies before finding the first with a joey is 0.7338.

Question 2d.
Use an appropriate limiting distribution to estimate the probability that at least 700 wallabies from a sample of 1200 have a joey.
Solution:
Initially, denote X as a binomial distribution as $X \sim Bi(1200, p)$. The central limit theorem can be utilised given both $np, n(1-p) \geq 5$.

$$np = 1200 \times 0.4836 = 580.32$$
$$n(1-p) = 1200 \times (1 - 0.4836) = 619.68$$

Since $np, n(1-p) > 5$, model via a normal distribution $\overline{X} \sim N(\mu, \sigma^2)$, solving for μ, σ:

$$\begin{aligned} \mu &= np \\ &= 1200 \times 0.4836 \\ &= 580.32 \\ \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{1200 \times 0.4836(1 - 0.4836)} \\ &\approx 17.311 \end{aligned}$$

Hence $\overline{X} \sim N(580.32, 17.311^2)$. $\Pr(X \geq 700) \approx 2.0 \times 10^{-12}$, which is negligible.

Question 2e.
The gestation period (that is, the time from conception to birth) of wallabies is, on average, 33 days, with a standard deviation of 2 days. Assuming that the gestation period is normally distributed, what is the probability that a particular joey had a gestation period of within 3 days of the mean?
Solution:
Denote X as a normal distribution $X \sim N(\mu, \sigma^2)$ where $\mu = 33, \sigma = 2$. To find the probability that a joey's gestation period is within 3 days of the mean, i.e., $\Pr(30 \leq X \leq 36)$, the bounds will be standardised using Z-scores given a standard normal distribution:

$$\begin{aligned} Z_X &= \frac{X - \mu}{\sigma} \\ Z_{30} &= \frac{30 - 33}{2} = -1.5 \\ Z_{36} &= \frac{36 - 33}{2} = 1.5 \end{aligned}$$

Hence $\Pr(30 \leq X \leq 36) = \Pr(-1.5 \leq Z \leq 1.5) = 0.8663$

Question 2f.

From birth, joeys spend an average of 280 days in their mother's pouch. To answer the following, you may assume that the time a joey spends in the pouch is exponentially distributed.

Question 2fi.

What is the probability that a joey spends more than 320 days in their mother's pouch?

Solution:

Let T~Exponential(λ),

$$\mathbb{E}(\lambda) = \frac{1}{\lambda}$$
$$280 = \frac{1}{\lambda}$$
$$\therefore \lambda = \frac{1}{280}$$

The probability that a joey spends more than 320 days in the pouch is:

$$\Pr(T > t) = e^{-\lambda t}$$
$$\Pr(T > 320) = e^{-\frac{320}{280}}$$
$$= e^{-1.1429}$$
$$\approx 0.3189$$

The probability the joey spends more than 320 days in the pouch is 0.3189.

Question 2fii.

You have determined from the size of the joey that it has been in the pouch more than 120 days. What is the probability that it has been in the pouch for more than 160 days?

Solution:

Recall the memoryless property of the exponential distribution:

$$\Pr(T > a | T > b) = \Pr(T > (a - b))$$
$$\Pr(T > 160 | T > 120) = \Pr(T > 40)$$

To calculate $\Pr(T > 40)$, use previous method:

$$\Pr(T > t) = e^{-\lambda t}$$
$$\Pr(T > 40) = e^{-\frac{40}{280}}$$
$$= e^{-0.142857}$$
$$\approx 0.8669$$

Therefore, the probability that the joey has been in the pouch more than 160 days given it has already spent 120 days or more is 0.8669.