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MXB101 Problem Solving Task 1 (10%)

Question 1.

Using rules of set theory and probability theory, prove the following statements are true:

(a) Prove $\overline{(\bar{A} \cup \bar{B})} \cup \overline{(\bar{A} \cup B)} = A$.

(b) Prove $(A \cup B) \cap \overline{(A \cap B)} = (A \cap \bar{B}) \cup (\bar{A} \cap B)$.

(c) Prove that if $A \cap B = \phi$, then $\Pr(A) \leq \Pr(\bar{B})$.

(d) Prove that if $\Pr(A) = \Pr(B) = \Pr(A \cap B)$, then $\Pr\big((A \cap \bar{B}) \cup (\bar{A} \cap B)\big) = 0$.

(e) Prove that if $\Pr(A) = \Pr(B) = \Pr(A \cap B)$, then $\Pr\big((A \cap \bar{B}) \cup (\bar{A} \cap B)\big) = 0$.

(f) Show that the addition rule for four events, A, B, C, D is given by:

$$\begin{aligned}\Pr(\square \cup \square \cup \square \cup \square)\\ &= \Pr(A) + \Pr(B) + \Pr(C) + \Pr(D) - \Pr(A \cap B) - \Pr(A \cap C)\\ &- \Pr(A \cap D) - \Pr(B \cap C) - \Pr(B \cap D) - \Pr(C \cap D)\\ &+ \Pr(A \cap B \cap C) + \Pr(A \cap B \cap D) + \Pr(B \cap C \cap D)\\ &+ \Pr(A \cap C \cap D) - \Pr(A \cap B \cap C \cap D)\end{aligned}$$

Question 2.

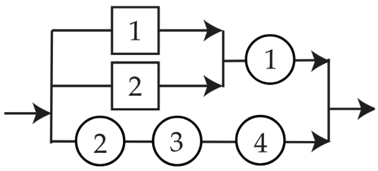
Your company has ordered electrical components from two different manufacturers. In the last delivery, your company received two boxes: one containing 1,000 components from manufacturer 1, and the other containing 2,000 components from manufacturer 2. Your quality control team tested every component and found that both boxes contain exactly 100 faulty components. Unfortunately they also removed all the labels from the boxes, so you cannot tell which box is from which manufacturer. Given this information, answer the following (justify your answers):

(a) Suppose you randomly pick a box by tossing a fair coin, then draw two components from that box. What is the probability that both components are faulty?

(b) Given the process in (a), if both components are faulty, what is the probability that the box selected is from manufacturer 1?

Question 3.

Consider the circuit below made of two types of components (type 1 are boxes, and type 2 are circles). Let W_i be the event that the i-th type 1 component functions, and let H_k be the event that the k-th type 2 component functions. Suppose the probability that a type 1 component functions is p, and the probability that a type 2 component functions is q. You may assume independence of all components regardless of type.



(a) Let S be the event that the circuit functions. Describe the event S in terms of W_i and H_k using set operations.

(b) Show that $\Pr(S) = (p^2 - 2p)q^4 + q^3 + (2p - p^2)q$.

(c) Would $p > q$ or $p < q$ lead to a more stable circuit design? Explain why.

Solutions

Question 1a.

Prove $\overline{(\bar{A} \cup \bar{B})} \cup \overline{(\bar{A} \cup B)} = A$.
Solution:
By De Morgan's Law 1:

$$(\bar{A} \cup \bar{B}) = \overline{(A \cap B)}$$
$$\therefore \overline{(\bar{A} \cup \bar{B})} = \overline{\overline{(A \cap B)}}$$
$$= A \cap B$$

By De Morgan's Law 2:

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$
$$\therefore \overline{(\bar{A} \cup B)} = \bar{\bar{A}} \cap \bar{B}$$
$$= A \cap \bar{B}$$

$$\therefore \overline{(\bar{A} \cup \bar{B})} \cup \overline{(\bar{A} \cup B)} = (A \cap B) \cup (A \cap \bar{B})$$

Since $(A \cap B) \cap (A \cap \bar{B})$ are disjoint events, $(A \cap B) \cup (A \cap \bar{B}) = (A \cap B) + (A \cap \bar{B})$
$(A \cap B) + (A \cap \bar{B}) = A$ by law of total probability.

$$\therefore \overline{(\bar{A} \cup \bar{B})} \cup \overline{(\bar{A} \cup B)} = A. \text{QED}$$

Question 1b.

Prove $(A \cup B) \cap \overline{(A \cap B)} = (A \cap \bar{B}) \cup (\bar{A} \cap B)$.
Solution:
By De Morgan's Law:

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$
$$\therefore (A \cup B) \cap \overline{(A \cap B)} = (A \cup B) \cap (\bar{A} \cup \bar{B})$$

Let $(A \cup B) = C$, therefore by distributive law:

$$C \cap (\bar{A} \cup \bar{B}) = (C \cap \bar{A}) \cup (C \cap \bar{B})$$
$$\therefore (A \cup B) \cap (\bar{A} \cup \bar{B}) = \big((A \cup B) \cap \bar{A}\big) \cup \big((A \cup B) \cap \bar{B}\big)$$

Distribute each term using distributive laws:

$$(A \cup B) \cap \bar{A} = (A \cap \bar{A}) \cup (B \cap \bar{A})$$
$$\text{Complement rule: } A \cap \bar{A} = \phi$$
$$= \phi \cup (B \cap \bar{A})$$
$$= (B \cap \bar{A})$$
$$(A \cup B) \cap \bar{B} = (A \cap \bar{B}) \cup (B \cap \bar{B}), B \cap \bar{B} = \phi$$
$$= (A \cap \bar{B}) \cup \phi$$
$$= (A \cap \bar{B})$$

$$\therefore (A \cup B) \cap \overline{(A \cap B)} = (A \cap \bar{B}) \cup (\bar{A} \cap B). \text{QED}$$

Question 1c.

Prove that if $A \cap B = \phi$, then $\Pr(A) \leq \Pr(\bar{B})$.
Solution:
Given the complement of B; $\bar{B}$, consists of all elements not in B, and since A and B are disjoint events, that implies that all elements of A must not be in B and instead in $\bar{B}$, i.e., $A \subseteq \bar{B}$.

If $A \subseteq B$ implies $\Pr(A) \leq \Pr(B)$, by subset laws and monotonicity of probability, therefore given $A \cap B = \phi$, then $\Pr(A) \leq \Pr(\bar{B})$.

Question 1d.

Prove that if $\Pr(A) = \Pr(B) = \Pr(A \cap B)$, then $\Pr\left((A \cap \bar{B}) \cup (\bar{A} \cap B)\right) = 0$.
Solution:
If $\Pr(A) = \Pr(B) = \Pr(A \cap B)$, it also implies using inclusion-exclusion principle that $\Pr(A \cap B) = \Pr(A \cup B)$, therefore the two sets are equal and overlap.

Since $A = B = A \cap B$, therefore $A \cap \bar{B} = \phi$, and $\bar{A} \cap B = \phi$.

Since both events are disjoint, therefore $\Pr\left((A \cap \bar{B}) \cup (\bar{A} \cap B)\right) = 0$.

Question 1e.

Prove that if $\Pr(A) = \Pr(B) = 1$, then $\Pr(A \cap B) = 1$.
Solution:
If $\Pr(A) = \Pr(B) = 1$, and $\Pr(\Omega) = 1$, then $A, B \subseteq \Omega$. Using the addition rule:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

where $\Pr(A) = \Pr(B) = 1$, therefore

$$\Pr(A \cup B) = 1 + 1 - \Pr(A \cap B)$$
$$\Pr(A \cup B) = 2 - \Pr(A \cap B)$$
$$\therefore \Pr(A \cup B) + \Pr(A \cap B) = 2$$

Since $\Pr(\Omega) = 1$, and $0 \leq \Pr(E) \leq 1$, therefore $\Pr(A \cup B)$ and $\Pr(A \cap B) = 1$.

Therefore, if $\Pr(A) = \Pr(B) = 1$, then $\Pr(A \cap B) = 1$.

Question 1f.

Show that the addition rule for four events, A, B, C, D is given by:

$$
\begin{aligned}
\mathrm{Pr}(\square \cup \square \cup \square \cup \square) &= \mathrm{Pr}(A) + \mathrm{Pr}(B) + \mathrm{Pr}(C) + \mathrm{Pr}(D) - \mathrm{Pr}(A \cap B) - \mathrm{Pr}(A \cap C) - \mathrm{Pr}(A \cap D) \\
&- \mathrm{Pr}(B \cap C) - \mathrm{Pr}(B \cap D) - \mathrm{Pr}(C \cap D) + \mathrm{Pr}(A \cap B \cap C) + \mathrm{Pr}(A \cap B \cap D) \\
&+ \mathrm{Pr}(B \cap C \cap D) + \mathrm{Pr}(A \cap C \cap D) - \mathrm{Pr}(A \cap B \cap C \cap D)
\end{aligned}
$$

Solution:

Step 1: Compute $\mathrm{Pr}(A \cup B \cup C \cup D)$ where $P = A \cup B \cup C$

$$
\begin{aligned}
\mathrm{Pr}(A \cup B \cup C \cup D) &= \mathrm{Pr}(P \cup D) \{\textbf{addition rule}\} \\
&= \mathrm{Pr}(P) + \mathrm{Pr}(D) - \mathrm{Pr}(P \cap D) \\
&= \mathrm{Pr}(A \cup B \cup C) + \mathrm{Pr}(D) - \mathrm{Pr}\big((A \cup B \cup C) \cap D\big)
\end{aligned}
$$

Step 2: Compute $\mathrm{Pr}(A \cup B \cup C)$ where $Q = A \cup B$.

$$
\begin{aligned}
\mathrm{Pr}(A \cup B \cup C \cup D) &= \mathrm{Pr}(Q \cup C) + \mathrm{Pr}(D) - \mathrm{Pr}\big((A \cup B \cup C) \cap D\big) \{\textbf{addition rule}\} \\
&= \mathrm{Pr}(Q) + \mathrm{Pr}(C) - \mathrm{Pr}(Q \cap C) + \mathrm{Pr}(D) - \mathrm{Pr}\big((A \cup B \cup C) \cap D\big) \\
&= \mathrm{Pr}(A \cup B) + \mathrm{Pr}(C) - \mathrm{Pr}\big((A \cup B) \cap C\big) + \mathrm{Pr}(D) \\
&- \mathrm{Pr}\big((A \cup B \cup C) \cap D\big)
\end{aligned}
$$

Recall addition rule for 3 events and substitute:

$$= \mathrm{Pr}(A) + \mathrm{Pr}(B) + \mathrm{Pr}(C) + \mathrm{Pr}(D) - \mathrm{Pr}(A \cap B) - \mathrm{Pr}\big((A \cup B) \cap C\big) - \mathrm{Pr}\big((A \cup B \cup C) \cap D\big)$$

Step 3: Represent $\mathrm{Pr}\big((A \cup B) \cap C\big)$ as $\mathrm{Pr}\big((A \cap C) \cup (B \cap C)\big) \{\textbf{distributive law}\}$

$$\mathrm{Pr}\big((A \cap C) \cup (B \cap C)\big) = \mathrm{Pr}(A \cap C) + \mathrm{Pr}(B \cap C) - \mathrm{Pr}(A \cap B \cap C) \ \{\textbf{addition rule}\}$$

$$
\begin{aligned}
\mathrm{Pr}(\square \cup \square \cup \square \cup \square) &= \mathrm{Pr}(A) + \mathrm{Pr}(B) + \mathrm{Pr}(C) + \mathrm{Pr}(D) - \mathrm{Pr}(A \cap B) - \mathrm{Pr}(A \cap C) - \mathrm{Pr}(B \cap C) \\
&+ \mathrm{Pr}(A \cap B \cap C) - \mathrm{Pr}\big((A \cup B \cup C) \cap D\big)
\end{aligned}
$$

Step 4: Represent $\mathrm{Pr}\big((A \cup B \cup C) \cap D\big)$ as $\mathrm{Pr}\big((A \cap D) \cup (B \cap D) \cup (C \cap D)\big) \{\textbf{distributive law}\}$

$$
\begin{aligned}
\mathrm{Pr}\big((\square \cap \square) \cup (\square \cap \square) \cup (\square \cap \square)\big) &= (\mathrm{Pr}(A \cap D) + \mathrm{Pr}(B \cap D) + \mathrm{Pr}(C \cap D) - \mathrm{Pr}(A \cap B \cap D) - \mathrm{Pr}(A \cap C \cap D) \\
&- \mathrm{Pr}(B \cap C \cap D) + \mathrm{Pr}(A \cap B \cap C \cap D))
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Pr}(\square \cup \square \cup \square \cup \square) &= \mathrm{Pr}(A) + \mathrm{Pr}(B) + \mathrm{Pr}(C) + \mathrm{Pr}(D) - \mathrm{Pr}(A \cap B) - \mathrm{Pr}(A \cap C) - \mathrm{Pr}(B \cap C) \\
&+ \mathrm{Pr}(\square \cap \square \cap \square) \\
&- (\mathrm{Pr}(A \cap D) + \mathrm{Pr}(B \cap D) + \mathrm{Pr}(C \cap D) - \mathrm{Pr}(A \cap B \cap D) - \mathrm{Pr}(A \cap C \cap D) \\
&- \mathrm{Pr}(B \cap C \cap D) + \mathrm{Pr}(A \cap B \cap C \cap D))
\end{aligned}
$$

$$
\begin{aligned}
\therefore \mathrm{Pr}(\square \cup \square \cup \square \cup \square) &= \mathrm{Pr}(A) + \mathrm{Pr}(B) + \mathrm{Pr}(C) + \mathrm{Pr}(D) - \mathrm{Pr}(A \cap B) - \mathrm{Pr}(A \cap C) - \mathrm{Pr}(A \cap D) \\
&- \mathrm{Pr}(B \cap C) - \mathrm{Pr}(B \cap D) - \mathrm{Pr}(C \cap D) + \mathrm{Pr}(A \cap B \cap C) + \mathrm{Pr}(A \cap B \cap D) \\
&+ \mathrm{Pr}(B \cap C \cap D) + \mathrm{Pr}(A \cap C \cap D) - \mathrm{Pr}(A \cap B \cap C \cap D)
\end{aligned}
$$

Question 2a.

Suppose you randomly pick a box by tossing a fair coin, then draw two components from that box. What is the probability that both components are faulty?

Solution:

Context:

Box 1 contains 1,000 components from manufacturer 1 with 100 faulty components.
Box 2 contains 2,000 components from manufacturer 2 with 100 faulty components.

Let B_{iT} denote total components from Box $i, i = 1$ or 2.
Let B_{iF} denote number of faulty components from Box i.
Let $F_{i,j}$ denote the event of j-th draw from Box i is faulty. F_i denotes both faulty.
Let E denote the event of either draw from either box is faulty.

<u>Box 1:</u>
Faulty component from Box 1 first time:

$$\Pr(F_{1,1}) = \frac{B_{1F}}{B_{1T}} = \frac{100}{1000} = 0.1$$

Faulty component from Box 1 second time:

$$\Pr(F_{1,2}) = \frac{B_{1F} - 1}{B_{1T} - 1} = \frac{99}{999} = 0.0\overline{9}$$

Probability of both (disjoint, assume independence):

$$\begin{aligned}\Pr(F_1) &= \Pr(F_{1,1}) \times \Pr(F_{1,2}) \\ &= 0.1 \times 0.0\overline{9} = 0.00\overline{9}\end{aligned}$$

<u>Box 2:</u>
Faulty component from Box 2 first time:

$$\Pr(F_{2,1}) = \frac{B_{2F}}{B_{2T}} = \frac{100}{2000} = 0.05$$

Faulty component from Box 2 second time:

$$\Pr(F_{2,2}) = \frac{B_{2F} - 1}{B_{2T} - 1} = \frac{99}{1999} \approx 0.0495$$

Probability of both (disjoint, assume independence):

$$\begin{aligned}\Pr(F_2) &= \Pr(F_{2,1}) \times \Pr(F_{2,2}) \\ &= 0.05 \times 0.0495 = 0.002475\end{aligned}$$

<u>Total probability:</u>
A fair coin is tossed, therefore there is a 50/50 chance of choosing either box. Given 50% chance of picking either component, therefore:

$$\begin{aligned}\Pr(E) &= \frac{1}{2}\Pr(F_1) + \frac{1}{2}\Pr(F_2) \\ &= \frac{1}{2} \times 0.00\overline{9} + \frac{1}{2} \times 0.002475 \\ &= 0.006238\end{aligned}$$

Hence, the probability of both components are faulty, is 0.006238.

Question 2b.

Given the process in (a), if both components are faulty, what is the probability that the box selected is from manufacturer 1?
Solution:
Let C_i denote the probability of choosing either Box i - $\Pr(C_1) = \Pr(C_2) = 0.5$

Using Bayes' Theorem:

$$\Pr(C_1|E) = \frac{\Pr(E|C_1) \times \Pr(C_1)}{\Pr(E)}$$

- $\Pr(E|C_1) = \Pr(F_1) = 0.00\overline{9}$.
  - The probability of both draws being faulty from either box, given that Box 1 is chosen, is the probability that both components from Box 1 are faulty.
- $\Pr(C_1) = \Pr(C_2) = 0.5$
  - Probability of choosing either box is 0.5.
- $\Pr(E) = 0.006238$
  - Probability that from either box, both components are faulty.

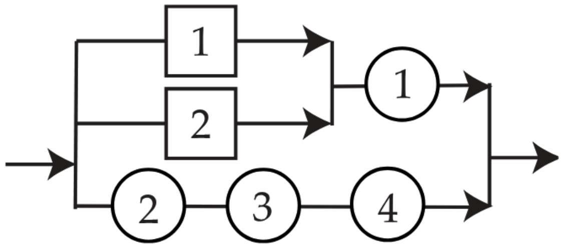
$$\therefore \Pr(C_1|E) = \frac{0.00\overline{9} \times 0.5}{0.006238}$$

$$= 0.8015$$

Therefore there is approximately an 80% chance that if both components are faulty, that the box was from manufacturer 1.

Question 3.

Consider the circuit below made of two types of components (type 1 are boxes, and type 2 are circles). Let W_i be the event that the i-th type 1 component functions, and let H_k be the event that the k-th type 2 component functions. Suppose the probability that a type 1 component functions is p, and the probability that a type 2 component functions is q. You may assume independence of all components regardless of type.



Question 3a.

Let S be the event that the circuit functions. Describe the event S in terms of W_i and H_k using set operations.

Solution:

Let: S be the circuit function, W_i denotes the i-th *box* (type 1) event, H_k denotes the k-th *circle* (type 2) event.

$$\therefore S = \big((W_1 \cup W_2) \cap H_1\big) \cup (H_2 \cap H_3 \cap H_4)$$
$$\therefore S = \big((W_1 \cup W_2) \cap H_1\big) \cup H_2H_3H_4$$

Question 3b.

Show that $\Pr(S) = (p^2 - 2p)q^4 + q^3 + (2p - p^2)q$.

Solution:

Suppose $\Pr(W_i) = p, \Pr(H_k) = q$

Step 1: <u>Upper path:</u> To calculate $\Pr\big((W_1 \cup W_2) \cap H_1\big)$;

- $\Pr(W_1 \cup W_2) = 1 - (1-p)^2, \Pr(H_1) = q$.
- Therefore $\Pr\big((W_1 \cup W_2) \cap H_1\big) = \Pr(W_1 \cup W_2)\Pr(H_1) = q(1-(1-p)^2)$

Step 2: <u>Lower path:</u> $\Pr(H_2H_3H_4) = \Pr(H_2)\Pr(H_3)\Pr(H_4) = q^3$

Step 3: Total probability of S using addition rule:

$$\begin{aligned}
\Pr(S) &= \Pr(\text{Upper}) + \Pr(\text{Lower}) - \Pr(\text{Both}) \\
\therefore \Pr(S) &= (1-(1-p)^2)q + q^3 - (1-(1-p)^2)q^4 \\
&= q^3 + (1-(1-p)^2)(q - q^4) \\
&= q^3 + \big(1-(1-2p+p^2)\big)(q-q^4) \\
&= q^3 + (2p - p^2)(q - q^4) \\
&= q^3 + (2p-p^2)q - (2p-p^2)q^4 \\
&= q^3 + (2p-p^2)q + p^2q^4 - 2pq^4 \\
&= q^3 + (2p-p^2)q + (p^2-2p)q^4 \\
\therefore \Pr(S) &= (p^2-2p)q^4 + q^3 + (2p-p^2)q
\end{aligned}$$

Question 3c.

Would $p > q$ or $p < q$ lead to a more stable circuit design? Explain why.

Solution:

Observing visually, the top path of the circuit contains a parallel component of Type 1 in series with Type 2, and the bottom is a series system of Type 2 components.

- Using a parallel system will allow for higher reliability, since the two events are independent; if 1 fails, the other path will be utilised.
- Using a series system is less reliable, if one fails then all fail in that system. Hence, if $p < q$, and a component in the lower system path fails, the circuit it less reliable as probability of p succeeding is less.

Therefore, $p > q$ leads to a more stable circuit design.