Preliminaries

ERRORS

Definition: (absolute/relative): Let $\bar{x}$ be an approximation to the true value x. Absolute $= |x-\bar{x}|$, Relative $= \frac{|x-\bar{x}|}{|x|}$, $x \neq 0$.

FLOATING POINT ARITHMETIC

Definition: A floating point number system $\mathbb{F}(\beta,k,m,M) \subset \mathbb{R}$:

- β: base, k: digits in significand, m: min exp, M: max exp.

Definition: The floating point numbers $f \in \mathbb{F}(\beta,k,m,M)$ are those real numbers that are expressible in the form:

$$\boxed{f = \pm(d_1.d_2d_3\cdots d_n)_\beta \times \beta^e}$$

Where the exponent $e \in \mathbb{Z}$, $m \leq e \leq M$, $d_1.d_2d_3\cdots d_k$ is known as the significand, the d_i are base-β digits, and $d_1 \neq 0$ unless $f=0$ to ensure unique representation of f.

- $\beta=2$, binary, $\beta=10$, decimal, $\beta=16$, hex

- E.g. $\mathbb{F}(10,2,-1,1) = \{0,$

$$\pm 1.0\times 10^{-1}, \pm 1.1\times 10^{-1}, \ldots, \pm 9.9\times 10^{-1},$$
$$\pm 1.0\times 10^{0}, \pm 1.1\times 10^{0}, \ldots, \pm 9.9\times 10^{0},$$
$$\pm 1.0\times 10^{1}, \pm 1.1\times 10^{1}, \ldots, \pm 9.9\times 10^{1}\}$$

- If $fl(x)$ exists w/ $x < |\mathbb{F}|_{min}$, underflow $\rightarrow n=0$
  $x > |\mathbb{F}|_{max}$, overflow $\rightarrow n=\text{Inf}$.
- Roundoff error occurs
- Relative error produced by rounding: $\frac{|x-fl(x)|}{|x|} \leq \frac{1}{2}\beta^{1-k}$.

Definition: Unit roundoff u given $\mathbb{F}(\beta,k,m,M)$ is

$$\boxed{u = \frac{1}{2}\beta^{1-k}}$$

- **IEEE 754 standard**: $\mathbb{F}(2, 53, -1022, 1023)$
  - Unit roundoff: 1.11×10^{-16}
  - Largest: 1.80×10^{308}
  - Smallest: 2.23×10^{-308}
  - Special values: ± 0, $\pm \infty$, NaN

- **Catastrophic cancellation:** FP subtraction of two almost equal numbers

TAYLOR POLYNOMIAL

- Suppose $f(x)$ that can be differentiated n times at x_0

Definition: (Taylor) for degree n for f centered at x_0

$$\boxed{\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x-x_0)^n. \\ &= \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k. \end{aligned}}$$

TAYLORS THEOREM

Theorem: Suppose f is $n+1$ times differentiable on $[a,b]$ containing x_0, and P_n exist of degree n on f centered on x_0. Then $\forall x \in [a,b]$, $\exists c \in [x_0, x]$ such that

$$f(x) = P_n(x) + \underbrace{\frac{f^{n+1}(c)}{(n+1)!}(x-x_0)^{n+1}}_{R_n(x)}$$

- Truncation error: limiting an infinite sum

ERRORS

E.g 1a.) $x = 4.0$, $\tilde{x} = 4.1$

$$\text{Abs. error} = |x - \tilde{x}| = |4 - 4.1|$$
$$= 0.1$$
$$\text{Rel. error} = \frac{|x-\tilde{x}|}{|x|} = \frac{|4-4.1|}{|4|}$$
$$= 0.025$$

E.g 1b.) $x = 4.0 \times 10^{-3}$, $\tilde{x} = 4.1 \times 10^{-3}$

$$\text{Abs error} = |x - \tilde{x}| = |4.0\times10^{-3} - 4.1\times10^{-3}|$$
$$= 1.0 \times 10^{-4}$$
$$\text{Rel error} = \frac{|x-\tilde{x}|}{|x|} = \frac{|4.0\times10^{-3} - 4.1\times10^{-3}|}{|4.0\times10^{-3}|}$$
$$= 0.025$$

E.g. 1c) 1101.001_2 ?

$$x = \overset{3}{1}\overset{2}{1}\overset{1}{0}\overset{0}{1}.\overset{-1}{0}\overset{-2}{0}\overset{-3}{1}{}_2 \Rightarrow 1.101001 \times 2^3$$
$$= 1\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0 + 0\times2^{-1} + 0\times2^{-2} + 1\times2^{-3}$$
$$= 8 + 4 + 0 + 1 + 0 + 0 + 0.125$$
$$= 13.125, \quad = 1.3125 \times 10^1$$

E.g. 1d.) $13.125 \rightarrow \mathbb{F}(10, 2, -1, 1)$

Step 1: Express x in terms of $\beta = 10$

Step 2: Scientific form $\Rightarrow 1.3125 \times 10^1$

Step 3: Ensure $m \leq e \leq M \Rightarrow 1 \in [-1, 1]$

Step 4: Round to 2: $fl(x) = 1.3 \times 10^{-1}$

E.g. 1e.) $13.125 \Rightarrow \mathbb{F}(2, 5, -4, 4)$

Step 1: Express x in terms of $\beta = 2$

Step 2: Scientific form $\Rightarrow x = 1.101001_2 \times 2^3$

Step 3: Exponent satisfied

Step 4: Round: $fl(x) = 1.1010_2 \times 2^3$

$\beta\ \ h\ \ m\ \ M$

EXAMPLE: $\mathbb{F}(2, 53, -1022, 1023)$

$$u = \frac{1}{2}\beta^{1-h} = \frac{1}{2}2^{1-53} = \frac{1}{2}2^{-52} = 2^{-53} = 1.1102 \times 10^{-16}$$

$$\begin{aligned} L &= 1.11\ldots1_2 \times 2^{1023} \\ &= (1\times2^0 + 1\times2^{-1} + \cdots 1\times2^{-52}) \times 2^{1023} \\ &= \underbrace{(2^0 + 2^{-1} + \cdots + 2^{-52})}_{S} \times 2^{1023} \end{aligned}$$

$$S = 2^0 + 2^{-1} + 2^{-2} + \cdots + 2^{-52} \qquad (1)$$
$$2^{-1}S = 2^{-1} + 2^{-2} + \cdots + 2^{-52} + 2^{-53} \qquad (2)$$

(1) − (2):
$$\begin{aligned} S - 2^{-1}S &= 2^0 - 2^{-53} \\ (1-2^{-1})S &= 1 - 2^{-53} \\ S &= \frac{1-2^{-53}}{1-2^{-1}} \\ &= \frac{1-2^{-53}}{2^{-1}} \\ &= 2(1-2^{-53}) \\ &= 2 - 2^{-52} \end{aligned}$$

Hence $L = S \times 2^{1023} = (2-2^{-52}) \times 2^{1023} = 1.80 \times 10^{308}$

$$\begin{aligned} S &= 1.000\ldots0_2 \times 2^{-1022} \\ &= (1\times2^0 + 0\times2^{-1} + \cdots + 0\times2^{-52}) \times 2^{-1022} \\ &= 1\times2^{-1022} = 2^{-1022} \\ &= 2.23 \times 10^{-308} \end{aligned}$$

E.g. 1f.) $x = 3.34$, $y = 2.44$, $z = 4.56$, $x^2 - yz$

$x^2 = 3.34^2 = 11.1556$

$fl(x^2) = 11.2$

$yz = 2.44 \times 4.56 = 11.1264$

$fl(yz) = 11.1$

$fl(x^2 - yz) = 0.1$

Exact $= 0.0292$

EXAMPLE.) Rewrite the formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ such that there is no CC for $b^2 \gg 4ac$, $a=1$, $b=-(10^{-8}+10^{8})$, $c=1$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Since $b^2 \gg 4ac$, assume $b^2-4ac \approx b^2 \Rightarrow \sqrt{b^2-4ac} = \sqrt{b^2} = |b|$

$$\therefore \; x_1 = \frac{-b+|b|}{2a}, \quad x_2 = \frac{-b-|b|}{2a}$$

If $b>0$, $|b|=b$, $x_1 \approx \frac{b-b}{2a}$ experiences CC

If $b<0$, $|b|=-b$, $x_2 \approx \frac{b-b}{2a}$ experiences CC.

$a=1$, $b=-(10^{8}+10^{-8})$, $c=1$

Roots are $x_1 = 10^8$, $x_2 = 10^{-8}$

E.g., $ax_1^2 + bx_1 + c = 1 \times (10^8)^2 - (10^8+10^{-8}) \times 10^8 + 1$

$$= 10^{16} - 10^{16} - 1 + 1$$

$$= 0$$

Similarly, $ax_2^2 + bx_2 + c = 0$

Since $b = -(10^8+10^{-8}) < 0$, x_1 is good, x_2 experiences CC.

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = 10^8 \text{ (MATLAB)}$$

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = 1.4901 \times 10^{-8} \text{ (MATLAB)}$$

CC can be avoided by noting:

$$x_1x_2 = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$= \frac{b^2-b^2+4ac}{4a^2}$$

$$x_1x_2 = \frac{4ac}{4a^2} = \frac{c}{a} \Rightarrow x_2 = \frac{c}{ax_1}$$

W/ $a=1$, $c=1$, $x_1 = 10^8$

$x_2 = 10^{-8}$

DIFFERENTIAL EQUATIONS

i.) Integration: $\frac{dy}{dt} = t^2$

$$dy = t^2 dt$$
$$\int dy = \int t^2 dt$$
$$y(t) = \frac{1}{3}t^3 + C$$

ii.) Integrating factor: $\frac{dy}{dt} = t + y$

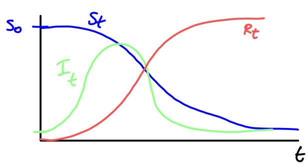
$$\frac{dy}{dt} - y = t$$
$$e^{-t}\frac{dy}{dt} - ye^{-t} = te^{-t}$$
$$\frac{d}{dt}(e^{-t}y) = te^{-t}$$
$$\int \frac{d}{dt}(e^{-t}y)dt = \int te^{-t}dt$$
$$e^{-t}y(t) = -te^{-t} - \int e^{t}dt$$
$$e^{-t}y(t) = -te^{-t} - e^{t} + C$$
$$y(t) = Ce^{t} - t - 1$$

iii.) Separation of variables: $\frac{dy}{dt} = ty$

$$dy = ty\,dt$$
$$\frac{1}{y}dy = t\,dt$$
$$\int \frac{1}{y}dy = \int t\,dt$$
$$\log_e(y) = \frac{1}{2}t^2 + C$$
$$y = e^{\frac{1}{2}t^2 + c}, \quad e^{c} = A$$
$$\therefore y = Ae^{\frac{1}{2}t^2}$$

SIR model: $\frac{dS}{dt} = -\beta IS$

$$\frac{dI}{dt} = \beta IS - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$



Ex.) Verify that $y = t^2 + \frac{1}{t}$ is solution to IVP $\frac{dy}{dt} = 3t - \frac{y}{t}$, $y(1) = 2$.

$\frac{dy}{dt} = 3t - \frac{y}{t}$

$y = t^2 + \frac{1}{t}$

LHS: $\frac{dy}{dt}$

$= \frac{d}{dt}\left(t^2 - \frac{1}{t}\right)$ ← t^{-1}

$= 2t - \frac{1}{t^2}$

RHS: $3t - \frac{y}{t}$

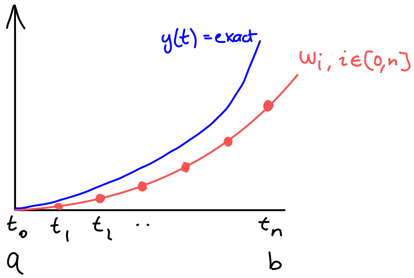
$= 3t - \frac{1}{t}\left(2t - \frac{1}{t^2}\right)$

$= 3t - t - \frac{1}{t^2}$

$= 2t - \frac{1}{t^2}$

LHS = RHS $\Rightarrow$ $y(t)$ is sol$^{\underline{n}}$ to the ODE

TIME DISCRETISATION SCHEME:



Divide $[a,b]$ into n subintervals, width $h=(b-a)/n$.

Define $t_i = a + ih$, $i \in [0,n]$, $t_0 = a$, $t_n = b$.

Aim: $y_i = y(t_i)$, $w_i \Rightarrow w_i \approx y_i$

EULER METHOD:

$$y(t) = P_1(t) + R_1(t)$$
$$= y(t_i) + y'(t_i)(t-t_i) + \frac{y''(c)}{2}(t-t_i)^2, \quad t_i \le c \le t$$

Evaluate $t = t_{i+1}$

$$y(t_{i+1}) = y(t_i) + y'(t_i)(t_{i+1}-t_i) + \frac{y''(c)}{2}(t_{i+1}-t_i)^2$$
$$y_{i+1} = y_i + hy'(t_i) + \frac{y''(c)}{2}h^2$$
$$= y_i + hy'(t_i) + \mathcal{O}(h^2)$$ "order h^2"

From ODE: $\frac{dy}{dt} = f(t,y) \Rightarrow y'(t) = f(t,y) = f(t,y(t))$

Hence, $y'(t_i) = f(t_i, y(t_i)) = f(t_i, y_i)$

Hence $y_{i+1} = y_i + hf(t_i, y_i) + \mathcal{O}(h^2)$
$$\approx y_i + hf(t_i, y_i)$$

w_i: $w_i \approx y_i$, $w_{i+1} \approx y_{i+1}$

$$w_{i+1} = w_i + hf(t_i, w_i), \quad i \in 0, 1, \dots, n-1$$

Algorithm: Start $w_0 = \alpha$ (initial cond.)

$i = 0$, $w_1 = w_0 + hf(t_0, w_0)$

$i = 1$, $w_2 = w_1 + hf(t_1, w_1)$

$\vdots$

$i = n-1$; $w_n = w_{n-1} + hf(t_{n-1}, w_{n-1})$

Ex.) Use Euler's method w/ $h=0.5$ to solve $\frac{dy}{dt} = 3t - \frac{y}{t}$, $1 \le t \le 2$

$y(1) = 2$

$f(t,y) = 3t - \frac{y}{t}$, $a=1$, $b=2$, $\alpha = 2$, $h = 0.5$

$t_0 = 1$, $w_0 = 2$, ($t_0 = a$)

$t_1 = 1.5$, $w_1 = w_0 + hf(t_0, w_0)$ ($t_1 = t_0 + h$)

$= 2 + 0.5 f(1,2)$

$= 2.5$ (Exact $y(1.5) = 2.9167$)

$t_2 = 2$, $w_2 = w_1 + hf(t_1, w_1)$

$= 2.5 + 0.5(1.5, 2.5)$

$= 3.9167$ (Exact: $y(2) = 4.5$)

Local error $O(h^{p+1})$, global error is $O(h^p)$

Q.1.) Solve initial value problem: $\frac{dy}{dt} = 1 + \frac{y}{t}$, $1 \le t \le 2$, $y(1) = 1$, $h = 0.5$

$f(t,y) = 1 + \frac{y}{t}$, $a=1$, $b=2$, $\alpha = 1$, $h = 0.5$

$t_0 = 1$, $w_0 = 1$

$t_1 = 1.5$, $w_1 = w_0 + hf(t_0, w_0)$

$= 1 + 0.5 f(1,1)$

$= 1 + 0.5 \times (1 + \frac{1}{1})$

$= 2$

$t_2 = 2$, $w_2 = w_1 + hf(t_1, w_1)$

$= 2 + 0.5 f(1.5, 2)$

$= 2 + 0.5 \times (1 + \frac{2}{1.5})$

$= 2.6667$

SECOND ORDER TAYLOR METHOD

Derivation: Taylor polynomial of $n=2$ for $y(t)$ centered $t=t_i$

$$y(t) = y(t_i) + y'(t_i)(t-t_i) + \frac{y''(t_i)}{2}(t-t_i)^2 + O((t-t_i)^3)$$

Evaluate $t = t_{i+1}$

$$y(t_{i+1}) = y(t_i) + y'(t_i)(t_{i+1}-t_i) + \frac{y''(t_i)}{2}(t_{i+1}-t_i)^2 + O((t_{i+1}-t_i)^3)$$

$$y_{i+1} = y_i + hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3)$$

From ODE $\left(\frac{dy}{dt} = y'(t) = f(t,y)\right)$ gives $y'(t_i) = f(t_i, y_i)$

Also, $y''(t) = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}f(t,y) = f'(t,y) = f'(t, y(t))$

$\Rightarrow y''(t_i) = f'(t_i, y(t_i)) = f'(t_i, y_i)$

Hence, $y_{i+1} = y_i + hf(t_i,y_i) + \frac{h^2}{2}f'(t_i,y_i) + O(h^3)$

$y_{i+1} \approx y_i + hf(t_i,y_i) + \frac{h^2}{2}f'(t_i,y_i)$

$$w_{i+1} = w_i + hf(t_i,w_i) + \frac{h^2}{2}f'(t_i,w_i), \quad i = 0,1,\dots,n-1$$

Ex.) Use 2nd order Euler's method w/ $h=0.5$ to solve $\frac{dy}{dt} = 3t - \frac{y}{t}$, $1 \le t \le 2$

$f(t,y) = 3t - \frac{y}{t}$, $a=1$, $b=2$, $\alpha = 2$, $h = 0.5$ $(n=2)$

$$\begin{aligned}
f'(t,y) &= \frac{d}{dt}f(t,y) \\
&= \frac{d}{dt}\left(3t - \frac{y}{t}\right) \\
&= \frac{d}{dt}3t - \frac{d}{dt}t^{-1}y \\
&= 3 - \left[\frac{d}{dt}(t^{-1})y + t^{-1}\frac{d}{dt}y\right] \\
&= 3 - \left[-t^{-2}y + t^{-1}\frac{dy}{dt}\right] \\
&= 3 - \left[-\frac{y}{t^2} + t^{-1}f(t,y)\right] \\
&= 3 + \frac{y}{t^2} - t^{-1}\left(3t - \frac{y}{t}\right) \\
&= \cancel{3+}\frac{y}{t^2} - \cancel{3+}\frac{y}{t^2} \\
&= \frac{2y}{t^2}
\end{aligned}$$

$t_0 = 1$, $w_0 = 2$

$t_1 = 1.5$, $w_1 = w_0 + hf(t_0,w_0) + \frac{h^2}{2}f'(t_0,w_0)$

$= 2 + 0.5f(1,2) + \frac{0.5^2}{2}f'(1,2)$

$= 3$ (Exact $y(1.5) = 2.9167$)

$t_2 = 2$, $w_2 = w_1 + hf(t_1,w_1) + \frac{h^2}{2}f'(t_1,w_1)$

$= 3 + 0.5f(1.5,3) + \frac{h^2}{2}f'(1.5,3)$

$= 4.5833$ (Exact: $y(2) = 4.5$)

MODIFIED EULER METHOD

Recall $f'(t_i, y_i) = \lim_{h \to 0} \frac{f(t_{i+1}, y_{i+1}) - f(t_i, y_i)}{h}$

For small $h \Rightarrow f'(t_i, y_i) \approx \frac{f(t_{i+1}, y_{i+1}) - f(t_i, y_i)}{h} + O(h)$

Derivation: 2^{nd} order: $y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2} f'(t_i, y_i) + O(h^3)$

To avoid $f'(t_i, y_i)$, use approximation.

$$f'(t_i, y_i) = \frac{f(t_{i+1}, y_{i+1}) - f(t_i, y_i)}{h} + O(h)$$

Substituting

absorbed into single $O(h^3)$

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2}\Big[\underbrace{\frac{f(t_{i+1}, y_{i+1}) - f(t_i, y_i)}{h} + O(h)}_{O(h^3)}\Big] + O(h^3)$$

$$= y_i + hf(t_i, y_i) + \frac{h}{2}\left[f(t_{i+1}, y_{i+1}) - f(t_i, y_i)\right] + O(h^3)$$

$$= y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1})\right] + O(h^3)$$

$$\approx y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1})\right]$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_{i+1})\right]$$

Problem is the formula for w_{i+1} involves itself. Use Euler's method $w_{i+1} = w_i + hf(t_i, w_i)$ to approximate w_{i+1} on the RHS

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right]$$

$$w_i + \frac{1}{2}\Big[\underbrace{hf(t_i, w_i)}_{k_1} + \underbrace{f(t_{i+1}, w_i + k_1)}_{k_2}\Big]$$

E.g.) $f(t,y) = 3t - \frac{y}{t}$, $a=1$, $b=2$, $\alpha=2$, $n=2$ $(h=0.5)$

$t_0 = 1$, $w_0 = 2$

$t_1 = 1.5$, $k_1 = hf(t_0, w_0)$
$= 0.5f(1,2)$
$= 0.5$

$k_2 = hf(t_1, w_0 + k_1)$
$= 0.5f(1.5, 2+0.5)$
$= 0.5f(1.5, 2.5)$
$= 1.4167$

$w_1 = w_0 + \frac{1}{2}(k_1 + k_2)$
$= 2.9583$

$t_2 = 2$, $k_1 = hf(t_1, w_1)$
$= 1.2639$

$k_2 = hf(t_2, w_1 + k_1)$
$= 0.5f(2, w_1 + 0.5)$
$= 1.9444$

$w_2 = w_1 + \frac{1}{2}(k_1 + k_2)$
$= 4.5625$

RUNGE KUTTA METHOD

- $w_0 = \alpha,\ w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),\ i = 0, 1, \dots, n-1$

$$\begin{cases} k_1 = hf(t_i, w_i) \\ k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right) \\ k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right) \\ k_4 = hf(t_i + h, w_i + k_3) \end{cases}$$

$1 \le t \le 2$

E.g.) $f(t,y) = 3t - \frac{y}{t}$, $a = 1$, $b = 2$, $\alpha = 2$, $n = 2$ $(h = 0.5)$

$t_0 = 1,\ w_0 = 2$

$t_1 = 1.5,\ w_1 = w_0 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$

$$k_1 = hf(t_0, w_0) = 0.5$$
$$k_2 = hf\left(t_0 + \frac{h}{2}, w_0 + \frac{k_1}{2}\right) = 0.9750$$
$$k_3 = hf\left(t_0 + \frac{h}{2}, w_0 + \frac{k_2}{2}\right) = 0.8800$$
$$k_4 = hf(t_0 + h, w_0 + k_3) = 1.2900$$

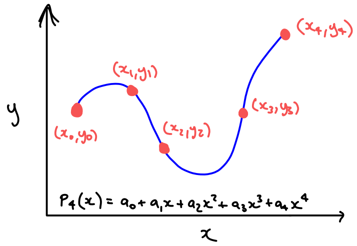
$\therefore w_1 = 2.9167$, (Exact: 2.9167)

$t_2 = 2,\ w_2 = w_1 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$

$\therefore w_2 = 4.5$ (Exact, 4.5)

INTERPOLATION

- A function interpolates points $\{(x_0,y_0),(x_1,y_1),\dots,(x_n,y_n)\}$ if it satisfies $f(x_0)=y_0, f(x_1)=y_1,\dots,f(x_n)=y_n$.



- Theorem: Let the abscissas $x_0,x_1,\dots,x_n$ be distinct and function values $y_0,y_1,\dots,y_n$ be given. There exists a unique interpolating polynomial P_n of degree (at most) n such that $P_n(x_i)=y_i$ for $i=0,1,\dots,n$., Lagrange, Newton divided difference, Newton forward difference

$\Rightarrow$ $n=0$: (x_0,y_0), $P_0(x)=y_0$

$n=1$: (x_0,y_0) and (x_1,y_1), $x_0\neq x_1$, $P_1(x)=y_0+\frac{y_1-y_0}{x_1-x_0}(x-x_0)$

Consider $n=1$, $(x_0,y_0),(x_1,y_1)$

Find $P_1(x)=a_0+a_1x$ interpolating 2 points

$$P_1(x_0)=y_0 \Rightarrow a_0+a_1x_0=y_0 \quad \dots(1)$$
$$P_1(x_1)=y_1 \Rightarrow a_0+a_1x_1=y_1 \quad \dots(2)$$

$$(1)-(2):\ \cancel{a_0}+a_1x_0-(\cancel{a_0}+a_1x_1)=y_0-y_1$$
$$a_1(x_0-x_1)=y_0-y_1$$
$$a_1=\frac{y_0-y_1}{x_0-x_1}=\frac{y_1-y_0}{x_1-x_0}$$

Sub a_1 into (1), solve for a_0

$$\Rightarrow a_0+\frac{y_1-y_0}{x_1-x_0}x_0=y_0$$
$$a_0=y_0-\frac{y_1-y_0}{x_1-x_0}x_0$$
$$a_0=\frac{x_1y_0-x_0y_1}{x_1-x_0}$$

Hence, $P_1(x)=\frac{x_1y_0-x_0y_1}{x_1-x_0}+\frac{y_1-y_0}{x_1-x_0}x$, How do we generalise the form of $P_1(x)$ to more data points? Solve the corresponding lin. eq. as we did above for 2 points.

Example, $n=2$, $P_2(x) = a_0 + a_1 x + a_2 x^2$ that interpolates $(x_0,y_0), (x_1,y_1), (x_2,y_2)$

$$\left.\begin{aligned} P_2(x_0) = y_0 &\Rightarrow a_0 + a_1 x_0 + a_2 x_0^2 = y_0 \\ P_2(x_1) = y_1 &\Rightarrow a_0 + a_1 x_1 + a_2 x_1^2 = y_1 \\ P_2(x_2) = y_2 &\Rightarrow a_0 + a_1 x_2 + a_2 x_2^2 = y_2 \end{aligned}\right\}$$

Could solve for a_0, a_1, a_2
(Topic 6)

Lagrange: rearranging $P_1(x)$:

$$\begin{aligned} P_1(x) &= \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0} x \\ &= \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0} + \frac{x y_1 - x y_0}{x_1 - x_0} \\ &= \frac{x_1 y_0 - x y_0}{x_1 - x_0} + \frac{x y_1 - x_0 y_1}{x_1 - x_0} \\ &= \frac{x_1 - x}{x_1 - x_0} y_0 + \frac{x - x_0}{x_1 - x_0} y_1 \\ &= \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1 \end{aligned}$$

$(x_0, y_0), (x_1, y_1)$

Rearrange $P_1(x)$ as follows $\frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1$

Rewrite as $P_1(x)$ as $P_1(x) = L_{1,0}(x) y_0 + L_{1,1}(x) y_1$

E.g.) Find interpolating polynomial $P_2(x)$ through $(x_0,y_0), (x_1,y_1), (x_2,y_2)$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$P_n(x) = L_{n,0}(x) y_0 + L_{n,1}(x) y_1 + \dots + L_{n,n}(x) y_n = \sum_{i=0}^{n} L_{n,i}(x) y_i$$

where

$$L_{n,i}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x-x_j}{x_i-x_j}$$

Note, $L_{n,i}(x_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

Example.

(a) Find the Lagrange form of the interpolating polynomial for the function $f(x) = \cos(x)$ and the abscissas $x_0 = 0$, $x_1 = \pi/2$, $x_2 = \pi/6$.

(b) Find an upper bound on the absolute error incurred when using this polynomial to approximate $\cos(x)$ for x between 0 and $\pi/2$.

(a) $x_0 = 0, \quad y_0 = f(x_0) = \cos(x_0) = \cos(0) = 1$

$x_1 = \frac{\pi}{2}, \quad y_1 = f(x_1) = \cos(x_1) = \cos\left(\frac{\pi}{2}\right) = 0$

$x_2 = \frac{\pi}{6} \quad y_2 = f(x_2) = \cos(x_2) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Find $P_2(x)$ that passes through $(0,1), \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-\frac{\pi}{2})(x-\frac{\pi}{6})}{(0-\frac{\pi}{2})(0-\frac{\pi}{6})} + \frac{(x-0)(x-\frac{\pi}{2})}{(\frac{\pi}{6}-0)(\frac{\pi}{6}-\frac{\pi}{2})} \times \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{12}{\pi^2}\left(x-\frac{\pi}{2}\right)\left(x-\frac{\pi}{6}\right) - \frac{9\sqrt{3}}{\pi^2} x\left(x-\frac{\pi}{2}\right)}$$

(b) Absolute error $= |f(x) - P_2(x)|$

$= |R_2(x)|$

$= \left|\frac{f'''(c)}{6}(x-0)\left(x-\frac{\pi}{2}\right)\left(x-\frac{\pi}{6}\right)\right|$

$= \left|\frac{\sin(c)}{6} x\left(x-\frac{\pi}{2}\right)\left(x-\frac{\pi}{6}\right)\right|$

$= \frac{|\sin(c)|}{6}\left|x\left(x-\frac{\pi}{2}\right)\left(x-\frac{\pi}{6}\right)\right|$

Absolute error $\leq \frac{1}{6}\left|x\left(x-\frac{\pi}{2}\right)\left(x-\frac{\pi}{6}\right)\right|$

$\leq \frac{1}{6} \max_{0 \leq x \leq \frac{\pi}{2}} \left|x\left(x-\frac{\pi}{2}\right)\left(x-\frac{\pi}{6}\right)\right|$

$\approx\leq 0.05$

NEWTON DIVIDED DIFFERENCE FORM

- $P_n(x)$ is expressed as $P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_n)$
- The interpolation conditions
  - $P_n(x_0) = y_0$; $P_n(x_1) = y_1$, ... $P_n(x_n) = y_n$.
  - $a_0 = y_0$
  - $a_0 + a_1(x_1 - x_0) = y_1$
  - $a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = y_2$

    $\vdots$
  - $a + a_1(x_n - x_0) + \cdots + a_n(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1}) = y_n$

Solve for a_0, a_1, a_2

$a_0 = y_0$

$a_0 + a_1(x_1 - x_0) = y_1$

$a_1(x_1 - x_0) = y_1 - a_0$

$a_1 = \frac{y_1 - a_0}{x_1 - x_0}$

$a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = y_2$

$a_2(x_2 - x_0)(x_2 - x_1) = y_2 - a_0 - a_1(x_2 - x_0)$

$a_2 = \frac{y_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_1 - x_0)}$

$a_2 = \frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_1 - x_0)}$

$$a_2 = \frac{\frac{y_2 - y_0}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\,\frac{x_2 - x_0}{x_2 - x_1}}{x_2 - x_0}$$

$$= \frac{\frac{y_2 - y_0}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\,\frac{x_2 - x_1 + x_1 - x_0}{x_2 - x_1}}{x_2 - x_0}$$

$$= \frac{\frac{y_2 - y_0}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\left[1 + \frac{x_1 - x_0}{x_2 - x_1}\right]}{x_2 - x_0}$$

$$= \frac{\frac{y_2 - y_0}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0} - \frac{y_1 - y_0}{x_2 - x_1}}{x_2 - x_0}$$

$$= \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

- **Theorem**: the zeroth divided difference of f wrt x_i is $f[x_i]$, defined by $\boxed{f[x_i]=y_i}$
- Theorem: the kth divided difference of f wrt $x_i, x_{i+1}, \dots, x_{i+k}$ is $f[x_i, x_{i+1}, \dots, x_{i+k}]$, defined by:

$$\boxed{f[x_i, x_{i+1}, \dots x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}}$$

Example. Express the divided differences $f[x_0, x_1]$, $f[x_1, x_2]$ and $f[x_0, x_1, x_2]$ in terms of the abscissa x_0, x_1, x_2 and function values y_0, y_1, y_2.

$$f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{y_1-y_0}{x_1-x_0}$$

$$f[x_1,x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{\frac{y_2-y_1}{x_2-x_1} - \frac{y_1-y_0}{x_1-x_0}}{x_2-x_0}$$

Using this new notation, the coefficients can be written succinctly:

$$\begin{aligned} a_0 &= f[x_0] \\ a_1 &= f[x_0, x_1] \\ a_2 &= f[x_0, x_1, x_2] \\ &\vdots \\ a_n &= f[x_0, x_1, x_2, \dots, x_n] \end{aligned}$$

and thus the interpolating polynomial can be expressed in the form:

$$\begin{aligned} P_n(x) &= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \dots \\ &\qquad + f[x_0,x_1,x_2,\dots,x_n](x-x_0)(x-x_1)\dots(x-x_{n-1}) \\ &= \sum_{k=0}^{n} f[x_0,x_1,\dots,x_k] \left(\prod_{i=0}^{k-1} (x-x_i) \right) \end{aligned}$$

Definition.
Given the distinct abscissas $x_0, x_1, \dots, x_n$ and function values $y_0, y_1, \dots, y_n$, the polynomial P_n defined by

$$P_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \dots, x_k] \left(\prod_{i=0}^{k-1} (x - x_i) \right)$$

is called the **Newton divided difference form of the interpolating polynomial**.

Example.

(a) Find the Newton divided difference form of the interpolating polynomial for the function $f(x) = \cos(x)$ and the abscissas $x_0 = 0$, $x_1 = \pi/2$, $x_2 = \pi/6$.

(b) Use the polynomial to approximate $\cos(\pi/3)$.

(c) With as little extra work as possible, include the abscissa $x_3 = \pi/4$ and re-compute the approximation of $\cos(\pi/3)$.

Divided differences are best processed in tabular format:

x_i	zeroth	first	second	third
x_0	$f[x_0]^{a_0}$	$f[x_0,x_1]^{a_1}$		
x_1	$f[x_1]$	$f[x_1,x_2]$	$f[x_0,x_1,x_2]^{a_2}$	$f[x_0,x_1,x_2,x_3]^{a_3}$
x_2	$f[x_2]$	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	
x_3	$f[x_3]$			

$$P_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_3)$$

Example

(a) $x_0 = 0$, $y_0 = 1$, $x_1 = \frac{\pi}{2}$, $y_1 = 0$, $x_2 = \frac{\pi}{6}$, $y_2 = \frac{\sqrt{3}}{2}$

x_i	y_i zeroth	first	second
0	1	-0.6366	-0.3636
1.5708	0	-0.8270	
0.5236	0.8660		

$\frac{0-1}{1.5708-0}$ $\qquad$ $\frac{0.8660-0}{0.5236-1.5736}$

Hence, $P_2(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$

$$= 1 - 0.6366(x-0) - 0.3636(x-0)(x-1.5708)$$

$$= 1 - 0.6366x - 0.3636x(x-1.5708)$$

(b) $P_2(x) = 1 - 0.6366x - 0.3636x(x - 1.5708)$

$\cos(x) \approx P_2(x), \quad x \in [0, \frac{\pi}{2}]$

$\therefore \cos(\frac{\pi}{3}) \approx P_2(\frac{\pi}{3})$

$= 0.5327$ (Exact: 0.5, abs error of 0.0327)

(c) $P_3(x) = P_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$

	x_i	y_i zeroth	first	second	third
	0	1			
			-0.6366		
	1.5708	0		-0.3636	
			-0.8270		0.1063
	0.5236	0.8660		-0.2801	
			-0.6070		
x_3	0.7854	0.7071			

$P_3(x) = P_2(x) + 0.1063x(x - 1.5708)(x - 0.5236)$

$\cos(x) \approx P_3(x), \quad x \in [0, \frac{\pi}{2}]$

$\therefore \cos(\frac{\pi}{3}) \approx 0.5022$, (Exact: 0.5, abs error = 0.0022

NEWTON FORWARD DIFFERENCE FORM

Definition: forward difference operator Δ is defined by $\Delta y_i = y_{i+1} - y_i$

Higher order:

$$\begin{aligned}\Delta^2 y_i &= \Delta(\Delta y_i) \\ &= \Delta(y_{i+1} - y_i) \\ &= \Delta y_{i+1} - \Delta y_i \\ &= y_{i+2} - 2y_{i+1} + y_i\end{aligned}$$

$$\begin{aligned}\Delta^3 y_i &= \Delta(\Delta^2 y_i) \\ &= \Delta(y_{i+2} - 2y_{i+1} + y_i) \\ &= \Delta y_{i+2} - \Delta 2y_{i+1} + \Delta y_i \\ &= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i\end{aligned}$$

$h = x_{i+1} - x_i$

Theorem. Given the equally-spaced abscissas $x_0, x_1, \ldots, x_n$ and function values $y_0, y_1, \ldots, y_n$, the divided difference notation and the forward difference notation are related by

$$f[x_0, x_1, \ldots, x_k] = \frac{\Delta^k y_0}{k!\, h^k},$$

where $h = x_{i+1} - x_i$ is the spacing between the abscissas.

Using the above theorem, and making the substitution $x = x_0 + sh$ into the Newton divided difference form, we obtain

$$\begin{aligned}P_n(x) &= y_0 + s\Delta y_0 + \frac{s(s-1)}{2}\Delta^2 y_0 + \ldots + \frac{s(s-1)(s-2)\ldots(s-n+1)}{n!}\Delta^n y_0 \\ &= \sum_{k=0}^{n} \binom{s}{k} \Delta^k y_0\end{aligned}$$

which expresses $P_n(x)$ in terms of the new variable $s = (x - x_0)/h$.

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From Theorem: $f[x_0, x_1, \dots, x_k] = \frac{\Delta^k y_0}{k!\, h^k}$

$\underline{k=0}$: LHS $= f[x_0] = y_0$

RHS $= \frac{\Delta^0 y_0}{0!\, h^0} = y_0$, LHS = RHS

$\underline{k=1}$: LHS $= f[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$

RHS $= \frac{\Delta y_0}{1!\, h^1} = \frac{\Delta y_0}{h} = \frac{y_1 - y_0}{h} = \frac{y_1 - y_0}{x_1 - x_0}$, LHS = RHS

$\underline{k=2}$: LHS $= f[x_0, x_1, x_2] = \dfrac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$

$$\begin{aligned}
\text{RHS} = \frac{\Delta^2 y_0}{2!\, h^2} &= \frac{y_2 - 2y_1 + y_0}{2h^2} \\
&= \frac{y_2 - y_1 - (y_1 - y_0)}{2h^2} \\
&= \frac{\frac{y_2 - y_1}{h} - \frac{y_1 - y_0}{h}}{2h} \\
&= \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}
\end{aligned}$$

, LHS = RHS

NDDF $\rightarrow$ NFDF $\quad (x_i = x_0 + ih)$

$$P_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \dots, x_k] \prod_{i=0}^{k-1} (x - x_i)$$

$\left(f[x_0, x_1, \dots, x_k] = \frac{\Delta^k y_0}{k!\, h^k} \right.$, $x = x_0 + sh$, $x_i = x_0 + ih$, $s = \frac{x - x_0}{h}$

$$\begin{aligned}
&= \sum_{k=0}^{n} \frac{\Delta^k y_0}{k!\, h^k} \prod_{i=0}^{k-1} [\cancel{x_0} + sh - (\cancel{x_0} + ih)] \\
&= \sum_{k=0}^{n} \frac{\Delta^k y_0}{k!\, h^k} \prod_{i=0}^{k-1} (s - i) h \\
&= \sum_{k=0}^{n} \frac{\Delta^k y_0}{k!\, h^k} \left[\prod_{i=0}^{k-1} (s - i) \right] \left[\prod_{i=0}^{k-1} h \right] \\
&= \sum_{k=0}^{n} \frac{\Delta^k y_0}{k!\, \cancel{h^k}} \left[\prod_{i=0}^{k-1} (s - i) \right] \cancel{h^k} \\
&= \sum_{k=0}^{n} \frac{\prod_{i=0}^{k-1} (s - i)}{k!} \Delta^k y_0 \\
&\boxed{= \sum_{k=0}^{n} \binom{s}{k} \Delta^k y_0}
\end{aligned}$$

$\Rightarrow$ Binomial coefficient, $\binom{s}{k} = \frac{s!}{k!(s-k)!}$

Example.

(a) Find the Newton forward difference form of the interpolating polynomial through the following points: $(0, 1)$, $(0.5, 1.6487)$, $(1, 2.7183)$.

(b) Given that these data come from the function $f(x) = e^x$, use the polynomial to approximate $\sqrt[3]{e}$, and compute the absolute error in the approximation.

(c) Include the points $(1.5, 4.4817)$, $(2, 7.3891)$, and repeat (a) and (b).

Process FD in tabular format

x_i	y_0	Δy_i	$\Delta^2 y_i$	
x_0	y_0			
		Δy_0		
x_1	y_1		$\Delta^2 y^0$	
		Δy_1		$\Delta^3 y_0$
x_2	y_2		$\Delta^2 y_1$	
		Δy_2		$\Delta^2 y_1 - \Delta^2 y_0$
x_3	y_3			
		$y_3 - y_2$	$\Delta y_2 - \Delta y_1$	

Example (a)

x_i	y_i	Δy_i	
0	1		
		0.6487	
0.5	1.6487		0.4209
		1.0696	
1	2.7183		

$$P_2(x) = \sum_{k=0}^{2} \binom{s}{k} \Delta^k y_0$$

$$= \binom{s}{0} y_0 + \binom{s}{1} \Delta y_0 + \binom{s}{2} \Delta^2 y_0$$

$$= y_0 + s\Delta y_0 + \frac{s(s-1)}{2} \Delta^2 y_0 \qquad s = \frac{x - x_0}{h}$$

$$= 1 + 0.6487s + \frac{s(s-1)}{2} \times 0.4209$$

$$= 1 + 0.6487s + 0.21045\, s(s-1), \quad s = \frac{x-0}{0.5} = 2x$$

$$= 1 + 0.6487 \times 2x + 0.2104 \times 2x(2x-1)$$

(b) $e^x \approx P_2(x)$ for $0 \le x \le 1$, $\sqrt[3]{e} = e^{\frac{1}{3}} \approx P_2\left(\frac{1}{3}\right)$

$P_2\left(\frac{1}{3}\right) = 1.3857$, [Exact = 1.3956]

$$\text{error} = \left|P_2\left(\tfrac{1}{3}\right) - e^{1/3}\right| = 0.0099$$

(c)

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
0	1	0.6487	0.4209	0.2729	0.1773
0.5	1.6487	1.0696	0.6938	0.4502	
1	2.7183	1.7634	1.1140		
1.5	4.4817	2.9074			
2	7.3891				

$$P_4(x) = P_2(x) + \binom{s}{3}\Delta^3 y_0 + \binom{s}{4}\Delta^4 y_0$$

$$= P_2(x) + \frac{s(s-1)(s-2)}{6}\,0.2729 + \frac{s(s-1)(s-2)(s-3)}{24}\,0.1773$$

$$= P_2(x) + \ldots$$

ROOTS OF NON LINEAR FUNCTIONS

$f(x) = ax+b$, $f(x) = 0 \Rightarrow ax+b=0$

$$ax = -b$$

$$x = \frac{-b}{a}$$

$f(x) = ax^2+bx+c$, $f(x)=0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$f(x) = ae^{bx} - d$, $f(x)=0$, $ae^{bx}-d=0$

$$ae^{bx} = d$$

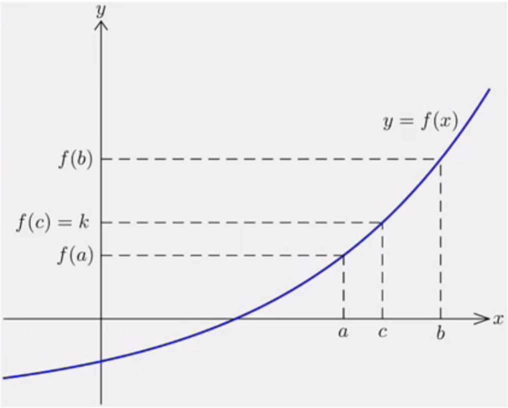
$$e^{bx} = \frac{d}{a}$$

$$bx = \log_e\left(\frac{d}{a}\right)$$

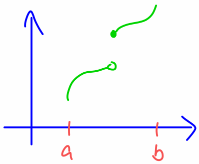
$$x = \frac{1}{b}\log_e\left(\frac{d}{a}\right), \quad b \neq 0$$

The **bisection method** is a simple method based on the Intermediate value theorem.

Theorem (Intermediate value theorem). Let f be a continuous function on the interval $[a, b]$, and let k be any number between $f(a)$ and $f(b)$ inclusive. Then there exists a number c in $[a, b]$ such that $f(c) = k$.

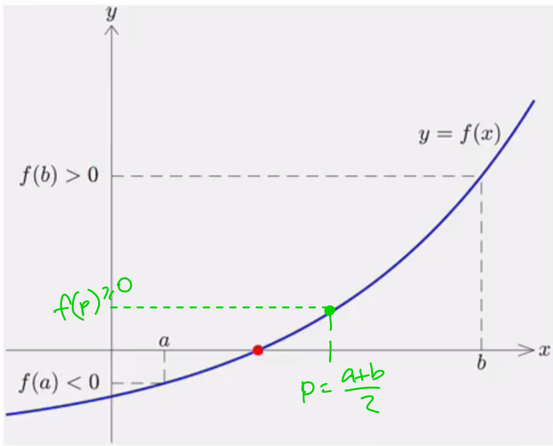


Discontinuous



Bisection method:

Corollary. Let f be a continuous function on the interval $[a, b]$, and let $f(a)$ and $f(b)$ have opposite signs. Then f has a root in $[a, b]$.



The algorithm is summarised below:

1. Find a bracketing for $f(x)$, i.e., find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs.
2. Find the midpoint p of the interval $[a, b]$

$$p = \frac{a+b}{2}.$$

3. Check the sign of $f(p)$
   (a) If $f(p)$ and $f(a)$ have the same sign then p becomes the new a
   (b) If $f(p)$ and $f(b)$ have the same sign then p becomes the new b
4. Go back to step 2 and repeat until a satisfactory solution is obtained.

Example. Use the bisection method to solve

$$0.9\cos(x) = \sqrt{x}$$

on the interval $[0, 1]$ correct to two decimal places.

$0.9\cos(x) = \sqrt{x}$, $0 \le x \le 1$

$0.9\cos(x) - \sqrt{x} = 0$ is continuous on $[0,1]$

$f(0) = 0.9\cos(0) - \sqrt{0}$

$= 0.9$ $(+)$

$f(1) = 0.9\cos(1) - \sqrt{1} \approx -0.51$ $(-)$

$p = \frac{a+b}{2}$, $= 0.5$

$f(a) > 0$, $f(b) < 0$

$f(p) \approx 0.08 > 0$ $[p \to a]$

$p' = \frac{a+b}{2} = 0.75$

$f(a) > 0$, $f(b) < 0$

$f(p) \approx -0.21 < 0$ $[p \to b]$

Iteration	a	b	$p = \frac{a+b}{2}$	f(a)	f(b)	f(p)
1	0	1	0.5	+	−	+
2	0.5	1	0.75	+	−	−
3	0.5	0.75	0.625	+	−	−
4	0.5	0.625				
⋮						
9	0.5703125	0.57421875	0.572265625			

$x \approx 0.57$ (2 dp), Approximation

```matlab
clc, clear all, close all

f = @(x) 0.9*cos(x)-sqrt(x);
a = 0;
b = 1;
signfa = sign(f(a));
signfb = sign(f(b));
N = 10;

for k = 1:N

    p = (a+b)/2;
    signfp = sign(f(p));

    if signfa == signfp
        a = p;
    elseif signfb == signfp
        b = p;
    else
        break;
    end
    disp(p)

end
```

Example. Find the fixed points of the function $g(x) = x^2 - 2$.

$g(x) = x^2 - 2 \implies g(x) = x$

$x = x^2 - 2$

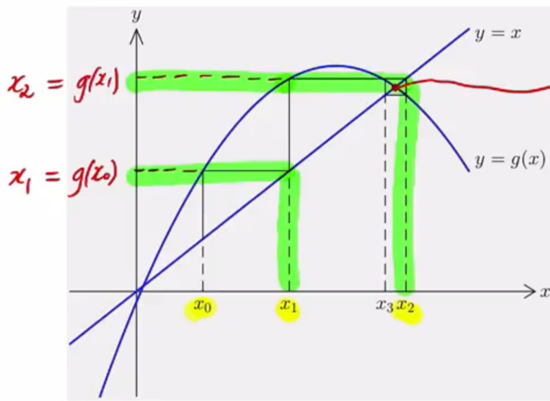
$0 = x^2 - x - 2$

$= (x-2)(x+1)$

$\therefore x = -1, 2$

Fixed point iteration

Graphical representation



We want to solve $x = g(x)$ which is equivalent to finding the x-coordinate of the point of intersection of $y = x$ and $y = g(x)$ (Equivalent to the solution of $f(x) = 0$).

We start with an initial iterate, or "initial guess" x_0, and compute $x_1 = g(x_0)$. Then, we compute $x_2 = g(x_1)$. And so on.

This generates a **sequence** $x_0, x_1, x_2, \ldots,$ that, under the right conditions, will **converge** to the fixed point and hence the solution of our original problem $f(x) = 0$.

$x_{n+1} = g(x_n)$ for $n = 0, 1, 2 \ldots,$ is fixed point iteration.

In summary, the process of applying fixed point iteration consists of:

1. rearranging $f(x) = 0$ into the form $x = g(x)$;
2. performing the iteration $x_{n+1} = g(x_n)$ for $n = 0, 1, 2, \ldots$ until a satisfactory solution is obtained *or* the solution escapes from the interval; if the latter occurs, switch to another rearrangement.

Example. Use fixed point iteration to solve

$$0.9\cos(x) = \sqrt{x}$$

on the interval $[0, 1]$ correct to two decimal places.

(1) $0.9\cos(x) = \sqrt{x} \Rightarrow \quad x = \underbrace{\cos^{-1}\left(\frac{\sqrt{x}}{0.9}\right)}_{g_1(x)}$

(2) $0.9\cos(x) = \sqrt{x} \Rightarrow \quad \underbrace{0.81\cos^2(x)}_{g_2(x)} = x$

$x_0 = 0.5$, initial guess $\longrightarrow$ $x_0 = 0.5$

$g(x) = g_1(x)$: $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$

n	x_n	$g(x_n)$
0	0.5	0.6670
1	0.6670	0.4436
2	0.4436	0.7500
3	0.7500	0.2755
4	0.2755	0.9481
5	0.9481	undefined

$g(x) = g_2(x)$, $x_{n+1} = g_2(x_n)$, $n = 0, 1, 2, \ldots$

n	x_n	$g(x_n)$
0	0.5	0.6238
1	0.6238	0.5336
2	...	0.6004
3	...	0.5514
		0.5878
		$\vdots$
$n = 9 \Longrightarrow$		0.5748

$\therefore \; 0.9\cos(x) = \sqrt{x} \quad \approx 0.57 \;$ (2dp)

1. is continuous on $[0,1]$
2. $g(x) \in [0,1] \; \forall \; x \in [0,1]$, holds true, $|\cos(x)| \leq 1$, $\cos^2(x) \in [0,1]$, $\therefore g(x) \in [0,1] \; \forall \; x \in [0,1]$
3. $g'(x) = -1.62\cos(x)\sin(x)$, differentiable.
4. $|g'(x)| = 1.62|\cos(x)\sin(x)| = 1.62 \times \left|\frac{1}{2}\sin(2x)\right| = 0.81\,|\sin(2x)| \leq 0.81 \times 1 = 0.81$

NEWTON'S METHOD

$$f(x)=0 \Rightarrow f(x_n)+f'(x_n)(x-x_n)=0$$

$$f'(x_n)(x-x_n) = -f(x_n)$$

$$x-x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Algo: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\vdots$

$[0,1]$

Eg) $0.9\cos(x) = \sqrt{x}$, $f(x)=0$, $0.9\cos(x) - \sqrt{x} = 0$

$f(x) = 0.9\cos(x) - \sqrt{x}$, $f'(x) = -0.9\sin(x) - \frac{1}{2\sqrt{x}}$

Initial: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, let $x_0 = 0.5$

$\therefore x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

n	x_n	x_{n+1}
0	0.5	0.5726
1	0.5726	0.5724
2	0.5724	0.5724

$\therefore x \approx 0.5724$