GROUP PORTFOLIO 1
Calculus and Differential Equations

MXB105 Assessment Task 2A (15%)
Semester 1, 2025
Queensland University of Technology

Group 15
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Team Charter

Overview

During the semester you will work in a group for Assessment Task 2. To prepare your team for success, you are asked to complete the following details, which will be a record of your group's planning, and the tasks assigned to each group member. In addition, this charter will summarise meeting times, achievements and individual contributions.

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PLEASE COMPLETE THE FOLLOWING:

A. Team Number	> MXB105 Group 15

B. Contact Details

Full name	Student number	Email address	Mobile (optional)
Maxwell Hanks	N12405710	Maxwell.hanks@connect.qut.edu.au	0434 927 615
Jonathan Cox	N12202347	J34.cox@connect.qut.edu.au	
Josh Muir	N12017876	Jd.muir@connect.qut.edu.au	0401207789

C. Team Expectations

i. What individual skills and expertise can each member bring to the team? ii. How do you expect team members to behave towards each other? iii. How much time will you devote to this assignment each week? iv. What final grade would you be satisfied with?	i. Maxwell can bring MATLAB skills, as well as utilising CMU fonts for a more appropriate looking document. Jonathan can bring the use of wolfram alpha to confirm the accuracy and reasonableness of solutions. Finally, Josh can provide usage of Excel to verify trends and relationships of elevation changes for Problem 1. ii. Team members will respect each other and ensure that everyone agrees on any changes to the document iii. As a group, we hope to spend 5-10 hours each week working on the task. iv. As a group, we are all aiming for a 7 (high distinction)

D. Working Together

Leadership structure - Will you have a team leader? Rotating leadership (i.e. each person has a turn)? Shared leadership? What team roles are needed?	All team members will equally coordinate roles and responsibilities, sharing their updates to the document/any findings.
Decision-making - How will decisions be made? By consensus? Majority rules?	Decisions can be made independently for people working on specific sections, with group approval of major factors like graphs, etc.
Conflict - How will you handle conflict in the team?	We will try to work independently on individual sections, and communicate changes made to the document or any additions to be made. By doing this, we will try to avoid conflict in the team, if conflict does occur, we will communicate the issue and attempt to resolve them
Submission - Who will submit the final portfolio reports?	Maxwell will submit the portfolio reports and team charter.

E. Team Procedures

When, why and where will you meet?	Discord will be used as a platform to meet on to discuss the approach and solutions to the task. Updates will be made throughout the week.
How will you manage: - Setting agendas for each meeting? - Following the agenda during meetings? - Recording and distributing meeting actions?	While calls may not be made, there will be constant communication in Teams and Discord to ensure all tasks are being completed. If necessary, calls may be used in AST 2 Part B.
How will you use MS Teams to communicate and share files?	Microsoft teams will be used for collaborative document sharing, as well as any notifying other members about any major updates in the document/findings.

F. Brainstorm

Work in your team to complete an initial brainstorm of what needs to be done to complete the assessment task. Summarise your team's thoughts below.

- To complete the task, the two problems will be split between the three team members
- Problem 1 can be completed by Josh (Excel will be utilised for zone mapping for Problem 1.)
- Problem 2 can be shared with Maxwell and Jonathan (Maxwell can handle MATLAB and report writing, Jonathan can deal with calculations.)
- Visualisations are required for Problem 2, which can be completed by Maxwell using MATLAB
- Everyone will collaborate any references found and add them to bibliography using Vancouver referencing.

G. Team Plan

List your team's tasks in chronological order (*add as many rows as needed*):

Task description	Who is responsible?	Due?	Status: *Completed?*
Transferring questions/headings from task to final document	Maxwell	01/04/2025	Completed
Section 1.1 – Contour map analysis	Josh	05/04/2025	Completed
Section 1.2 – Elevation changes at A	Josh	07/04/2025	Completed
Section 2.1 – Initial Context	Maxwell	09/04/2025	Completed
Section 2.2 – 3D Visualisation	Maxwell	12/04/2025	Completed
Section 2.3 – Partial Derivative Calculations	Jonathan	13/04/2025	Completed
All derivative calculations	Jonathan	14/04/2025	
Section 2.4 – Probe Landing Height	Jonathan	15/04/2025	Completed
Section 2.5/6 – Probe properties (direction)	Maxwell	16/04/2025	Completed
Section 2.7 – Path Visualisation	Maxwell	18/04/2025	Completed
References/Appendices	Maxwell	20/04/2025	Completed

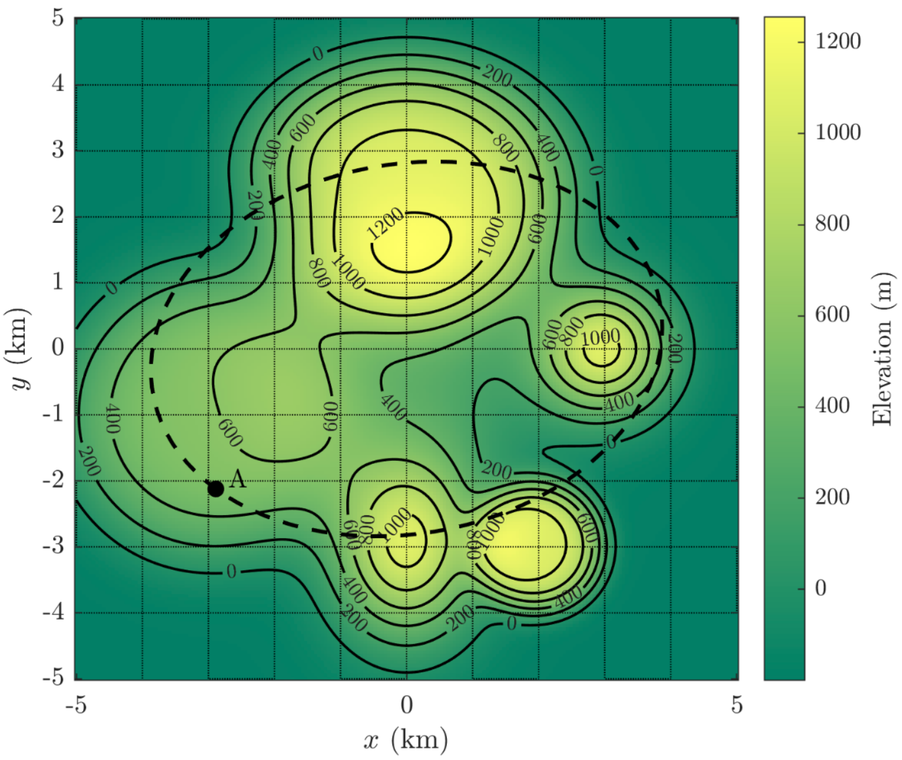
H. Overall contribution summary

Summarise each team member's contribution as a percentage of the submitted work:

Group member's name	**Percentage contribution**
Maxwell Hanks	40%
Jonathan Cox	30%
Josh Muir	30%

Problem 1 – Contour map analysis

You are a team of mathematicians who have been provided with the following contour map of a remote area and asked to report on the topography of the area. Contours are evenly spaced an elevation of 200m apart and the sea level is at an elevation of 0m.



In your report, you need to

- Describe the topography of the area, including identifying the approximate location of:
  - Any hilltops
  - Any saddle points
  - Any areas that may contain fresh water or seawater
  - Any areas of terrain that are particularly steep
  - Any area of terrain that are relatively flat
- Describe the change in elevation that would be experienced by someone traversing the area if they start at point A marked on the contour map and travel along the dashed path in an anticlockwise direction.

Make sure you clearly explain your reasoning.

Problem 1 Solutions

1.1 – Area topography

- Hilltops (local maxima) [Black]
  - Hilltop A (0, -3): Top of hill is relatively small, maximum elevation 1000m
  - Hilltop B (2, -3): Top of hill is very flat, maximum elevation 1000m
  - Hilltop C (3, 0): Top of hill is very small, maximum elevation 1000m
  - Hilltop D (0, 1.5): Top of hill is moderately flat, maximum elevation 1200m

- Saddle points (sudden troughs between hilltops whereby a minimum occurs in one direction and maximums occur perpendicular to that direction; the slope in 3D space is 0 (1)) [Red]
  - (0.9, -3): Saddle point between Hilltops A and B
  - (2, 1): Saddle point between Hilltops C and D

- Freshwater or Seawater [Blue]
  - Since there is no portion of the map where it would be possible for freshwater to occur (no place where a closed contour of 0 occurs within the boundaries of the already existing 0 contour), it indicates that freshwater does not occur on this island, instead, any region along and outside of the 0-contour boundary can be assumed to be seawater.

- Steep terrain [Orange]
  - Area around Hilltop B: The north (2, -2.2), south (2, -3.8), and eastern (2.75, -3) sides of this hilltop are incredibly steep possessing very tightly packed contours.
  - Area around Hilltop C: The northeastern (3.5, 0.75), southern (3, -0.75), and eastern (3.75, 0) sides of this hilltop are incredibly steep with slightly less packed contours than the area around hilltop B.
  - Area around Hilltop D: The northern and eastern sides of this hilltop possess tightly packed contours; it should be noted that the northern contours are significantly denser than the eastern contours.

- Flat terrain [Purple]
  - Extremely gentle incline (1, -0.5): The area between the 400 and 200 contour line could be considered flat terrain considering how far apart the two contour lines are from each other.
  - Flat peak (-2, -0.75): A small, raised portion of land that is extremely flat at its peak.
  - Very gentle incline (-3.8, -1): An area to the west of the flat peak whereby a vast gap between the 400 and 600 contour line occurs.

An annotated version of the topographical map has been provided to highlight and visualise these features:

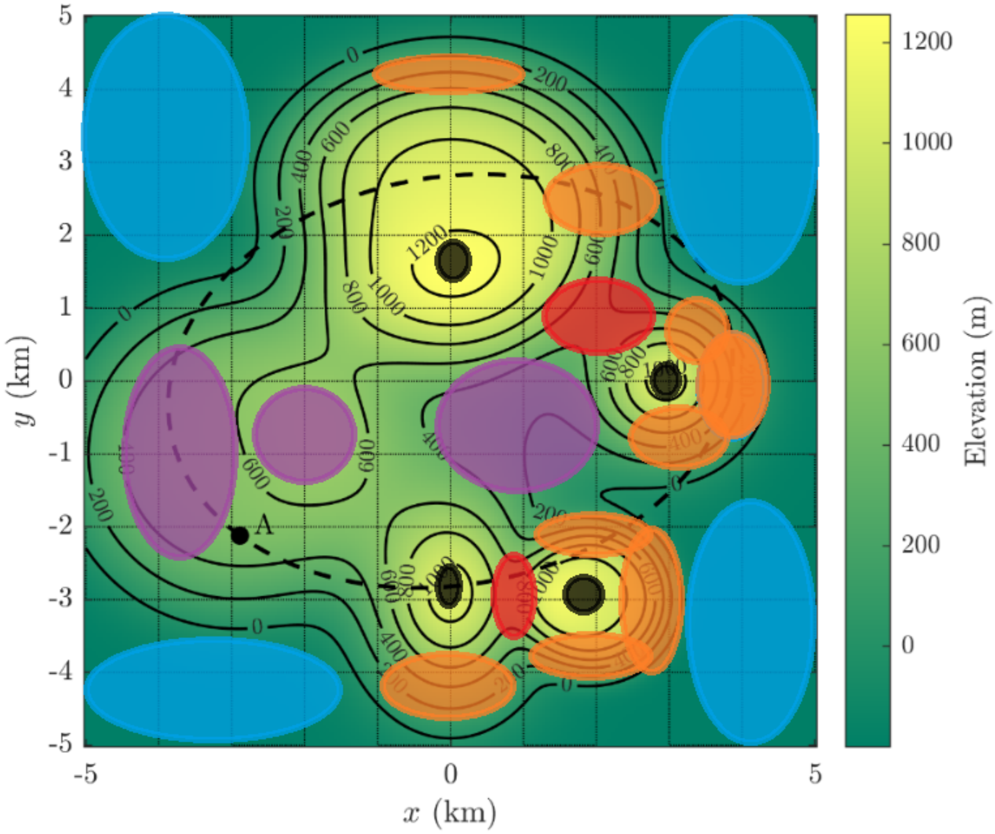


Figure 1.1 – Mapping zones of surface

1.2 – Elevation changes at path of point A

Assuming an anticlockwise path, the change in elevation experienced by someone traversing the dotted line from point A would be as follows:

1. From Point A (-3, -2), to (-1.5, -2.5)
    a. Elevation drops from 500m to 300m
    b. Change in elevation is slow (gentle slope)

2. From (-1,5, -2.5), to (0, -3) [
    a. Elevation rises from 300m to 1000m
    b. Change in elevation is very fast (steep slope)

3. From (0, -3), to (0.75, -2.5)
    a. Elevation drops from 1000m to 700m
    b. Change in elevation is slightly fast (moderate slope)

4. From (0.75, -2.5), to (1.5, -2.5)
    a. Elevation rises from 700m to ~950m
    b. Change in elevation is very fast (steep slope)

5. From (1.5, -2.5), to (2.5, -2)
    a. Elevation drops from ~950m to 100m
    b. Change in elevation is extremely fast (very steep slope)

6. From (2.5, -2), to (2.75, -1.75)
    a. Elevation drops from 100m to 0m, indicating that the traveller is now in the ocean (assuming that 0m is sea level)
    b. Change in elevation is moderate (moderate slope)

7. From (2.75, -1.75), to (3.5, -1)
    a. Elevation rises from 0m to 200m, indicating that the traveller has exited the ocean
    b. Change in elevation is slow (gentle slope)

8. From (3.5, -1), to (3.8, 0)
    a. Elevation rises from 200m to 400m
    b. Change in elevation is fast (steep slope)

9. From (3.8, 0), to (3.5, 1.5)
    a. Elevation drops from 400m to 0m, indicating the traveller is in the ocean again
    b. Change in elevation is fast (moderately steep slope)

10. From (3.5, 1.5), to (0, 2.8)
    a. Elevation rises from 0m to 1000m
    b. Change in elevation is very fast (highly steep slope)

11. From (0, 2.8), to (-2.9, 1.5)
    a. Elevation drops from 1000m to 0m, indicating the traveller is in the ocean again
    b. Change in elevation is slightly fast (slightly steep slope)

12. From (-2.9, 1.5), to (-3.9, 0)
    a. Elevation rises from 0m to 400m
    b. Change in elevation is moderate (moderate slope)

13. From (-3.9, 0), to (-3, -2 [Point A])
    a. Elevation has no notable change, staying at 400m
    b. Change in elevation is negligible

To generate a graphical representation of the elevation change, with respect to total distance travelled along the path of point A, the Euclidean distance between each point must be calculated. To derive this formula, first consider 2 points on a cartesian plane, create a line between these points, and calculate the length of the hypotenuse using the Pythagorean theorem, defined as:

$$c^2 = a^2 + b^2 \ (3)$$

In this case, $a = x_2 - x_1$, $b = y_2 - y_1$, $c = D$

Solving for D:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This formula can be applied since the topographical map is represented in 2 dimensions, although, the total distance travelled will contain a margin of error since it is an estimation of the path travelled as a series of straight lines whereas the path itself is an ellipse. Utilising the SUM and SQRT functions in excel, a table of distances between each point can be generated and then, from this data, a cumulative sum of each value can provide a total distance travelled for each point defined above. An example of the function implementation using the first 4 points can be found below:

x	y
-3	-2
-1.5	-2.5
0	-3
0.75	-2.5

Implement derived distance formula as an excel function:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In Excel:

=(SQRT((A3-A2) ^2) +(B3-B2)^2)

(assuming the above table corresponds to the top left of an excel spreadsheet where x would be in cell A1)

x	y	Displacement between each point (km)	Excel Formula
-3	-2	1.75	=(SQRT((A3-A2) ^2) +(B3-B2)^2)
-1.5	-2.5	1.75	=(SQRT((A4-A3) ^2)+(B4-B3)^2)
0	-3	1	=(SQRT((A5-A4) ^2)+(B5-B4)^2)
0.75	-2.5	0.75	=(SQRT((A6-A5) ^2)+(B6-B5)^2)

To calculate the total distance travelled from the starting point A to the 4th point, a sum of the displacement between the 1st and 2nd point must be taken first, then, the sum of the next displacement between points and the previous sum must be taken for each point. In Excel:

=SUM(D2, C3)

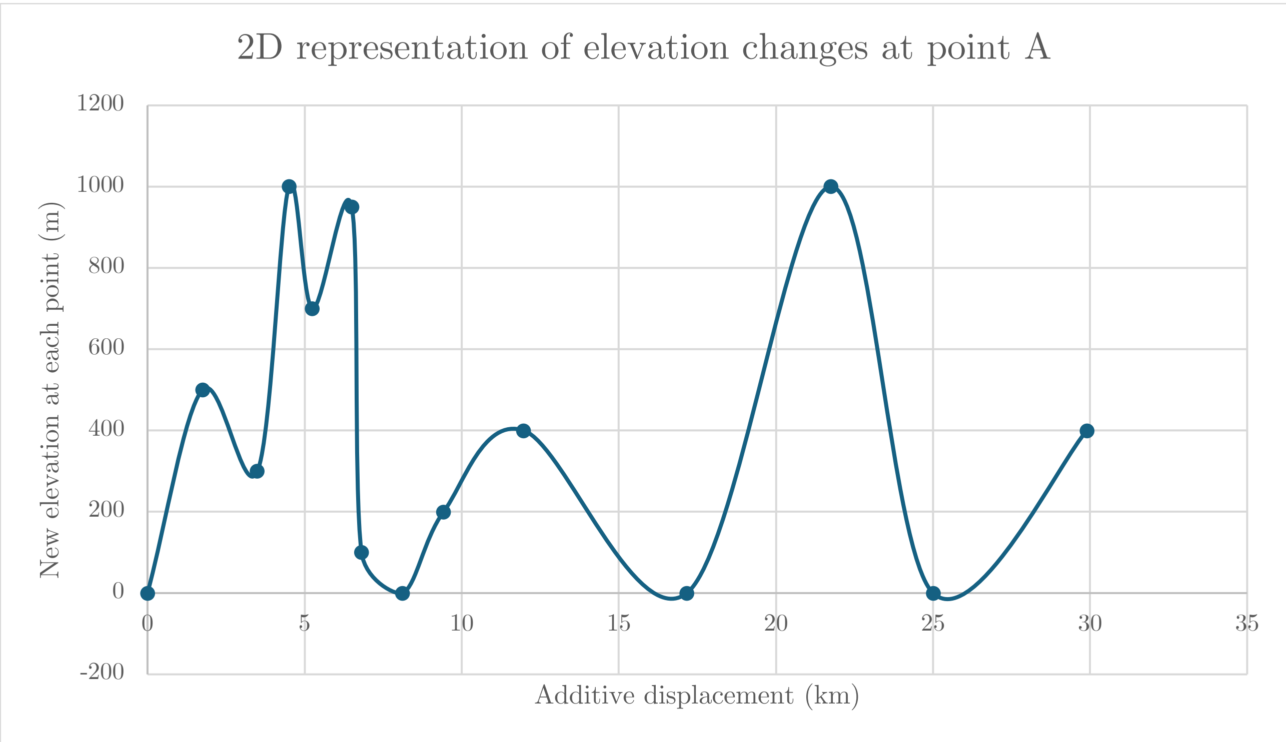
(where D2 is a cell that contains the initial displacement between point A and the second point)

x	y	Displacement between each point (km)	Additive displacement between each point (km)	Excel Formula
-3	-2	1.75	1.75	N/A
-1.5	-2.5	1.75	3.5	=SUM(D2, C3)
0	-3	1	4.5	=SUM(D3, C4)
0.75	-2.5	0.75	5.25	=SUM(D4, C5)

Finally, the last column in the table will contain the new elevation at each point:

x	y	Displacement between each point (km)	Additive displacement between each point (km)	New elevation at each point (m)
-3	-2	1.75	1.75	500
-1.5	-2.5	1.75	3.5	300
0	-3	1	4.5	1000
0.75	-2.5	0.75	5.25	700

From this data, a 2D representation of the change in elevation along the path on Point A can be plotted to visualise the change in elevation.



Problem 2 – Exoplanet and drone modelling

Your team has also been asked to provide a report on the topography of a mountain on a newly discovered exoplanet. You are told that the height of the mountain above the exoplanet's surface is approximately

$$h(x, y) = \exp\left(-\frac{4}{5}(x^2 + y^2)^2 - \frac{2}{3}x^3 + x^2 + y^2\right)$$

where (x, y) are the spatial coordinates in km, and the height $h(x, y)$ is also measured in km. The approximation is valid for $-2 \leq x \leq 2, -2 \leq y \leq 2$.

Scientists are planning to send a probe to land on the exoplanet at coordinates $(x, y) = (-\frac{1}{2}, \frac{1}{2})$.

In your report, you need to:

- Present at least one visualisation of the topography of the mountain.
- Mathematically analyse all critical points of $h(x, y)$ to determine their location and height. Classify critical points as local maxima, local minima or saddle points where possible.
- If a probe lands on the exoplanet at $(x, y) = (-\frac{1}{2}, \frac{1}{2})$, determine the height at which it lands.
- What is the direction of steepest ascent at the point where the probe lands? What is the slope of the mountain in this steepest ascent direction?
- If the probe needs to travel around the mountain at the height at which it lands, determine in what two directions it could move from its starting point at $(-\frac{1}{2}, \frac{1}{2})$.Write down an equation involving x and y that describes the probe's path around the mountain at constant height.
- Finally, present a visualisation of the path the probe would travel around the mountain.

You need to present your findings in a logical and clear way, with appropriate mathematical derivations, reasoning and diagrams. You are allowed to use mathematical software (e.g., MATLAB, Python, Julia, R, or Mathematica) for generating figures. When doing so, include the code you used to generate the figure in an appendix of your report. Mathematical derivations can appear in the main body of the report or in the appendix, as appropriate.

Problem 2 Solutions

2.1 – Initial context

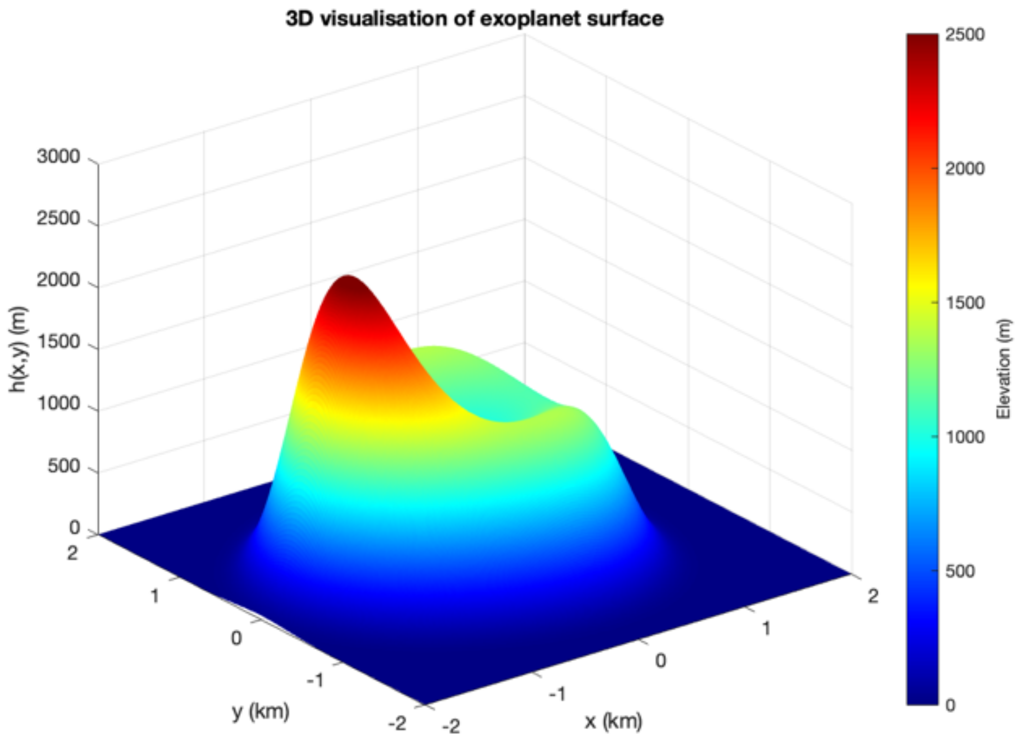
For this report, the topography of a mountain on a newly discovered planet will be analysed, and critical points will be classified by using various calculus methods, such partial differentiation, directional derivatives, etc. Scientists are planning to send a probe to land on the exoplanet at coordinates $(x, y) = (-\frac{1}{2}, \frac{1}{2})$. The height of the landing will be calculated, and be allocated to a contour on the graph, where the path of flight can be modelling using a multivariable function $\boldsymbol{k}(x, y)$. As well as that, other properties will be assessed such as slope vectors: $\nabla f(x_0, y_0)$, $\|\nabla f(x_0, y_0)\|$, etc.

2.2 – 3-dimensional visualisation

Let the height of the surface of the exoplanet be modelled by the function $f: \mathbb{R}^2 \to \mathbb{R}$.

$$h(x, y) = \exp\left(-\frac{4}{5}(x^2 + y^2)^2 - \frac{2}{3}x^3 + x^2 + y^2\right)$$

For $x, y \in [-2,2]$ respectively, where $\exp f(x)$ is defined as $e^{f(x)}$. Using MATLAB computational software, a 3D visualisation can be rendered for the function.



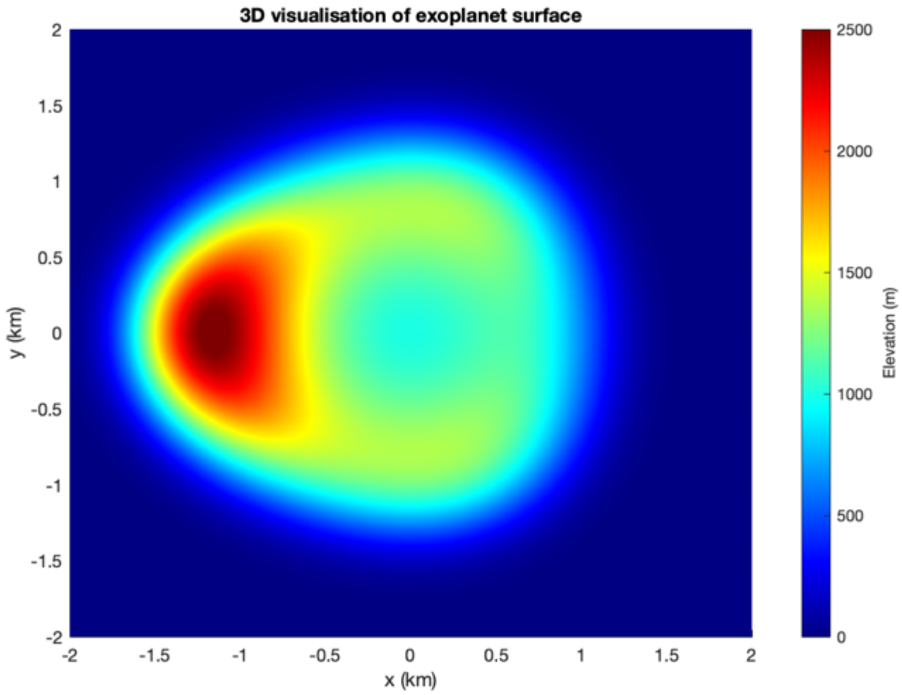


Figure 2.1 – 3D visualisation of exoplanet surface

Shown in Figure 2.1 is a 3D visualisation as well as a birds-eye-view of the xy-plane with a colourmap representing the height of the function in metres. A topological map with contours can be generated, with intervals of 200m up to 2000m for elevation, and $x, y \in [-2{,}2]$ as shown in Figure 2.2. Some initial observations are that the maximum height occurs at $y = 0$, and a local minimum at the "origin".

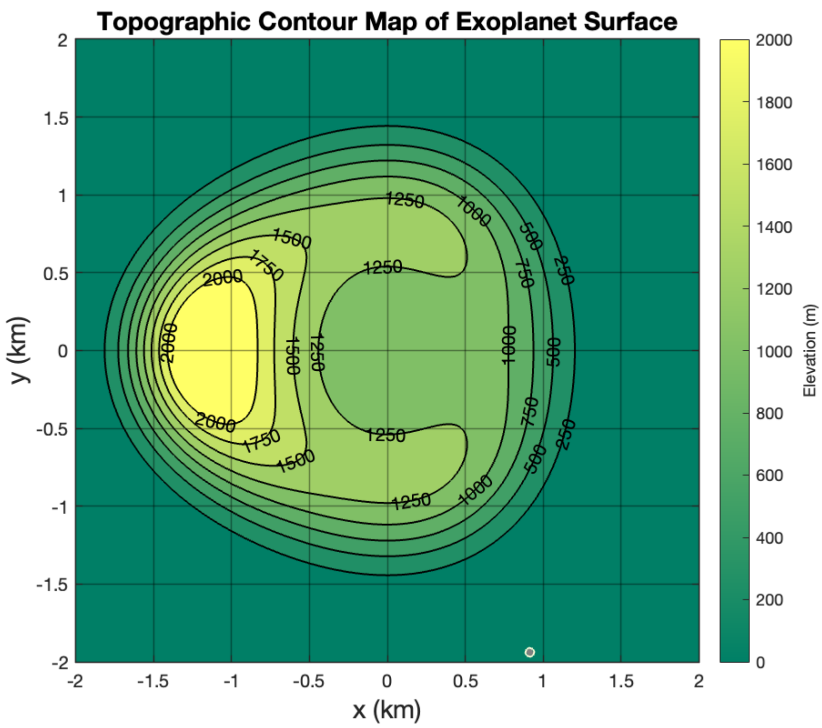


Figure 2.2 – Contour map of exoplanet

2.3 – Critical points analysis

2.3.1 – Partial differentiation

Critical points for a multivariable function are defined as either a relative minimum, relative maximum, or a saddle point. To determine the critical points and their nature, various first and second partial derivatives of the function must be considered with respect to variables x, y. The position of such points can be determined with methods such as considering a function D – the discriminant of the function - the determinant of the Hessian matrix $\mathbf{H}(f)$. (3). Given the height function:

$$h(x,y) = \exp\left(-\frac{4}{5}(x^2+y^2)^2 - \frac{2}{3}x^3 + x^2 + y^2\right)$$

For simplicity, let us define a new function $f(x, y)$. Whereby,

$$f(x,y) = -\frac{4}{5}(x^2+y^2)^2 - \frac{2}{3}x^3 + x^2 + y^2$$

Such that $h(x,y) = \exp(f(x,y))$.Using chain rule, the partial derivatives of h are defined as:

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x}h(x,y),$$

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial y}h(x,y)$$

Since the critical points of $f(x, y)$ will match those of $h(x, y)$ on the xy-plane, partial derivatives of h will be denoted as a product of the partial derivative of f and the height function.

Differentiate $f(x, y)$ with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{4}{5}(x^2+y^2)^2 - \frac{2}{3}x^3 + x^2 + y^2\right]$$
$$\therefore \frac{\partial f}{\partial x} = -\frac{16}{5}x(x^2+y^2) - 2x^2 + 2x$$

Differentiate $f(x, y)$ with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[-\frac{4}{5}(x^2+y^2)^2 - \frac{2}{3}x^3 + x^2 + y^2\right]$$
$$\therefore \frac{\partial f}{\partial y} = -\frac{16}{5}y(x^2+y^2) + 2y$$

First partial derivatives:

$$\frac{\partial f}{\partial x} = -\frac{16}{5}x(x^2+y^2) - 2x^2 + 2x,$$
$$\frac{\partial f}{\partial y} = -\frac{16}{5}y(x^2+y^2) + 2y$$

The three second partial derivatives of f with respect to x and/or y are defined as:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

The partials $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are denoted equal through equality of mixed partial theory: If f is defined in some domain $D \subset \mathbb{R}^2$, and $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous throughout D, therefore $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Applying the same principles previously, the following values are classified as:

$$\frac{\partial^2 f}{\partial x^2} = -\frac{16}{5}(3x^2+y^2) - 4x + 2$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{32}{5}xy$$
$$\frac{\partial^2 f}{\partial y^2} = -\frac{16}{5}(x^2+3y^2) + 2$$

With both first derivatives and three second partial derivatives, various tests can be used to determine nature of critical points, such as the “Second Partials Test”

2.3.2 – Determining critical points

The gradient vector of a function f is defined as $\boldsymbol{\nabla} f = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the x and y directions respectively. Then the critical points of the function f are the points (x_n, y_n) such that $\boldsymbol{\nabla} f(x_n, y_n) = \mathbf{0}$ or $\boldsymbol{\nabla} f(x_n, y_n)$ does not exist.

From section 2.3.1 it is known:

$$\frac{\partial f}{\partial x} = -\frac{16}{5}x(x^2+y^2) - 2x^2 + 2x$$
$$\frac{\partial f}{\partial y} = -\frac{16}{5}y(x^2+y^2) + 2y$$
$$\therefore \boldsymbol{\nabla} f(x,y) = \left(-\frac{16}{5}x(x^2+y^2) - 2x^2 + 2x\right)\boldsymbol{i} + \left(-\frac{16}{5}y(x^2+y^2) + 2y\right)\boldsymbol{j}$$

Since both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous functions there is no point (x_0, y_0) where values for the derivatives don't exist. Therefore, to find the coordinates of the critical points of the function all pairs of numbers (x_n, y_n) must be found such that:

$$\nabla f(x_n, y_n) = 0$$
$$\therefore \frac{\partial}{\partial x}\bigg|_{x_n, y_n} = -\frac{16}{5}x_n(x_n^2 + y_n^2) - 2x_n^2 + 2x_n = 0$$
$$\frac{\partial f}{\partial y}\bigg|_{x_n, y_n} = -\frac{16}{5}y_n(x_n^2 + y_n^2) + 2y_n = 0$$

Using Wolfram Alpha to solve for x_n and y_n gives the (x, y) pairs that make up the critical points C_n of the function f:

$$C_1 = (x_1, y_1) = (0, 0)$$
$$C_2 = (x_2, y_2) = \left(0, -\frac{\sqrt{10}}{4}\right)$$
$$C_3 = (x_3, y_3) = \left(0, \frac{\sqrt{10}}{4}\right)$$
$$C_4 = (x_4, y_4) = \left(-\frac{5 + \sqrt{185}}{16}, 0\right)$$
$$C_5 = (x_5, y_5) = \left(-\frac{5 - \sqrt{185}}{16}, 0\right)$$

2.3.3 – Determining nature of critical points/Second partials test

Theorem 2.3: Let f be a function of two variables with continuous second-order partial derivatives in a neighbourhood of a critical point (x_0, y_0), and let

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

Where all derivatives are evaluated at (x_0, y_0). Then for:

a. $D < 0 \implies f$ has a saddle point at (x_0, y_0).
b. $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0 \implies f$ has a relative minimum at (x_0, y_0).
c. $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0 \implies f$ has a relative maximum at (x_0, y_0).
d. $D = 0 \implies$ no conclusion can be drawn

$$D = \left(-\frac{16}{5}(3x^2 + y^2) - 4x + 2\right)\left(-\frac{16}{5}(x^2 + 3y^2) + 2\right) - \left(-\frac{32}{5}xy\right)^2$$
$$D_n = \frac{4}{25}(192x^4 + 80x^3 + 384x^2y^2 - 160x^2 + 240xy^2 - 50x + 192y^4 - 160y^2 + 25)$$
$$= x_n\left(x_n\left(x_n\left(\frac{768x_n}{25} + \frac{64}{5}\right) + \frac{(1536y_n^2)}{25} - \frac{128}{5}\right) + \frac{192y_n^2}{5} - 8\right) + \left(\frac{768y_n^2}{25} - \frac{128}{5}\right)y_n^2$$
$$+ \ 4$$

For $(x, y) = (0, 0)$:

$$D_1 = 4 > 0$$
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{0,0} = 2 > 0$$

therefore (0, 0) is a relative minimum.

For $(x, y) = \left(0, -\frac{\sqrt{10}}{4}\right)$

$$D_2 = \frac{10}{16}\left(\frac{768\left(\frac{10}{16}\right)}{25} - \frac{128}{5}\right) + 4 = 0$$

Therefore, no conclusion can be drawn for $\left(0, -\frac{\sqrt{10}}{4}\right)$. From a visual inspection of the graph, this point is a saddle point.

For $(x, y) = \left(0, \frac{\sqrt{10}}{4}\right)$

$$D_3 = \frac{10}{16}\left(\frac{768\left(\frac{10}{16}\right)}{25} - \frac{128}{5}\right) + 4 = 0$$

Therefore, no conclusion can be drawn for $\left(0, \frac{\sqrt{10}}{4}\right)$. From a visual inspection of the graph, this point is a saddle point.

For $(x, y) = \left(-\frac{5+\sqrt{185}}{16}, 0\right)$

$$D_4 = \frac{4}{25}\left(192\left(-\frac{5+\sqrt{185}}{16}\right)^4 + 80\left(-\frac{5+\sqrt{185}}{16}\right)^3 - 160\left(-\frac{5+\sqrt{185}}{16}\right)^2 - 50\left(-\frac{5+\sqrt{185}}{16}\right)\right) \approx 10.707 > 0$$
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{-\frac{5+\sqrt{185}}{16},0} \approx -6.325 < 0$$

Therefore $\left(-\frac{5+\sqrt{185}}{16}, 0\right)$ is a relative maximum.

For $(x, y) = \left(-\frac{5-\sqrt{185}}{16}, 0\right)$

$$D_5 = \frac{4}{25}\left(192\left(-\frac{5-\sqrt{185}}{16}\right)^4 + 80\left(-\frac{5-\sqrt{185}}{16}\right)^3 - 160\left(-\frac{5-\sqrt{185}}{16}\right)^2 - 50\left(-\frac{5-\sqrt{185}}{16}\right)\right) \approx -7.145 < 0$$
$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{-\frac{5-\sqrt{185}}{16},0} \approx -2.925 < 0$$

Therefore $\left(-\frac{5-\sqrt{185}}{16}, 0\right)$ is a relative maximum.

2.4 – Probe landing height

To find the height of the function at the point $(-\frac{1}{2}, \frac{1}{2})$, substitute the coordinates into the function:

$$h(x, y) = \exp\left(-\frac{4}{5}(x^2 + y^2)^2 - \frac{2}{3}x^3 + x^2 + y^2\right)$$

Substitute:

$$\begin{aligned} h\left(-\frac{1}{2}, \frac{1}{2}\right) &= \exp\left(-\frac{4}{5}\left(\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)^2 - \frac{2}{3}\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) \\ &= \exp\left(\frac{23}{60}\right) \approx 1.467167\text{km} \end{aligned}$$

Therefore, the height the probe will land at is 1.467km above the surface.

2.5 – Steepest ascent landing properties

The direction of the steepest slope/ascent can be calculated by using the gradient vector and some key properties associated with it. Let the gradient vector be defined as such:

$$\boldsymbol{\nabla} f(x, y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Theorem 2.5: The maximum slope of $f(x, y)$ at (x_0, y_0) is in the direction $\boldsymbol{\nabla} f(x_0, y_0)$. The maximum slope is $\|\boldsymbol{\nabla} f(x_0, y_0)\|$. The minimum slope of $f(x, y)$ at (x_0, y_0) is in the direction $-\boldsymbol{\nabla} f(x_0, y_0)$. The minimum slope is $-\|\boldsymbol{\nabla} f(x_0, y_0)\|$.

Since the probe lands at $P(x_0, y_0) = (-\frac{1}{2}, \frac{1}{2})$, the maximum slope can be calculated:

$$h(x, y) = \exp\left(-\frac{4}{5}(x^2 + y^2)^2 - \frac{2}{3}x^3 + x^2 + y^2\right)$$

$$\boldsymbol{\nabla} h(x, y) = \left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right\rangle \to \boldsymbol{\nabla} h(x_0, y_0) = \left\langle \left.\frac{\partial h}{\partial x}\right|_{(x_0, y_0)}, \left.\frac{\partial h}{\partial y}\right|_{(x_0, y_0)} \right\rangle$$

Let $(x_0, y_0) = (-\frac{1}{2}, \frac{1}{2})$:

$$\begin{aligned} &\boldsymbol{\nabla} h\left(-\frac{1}{2}, \frac{1}{2}\right) = \left\langle \left.\frac{\partial h}{\partial x}\right|_{(-\frac{1}{2}, \frac{1}{2})}, \left.\frac{\partial h}{\partial y}\right|_{(-\frac{1}{2}, \frac{1}{2})} \right\rangle \\ &= \exp\left(\frac{23}{60}\right)\left\langle \left[-\frac{16}{5}x(x^2 + y^2) - 2x^2 + 2x\right]_{(-\frac{1}{2}, \frac{1}{2})}, \left[-\frac{16}{5}y(x^2 + y^2) + 2y\right]_{(-\frac{1}{2}, \frac{1}{2})} \right\rangle \\ &= \langle -1.027{,}0.293 \rangle \end{aligned}$$

Therefore, the direction vector is $\langle -1.027{,}0.293 \rangle$. Alternatively, $\boldsymbol{\nabla} h$ can be denoted as $\langle -\frac{7}{10}, \frac{1}{5} \rangle = \langle -7{,}2 \rangle$ by factoring out $\exp(\frac{23}{60})$, since the vector is just a direction vector. Next, the magnitude of the slope vector $\|\boldsymbol{\nabla} h(x_0, y_0)\|$ will be determined:

$$\left\|\boldsymbol{\nabla} h\left(-\frac{1}{2}, \frac{1}{2}\right)\right\| = \sqrt{\left(-\frac{7}{10} \cdot \exp\left(\frac{23}{60}\right)\right)^2 + \left(\frac{1}{5} \cdot \exp\left(\frac{23}{60}\right)\right)^2} = 1.06811$$

Therefore, the steepest ascent is 1.06811 in the direction $\langle -7{,}2 \rangle$.

2.6 – Probe direction and path properties

The next part of the solution investigates the two directions the probe can follow at the same level/height it landed. Using $\boldsymbol{\nabla} h = \langle -7{,}2 \rangle$, a tangent line can be formed:

$$\boldsymbol{\nabla} h\left(-\frac{1}{2},\frac{1}{2}\right)\cdot\left(x+\frac{1}{2},y-\frac{1}{2}\right)=0$$
$$y=\frac{7}{2}x+\frac{9}{4}$$

Therefore, at the point $(-\frac{1}{2},\frac{1}{2})$ with height $\exp(\frac{23}{60})$, the two directions the probe can travel in is $\langle 2{,}7\rangle, \langle -2,-7\rangle$. Since the probe lands at a height $\exp(\frac{23}{60})$ km, a level curve can be formed at that height to form an equation in terms of x, y. Hence, let $h(x,y) = \exp(\frac{23}{60}) = \boldsymbol{k}(x,y)$

$$\begin{aligned}\boldsymbol{k}(x,y) &= \exp\left(-\frac{4}{5}(x^2+y^2)^2-\frac{2}{3}x^3+x^2+y^2\right)-h(x,y)\\ &= -\frac{4}{5}(x^2+y^2)^2-\frac{2}{3}x^3+x^2+y^2-\frac{23}{60}\end{aligned}$$

Hence, this equation describes the motion of the probe at any point (x_i, y_i) at a height of $\exp(\frac{23}{60})$ by the level curve k.

2.7 – Visualisation of probe path

Using the implicit function on MATLAB, the contour/level curve can be graphed:

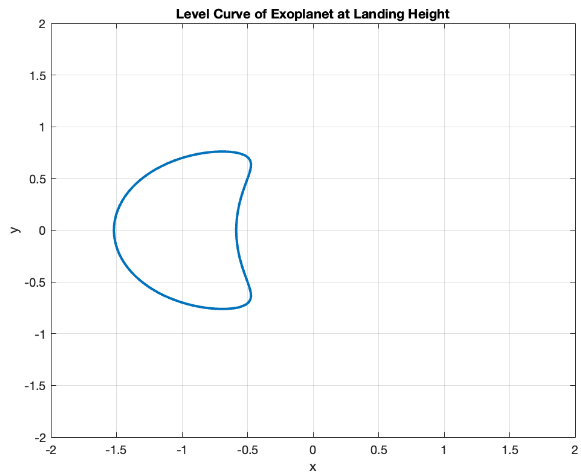


Figure 2.3 – Level curve of exoplanet at $\exp(\frac{23}{60})$

This is the path of the level curve $\boldsymbol{k}(x,y)$ of landing height $\exp(\frac{23}{60})$. As found in Section 2.6, the probe at point $(-\frac{1}{2},\frac{1}{2})$ can travel in a direction along $\langle 2{,}7\rangle, \langle -2,-7\rangle$ along a tangent line $y=\frac{7}{2}x+\frac{9}{4}$.

Bibliography

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Appendices

MATLAB CODE: Figure 2.1

```
x = linspace(-2, 2, 400);
y = linspace(-2, 2, 400);

[X, Y] = meshgrid(x, y);

H_km = exp(- (4/5) * (X.^2 + Y.^2).^2 - (2/3) * X.^3 + X.^2 + Y.^2);
H_m = H_km * 1000

figure;
surf(X, Y, H_m, 'EdgeColor', 'none');
colormap jet;
xlabel('x (km)')
ylabel('y (km)')
zlabel('h(x,y) (m)')
title('3D visualisation of exoplanet surface')
c = colorbar;
c.Label.String = 'Elevation (m)';
caxis([0 2500]);
```

MATLAB CODE: Figure 2.2

```
x = linspace(-2, 2, 200);
y = linspace(-2, 2, 200);
[X, Y] = meshgrid(x, y);

H_km = exp(- (4/5) * (X.^2 + Y.^2).^2 - (2/3) * X.^3 + X.^2 + Y.^2);

H_m = H_km * 1000;

figure;
[~, h] = contourf(X, Y, H_m, 0:250:2000);
colormap(summer);
hold on;

[C, h_cont] = contour(X, Y, H_m, 0:250:2000, 'k', 'LineWidth', 1);
clabel(C, h_cont, 'manual');

xlabel('x (km)', 'FontSize',14);
ylabel('y (km)', 'FontSize',14);
title('Topographic Contour Map of Exoplanet Surface', 'FontSize',14);
c = colorbar;
c.Label.String = 'Elevation (m)';
caxis([0 2000]);

axis equal tight;
grid on;

ax = gca;
ax.GridColor = [0, 0, 0];
ax.GridAlpha = 0.4;
ax.LineWidth = 0.75;
ax.Layer = 'top';
```

MATLAB CODE: Figure 2.3

```
f = @(x,y) -4/5*(x.^2 + y.^2).^2 - 2/3*x.^3 + x.^2 + y.^2 - 23/60;

figure;

fimplicit(f, [-2 2 -2 2], 'LineWidth', 2);

xlabel('x');
ylabel('y');
title('Level Curve of Exoplanet at Landing Height');

grid on;
```