# GROUP PORTFOLIO 2
# Calculus and Differential Equations

# MXB105 Assessment Task 2B (10%)
# Semester 1, 2025
# Queensland University of Technology

Group 15
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# Overall contribution summary

| Group member | Contributions | Percentage contribution |
|---|---|---|
| Maxwell Hanks | Problem 1, all qualitative analysis and graphs (Section 2.2, 2.4, 3.2, 3.4) | 40% |
| Jonathan Cox | Problem 2 – modifications, solution, limitations | 30% |
| Josh Muir | Problem 3 – critical harvesting rate, solution, maximum harvest rate | 30% |

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# 1 – Initial model for population increase

## 1.1 – Context and analysis

A company has recently opened a new salmon farm and requires an optimal number of salmon to harvest from their farm each year. Initially, there were 10,000 salmon, and 3 years later there are an estimated 200,000 salmon. The following first-order differential equation:

$$\frac{du}{dt} = \lambda u(t)$$

Equation 1 – Initial model

Can be a suitable equation for modelling population growth, due to the exponential nature of the population growth. This equation describes: the rate of change of population $u(t)$ over time $t$ is proportional the current population. A differential equation in this form after using integration methods, should output a general solution in the form of an exponential $Ae^{\lambda t}$.

## 1.2 – Solution of initial model

The differential equation for the initial model can be solved using an integrating factor.

$$\begin{aligned}
\frac{du}{dt} &= \lambda u(t) \\
du &= \lambda u(t) dt \\
\frac{1}{u(t)} du &= \lambda dt \\
\int \frac{1}{u(t)} du &= \int \lambda dt \\
\ln|u(t)| &= \lambda t + C \\
u(t) &= e^{\lambda t + C}
\end{aligned}$$

Using an initial condition $e^{\lambda t+C} = Ae^{\lambda t}$ where $A = e^C$ can be used, hence: $u(t) = Ae^{\lambda t}$. Substituting our initial population of 10,000 and population after 3 years of 200,000 will allow to find both unknowns of $A$ and $\lambda$. ($u = 10 = 10{,}000$ salmon).

$$\begin{aligned}
u(0) = 10 &= Ae^{\lambda(0)} \\
10 &= A
\end{aligned}$$

Substituting $A$ into eq. 1:

$$\begin{aligned}
u(t) &= 10(e^{\lambda t}) \\
u(3) = 200 &= 10(e^{\lambda(3)}) \\
20 &= e^{3\lambda} \\
\ln(20) &= 3\lambda \\
\therefore \lambda &= \frac{1}{3}(\ln(20)) \approx 0.998577 \\
\therefore u(t) &= 10e^{0.998577t}
\end{aligned}$$

## 1.3 – Visualisation of initial model

This function represents an exponential growth model, where since $\lambda$ is very close to 1, the curve is resemblant of a true exponential model with a magnitude of 10, and can be graphed as follows:

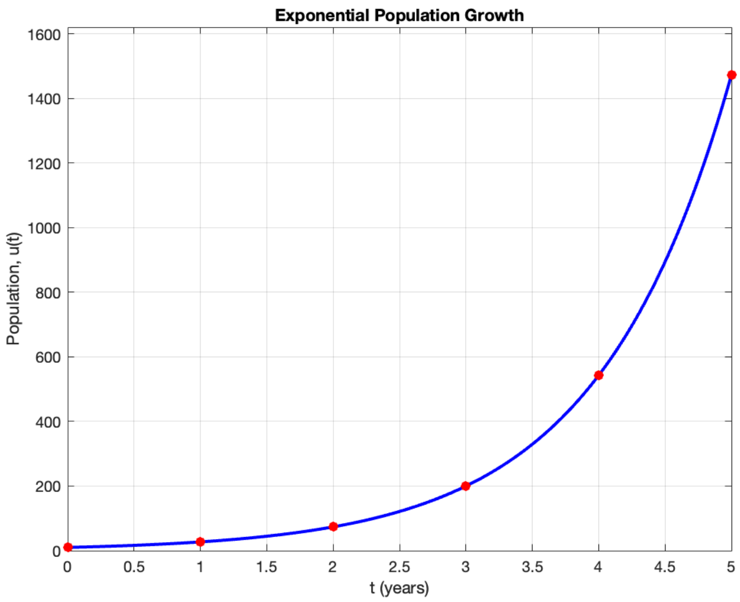


Figure 1 – Population growth for model 1

Visually, this graph represents the exact solution of the differential equation previously noted. This shows total population in 1000s against time, and it can be observed that after 4.5 years there are over 1 million salmon in the pen. After 10 years, the population would increase up to 217 million using the initial model.

## 1.4 – Limitations of the model

There are a few limitations that have to be addressed with an exponential model, such as there is no limit or maximum value like there is in a logistic function, exponential growth continues to rapidly increase. While these exponential growth models can be used in e.g., bacterial growth, exponential functions are not suitable for animal populations in the long term with a growth factor of this magnitude.

Growth over 3 years also does not indicate similar birthing/survival rates for more years to come. This exponential distribution excludes any external factors to be accounted for that may affect population such as water pollution, predation, disease, lack of food, etc. Therefore, this model only accounts for an "unlimited environment".

Taking the size of the cylindrical pen into consideration, there will become a point in time where overpopulation will begin to occur, hence an exponential model is not suitable for long term population modelling.

# 2 – Model with carrying capacity

## 2.1 – Modifications to initial model

Since space and food get used up, fewer salmon survive to adulthood. Taking this into account, a refined model for the population of salmon over a longer time is given by:

$$\frac{du}{dt} = \lambda u(t)\left(1 - \frac{u(t)}{K}\right)$$

Equation 2 – Carrying capacity model

Where $K$ is a new positive parameter referred to the carrying capacity (in thousands of salmon). Introducing the right-hand term $\left(1 - \frac{u(t)}{K}\right)$ turns this differential equation from exponential growth to logistic growth, with a general solution in the form:

$$u(t) = \frac{K}{1 + e^{-\lambda(t-C)}}$$

Where a new term $C$ is the $t$ value of the sigmoid midpoint (1). Using a logistic model for population growth more accurately describes population dynamics, due to the maximum value of salmon $K$ which hence removes overpopulation.

## 2.2 – Qualitative analysis

### 2.2.1 – Fixed points and stability

The fixed points of a differential equation are the solutions to the equation:

$$\frac{du}{dt} = 0$$

For the salmon population model, the stable points will be the solutions to:

$$\lambda u(t)\left(1 - \frac{u(t)}{K}\right) = 0 \rightarrow u(t) = 0, K$$

To find the stability of these fixed points, the values around those points are considered. The value of $K$ does not affect the results of these inequalities and as this is a population growth model, the value of $\lambda$ is positive ($\lambda \approx 1$).

For $u(t) = \lim_{c\to 0^+} c \rightarrow \lim_{c\to 0^+} \lambda c(1 - \frac{c}{K}) > 0$, therefore, for $u(t) > 0$, $\frac{du}{dt}$ is increasing, hence, for $u(t) < 0$ , $\frac{du}{dt}$ is decreasing. For $u(t) = \lim_{c\to 0^+} K + c \rightarrow \lim_{c\to 0^+} \lambda(K+c)(-\frac{c}{K}) < 0$ , therefore, for $u(t) < K$, $\frac{du}{dt}$ is decreasing, hence, for $u(t) > K$, $\frac{du}{dt}$ is increasing. Based on this analysis of the fixed points of $\frac{du}{dt}$, $K$ is a stable point of the differential equation and 0 is unstable.

### 2.2.2 – Phase line and solution curves

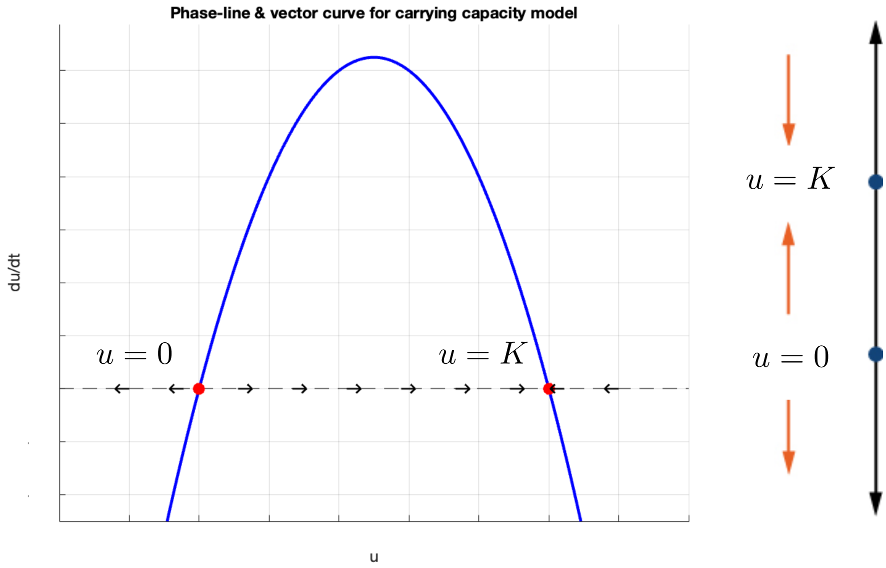


Figure 2 – phase line and solution curve for carrying capacity model

The following diagram shows a phase line diagram with a solution curve, accounting for 2 variables $\lambda$ and $K$. For this graph, the intercepts can be seen as $u(t) = 0, K$. As previously determined based on the qualitative analysis, the stability of the fixed points are 0 as unstable, and $K$ as stable. This can be seen based on the direction of the arrows – where left pointing arrows indicate $\frac{du}{dt}$ decreasing, and right-pointing arrows indicate $\frac{du}{dt}$ increasing. Visually, arrows are pointing away from $u(t) = 0$, hence unstable, and arrows point towards $u(t) = K$, hence stable.

## 2.3 – Solution of carrying capacity model

The model can now be solved via means of integration techniques such as partial fraction decomposition.

$$\frac{du}{dt} = \lambda u(t)\left(1 - \frac{u(t)}{K}\right)$$
$$\frac{dt}{du} = \frac{1}{\lambda u(t)\left(1 - \frac{u(t)}{K}\right)}$$
$$\frac{dt}{du} = \frac{1}{\lambda u(t)} + \frac{1}{\lambda(u(t) - K)}$$
$$\int dt = \int \frac{1}{\lambda u(t)} + \frac{1}{\lambda(u(t) - K)} du$$
$$t = \frac{1}{\lambda}\left(\ln\left|\frac{u(t)}{K - u(t)}\right|\right) + C$$
$$u(t) = \frac{Ke^{\lambda(t-C)}}{1 + e^{\lambda(t-C)}}$$
$$u(t) = \frac{K}{1 + e^{-\lambda(t-C)}}$$

Hence, as stated in the initial context, this differential equation has a general solution in the form of a logistic equation. Solving for parameters $K, \lambda$ and $C$ will allow for an exact solution and a graph to be generated.

To find our parameter $K$, the condition that a carrying capacity of 3 salmon per cubic metre must be considered, by using the dimensions provided for the size of the salmon pen. Given the pen is cylindrical:

$$V = \pi r^2 h$$

We are given circumference and not the radius, hence

$$\begin{aligned} C &= 2\pi r \\ r &= \frac{C}{2\pi} \\ V &= \pi \left(\frac{C}{2\pi}\right)^2 h \end{aligned}$$

Substituting values:

$$\begin{aligned} V &= \pi \times \left(\frac{167}{2\pi}\right)^2 \times 45 \\ \therefore V &\approx 99870.12468 \text{ m}^3 \end{aligned}$$

Assuming 3 salmon per cubic metre means a maximum of 299,610.374 salmon, hence $K \approx 300$ (for simplicity sake and avoiding decimal places). Therefore, the solution is:

$$u(t) = \frac{300}{1 + e^{-\lambda(t-C)}}$$

Using the initial values $u(0) = 10, u(3) = 200$, simultaneous equations can be formed and hence find parameters $\lambda$ and $C$.

$$\begin{aligned} u(0) &= \frac{300}{1 + e^{-\lambda(0-C)}} \\ 10 &= \frac{300}{1 + e^{\lambda C}} \\ 10(1 + e^{\lambda C}) &= 300 \\ 1 + e^{\lambda C} &= 30 \\ e^{\lambda C} &= 29 \\ \therefore u(t) &= \frac{300}{1 + 29e^{-\lambda t}} \end{aligned}$$

Using $u(3) = 200$

$$\begin{aligned} u(3) &= \frac{300}{1 + 29e^{-\lambda(3)}} \\ 200 &= \frac{300}{1 + 29e^{-3\lambda}} \end{aligned}$$

$$e^{-3\lambda} = \frac{1}{58}$$
$$-3\lambda = \ln\frac{1}{58}$$
$$\lambda = -\frac{1}{3}\ln\frac{1}{58}$$
$$\therefore \lambda \approx 1.35433$$

Therefore $\lambda = 1.35433$. Given $e^{\lambda C} = 29$, $C$ can be solved

$$e^{-1.35433(C)} = 29$$
$$\therefore C \approx 2.48788$$

Hence the final solution is

$$\therefore u(t) = \frac{300}{1 + e^{-1.35433(t-2.48788)}}$$

## 2.4 – Visualisation of carrying capacity model

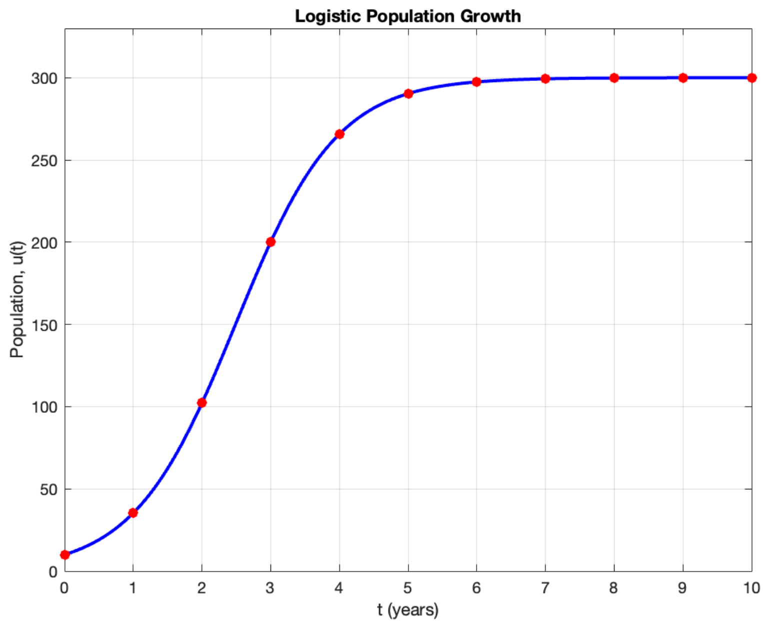


Figure 3 – Population growth for model 2

The following diagram shows the population growth model, that considers the limiting factor of the population limit. The graph has time as the independent variable and population, in thousands of fish, as the dependant variable. The population of fish starts out at 10000, and continues to follow an apparent exponential growth until, after 3 days, the population growth starts to slow, and appears to follow a logarithmic growth, with an asymptote along the line $u(t) = 300$.

## 2.5 – Limitations of carrying capacity model

There are a few limits to be addressed with the carrying capacity model, such as that it doesn't consider the harvesting of fish that would affect the population in a real-world situation. By introducing a harvesting rate to the model, it can simulate the real-world population growth with greater accuracy.

This solution also assumes that there are no environmental influences on the birthing/survival rate of the population, such as predation, disease, pollution and weather. Therefore, this model only accounts for population growth in a controlled environment.

# 3 – Extended model with harvesting

## 3.1 – Critical harvesting rate

Consider a differential equation where a harvesting rate has been added:

$$\frac{du}{dt} = \lambda u(t)\left(1 - \frac{u(t)}{K}\right) - h$$

Equation 3 – Extended harvesting model

Firstly, rewrite the fixed-point equation:

$$\begin{aligned}\frac{du}{dt} = 0 \rightarrow \lambda u(t)\left(1 - \frac{u(t)}{K}\right) - h = 0 \\ -\frac{\lambda u(t)^2}{K} + \lambda u(t) - h = 0 \\ -\frac{K}{\lambda}\left(-\frac{\lambda u(t)^2}{K} + \lambda u(t) - h\right) = 0 \\ u(t)^2 - Ku(t) + \frac{hK}{\lambda} = 0\end{aligned}$$

Provided the harvesting rate is small enough, there are two fixed points of this differential equation.

To find the critical point where there are no longer two fixed points, consider the discriminant of the function, $\Delta = K^2 - 4\frac{hK}{\lambda}$. There are two distinct fixed points for $\Delta > 0$, so hence the critical harvesting rate $h^*$ occurs when $\Delta = 0$.

$$K^2 - 4\frac{h^*K}{\lambda} = 0 \rightarrow h^* = \frac{\lambda K}{4}$$

Given our fixed carrying capacity of $K = 300$, and growth rate $\lambda \approx 1.35433$, our critical harvesting rate is given by: $h^* \approx 101.5$, beyond this harvesting rate, population will not self-sustain and lead to extinction.

# 3.2 – Qualitative analysis

## 3.2.1 – Harvesting rate of $h = 100$

Given the differential equation:

$$\frac{du}{dt} = \lambda u(t)\left(1 - \frac{u(t)}{K}\right) - h$$

Qualitative analysis can be conducted to find the fixed points and the nature. Letting the function equal zero.

$$\lambda u(t)\left(1 - \frac{u(t)}{K}\right) - h = 0$$

Hence,

$$\begin{aligned}
\lambda u(t) - \frac{\lambda u(t)^2}{K} - h &= 0 \\
-\frac{\lambda u(t)^2}{K} + \lambda u(t) - h &= 0 \\
u(t)^2 - Ku(t) + \frac{hK}{\lambda} &= 0
\end{aligned}$$

Substituting into the quadratic formula:

$$\begin{aligned}
u_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
u_{1,2} &= \frac{K \pm \sqrt{K^2 - \frac{4hK}{\lambda}}}{2}
\end{aligned}$$

The number of real fixed points depends on the discriminant, $\Delta = K^2 - \frac{4hK}{\lambda}$. If $\Delta > 0$, there are two fixed points, if $\Delta = 0$, there is one fixed point, if $\Delta < 0$, there are no real fixed points (only complex roots, no steady-state) solution. Assuming $u_1 < u_2$, there are two steady points.

Substituting the values for $K \approx 300, \lambda \approx 1.35433, h = 100$:

$$\begin{aligned}
u_{1,2} &= \frac{300 \pm \sqrt{90{,}000 - \frac{4 \cdot 100 \cdot 300}{1.35433}}}{2} \\
u_1 &= 131.323, u_2 = 168.677
\end{aligned}$$

Hence, the values for where there is a fixed point is 131.323, 168.677. This harvesting rate is not appropriate, since these bounds are quite close to each other and $h = 100$ is close to the critical harvesting rate of 101.5, where a larger harvesting rate would result in eventual extinction due to overharvesting. A more appropriate harvesting rate would be more conservative, greater than 0 and less than 50.

### 3.2.2 – Harvesting rate of $h = 35$

Using the same processes, substituting the values for $K \approx 300, \lambda \approx 1.35433$, with a new value of $h = 35$:

$$u_{1,2} = \frac{300 \pm \sqrt{90{,}000 - \frac{4 \cdot 35 \cdot 300}{1.35433}}}{2}$$
$$u_1 = 28.562, u_2 = 271.437$$

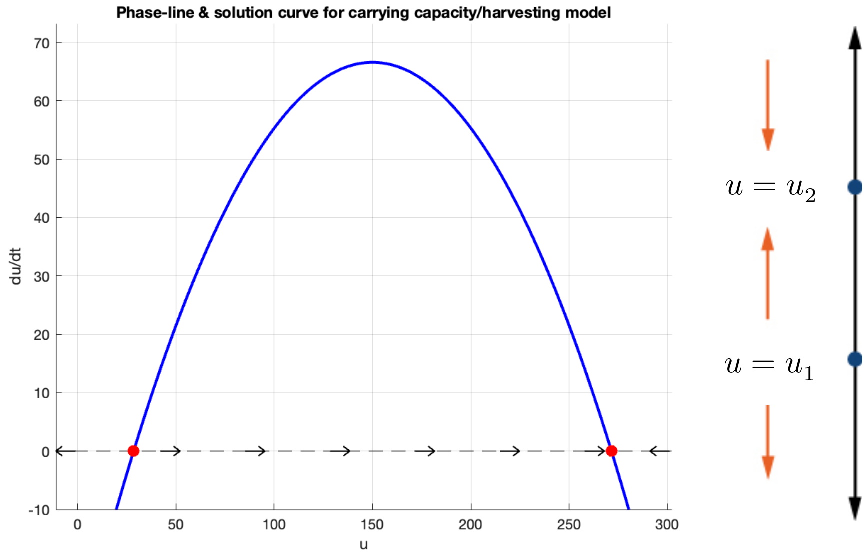


Figure 4 – Phase line and solution curve for harvesting model

It is notable that this graph forms a similar shape to the phase line and solution curve of the logistic growth model, however the intercepts rather than being 0 and K, are now given by the roots to the curve $u(t)^2 - Ku(t) + \frac{hK}{\lambda} = 0$, yielding $u_1 = 28.562, u_2 = 271.437$. The stability of the points also remains unchanged, where $u_1$ is unstable and $u_2$ is stable.

This harvesting rate of 35 is much more suitable than $h = 100$, since there is no possibility of extinction, but still manages to mitigate the issue of overpopulation.

## 3.3 – Solution of extended model

Recall the differential equation:

$$\frac{du}{dt} = \lambda u(t)\left(1 - \frac{u(t)}{K}\right) - h$$
$$\frac{du}{dt} = -\frac{\lambda u^2}{K} + \lambda u - h$$

The roots of $u$ can be calculated:

$$u_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\frac{\lambda h}{K}}}{2\left(-\frac{\lambda}{K}\right)}$$
$$\therefore \frac{du}{dt} = -\frac{\lambda}{K}(u-u_1)(u-u_2)$$
$$\frac{1}{(u-u_1)(u-u_2)}du = -\frac{\lambda}{K}dt$$

Let $a = -\frac{\lambda}{K}$

$$\int \frac{1}{(u-u_1)(u-u_2)}du = \int a dt$$
$$\int \frac{A}{u-u_1} + \frac{B}{u-u_2}du = a\int dt$$
$$\frac{A}{u-u_1} + \frac{B}{u-u_2} = \frac{1}{(u-u_1)(u-u_2)}$$
$$A(u-u_2) + B(u-u_1) = 1$$
$$(A+B)u - (Au_2 + Bu_1) = 1$$
$$A + B = 0, \qquad -(Au_2 + Bu_1) = 1$$
$$A = -B \to -\big((-B)u_2 + Bu_1\big) = 1$$
$$B(u_2 - u_1) = 1$$
$$B = \frac{1}{u_2 - u_1}$$
$$A = -B \to A = \frac{1}{u_1 - u_2}$$
$$\therefore \frac{1}{u_1-u_2}\left[\int \frac{1}{u-u_1}du - \int \frac{1}{u-u_2}du\right] = a\int dt$$
$$\frac{1}{u_1-u_2}(\ln|u-u_1| - \ln|u-u_2|) = at + C$$
$$\frac{1}{u_1-u_2}\left(\ln\left|\frac{u-u_1}{u-u_2}\right|\right) = at + C$$
$$\ln\left|\frac{u-u_1}{u-u_2}\right| = a(u_1-u_2)(t-t_0)$$

Let $t_0 = 3$ given $u(3) = 200$

$$\ln\left|\frac{u-u_1}{u-u_2}\right| = a(u_1-u_2)(t-3)$$
$$\frac{u-u_1}{u-u_2} = e^{a(u_1-u_2)(t-3)}$$

Let $\beta = a(u_1 - u_2)$, let $H = \frac{u(3)-u_1}{u(3)-u_2}$

$$\frac{u-u_1}{u-u_2} = He^{\beta(t-3)}$$
$$u - u_1 = He^{\beta(t-3)}(u-u_2)$$
$$u - He^{\beta(t-3)}u = u_1 - He^{\beta(t-3)}u_2$$
$$u(1 - He^{\beta(t-3)}) = u_1 - He^{\beta(t-3)}u_2$$
$$\therefore u(t) = \frac{u_1 - He^{\beta(t-3)}u_2}{1 - He^{\beta(t-3)}}$$

This can now be evaluated by substituting the following values:

$$\lambda = 1.35433, K = 300, h = 35, u_1 = 28.562, u_2 = 271.437$$

$$a = -\frac{\lambda}{K} = -\frac{1.35433}{300} \approx -0.004514$$
$$\beta = a(u_1 - u_2) = -0.004514(28.562 - 271.437) \approx 1.10085175$$
$$H = \frac{u(3) - u_1}{u(3) - u_2} = \frac{200 - 28.562}{200 - 271.437} \approx -2.4028760845$$

$$\therefore u(t) = \frac{28.562 + 652.229e^{1.10085175(t-3)}}{1 + 2.403e^{1.10085175(t-3)}}$$

## 3.4 – Visualisation of extended model

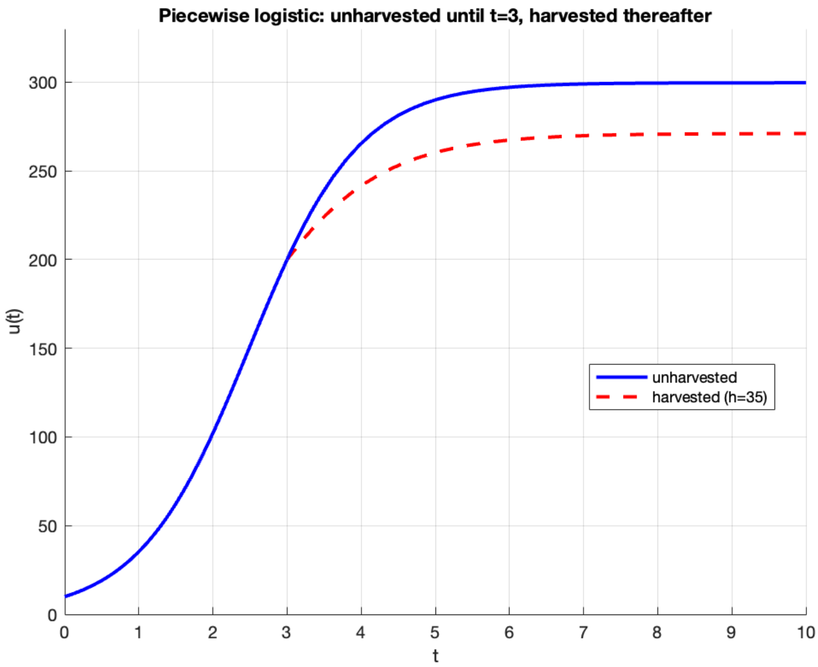


Figure 5 – Population growth for model 3

In Figure 5, both the unharvested equation and the harvested function where $h = 35$ after 3 years is plotted. It is observed that the harvested curve is a similar shape to the original logistic function, however the population plateaus around approximately 271 instead of the 300 carrying capacity value.

## 3.5 – Predicted maximum harvest rate

As calculated previously, the critical harvesting rate and hence the maximum harvesting rate is $h^* \approx 101.5$ , since harvesting over this value will result in overharvesting and hence extinction. However, harvesting at exactly this value is not the best strategy either, since it fails to consider the initial drop in population from overharvesting before a new equilibrium forms. While beneficial in the short-term, it leaves the farmer with less salmon overall in the long-term as shown below:

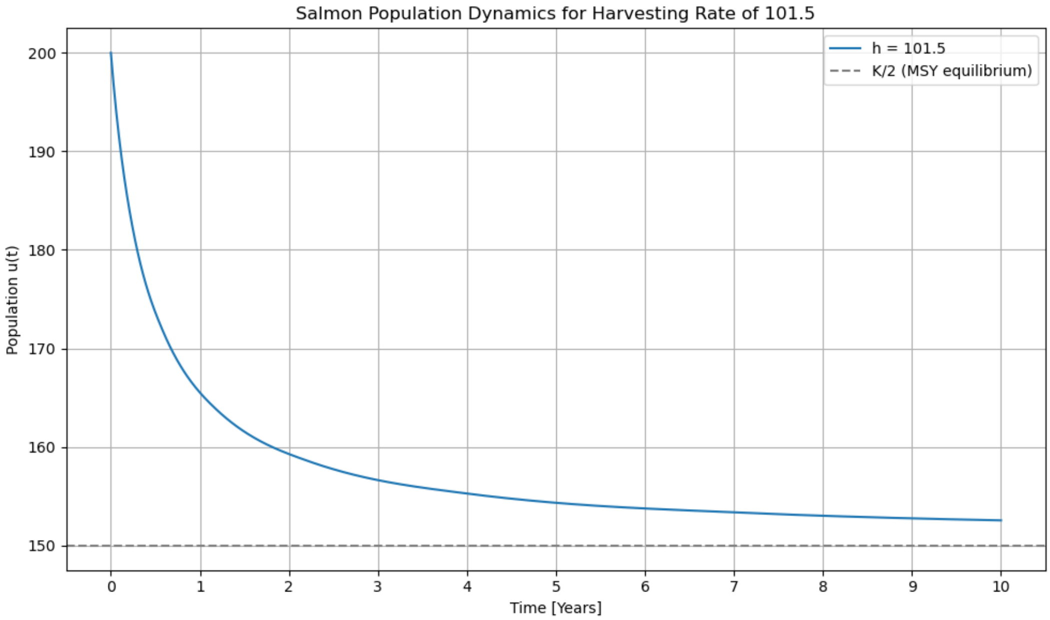


Figure 6 – Salmon population dynamic for critical harvesting rate

(MSY = maximum sustainable yield, the maximum number of salmon that can be taken from the initial population before the population cannot establish a new equilibrium and moves towards extinction)

We can visualise a proof of the $h_{crit}$ value being the maximum harvesting rate the farmer can sustain via alterations to the above graph:

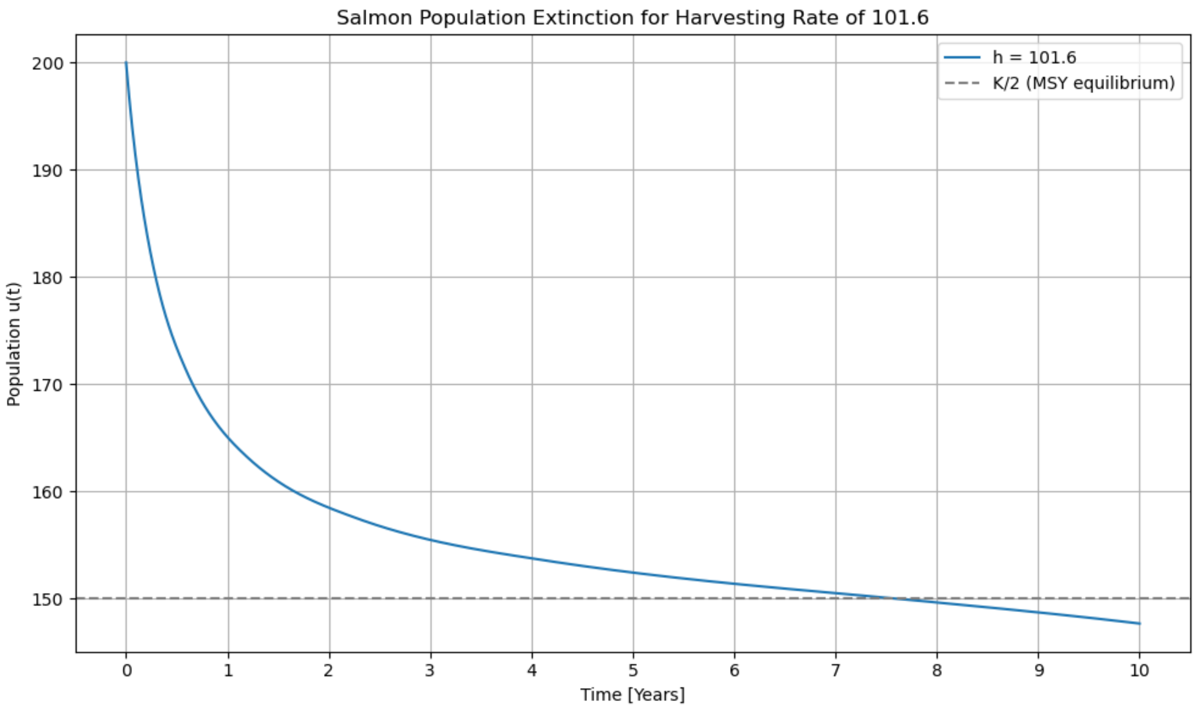


Figure 7 – Salmon population for harvesting rate above critical harvesting rate

This requires the farmer to either operate at extremely low harvesting rates until the population is brought back up or keep the new equilibrium and potentially crash the population to extinction given any inaccuracy in the variance of the population. An approximate solution to this problem is to ensure:

$$h < h_{crit}$$

However, if we want to find a harvesting rate which maintains the population for any given initial population value, we must solve for $\frac{du}{dt} = 0$. Assuming this is 200, we can solve the differential equation for $h$:

$$0 = \lambda u \cdot \left(1 - \frac{u}{K}\right) - h$$
$$0 = 1.35348 \cdot 200 \cdot \left(1 - \frac{200}{300}\right) - h$$
$$h = 1.35348 \cdot 200 \cdot \left(1 - \frac{2}{3}\right)$$
$$h = 90.232$$

This method of solving for $h$ ensures a stable harvesting rate that will not significantly impact the population of salmon for any provided initial population. We can visualise the stability of this harvesting rate against others graphically for the initial population 200:

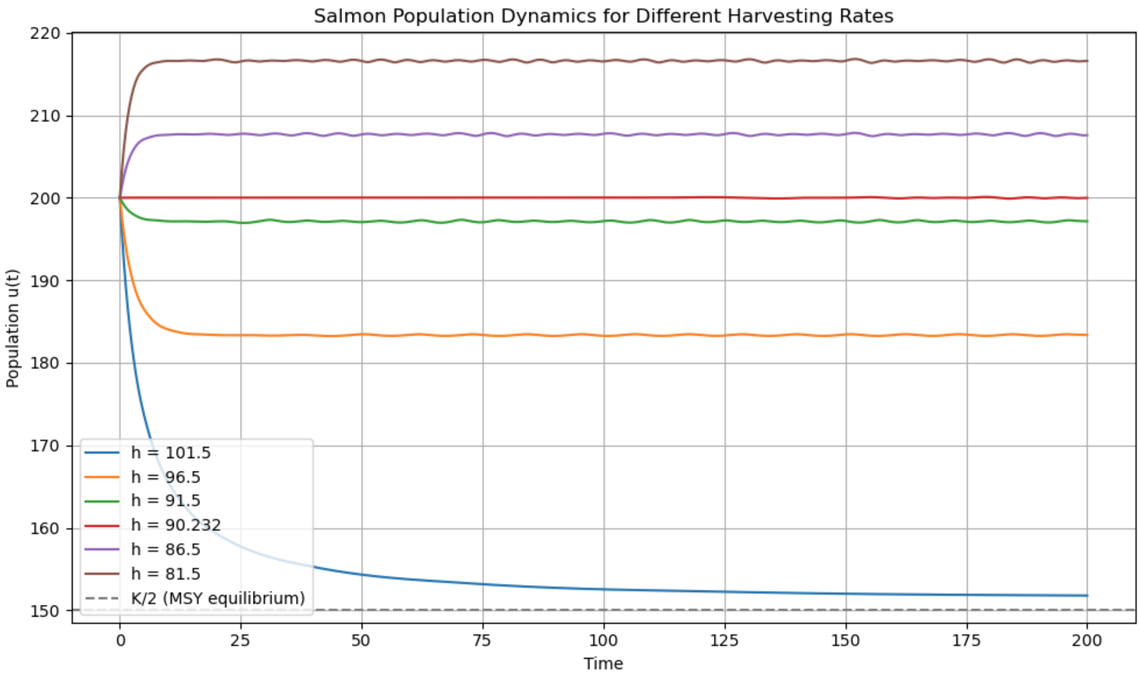


Figure 8 – Salmon population dynamic for various harvesting rates

Given that at a certain harvesting rate the population plateaus around a certain value, it is logical to assume that if the farmer wants an increase or decrease in population, they will only need to alter the harvesting rate to do so. It is simple to derive a function to this effect, considering that these changes occur over time:

$$h(t) = \begin{Bmatrix} h_{base}, & t < t_1 \\ h_{peak}, & t_1 \leq t < t_2 \\ h_{base}, & t \geq t_2 \end{Bmatrix}$$

We can run an example simulation of this piecewise function using our previously derived $h$ value of 90.232, where we increase it to 101.5 at its peak. Representing it graphically using a set of initial values: $t_1 = 20$, $t_2 = 40$, $h_{base} = 90.232$, $h_{peak} = 95.1$:

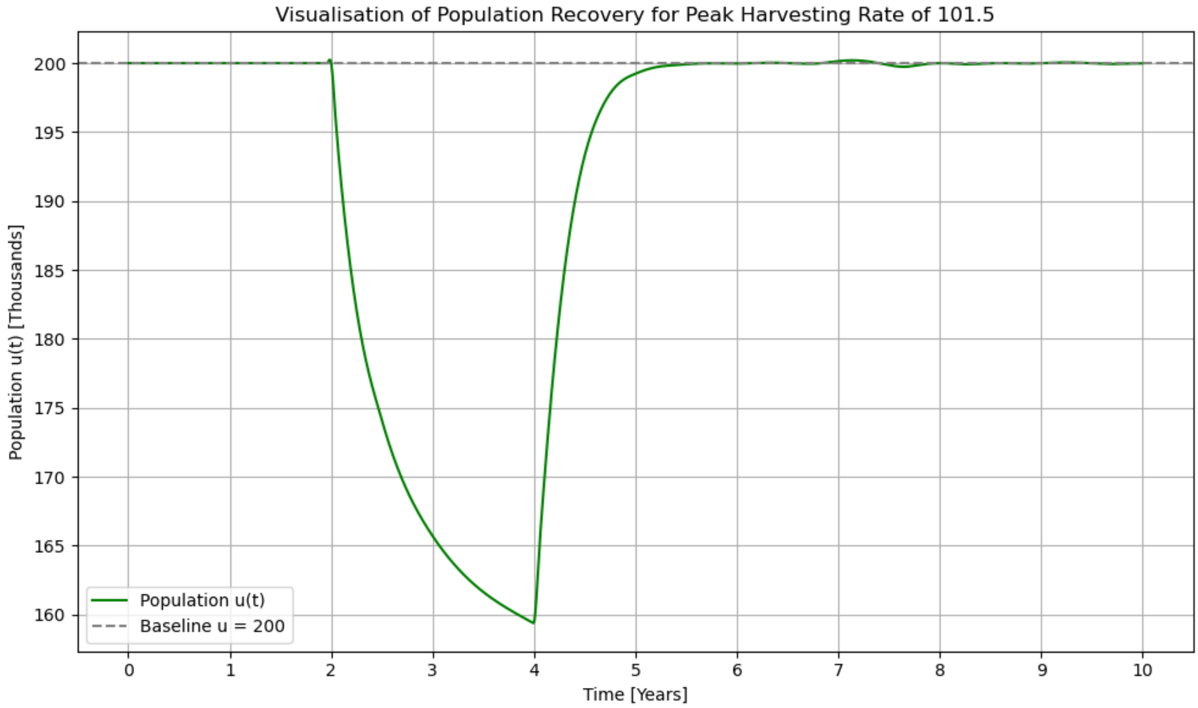


Figure 9 – Population recovery via differing harvest rates

# Bibliography

1. Strang G, Herman E. 8.4: The Logistic Equation [Internet]. Mathematics LibreTexts. Available from: https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/08%3A_Introduction_to_Differential_Equations/8.04%3A_The_Logistic_Equation

# Appendices

## Appendix 1

### Figure 1

```
a = (log(20))/3;
t = linspace(0, 5, 1000);

u = 10 * exp(a * t);

figure;
plot(t, u, 'b-', 'LineWidth', 2);
hold on;

t_pts = 0:5;
u_pts = 10 * exp(a * t_pts);
plot(t_pts, u_pts, 'ro', 'MarkerFaceColor', 'r', 'MarkerSize', 6);

xlabel('t (years)');
ylabel('Population, u(t)');
title('Exponential Population Growth');
grid on;
xlim([0 5]);
ylim([0 max(u_pts)*1.1]);

hold off;
```

## Figure 2

```
lambda = 1.35433;
K      = 300;

u = linspace(-2, K+2, 500);
f = lambda * u .* (1 - u./K);

figure('Color','w'), hold on

plot(u, f, 'b-', 'LineWidth', 2)
plot(u, zeros(size(u)), 'k--')

plot([0, K], [0, 0], 'ro', 'MarkerSize', 8, 'MarkerFaceColor','r')

xlabel('u'), ylabel('du/dt')
title('Phase-line & solution curve for carrying capacity model')
axis tight

u_arrows = linspace(-1, K+1, 8);
f_arrows = lambda * u_arrows .* (1 - u_arrows./K);
arrow_length = 0.1;

for i = 1:numel(u_arrows)
    dir = sign(f_arrows(i));
    quiver( u_arrows(i), 0, ...
            dir*arrow_length, 0, ...
            'MaxHeadSize', 1, ...
            'AutoScale','off', ...
            'Color','k', ...
            'LineWidth',1 )
end

ylim([-10, max(f)*1.1])
grid on
hold off
```

## Figure 3

```
a = - log(1/58)/3;
C = 2.48788;
t = linspace(0, 10, 1000);

u = 300./(1 + 29 * exp(-a*t));

figure;
plot(t, u, 'b-', 'LineWidth', 2);
hold on;

t_pts = 0:10;
u_pts = 300./(1 + 29 * exp(-a*t_pts));
plot(t_pts, u_pts, 'ro', 'MarkerFaceColor', 'r', 'MarkerSize', 6);

xlabel('t (years)');
ylabel('Population, u(t)');
title('Logistic Population Growth');
grid on;
xlim([0 10]);
ylim([0 max(u_pts)*1.1]);

hold off;
```

## Figure 4

```
lambda = 1.35433;
K      = 300;
h      = 35;

u = linspace(-2, K+2, 500);
f = lambda * u .* (1 - u./K) - h;

figure('Color','w'), hold on

plot(u, f, 'b-', 'LineWidth', 2)
plot(u, zeros(size(u)), 'k--')

plot([28.562, 271.437], [0, 0], 'ro', 'MarkerSize', 8,
'MarkerFaceColor','r')

xlabel('u'), ylabel('du/dt')
title('Phase-line & solution curve for carrying capacity/harvesting
model')
axis tight

u_arrows = linspace(-1, K+1, 8);
f_arrows = lambda * u_arrows .* (1 - u_arrows./K) - h;
arrow_length = 10;

for i = 1:numel(u_arrows)
    dir = sign(f_arrows(i));
    quiver( u_arrows(i), 0, ...
            dir*arrow_length, 0, ...
            'MaxHeadSize', 1, ...
            'AutoScale','off', ...
            'Color','k', ...
            'LineWidth',1 )
end

ylim([-10, max(f)*1.1])
grid on
hold off
```

## Figure 5

```
lambda = 1.35433;
K      = 299.610;
h      = 35;

t1_span = [0 3];
t2_span = [3 10];
t3_span = [3 10];

u0_1 = 10;
u0_2 = 200;

ode_unharvested = @(t,u) lambda*u.*(1 - u./K);
ode_unharvestedafter = @(t,u) lambda*u.*(1 - u./K);
ode_harvested   = @(t,u) lambda*u.*(1 - u./K) - h;

[t1, u1] = ode45(ode_unharvested, t1_span, u0_1);
[t2, u2] = ode45(ode_harvested, t2_span, u0_2);
[t3, u3] = ode45(ode_unharvestedafter, t3_span, u0_2);

figure;
hold on;
plot(t1, u1, 'b-', 'LineWidth', 2);
plot(t2, u2, 'r--', 'LineWidth', 2);
plot(t3, u3, 'b-', 'LineWidth', 2);
hold off;

grid on;
xlabel('t');
ylabel('u(t)');
xlim([0 10]);
ylim([0 330]);
title('Piecewise logistic: unharvested until t=3, harvested thereafter');
legend({'unharvested','harvested (h=35)'}, 'Location','best');
```

# Appendix 2

## Figure 6, 7 and 8

```
import numpy
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

lambda_ = 1.35433
K = 300
h_crit = (lambda_ * K) / 4
initial_u = 200
t_span = (0, 100)
t_eval = numpy.linspace(*t_span, 1000)
h_values = []

def logistic_harvest(t, u, lambda_, K, h):
    return lambda_ * u * (1 - (u / K)) - h

plt.figure(figsize=(10, 6))
for h in h_values:
    sol = solve_ivp(logistic_harvest, t_span, [initial_u], t_eval=t_eval,
args=(lambda_, K, h))
    plt.plot(sol.t, sol.y[0], label=f'h = {h}')

plt.axhline(K / 2, color='gray', linestyle='--', label='K/2 (MSY equilibrium)')
plt.title('Salmon Population Extinction for Harvesting Rate of 101.6')
plt.xticks(numpy.linspace(0, 100, 11), labels=[str(i) for i in range(11)])
plt.xlabel("Time [Years]")
plt.ylabel('Population u(t)')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```

## Figure 9

```
def compensated_harvesting_rate(t, h_base=None, h_peak=None, t1=None, t2=None):
    if t < t1:
        return h_base
    elif t1 <= t < t2:
        return h_peak
    else:
        return h_base

compensated_h_vectorized = numpy.vectorize(compensated_harvesting_rate)

def extinction_event(t, u, lambda_, K):
    return u[0]

def logistic_with_compensated_h(t, u, lambda_, K):
    h_t = compensated_harvesting_rate(t)
    return lambda_ * u * (1 - u / K) - h_t

sol_comp = solve_ivp(
    logistic_with_compensated_h,
    t_span,
    [initial_u],
    t_eval=t_eval,
    args=(lambda_, K),
    events=extinction_event)

u_comp = numpy.clip(sol_comp.y[0], 0, None)
h_comp = compensated_h_vectorized(sol_comp.t)

extinction_event.terminal = True
extinction_event.direction = -1
plt.figure(figsize=(10, 6))
plt.plot(sol_comp.t, u_comp, label='Population u(t)', color='green')
plt.axhline(initial_u, color='gray', linestyle='--', label='Baseline u = 200')
plt.xticks(numpy.linspace(0, 100, 11), labels=[str(i) for i in range(11)])
plt.xlabel("Time [Years]")
plt.ylabel('Population u(t) [Thousands]')
plt.title('Visualisation of Population Recovery for Peak Harvesting Rate of 105.1')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```