MXB105

Lecture Notes

M A X W E L L
H A N K S

CALCULUS AND DIFFERENTIAL EQUATIONS

Integration Techniques

SINE SUBSTITUTION

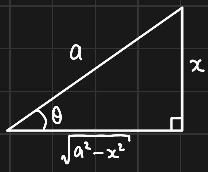
- When an integral contains $(a^2-x^2)^n$; Sine substitution
  - Recall $\sin^2\theta+\cos^2\theta=1$

$(a^2-x^2)^n \rightarrow$ Let $x=a\sin\theta$

$\therefore a^2-x^2 = a^2-a^2\sin^2\theta$

$= a^2\cos^2\theta$

$\left[\text{Domain: } \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\ a\in[-x,x]\right]$



↳ E.g, $\int \frac{1}{\sqrt{9-x^2}}dx$

Let $x=3\sin\theta$, $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\frac{dx}{d\theta}=3\cos\theta$, $dx=3\cos\theta\, d\theta$

$\therefore \int \frac{1}{\sqrt{9-x^2}}dx = \int \frac{1}{\sqrt{9-9\sin^2\theta}}3\cos\theta\, d\theta$

$= \int \frac{3\cos\theta}{\sqrt{9\cos^2\theta}}d\theta$

$= \int 1\, d\theta = \theta + c = \sin^{-1}\left(\frac{x}{3}\right)+C$

TAN SUBSTITUTION

- When an integral contains $(a^2+x^2)^n$; Tan substitution

$(a^2+x^2)^n \rightarrow$ Let $x=a\tan\theta$

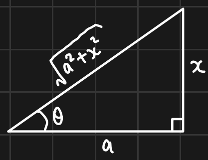
$\therefore a^2+x^2 = a^2+a^2\tan^2\theta$

$= a^2\sec^2\theta$

$\left[\text{Domain: } \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\ a\in[-x,x]\right]$

Recall $\cos^2\theta+\sin^2\theta=1$

$\Rightarrow 1+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta} \Rightarrow 1+\tan^2\theta=\sec^2\theta$



↳ E.g, $\int \frac{1}{(1+x^2)^2}dx$

Let $x=\tan\theta$, $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\frac{dx}{d\theta}=\frac{1}{\cos^2\theta}=\sec^2\theta$

$\therefore dx=\sec^2\theta$

$\therefore \int \frac{1}{(1+x^2)^2}dx = \int \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta$

$= \int \frac{\sec^2\theta}{\sec^4\theta}d\theta = \int \frac{1}{\sec^2\theta}d\theta$

$= \int \cos^2\theta\, d\theta = \int\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta$

$= \frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)+C$

$= \frac{1}{2}\tan^{-1}(x)+\frac{x}{2(1+x^2)}+C$

SEC SUBSTITUTION

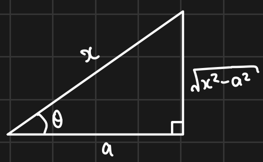
- When an integral contains $(x^2-a^2)^n$; Sec substitution

$(x^2-a^2)^n \rightarrow$ Let $x=a\sec\theta$

$\therefore x^2-a^2=a^2\sec^2\theta-a^2$

$= a^2\tan^2\theta$

$\left[\text{Domain: } \begin{cases}\forall x\geqslant a,\ \theta\in\left[0,\frac{\pi}{2}\right) \\ \forall x\leqslant -a,\ \theta\in\left(\frac{\pi}{2},\pi\right]\end{cases}\right]$



↳ E.g, $\int \frac{1}{\sqrt{x^2-16}}dx$, $a=4$, sec sub

Let $x=4\sec\theta$

$\frac{dx}{d\theta}=\frac{-4}{(\cos\theta)^2}(-\sin\theta)=\frac{4\sin\theta}{\cos^2\theta}=4\tan\theta\sec\theta$

$\therefore \int \frac{1}{\sqrt{x^2-16}}dx = \int \frac{1}{\sqrt{16\sec^2\theta-16}}4\tan\theta\sec\theta\, d\theta$

$= \int \frac{1}{4\tan\theta}4\tan\theta\sec\theta\, d\theta$

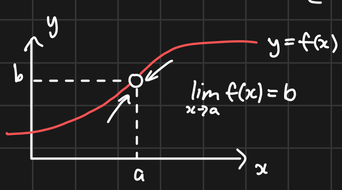
$= \int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta|+C$

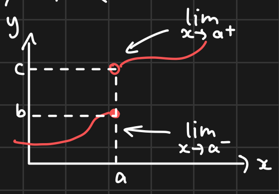
$= \ln\left|\frac{x}{4}+\frac{\sqrt{x^2-16}}{4}\right|+C$

Limits, Continuity, Differentiability

LIMIT

- $\lim_{x \to a} f(x)$ iff $f(x) \longrightarrow$ a single finite value as $x \to a$
- Three ways a limit can fail to exist;
  1. The function is unbounded; not a finite value
     - E.g., $\lim_{x\to a} f(x) = \infty$
  2. Two one-sided limits are not equal;
     - E.g., $\lim_{x\to a^+} f(x) = c \neq b = \lim_{x \to a^-} f(x)$
  3. Oscillations;
     - E.g., $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$





L'HÔPITAL'S RULE

- If $\lim_{x\to c} \frac{f(x)}{g(x)}$ produces an indeterminant $\left(\frac{0}{0}, \frac{\pm\infty}{\pm\infty}\right)$;

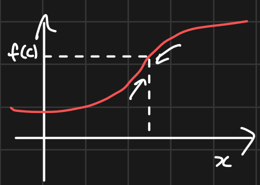
$$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},\ c \in \mathbb{R}}$$

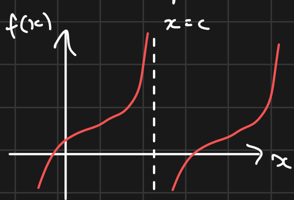
CONTINUITY

- A function $f(x)$ is continuous at a point $x = c$ iff;

$$\boxed{\lim_{x\to c} f(x) = f(c)}$$

  - Piecewise can be considered discontinuous
  - Two sided limits or undefined points





- $f(x)$ is continuous on (a,b) if x is cont $\forall x \in (a,b)$
- $f(x)$ is continuous on $[a,b]$ if $f(x)$ is;
  1. Continuous on (a,b)
  2. R cont. at a; $\lim_{x\to a^+} f(x) = f(a)$
  3. L cont. at b; $\lim_{x\to b^-} f(x) = f(b)$
  4. Continuous on $(-\infty,\infty)$; $x \in \mathbb{R}$; cont everywhere.

INTERMEDIATE VALUE THEOREM

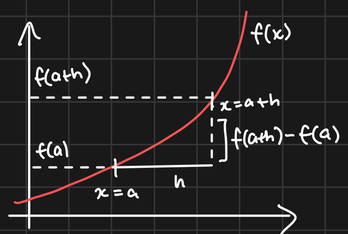
- If $f(x)$ is continuous on $[a,b]$ and $k \in [f(a), f(b)]$, inclusive, then $\exists x \in [a,b]$ such that $f(x) = k$

DIFFERENTIABILITY

- A function $f(x)$ is differentiable at $x = a$ iff $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. If so, the derivative;

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.}$$

- f is differentiable on (a,b) if f is diff. $\forall c \in (a,b)$



↳ E.g., Consider $f(x) = |x|$; $= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

Consider $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

From above; $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \frac{x-0}{x-0} = 1$

From below; $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \frac{-x-0}{x-0} = -1$

Thus; $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ DNE,
f is not differentiable at $x=0$.

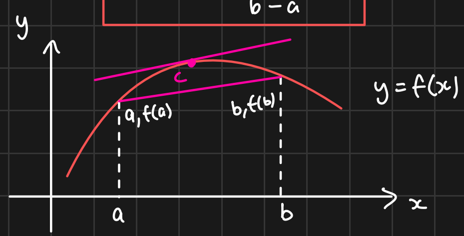
DIFFERENTIABILITY IMPLIES CONTINUITY

- $f(x)$ is differentiable at $x = x_0$
  $\Rightarrow$ continuous at $x = x_0$ (Not dif $\Rightarrow$, Not cont)
- Weierstrass function; continuous, not diff.

MEAN VALUE THEOREM

- If f is cont on $[a,b]$ and diff on (a,b), $\exists c \in (a,b)$;

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$



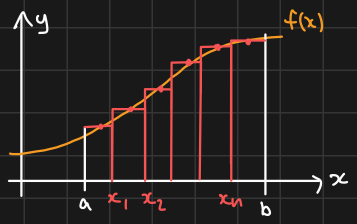
Definite Integrals/Taylor Maclaurin Polynomials

DEFINITE INTEGRALS

- Can be defined as limit of Riemann sums.
  - A = area under $f(x)$ on interval $[a,b]$

$$\boxed{A \approx \sum_{k=1}^{n} f(x_k)\Delta x_k, \quad \Delta x_k = \frac{b-a}{n}}$$

  - n - number of rectangles
  - x_k - centre, Δx_k - width



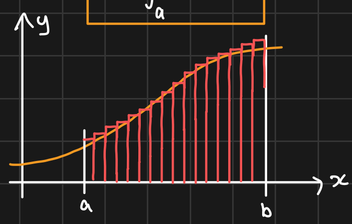
- More rectangles approximate A more closely:

$$\boxed{A = \int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k}$$

"Riemann Sum" integral

- If f is continuous on $[a,b]$, then area is defined:

$$\boxed{A = \int_a^b f(x)\,dx}$$



- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

FUNDAMENTAL THEOREM OF CALCULUS

- If $f(x)$ is continuous on $[a,b]$, and F is any antiderivative f on $[a,b]$ then;

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) \equiv F(x)}$$

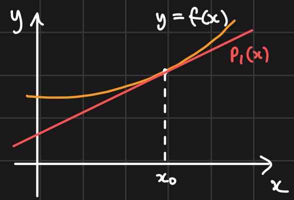
- If a is any point on the interval;

$$\boxed{F(x) = \int_a^x f(t)\,dt}$$

$$\boxed{\begin{aligned} f(x) &= \frac{d}{dx}\int_a^x f(t)\,dt \\ f(-x) &= \frac{d}{dx}\int_x^a f(t)\,dt, \quad a \in \mathbb{R} \end{aligned}}$$

TAYLOR POLYNOMIALS

- Local constant approximation; $f(x) \approx f(x_0) = P_0(x)$
- Local linear approximation; $f(x) \approx f(x_0) + f'(x_0)(x-x_0) = P_1(x)$



$P_1(x_0) = f(x_0)$
$P_1'(x_0) = f'(x_0)$

- Local quadratic approximation.

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 = P_2(x)$$

- If f can be differentiated n times at x_0, nth Taylor;

$$\boxed{P_n(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n.}$$

- Where error in this approximation; $s \in [x, x_0]$

$$\boxed{R_n(x) = f(x) - P_n(x) = \frac{f^{n+1}(s)}{(n+1)!}(x-x_0)^{n+1}.}$$

MACLAURIN POLYNOMIAL

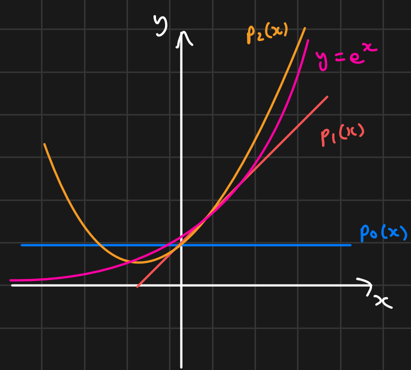
- Maclaurin polynomial; approximation where $x_0 = 0$, hence.

$$\boxed{P_n(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n}$$

↳ E.g., Find the Maclaurin polynomial of e^x; P_n, $n \in [0,3]$

$f(x) = e^x$, $f'(x) = e^x$, ... $f^n(x) = e^x$
$f(0) = 1$, $f'(0) = 1$, ... $f^n(0) = 1$

$P_0 = f(0) = 1$
$P_1 = f(0) + f'(0)x = 1 + x$
$P_2 = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 = 1 + x + \frac{1}{2}x^2$
$P_3 = f(0) + \dots + \frac{1}{3!}f'''(0)x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$



Taylor and Maclaurin Series

SERIES

- Recall Taylor polynomial of deg. n for $f(x)$; x_0

$$\boxed{p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^n(x_0)}{n!}(x-x_0)^n.}$$

- If f has derivatives at $x = x_0$, Taylor series:

$$\boxed{p(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \cdots}$$

- If $x = x_0$, a Maclaurin series exists:

$$\boxed{p(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = f(0) + f'(0)x + \cdots}$$

↳ E.g., Find the Maclaurin series for $f(x) = \cos(x)$

Let $p(x) = C_0 + C_1 x + C_2 x^2 + \cdots C_n x^n$

where $C_n = \frac{f^n(0)}{n!}$

$f(x) = \cos(x)$, $f(0) = 1$

$f'(x) = -\sin(x)$, $f'(0) = 0$

$f''(x) = -\cos(x)$, $f''(0) = -1$

$f'''(x) = \sin(x)$, $f'''(0) = 0$

$f^n(0) = 0 \ \forall n \in \mathbb{O}$; $f^{2n+1}(0) = 0$, $n \in \mathbb{Z}$

$f^{2n}(0) = (-1)^n$

$$\therefore \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

CONVERGENCE

- Taylor/Maclaurin series are power series in $(x - x_0)$

$$\boxed{\sum_{k=0}^{\infty} C_k (x-x_0)^k = C_0 + C_1(x-x_0) + \cdots}$$

- For any power series $\sum_{k=0}^{\infty} C_k(x-x_0)^k$, exactly one is true:
  1. Converges only for $x = x_0$
  2. Converges absolutely $\forall x \in \mathbb{R}$
  3. Converges absolutely $\forall x \in [x_0 - R, x_0 + R]$ (open interval)
     - Diverges for $(x < x_0 - R) \cup (x > x_0 + R)$
     - May con/diverge on bound

CONVERGENCE TEST

- Let $\sum u_k$ be a non-zero series; suppose

$$\boxed{\rho = \lim_{k \to +\infty} \frac{|u_{k+1}|}{|u_k|}}$$

(a) Converges if $\rho < 1$

(b) Diverges if $\rho \in (1, +\infty)$

(c) Inconclusive if $\rho = 1$

ALTERNATING SERIES TEST

- If $a_k > 0$ for $k \in \mathbb{Z}$, then $a_1 - a_2 + a_3$ / $-a_1 + a_2 - \ldots$ converge if both:

(a) $a_1 \geq a_2 \geq a_3$ ($a_k \geq a_{k+1}$)

(b) $\lim_{k \to +\infty} a_k = 0$

↳ E.g., Determine the interval of convergence for $f(x) = e^x$.

For $f(x) = e^x$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio test:

$$\rho = \lim_{k \to \infty} \frac{\left|\frac{x^{k+1}}{(k+1)!}\right|}{\left|\frac{x^k}{k!}\right|} = \lim_{k \to \infty} \left|\frac{x^{k+1} k!}{x^k (k+1)!}\right|$$

$$= \lim_{k \to \infty} \left|\frac{x}{k+1}\right| = 0, < 1 \ \forall x \in \mathbb{R}$$

Since $\rho < 1$, series converges for all x, $(-\infty, \infty)$.

CONVERGENCE REMAINDER

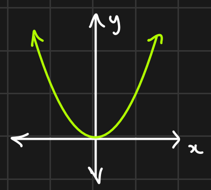
- To prove convergence of Taylor series to $f(x)$, prove that: $\lim_{n \to \infty} R_n(x) = 0$;

$$\boxed{R_n(x) = f(x) - \sum_{k=0}^{\infty} \frac{f^k(x_0)}{k!}(x-x_0)^k = \frac{f^{n+1}(s)}{(n+1)!}(x-x_0)^{n+1}, \ s \in (x_0, x)}$$

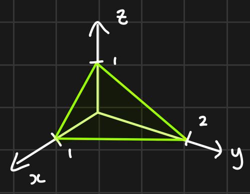
Multivariable Calculus

MULTIVARIABLE

- Single variable, let f be defined $f:\mathbb{R}\to\mathbb{R}$, e.g., $f(x)=x^2$
- Multivariable, let f be defined $f:\mathbb{R}^n\to\mathbb{R}$, e.g., $f(x,y)=1-x-\frac{1}{2}y$



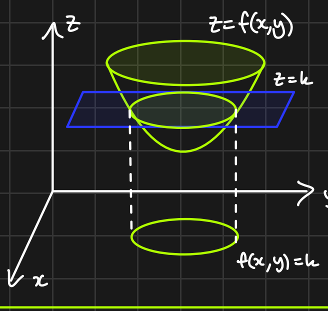
$f:\mathbb{R}\to\mathbb{R}$



$f:\mathbb{R}^3\to\mathbb{R}$

LEVEL CURVES

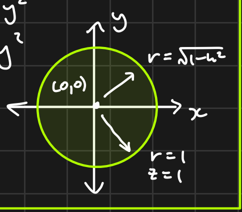
- Cut the surface $z=f(x,y)$ with the horizontal plane $z=k$, then at all points on the intersection, $f(x,y)=k$.
- Projection of intersection on xy-plane is level curve



↳ E.g., Draw the level curves of $z=\sqrt{1-x^2-y^2}$

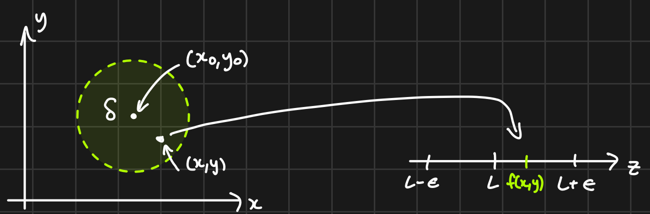
Let $z=k$, $k=\sqrt{1-x^2-y^2} \Rightarrow k^2=1-x^2-y^2$

$r=\sqrt{1-k^2}$, $k\in[0,1]$

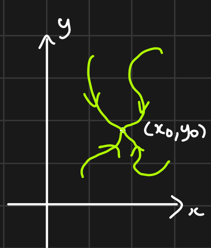


LIMITS

- Notation: $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=L$ or $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)\to L$
- Formal definition: $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=L$ if given any $\epsilon>0$, $\exists\,\delta>0$ such that $|f(x,y)-L|<\epsilon$ whenever $\sqrt{(x-x_0)^2+(y-y_0)^2}\in[0,\delta]$



- Consequences: $f(x,y)\to L$, $(x,y)\to(x_0,y_0)$ on any smooth curve.
  - If $f(x,y)$ has different limits, LDNE.



CONTINUITY

- A function $f(x,y)$ is continuous at (x_0,y_0) iff

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)}$$

- Cont. functions:
  - Sum/diff/product of cont. func. $(3x^2y^5)$
  - Quotient (den $\neq 0$) $\left(\frac{x^2}{y^3}\right)$
  - Composition function. $(\sin(3x^2y^5))$.

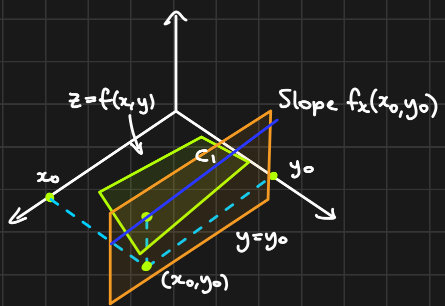
PARTIAL DIFFERENTIATION

- Partial derivative of f w.r.t x at (x_0,y_0):

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}}$$

- Partial derivative of f w.r.t y at (x_0,y_0):

$$\boxed{\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0) = \lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}}$$

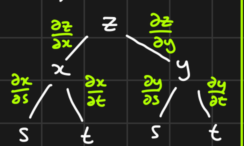


CHAIN RULE

- Recall chain rule for $y(x(t))$: $\boxed{\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}}$

- Example: $z=\sin 2x\cos 3y$ $(x=s+t,\ y=s-t)$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$



$$\frac{\partial z}{\partial x}=2\cos 2x\cos 3y$$

$$\frac{\partial z}{\partial y}=-3\sin 3y\sin 2x$$

$$\frac{\partial x}{\partial s}=\frac{\partial y}{\partial s}=1 \Rightarrow \frac{\partial z}{\partial s}=2\cos 2x\cos 3y-3\sin 3y\sin 2x.$$

Directional Derivatives/Critical Points

PARTIAL DERIVATIVES

- A function of two variables, x,y, is a rule $f:\mathbb{R}^2 \to \mathbb{R}$ that assigns a unique real number $f(x,y)$ to each point (x,y) within some set D in the xy-plane.
- $\frac{\partial f}{\partial x}$ – PD of f wrt x. (f_x)
- $\frac{\partial f}{\partial y}$ – PD of f wrt y. (f_y)

HIGHER-ORDER PDs

- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$ (f_{xx}, f_{yy})
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ (f_{yx}, f_{xy})
- Equality of mixed partials; if f is defined in some domain $D \subseteq \mathbb{R}^2$, and f_{xy} and f_{yx} are continuous throughout D, then $f_{yx} = f_{xy}$ $\forall (x,y) \in D$.

DIRECTIONAL DERIVATIVE

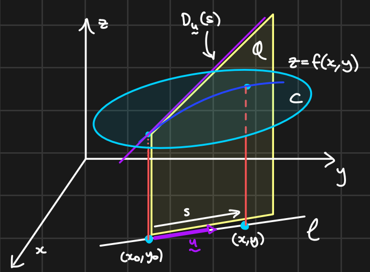
- If $f(x,y)$ is a function of two variables, x and y;

$$\boxed{\underset{\sim}{u} = u_1 \underset{\sim}{i} + u_2 \underset{\sim}{j}}$$

is a unit vector, then the directional derivative of f in the direction of $\underset{\sim}{u}$ at (x_0,y_0) is $D_u f(x_0,y_0)$

- Provided that $f(x,y)$ is differentiable at (x_0,y_0), DDer:

$$\boxed{D_{\underset{\sim}{u}} f(x_0,y_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} u_1 + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} u_2}$$



GRADIENT VECTOR

- If f is a function x,y, gradient of f is:

$$\boxed{\vec{\nabla} f(x,y) = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j}}$$

- Directional derivative in terms of gradient vector.

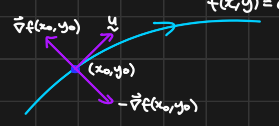
$$D_{\underset{\sim}{u}} f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$$

- But, $(f_x(x_0,y_0), f_y(x_0,y_0))$ is gradient vector $\vec{\nabla} f(x_0,y_0)$

$\therefore$ $\boxed{D_{\underset{\sim}{u}} f(x_0,y_0) = \|\vec{\nabla} f(x_0,y_0)\| \cos\theta}$

GRADIENT VECTOR

- Maximum slope of $f(x_0,y_0)$ points towards $\vec{\nabla} f(x_0,y_0) \Rightarrow \|\vec{\nabla} f(x_0,y_0)\|$
- Minimum slope of $f(x_0,y_0)$ points towards $-\vec{\nabla} f(x_0,y_0) \Rightarrow -\|\vec{\nabla} f(x_0,y_0)\|$.
- Assuming $\vec{\nabla} f(x_0,y_0) \neq 0$, $\vec{\nabla} f(x_0,y_0)$ is a normal vector to $f(x_0,y_0)$



CRITICAL POINTS

- $f(x)$ has a critical point at $f'(x)=0$, or not differentiable.
  - Minimum: $f'(x)=0$, $f''(x)>0$
  - Maximum: $f'(x)=0$, $f''(x)<0$
  - Inconclusive: $f'(x)=f''(x)=0$
- For $f(x_0,y_0)$ to have a critical point, $\vec{\nabla} f(x_0,y_0)=0$ or DNE.

CLASSIFYING CRITICAL POINTS

- Let f be a function of two variables with continuous 2nd PDs in a neighbourhood of a critical point (x_0,y_0);

$$\boxed{D = \det|\vec{H}| = \det\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2}$$

- $D<0$, saddle point
- $D>0$, $\frac{\partial^2 f}{\partial x^2}>0$, rel. minimum
- $D>0$, $\frac{\partial^2 f}{\partial x^2}<0$, rel. maximum
- $D=0$, inconclusive.

Double and Triple Integrals

DOUBLE INTEGRALS

- Recall a Riemann Sum integral $\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$.
- Net area A of $f \in [a,b]$: $A = \int_a^b f(x)\,dx$.

- If f is a function of two variables that is continuous and nonnegative on a region R on xy-plane, volume of the solid enclosed between the surface $z = f(x,y)$ and R is defined to:

$$\boxed{V = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \equiv \iint_R f(x,y)\,dA}$$

- If f is nonnegative on R, V is volume.
- If f is $+/-$, difference volume or net volume.

PROPERTIES

- $\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA$
- $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$
- $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA \Rightarrow R = R_1 + R_2$
- $\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy \quad (x\in(a,b), y\in(c,d))$

E.g., Evaluate $\iint_R 40 - 2xy\,dA$, $R = \{(x,y) : x\in[1,3], y\in[2,4]\}$

$$\iint_R 40-2xy\,dA = \int_2^4\int_1^3 (40-2xy)\,dx\,dy$$

$$= \int_2^4 [40x - x^2y]_{x=1}^{x=3}\,dy$$

$$= \int_2^4 (120 - 9y - 40 + y)\,dy = \int_2^4 (80-8y)\,dy$$

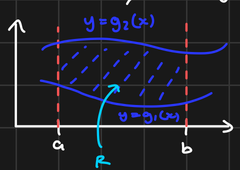
$$= [80y - 4y^2]_{y=2}^{y=4}$$

$$= 112.$$

NONRECTANGULAR REGIONS

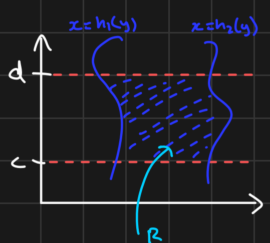
- The limits of integration may not be constant, but $f(x,y)$.
- Bounded by $x=a$, $x=b$, bounded below and above by $y=g_1(x)$, $y=g_2(x)$, $g_1(x) \le g_2(x)\ \forall x\in[a,b]$:

$$\iint_R f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



- Bounded below/above $y=c$, $y=d$, bounded $x=h_1(y)$, $x=h_2(y)$, $h_1(y)\le h_2(y)$

$$\iint_R f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$



TRIPLE INTEGRALS

- A triple integral of $f(x,y,z)$ is defined over a closed solid region G in an xyz-coordinate system:

$$\boxed{\iiint_G f(x,y,z)\,dV = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k.}$$

E.g., $\iiint 12xy^2z^3\,dV$, $G \in [(x,y,z), x\in[-1,2], y\in[0,3], z\in[0,2]]$

$$\int_0^2\int_0^3\int_{-1}^2 12xy^2z^3\,dx\,dy\,dz$$

$$= \int_0^2\int_0^3 [6x^2y^2z^3]_{x=-1}^{x=2}\,dy\,dz$$

$$= \int_0^2\int_0^3 18y^2z^3\,dy\,dz$$

$$= \int_0^2 [6y^3z^3]_{y=0}^{y=3}\,dz$$

$$= \int_0^2 162z^3\,dz$$

$$= \left[\frac{162z^4}{4}\right]_{z=0}^{z=2}$$

$$= 664$$

Vector-Valued Functions

VVF

- A VVF: $f: \mathbb{R} \to \mathbb{R}^n$ is a function that maps a number to a vector, using notation, $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$, or; $\underset{\sim}{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $t \in \mathbb{R}$.

CALCULUS OF VVF

- For $\underset{\sim}{r}(t) = \langle x(t), y(t), z(t) \rangle$, domain is set t for all defined.
  - E.g., $\underset{\sim}{r}(t) = (\sqrt{t}, \ln(1-t), e^t)$.
    - $x(t): D = \{t : t \geq 0\}$
    - $y(t): D = \{t : t < 1\}$
    - $z(t): D = \{t : t \in \mathbb{R}\}$
  - Domain of $\underset{\sim}{r}(t) = 0 \leq t < 1$, $t \in [0, 1)$
- Orientation of $\underset{\sim}{r}(t)$, direction of increasing parameter.

LIMIT

$$\boxed{\lim_{t \to a} \underset{\sim}{r}(t) = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right\rangle.}$$

- $\lim_{t \to a} \underset{\sim}{r}(t) = \vec{L}$ iff $\lim_{t \to a} \| \underset{\sim}{r}(t) - \vec{L} \| = 0$.

CONTINUITY

- $\underset{\sim}{r}(t)$ is continuous at $t = a$ iff $\lim_{t \to a} \underset{\sim}{r}(t) = \underset{\sim}{r}(a)$.
  - Each component must be continuous.

DIFFERENTIATION

- $\underset{\sim}{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- Indefinite; $\int \underset{\sim}{r}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$
  - Own constant of integration ($C_1, C_2, \ldots$)
- Definite: $\int_a^b \underset{\sim}{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$

VVF OF A LINE

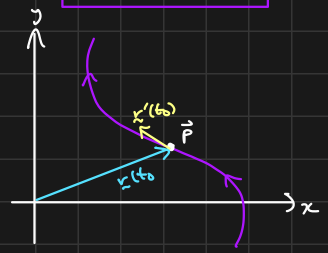
- Line is determined by point and vector. Passes through $\vec{P}_0(x_0, y_0, z_0)$ and parallel to $\underset{\sim}{v} = \langle a, b, c \rangle$:

$$\boxed{\underset{\sim}{r}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{P}_0 + t\underset{\sim}{v}}$$

TANGENT LINE

- When $\underset{\sim}{r}(t)$ is diff at t_0 and $\underset{\sim}{r}'(t_0)$ is nonzero, $\underset{\sim}{r}'(t_0)$ is tangent to $\underset{\sim}{r}(t)$ at t_0, points in direction increasing param.

$$\boxed{\underset{\sim}{L}(t) = \underset{\sim}{r}(t_0) + t\,\underset{\sim}{r}'(t_0)}$$

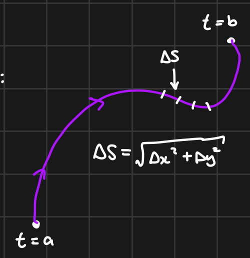


ARC LENGTH

- Distance along VVF between two points.
- Arc length S of $\underset{\sim}{r}(t)$ between $t = a$, $t = b$:

$$\boxed{S = \int_a^b \| \underset{\sim}{r}'(t) \| \, dt}$$

  - 2D: $\| \underset{\sim}{r}'(t) \| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
  - 3D: $\| \underset{\sim}{r}'(t) \| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$



First-Order ODEs

DE TERMINOLOGY

- A DE is an equation involving derivatives of one or more unknown dependent variables, wrt. to ind. var.
- E.g., $\frac{dy}{dt} = 3y$, $\frac{dv}{dt} = -mg$
- An ODE is single variable: $\frac{dy}{dt} = 4ty$
- A PDE is multivariable: $u = u(x,y) \Rightarrow \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$
- The order is the highest derivative:
  - 2nd order: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$
  - 4th order: $y''''(t) + y''(t) = 4t$
- An autonomous DE does not depend explicitly on ind. var.
  - Autonomous: $\frac{dy}{dt} = 4y$
  - Nonautonomous: $\frac{dy}{dt} = yt$
- A linear DE does not have products of dependent variable with itself or derivatives, general form:

$$\boxed{a_n(t)\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = F(t)}$$

  - Linear: $\frac{dy}{dt} = 2y$, $t\frac{dy}{dt} - y = \sin(t)$
  - Nonlinear: $\frac{dy}{dt} = \sin(y)$, $y^3\frac{d^3y}{dt^3} = y$
- A solution is a function that satisfies the equation

SEPARABLE DEs

- A separable 1st order ODE is in the form:

$$\boxed{\frac{dy}{dx} = p(x)q(y) \text{ or } \frac{1}{q(y)}\frac{dy}{dx} = p(x)}$$

SPECIAL CASE – DIRECT INTEGRATION

- The even simpler form $\frac{dy}{dx} = g(x)$ can be solved $y(x) = \int g(x)dx$
  - General solution, has constant of int. $(+C)$ or particular solution if initial condition is given $(y(x_0) = y_0)$

SOLVING A 1st ORDER EQUATION

- $\frac{1}{q(y)}\frac{dy}{dx} = p(x) \Rightarrow \int \frac{1}{q(y)}\frac{dy}{dx}dx = \int p(x)dx$

  $\int \frac{1}{q(y)}dy = \int p(x)dx$

INTEGRATING FACTOR METHOD

- IFM used for equations in the form:

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)}$$

- The integrating factor is given by $\boxed{I(x) = e^{\int p(x)dx}}$
- The solution is:

$$\boxed{y(x) = \frac{1}{I(x)}\int I(x)q(x)dx}$$

QUALITATIVE ANALYSIS

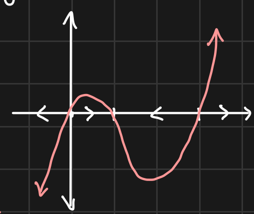
- With QA, understand behaviour of solutions of DE.
- We look for fixed/equilibrium points, sketch solution curves, draw a phase line diagram. For 1st auto-DEs:

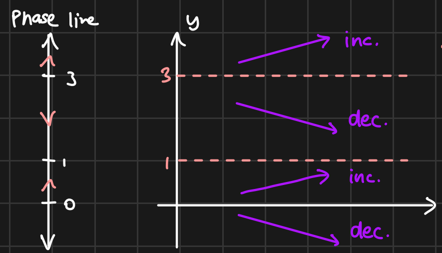
$$\boxed{\frac{dy}{dt} = f(y)}$$

- To draw a phase line diagram for $y'(t) = f(y)$
  1. Compute the fixed point, which are values of y for $f(y) = 0$.
  2. Draw $f(y)$
     a. If $f(y) > 0$, $y'(t) > 0$, $y(t)$ increases
     b. If $f(y) < 0$, $y'(t) < 0$, $y(t)$ decreases
  3. Determine stability of the fixed point
  4. Draw the fixed points on a phase line and indicate stability.

EXAMPLE: $\frac{dy}{dt} = y(1-y)(3-y)$

1. Fixed points: $y(1-y)(3-y) = 0$, $y = 0, 1, 3$ are fixed
2. Sketch:
3. Stability: unstable at 0, stable at 1, unstable at 3.





EXACT DEs

- E.g., consider $x^2y^2 = 2$, derive/form DE $2xy^2 + 2x^2y\left(\frac{dy}{dx}\right) = 0$
- The general form of an exact DE is: $\boxed{\psi(x, y(x)) = C}$
  - Take derivative wrt x:

$$\boxed{\frac{d\psi(x,y(x))}{dx} = \frac{\partial\psi(x,y)}{\partial x} + \frac{\partial\psi(x,y)}{\partial y}\frac{dy}{dx} = 0}$$

  - Or:

$$\boxed{P(x,y) + Q(x,y)\frac{dy}{dx} = 0 \Rightarrow P(x,y) = \psi_x,\ Q(x,y) = \psi_y}$$

  - By equality of mixed partials: $\psi_{xy} = \psi_{yx} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
- Hence, the DE is exact if $P_y = Q_x \left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\right)$
- We find $\psi(x,y)$ as

$$\boxed{\psi(x,y) = \int P(x,y)dx + f(y) = \int Q(x,y)dy + g(x)}$$

Second-Order ODEs

TERMINOLOGY

- Second-order DE in form:

$$\boxed{y''(x) = f(y'(x), y(x), x).}$$

- Linear second-order DE in form:

$$\boxed{a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y = f(x)}$$

  - $f(x) = 0$, homogeneous
  - $f(x) \neq 0$, non-homogeneous
- An initial value problem specifies the value for y and y' at a single value of the independent variable:

$$\boxed{y(x_0) = y_0,\ y'(x_0) = y_1}$$

- A boundary value problem specifies the value for y at 2 values for the independent variable.

$$\boxed{y(x_0) = y_0, y(x_1) = y_1}$$

FINDING SOLUTIONS

- Theorem: Superposition of solutions for homogeneous equations:
  - Consider a homo-2^{nd} ODE: $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$
  - If $y_1(x), y_2(x), \dots, y_n(x)$ are solutions:
  - $y(x) = c_1y_1(x) + c_2y_2(x) + \dots + c_ny_n(x)$ is also a solution.
- Theorem: General Solution for homogeneous equations
  - Consider a 2^{nd}-ODE w/ $a_i(x)$ cont on open interval I.
  - This has a fundamental set of solutions on I. (2 lin ind.)
  - General sol: $y(x) = c_1y_1(x) + c_2y_2(x)$, $c_1, c_2 \in \mathbb{R}$

REDUCTION OF ORDER

- Suppose one solution already exists $y_1(x)$, $y_2(x)$ is:

$$\boxed{y_2(x) = v(x)y_1(x)}$$

- New DE $v(x)$ is formed.

2^{nd}-ORDER ODES w/ CONSTANT COEFFICIENTS

- When linear 2^{nd}-ODE has constant coeffs:

$$a_2y''(x) + a_1y'(x) + a_0y(x) = 0$$

- What functions can $y(x)$ be, such that sum of terms is 0?
- TRIAL SOLUTION:
  - Step 1: Let $y = e^{mx}$ $(y' = me^{mx},\ y'' = m^2e^{mx})$
  - Step 2: Substitute: $(a_2m^2e^{mx} + a_1me^{mx} + a_0e^{mx} = 0)$
  - Step 3: Solve characteristic equation $(a_2m^2 + a_1m + a_0 = 0)$

$$m = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}$$

1. If $a_1^2 > 4a_0a_2\ (\Delta > 0)$, $y(x) = c_1e^{m_1x} + c_2e^{m_2x}$
2. If $a_1^2 = 4a_0a_2\ (\Delta = 0)$, $y(x) = c_1e^{mx} + c_2xe^{mx}$
3. If $a_1^2 < 4a_0a_2\ (\Delta < 0)$, $y(x) = e^{ax}(c_1\cos(bx) + c_2\sin(bx))$
   - $m \in \mathbb{C}$: $m = a \pm bi \Rightarrow a = Re(m),\ b = Im(m)$

NONHOMOGENEOUS 2^{nd}-ODE w/ CONSTANT COEFFS

- For the ODE $a_2y'' + a_1y' + a_0y = f(x)$.
- Homogeneous solution $f(x) = 0 \Rightarrow y_H(x)$
- Suppose a particular solution, $y_p(x)$,

$$\boxed{a_2y_p'' + a_1y_p' + a_0y_p = f(x)}$$

- Is $y_H(x) + y_p(x)$ also a solution to the DE?

$$a_2(y_H''(x) + y_p''(x)) + a_1(y_H'(x) + y_p'(x)) + a_0(y_H(x) + y_p(x))$$
$$= (a_2y_H'' + a_1y_H' + a_0y_H) + (a_2y_p'' + a_1y_p' + a_0y_p)$$

- General solution $a_2y'' + a_1y' + a_0y = f(x) \Rightarrow y(x) = \overbrace{y_H(x)}^{0} + \overbrace{y_p(x)}^{c}$

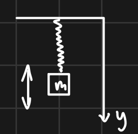
METHOD OF UNDETERMINED COEFFS

$f(x)$	$y_p(x)$
Constant	A
Linear	$A_1x + A_0$
Polynomial 'n'	$\sum_i^n A_ix^i$
Exponential	Ae^{kx}
$\cos/\sin(\omega x)$	$A_0\cos(\omega x) + A_1\sin(\omega x)$
$e^{kx}\cos/\sin(\omega x)$	$e^{kx}(A_0\cos(\omega x) + A_1\sin(\omega x))$

HOMOGENEOUS SOLUTION AS FORCING TERM

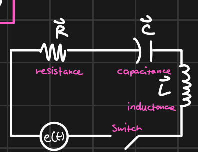
- Consider $y'' - y = e^{-x}$; $y'' - y = 0 \Rightarrow y_H = c_1e^{x} + c_2e^{-x}$
- $y_p = Ae^{-x}$ does not work, e^{-x} part of homogeneous solution
- Multiply by factor of $x \Rightarrow Axe^{-x}$

2^{nd} ODE APPLICATIONS



- Spring/mass systems:
  - $\vec{F} = m\vec{a} \Rightarrow F = m\frac{d^2y}{dt^2}$
  - $F = F_s + F_d + f(t)$ $\left(f(t) - ky(t) - \gamma y'(t) = m\frac{d^2y}{dt^2}\right)$

$$\boxed{my''(t) + \gamma y'(t) + ky(t) = f(t)}$$



- RLC Circuits; q = charge $\Rightarrow \frac{dq}{dt} = i$ (current)
- Resistor: iR, Capacitor: $\frac{q}{c}$, Inductor: $L\frac{di}{dt}$
  - $L\frac{di}{dt} + iR + \frac{q}{c} = e(t)$

$$\boxed{Lq'' + Rq' + \frac{1}{c}q = e(t)}$$