INDUCTIONFEST
Solutions

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Example 1.

Show that for every positive integer n, $1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot n! = (n+1)! - 1$.

Proof. Let $p(n) :=$ "$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$". Since we want to show $p(n) \forall n \in \mathbb{N}^+$, we proceed by induction on $n \geq 1$.

Base case: Let $n = 1$. Then, $1 \cdot 1! = 1 = 2 - 1 = (1+1)! - 1$. Thus, $p(1)$.

Inductive hypothesis: Let $n \geq 1$. Assume $p(n)$.

Inductive step: We show IH $\implies p(n+1)$ in the following:

$$
\begin{aligned}
& 1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot n! = (n+1)! - 1 && \text{(IH)} \\
\implies & 1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot n! + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)! \\
\implies & 1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot n! + (n+1)(n+1)! = (n+1+1)! - 1 && \text{(group like terms on RHS)} \\
\implies & 1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot n! + (n+1)(n+1)! = (n+2)! - 1 \\
\implies & p(n+1)
\end{aligned}
$$

Thus by PMI, $p(n) \forall n \in \mathbb{N}^+$. $\square$
Note: we could have also defined $p(n) :=$ " $\sum_{i=1}^{n} i * i!$" as this is equivalent. Using either the summation or the $\cdots$ notation would be fine for this problem.

Example 2.

Let h be a real number, $h \geq -1$. Show that for every positive integer n,

$$1 + nh \leq (1+h)^n$$

.

Proof. Let $p(n) :=$ "$1 + nh \leq (1+h)^n$". Since we want to show $p(n) \forall n \in \mathbb{N}^+$, we proceed by induction on $n \geq 1$.

Base case: Let $n = 1$. Then,

$$\begin{aligned}
&1 + h = 1 + h \\
\Longrightarrow\ &1 + h \leq 1 + h \\
\Longrightarrow\ &1 + 1 \cdot h \leq (1+h)^1
\end{aligned}$$

Thus, $p(1)$.

Inductive hypothesis: Let $n \geq 1$. Assume $p(n)$.

Inductive step: We show IH $\implies p(n+1)$ in the following:

$$\begin{aligned}
&1 + nh \leq (1+h)^n && \text{(IH)} \\
\Longrightarrow\ &(1+nh)(1+h) \leq (1+h)^{n+1} && (h \geq -1 \implies h + 1 \geq 0) \\
\Longrightarrow\ &1 + h + nh + nh^2 \leq (1+h)^{n+1} \\
\Longrightarrow\ &1 + h(1+n) + nh^2 \leq (1+h)^{n+1} \\
\Longrightarrow\ &1 + h(1+n) \leq (1+h)^{n+1} && (h^2 \geq 0 \wedge n \geq 0 \implies nh^2 \geq 0) \\
\Longrightarrow\ &p(n+1)
\end{aligned}$$

Thus by PMI, $p(n) \forall n \in \mathbb{N}^+$. $\square$

Example 3.

Show that for every positive integer n, $21|\left(4^{n+1}+5^{2n-1}\right)$.

Proof. Let $p(n) :=$ "$21|\left(4^{n+1}+5^{2n-1}\right)$". Since we want to show $p(n)\forall n \in \mathbb{N}^+$, we proceed by induction on $n \geq 1$.

Base case: Let $n = 1$. Then,

$$\begin{aligned}
&4^{1+1}+5^{2*1-1} \\
=&4^2+5^1 \\
=&16+5 \\
=&21 \\
\implies& 21|(4^{1+1}+5^{2*1-1}) && (21|21)
\end{aligned}$$

Thus, $p(1)$.

Inductive hypothesis: Let $n \geq 1$. Assume $p(n)$.

Inductive step: We show IH $\implies p(n+1) :=$ "$21|\left(4^{n+2}+5^{2n+1}\right) = 4\cdot 4^{n+1}+25\cdot 5^{2n-1}$" in the following:

$$\begin{aligned}
&21|\left(4^{n+1}+5^{2n-1}\right) && \text{(IH)} \\
\implies& \exists k \in \mathbb{Z} \text{ such that } 21k = 4^{n+1}+5^{2n-1} && \text{(definition of } |) \\
\implies& 4^{n+1} = 21k - 5^{2n-1} \\
\implies& 4\cdot 4^{n+1}+25\cdot 5^{2n-1} = 4\cdot(21k-5^{2n-1})+25\cdot 5^{2n-1} && \text{(substitution)}
\end{aligned}$$

$$\begin{aligned}
&4\cdot(21k-5^{2n-1})+25\cdot 5^{2n-1} \\
=&4\cdot 21k - 4\cdot 5^{2n-1}+25\cdot 5^{2n-1} \\
=&4\cdot 21k - 21\cdot 5^{2n-1} \\
=&21(4k-5^{2n-1}) \\
\implies& 21|4\cdot(21k-5^{2n-1})+25\cdot 5^{2n-1} \\
\implies& 21|4\cdot 4^{n+1}+25\cdot 5^{2n-1} && \text{(substitution)} \\
\implies& p(n+1)
\end{aligned}$$

Thus by PMI, $p(n)\forall n \in \mathbb{N}^+$. $\square$

Example 4.

Consider the sequence $\{a_n\}$ defined by $a_0 = 2, a_1 = 1$, and $a_n = 3a_{n-1} + a_{n-2}$ when $n \geq 2$. First show that for all natural numbers n, $3 \nmid a_n$. As a second problem, show that a_n is even if and only if n is a multiple of 3.

Part 1

Proof. Let $p(n) :=$ "$3 \nmid a_n$". We want to show p(n) $\forall n \in \mathbb{N}$. we proceed by strong induction on $n \geq 0$.

Base cases:

- Let $n = 0$. Then, $a_0 = 2$. Indeed, $3 \nmid 2$. Thus, $p(0)$.
- Let $n = 1$. Then, $a_1 = 1$. Indeed, $3 \nmid 1$. Thus, $p(1)$.

Inductive hypothesis: Let $n \geq 1$. Assume $p(k)$ for all $0 \leq k \leq n$.

Inductive step: We show IH $\implies$ $p(n+1) :=$ "$3 \nmid a_{n+1} = 3a_n + a_{n-1}$".

First note that since $n - 1 \leq n$ we can invoke the IH on $n - 1$ and see that $3 \nmid a_{n-1}$, so $a_{n-1} \neq 3k$ for any $k \in \mathbb{Z}$. Then by the division theorem, there is a unique way to write $a_{n-1} = 3k + r, k \in \mathbb{Z}, r \in \{1, 2\}$.

We can then see that

$$a_{n+1} = 3a_n + a_{n-1} = 3a_n + 3k + r = 3(a_n + k) + r.$$

The division theorem tells us that there is a unique way to write $a_{n+1} = 3b + t$, and since $r = t \in \{1, 2\}$, it must be the case that $3 \nmid a_{n+1}$. Thus, $p(n+1)$.

Thus by PMI, $p(n) \forall n \in \mathbb{N}$. $\square$

Part 2

Proof. We want to show that a_n even $\iff$ n multiple of 3. We show that $3|n \implies 2|a_n$ and $3 \nmid n \implies 2 \nmid a_n$ for all $n \in \mathbb{N}$.

Let $p(n) :=$ "$3|n \implies 2|a_n$ and $3 \nmid n \implies 2 \nmid a_n$".

We want to show p(n) $\forall n \in \mathbb{N}$. we proceed by strong induction on $n \geq 0$.

Base cases:

- Let $n = 0$. Then, $3|n$ and $a_0 = 2$. Indeed, $2|2$. Thus, $p(0)$.
- Let $n = 1$. Then, $3 \nmid n$ and $a_1 = 1$. Indeed, $2 \nmid 1$. Thus, $p(1)$.

Inductive hypothesis: Let $n \geq 1$. Assume $p(k)$ for all $0 \leq k \leq n$.

Inductive step: We show IH $\implies p(n+1)$.

We now case on if $3|(n+1)$:

- Assume $3|(n+1)$. We show $2|a_{n+1}$. By the definition of divides, we know that $\exists k \in \mathbb{Z}$ such that $n+1 = 3k$. Thus $n = 3k-1$ and $n-1 = 3k-2$, so we can see that $3 \nmid n$ and $3 \nmid (n-1)$. Since $0 \leq n, n-1 \leq n$, we can invoke the IH and see that $2 \nmid a_n$ and $2 \nmid a_{n-1}$. Thus by the division theorem, $\exists \ell, m \in \mathbb{Z}$ such that $a_n = 2\ell+1$ and $a_{n-1} = 2m+1$. Then,

  $$a_{n+1} = 3a_n + a_{n-1} = 3(2\ell+1) + 2m + 1 = 6\ell + 2m + 4 = 2(3\ell + m + 2).$$

  By the definition of divides, $2|a_{n+1}$, so $p(n+1)$ in this case.

- Assume $3 \nmid (n+1)$. We show $2 \nmid a_n$. By the division theorem, there is a unique way to write $n+1 = 3k+r, k \in \mathbb{Z}, r \in \{1,2\}$.

  We now case on r:

  a) Let $r = 1$. Then $n+1 = 3k$, and so $n = 3k$ and $n-1 = 3k-1$. Note that this means that $3|n$ and $3 \nmid (n-1)$. Since $0 \leq n, n-1 \leq n$, we can invoke the IH and see that $2 \mid a_n$ and $2 \nmid a_{n-1}$. Thus by the division theorem, $\exists \ell, m \in \mathbb{Z}$ such that $a_n = 2\ell$ and $a_{n-1} = 2m+1$. Then,

  $$a_{n+1} = 3a_n + a_{n-1} = 3(2\ell) + 2m + 1 = 6\ell + 2m + 1 = 2(3\ell + m) + 1.$$

  Thus $2 \nmid a_{n+1}$, so $p(n+1)$.

  b) Let $r = 2$. Then $n+1 = 3k+2$, and so $n = 3k+1$ and $n-1 = 3k$. Note that this means that $3|(n-1)$ and $3 \mid (n-1)$. Since $0 \leq n, n-1 \leq n$, we can invoke the IH and see that $2 \mid a_{n-1}$ and $2 \nmid a_n$. Thus by the division theorem, $\exists \ell, m \in \mathbb{Z}$ such that $a_{n-1} = 2\ell$ and $a_n = 2m+1$. Then,

  $$a_{n+1} = 3a_n + a_{n-1} = 3(2m+1) + 2\ell = 6m + 2\ell + 3 = 2(3\ell + m + 1) + 1.$$

  Thus $2 \nmid a_{n+1}$, so $p(n+1)$.

Note that either $3|(n+1)$ or $3 \nmid (n+1)$ must be true, so our cases are exhaustive. In every case, we have shown $p(n+1)$. Thus by PMI, $p(n) \forall n \in \mathbb{N}$. $\square$

Example 5.

Show that every positive integer n can be written as a sum of Fibonacci numbers, such that no number is repeated and no pair of adjacent Fibonacci numbers are used (for example, 1 can't be repeated and 5 and 8 can't be used in the same summation)

Proof. Let $p(n)$:= "n can be written as a sum of unique non-adjacent Fibonacci numbers". Since we want to show $p(n) \forall n \in \mathbb{N}^+$, we proceed by strong induction on $n \geq 1$.

Base case: Let $n = 1$. 1 is a Fibonacci number so the base case holds.

Inductive Hypothesis: Let $n \geq 1$. Assume $P(k)$ for all $1 \leq k \leq n$

Inductive step: we show IH $\implies P(n+1)$ in the following:

We would like to represent $n+1$ as a sum of unique Fibonacci numbers. The first number we should consider adding to this sum is the largest Fibonacci number not greater than $n+1$. Let F_ℓ be the largest fibonacci number with $F_\ell \leq n+1$. We case on if $F_\ell = n+1$:

Case 1: $F_\ell = n+1$

If $F_\ell = n+1$, we are done immediately as $n+1$ is itself a Fibonacci number

Case 2: $F_\ell < n+1$

Since $F_\ell < n+1$, we have:

$$n + 1 = F_\ell + [\text{other terms}]$$

In order to complete the sum, we need to write the difference $((n+1) - F_\ell)$ as a sum of unique non-adjacent Fibonacci numbers. Since $((n+1) - F_\ell) < n+1$, we can invoke IH and argue that $((n+1) - F_\ell)$ is the sum of some set of unique non-adjacent fibonacci numbers S. We can write this as:

$$((n+1) - F_\ell) = \sum_{F_i \in S} F_i$$

where F_i represents an arbitrary Fibonacci number in S. From here,

$$n + 1 = F_\ell + ((n+1) - F_\ell) = F_\ell + \sum_{F_i \in S} F_i$$

We now have that $n+1$ is the sum of fibonacci numbers. In order to prove that each fibonacci number is unique and we don't use a number adjacent to F_ℓ, we need to argue that $F_\ell \notin S$ and $F_{\ell-1} \notin S$, as our IH tells us that each $F_i \in S$ is unique and non-adjacent to other values in S.

The key here is that if F_ℓ is the largest Fibonacci less than or equal to $n+1$, it must be that $F_{\ell+1} > n+1$. Then we have that:

$$\begin{aligned}
& F_{\ell+1} > n+1 & \\
\implies & F_{\ell-1} + F_\ell > n+1 & \qquad F_{\ell-1} + F_\ell = F_{\ell+1} \\
\implies & F_{\ell-1} > (n+1) - F_\ell & \\
\implies & F_\ell > (n+1) - F_\ell & \qquad F_\ell = F_{\ell-1} + F_{\ell-2} > F_{\ell-1}
\end{aligned}$$

Note

$$\sum_{F_i \in S} F_i = ((n+1) - F_\ell) \implies F_i \leq ((n+1) - F_\ell) \ \forall \ F_i \in S$$

as all Fibonacci numbers are positive. Since

$$F_\ell > F_{\ell-1} > (n+1) - F_\ell \geq F_i \forall F_i \in S$$

, F_ℓ and $F_{\ell-1}$ cannot be in S $\square$.