Secretary Problem Variant Deep Dive

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1 Introduction

The secretary problem (someimes called the marriage problem) is a famous algorithmic puzzle with applications in social decision making. I've taken the initial problem setup from another great article[1] on the topic:

You are the HR manager of a company and need to hire the best secretary out of a given number N of candidates. You can interview them one by one, in random order. However, the decision of appointing or rejecting a particular applicant must be taken immediately after the interview. If nobody has been accepted before the end, the last candidate is chosen.

At this point, the following question is usually asked:

What strategy do you use to maximize the chances to hire the best applicant?

The well known solution to the problem (as N goes to infinity) is as follows:

1. Reject the first $\frac{N}{e}$ secretaries where e is Eulers number, and note down the best out of this group as init_{max}
2. Hire the first secretary after that point that is better than init_{max}

While this may be an interesting question, it falls short of the real world scenario we are trying to model here in our opinion. If the secretaries are randomly distributed, the second best secretary out of the N should also be a pretty good hire, and same for the third best. In fact, a strategy that picks the best secretary 10 percent of the time and the worst secretary every other time should really be deemed worse than one that picks the best secretary 9 percent of the time and the second best the other 89 percent.

Our previous musings lead to the idea of trying to maximize the best secretary ***in expectation***, where the best secretary of the group has the highest value of N, the second best $N-1$, and so on until the worst secretary with a score of 1. This may yield different results to the more famous setup where the probability of picking the best secretary is maximized.

2 Solution to the "Expected Secretary Problem"

2.1 Formal Setup/Ideas

Formally, we will consider a randomly distributed set of N secretaries, where each secretary has a score from 1 to N[2]. We will use the strategy of passing on some fixed number k secretaries then picking the next one

[1] I highly recommend reading that article if you are unfamiliar with the problem or it's original solution

[2] A similar problem with expectation was solved by Bearden, in which each secretary's score is independently uniformly distributed between 0 and 1. This is a similar, but not identical formulation since ours has exactly 1 person at each score level between 1 and N, which will yield some nice combinatorial results.

better than all those we passed on, with k_{opt} being the optimal value for highest expected return.

If we can write the expected value in terms of k, i.e.

$$\mathbb{E}[\text{Secretary chosen}] = f(k) \tag{1}$$

then we can simply differentiate with respect to k and find the max by setting the derivative equal to zero. Let's try that. From here on, k represents the number of secretaries we reject in part one of the algorithm

2.2 Expectation Calculation

To tackle this problem, we will use the law of total expectation, and partition our possibilities based on the score of $\text{init}_{\max}$, the best secretary from the initial rejected group. Notice that this value must be at least k, since the initial group has k secretaries:

$$\mathbb{E}[\text{Secretary chosen}] = \sum_{i=k}^{N} \mathbb{P}[\text{init}_{\max} = i]\mathbb{E}[\text{Secretary chosen}|\text{init}_{\max} = i] \tag{2}$$

From here, notice that calculating $\mathbb{E}[\text{Secretary chosen}|\text{init}_{\max} = i]$ is relatively easy.

- If $\text{init}_{\max} = N$, then the best secretary was in the rejecting group, and we are forced to choose the last secretary in the list of N. The last secretary is distributed randomly between the worst and second best in this case (since the best is in the initial k), so the expected value is $\frac{1+(N-1)}{2} = \frac{N}{2}$
- If $\text{init}_{\max} \neq N$, then the best secretary was not in the rejecting group, and any secretary better than $\text{init}_{\max}$ has the same probability of being the first one to be found after all rejections. As a result, the expected value is the average of all of their scores, or $\frac{(\text{init}_{\max}+1)+N}{2}$

These thoughts lead us to the new equation of:

$$\mathbb{E}[\text{Secretary chosen}] = \mathbb{P}[\text{init}_{\max} = N]\frac{N}{2} + \sum_{i=k}^{N-1} \mathbb{P}[\text{init}_{\max} = i]\frac{i+1+N}{2} \tag{3}$$

We can again simplify, since the probability that the best secretary of the first k is N (i.e. $\text{init}_{\max} = N$) is simply the probability that the best secretary (with score N) is in the first k. Since everyone is randomly distributed, this occurs with probability $\frac{k}{N}$:

$$\mathbb{P}[\text{init}_{\max} = N] = \mathbb{P}[\text{the top secretary is in the first } k] = \frac{k}{N} \tag{4}$$

We now have:

$$\mathbb{E}[\text{Secretary chosen}] = \frac{k}{N}\frac{N}{2} + \sum_{i=k}^{N-1} \mathbb{P}[\text{init}_{\max} = i]\frac{i+1+N}{2} = \frac{k}{2} + \sum_{i=k}^{N-1} \mathbb{P}[\text{init}_{\max} = i]\frac{i+1+N}{2} \tag{5}$$

What's left is to determine $\mathbb{P}[\text{init}_{\max} = i]$ in the other cases, and turn this whole thing into closed form. Here we will turn to combinatorics.

Note that the total number of ways to arrange N secretaries is $N!$. Let's now count the number of ways in which $\text{init}_{\max} = i$:

$$\underbrace{k}_{\text{put } i \text{ in the first } k} \quad * \quad \underbrace{(i-1) * \cdots * (i-(k-1))}_{\text{fill other } k-1 \text{ with secretaries worse than } i} \quad * \underbrace{(N-k) * \cdots * (1)}_{\text{fill rest}} \tag{6}$$

Since the secretaries are randomly distributed, each permutation has an equal probability, leading to a result of:

$$\mathbb{P}[\text{init}_{\max} = i] = \frac{k(i-1)\ldots(i-(k-1))((N-k)\ldots(1))}{N!} \tag{7}$$

From here we can finally write the expected value of the secretary chosen for an arbitrary stopping point k as

$$\frac{k}{2} + \sum_{i=k}^{N-1} \frac{k(i-1)\dots(i-(k-1))((N-k)\dots(1))}{N!} \frac{i+1+N}{2} \tag{8}$$

Luckily, the result simplifies nicely into

$$\frac{k}{2} + \frac{N-k}{2}\left(\frac{N+1}{N} + \frac{k}{k+1}\right) \tag{9}$$

The proof is a little long, but it can be found in Appendix A.1. Using the chain/quotient rules, the derivative with respect to k works out to be

$$\frac{\partial}{\partial k}\mathbb{E}[\text{Secretary chosen}] = \frac{1}{2} + \frac{-1}{2}\left(\frac{N+1}{N} + \frac{k}{k+1}\right) + \frac{N-k}{2}\frac{1}{(k+1)^2} \tag{10}$$

Finally, we can set this to zero to and solve for k to achieve a solution of

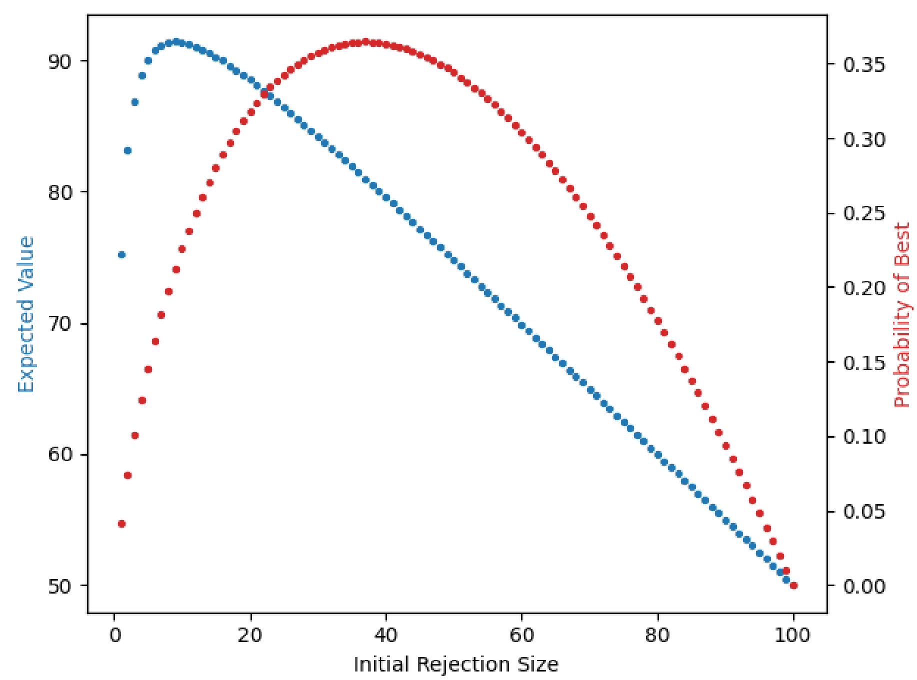
$$k_{\text{opt}} = \sqrt{N} - 1 \tag{11}$$

Details on this step are in Appendix A.5. Success! we now know that we should only reject the first $\sqrt{N}-1$ secretaries, then pick the next one that is greater than all those we rejected.

2.3 Visualizations/Intuition

In the original version of the problem, we had to reject the first $\frac{N}{e}$, or 34 percent of secretaries before beginning the next stage. This means that 34 percent of the time, the top secretary is in the rejection group and the algorithm fails! When optimizing for only the top secretary, we have to pick a random secretary with a relatively low expected score over 1/3 of the time.

This is more OK when the only goal is to try and maximize the probability of the highest secretary, but severely punishes our formulation where getting the second best secretary or third best is also a win.



Here we visualize the expected value for picking the rejection number at 1 through 100 for $N = 100$ as well as the probability of picking the best secretary. As predicted, the highest expected value is at $k = \sqrt{100} - 1 = 9$, with an expected secretary score of 91.405, which is pretty good!! At this value, the probability of picking the best secretary is only 21 percent, as opposed to the 36 percent which occurs if we wait 36 people instead of 9. Since we only wait for the first 9 secretaries, our algorithm "fails" and is forced to pick the last secretary only 9 percent of the time.

A nicer formulation of the expected value is as follows:

$$(n+1)\left(1 - \frac{k}{2n} - \frac{1}{2(k+1)}\right) \tag{12}$$

Here, notice that if k is too big, the first negative term will reduce the epxected value, while if k is too small, the second negative term will highly reduce the expected value.

A Appendix: Supplementary math

A.1 Simplification of Expectation formula

First, our ugly combinatorial term can actually be simplified as follows:

$$\frac{k(i-1)\dots(i-(k-1))((N-k)\dots(1))}{N!} = \frac{k}{N}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}} \tag{13}$$

with a proof Appendix A.2. Next, note that in every assignment, init_{max} must be a value between k and N. As a result, the sum of probabilities of $\text{init}_{\text{max}} = i$ for i between k and N must add to 1. We have that:

$$1 = \sum_{i=k}^{N} \mathbb{P}[\text{init}_{\text{max}} = i] = \sum_{i=k}^{N} \frac{k(i-1)\dots(i-(k-1))((N-k)\dots(1))}{N!} = \sum_{i=k}^{N} \frac{k}{N}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}} \tag{14}$$

moving around some terms, we have that:

$$\frac{N-k}{k} = \sum_{i=k}^{N-1} \frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}} \tag{15}$$

with the full derivation here being in Appendix A.3

Via a similar process where we reject the first $k+1$ secretaries, we arrive at

$$\frac{N(N-k)}{k+1} = \sum_{i=k}^{N-1} i\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}} \tag{16}$$

Full details are in Appendix A.4. Putting this all together, we get:

$$\begin{aligned}
&\mathbb{E}[\text{Secretary chosen}] \\
=&\frac{k}{2} + \sum_{i=k}^{N-1} \frac{k(i-1)\dots(i-(k-1))((N-k)\dots(1))}{N!}\frac{i+1+N}{2} \\
=&\frac{k}{2} + \sum_{i=k}^{N-1} \left(\frac{k}{N}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}}\right)\frac{i+1+N}{2} && \text{Equation 13} \\
=&\frac{k}{2} + \frac{k(N+1)}{2N}\left(\sum_{i=k}^{N-1}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}}\right) + \frac{k}{2N}\left(\sum_{i=k}^{N-1} i\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}}\right) \\
=&\frac{k}{2} + \frac{k(N+1)}{2N}\left(\frac{N-k}{k}\right) + \frac{k}{2N}\left(\frac{N(N-k)}{k+1}\right) && \text{Equations 15 and 16} \\
=&\frac{k}{2} + \frac{N-k}{2}\left(\frac{N+1}{N} + \frac{k}{k+1}\right)
\end{aligned}$$

A.2 Work for Equation 13

First, note that

$$\begin{aligned}
&\frac{k(i-1)\dots(i-(k-1))((N-k)\dots(1))}{N!}\\
=&\frac{k}{N}\frac{\frac{(i-1)!}{(i-k)!}(N-k)!}{(N-1)!}\\
=&\frac{k}{N}\frac{\frac{(i-1)!}{(i-k)!}}{\frac{(N-1)!}{(N-k)!}}\\
=&\frac{k}{N}\frac{\frac{(i-1)!}{(i-k)!(k-1)!}}{\frac{(N-1)!}{(N-k)!(k-1)!}}\\
=&\frac{k}{N}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}}
\end{aligned}$$

A.3 Work from Equation 14 to Equation 15

With this, let's continue:

$$\begin{aligned}
1 &= \sum_{i=k}^{N}\frac{k(i-1)\dots(i-(k-1))((N-k)\dots(1))}{N!}\\
&= \sum_{i=k}^{N}\frac{k}{N}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}} && \text{Appendix A.2}\\
\implies \frac{N}{k} &= \sum_{i=k}^{N}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}}\\
\implies \frac{N}{k} &= \sum_{i=k}^{N-1}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}}+1\\
\implies \frac{N-k}{k} &= \sum_{i=k}^{N-1}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}}
\end{aligned}$$

A.4 Proof of Equation 16

Let's consider rejecting the first $k+1$ secretaries, and finding the probability that secretary $i+1$ is the best secretary in this group. We can count the number of ways this happens:

$$\underbrace{k+1}_{\text{put } i+1 \text{ in the first } k+1} * \underbrace{(i)*\dots*(i-(k-1))}_{\text{fill other } k \text{ with secretaries worse than } i+1} * \underbrace{(N-(k+1))*\dots*(1)}_{\text{fill rest}} \tag{17}$$

Again we can denote the probability of the $i+1$th person being the best in the group as the quotient of this and $N!$, yielding:

$$\begin{aligned}
&\frac{(k+1)(i)\dots(i-(k-1))(N-(k+1)))\dots(1)}{N!}\\
=&\frac{(k+1)i}{k(n-k)}\frac{k(i-1)\dots(i-(k-1))((N-(k+1))\dots(1))}{N!}\\
=&\frac{(k+1)i}{k(n-k)}\frac{k}{N}\frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}} && \text{first part of A.2}
\end{aligned}$$

Again, since the sum of these probabilities from $i+1=k+1$ to $i+1=N$ covers all possible top rejecting people, we have that:

$$\sum_{i=k}^{N-1} \frac{(k+1)i}{k(n-k)} \frac{k}{N} \frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}} = 1 \tag{18}$$

This directly yields the result, or

$$\frac{N(N-k)}{k+1} = \sum_{i=k}^{N-1} \frac{\binom{i-1}{k-1}}{\binom{N-1}{k-1}} \tag{19}$$

A.5 Derivative full proof

$$\begin{aligned}
&\frac{1}{2} + \frac{-1}{2}\left(\frac{N+1}{N} + \frac{k}{k+1}\right) + \frac{N-k}{2}\frac{1}{(k+1)^2} = 0 \\
\Longrightarrow\ &\frac{N(k+1)^2 - (N+1)(k+1)^2 - Nk(k+1) + (N-k)N}{2N(k+1)^2} = 0 \\
\Longrightarrow\ &N(k+1)^2 - (N+1)(k+1)^2 - Nk(k+1) + (N-k)N = 0 \\
\Longrightarrow\ &-(N+1)k^2 - 2(N+1)k - (1-N^2) = 0 \\
\Longrightarrow\ &(N+1)k^2 + 2(N+1)k + (1-N^2) = 0 \\
\Longrightarrow\ &k = \frac{-2(N+1) \pm \sqrt{4(N+1)^2 - 4(N+1)(1-N^2)}}{2(N+1)} \\
\Longrightarrow\ &k = \frac{-2(N+1) \pm 2(N+1)\sqrt{1-(1-N)}}{2(N+1)} \\
\Longrightarrow\ &k = -1 \pm \sqrt{N}
\end{aligned}$$

Since $-1-\sqrt{N}$ is negative, the only positive solution is $-1+\sqrt{N}$