MAT 212

LINEAR ALGEBRA

The Hidden Connection: Exploring Dimensions of Vector Space through the Diagonal of Pascals Triangle

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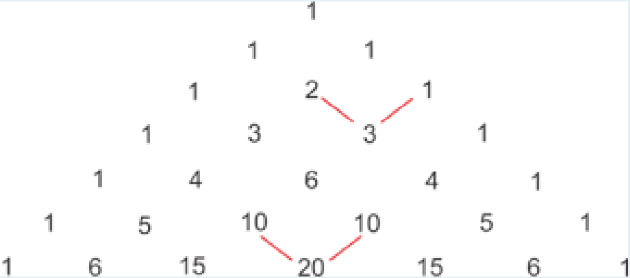
Table of Contents

1. Introduction
2. Space Of Multi-variable Polynomial of degree n or less
3. Space of Homogeneous polynomial of degree two in k variables.
4. Space of Cubic Matrices or Symmetric Tensor
5. Programming

Introduction to Pascals triangle

Pascal's triangle is a fascinating mathematical construct named after the French mathematician Blaise Pascal. It is a triangular array of numbers that holds immense significance in combinatorial mathematics.
Each number in the triangle is obtained by adding the two numbers directly above it, forming a symmetrical pattern.

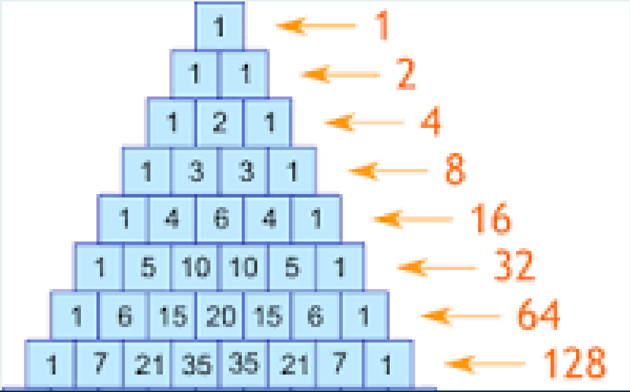


Introduction to Pascals triangle

Pascal's triangle reveals various intriguing properties, such as the binomial coefficients and the Fibonacci sequence. It finds applications in probability theory, algebra, and number theory. This triangular arrangement of numbers provides a visual representation of the mathematical relationships between them, offering a powerful tool for solving problems and exploring the intricacies of mathematical patterns.
Pascal's triangle holds some hidden patterns and relationships within its structure. Here are a few intriguing discoveries:

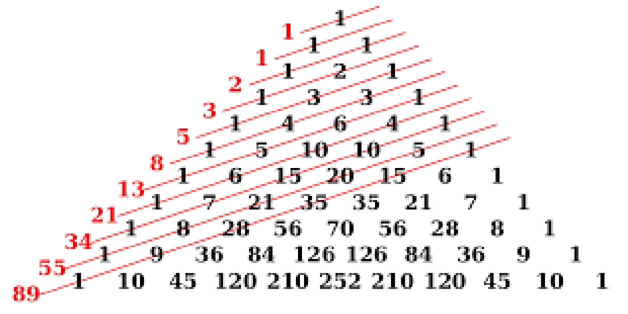
POWERS OF 2:

Powers of 2: The sum of the numbers in each row of Pascal's triangle is always a power of 2. For example, in the 5th row (1, 4, 6, 4, 1), the sum is 16, which is equal to 2^4.



Fibonacci

Pascal's triangle provides a visual representation of linear recurrence relations, which are equations that define a sequence based on previous terms. The triangle reveals pattern of Fibonacci sequence.



Space of Multi-variable Polynomials of degree n or less

Space of Multi-variable Polynomials of degree n or less in k-variables is denoted by $P_{\leq n}[x_k]$

Examples of those space vectors are :

- Space of Multi-variable Polynomials of degree n or less in two variables
- Space of Multi-variable Polynomials of degree n or less in three variables
- Space of Multi-variable Polynomials of degree n or less in four variables
- Space of Multi-variable Polynomials of degree n or less in five variables

POLYNOMIALS OF DEGREE N OR LESS IN TWO VARIABLES

Here we are going to consider polynomial in two variables while vary the degree. then we will study the Basis and dimension and see how it relates to pascal triangle.

- Polynomial of Degree one or less in two variables
  A Basis is $\{1,\ x_1,\ x_0\}$
  Dimension=3
- Polynomial of Degree two or less in two variables A Basis is $\{1,\ x_1,\ x_0,\ x_1^2,\ x_0^2,\ x_0 x_1\}$
  Dimension=6
- Polynomial of Degree three or less in two variables
  A Basis is $\{1,\ x_1,\ x_0,\ x_1^2,\ x_0^2,\ x_1^3,\ x_0^3,\ x_0 x_1,\ x_0 x_1^2,\ x_0^2 x_1\}$
  Dimension=10

Contd.

- Polynomial of Degree four or less in two variables

  $$A \ \ Basis \ is \left\{ \begin{array}{c} 1, \ x_1, \ x_0, \ x_1^2, \ x_0^2, \ x_1^3, \ x_0^3, \ x_1^4, \ x_0^4, \\ x_0x_1, \ x_0x_1^2, \ x_0^2x_1, \ x_0x_1^3, \ x_0^3x_1, \ x_0^2x_1^2 \end{array} \right\}$$

  Dimension=15

Discovery

If we study the sequence the dimension of the vector space above we will see that it follows the sequence of the third diagonal of a pascal triangle. So, we can say that the third diagonal of a Pascal triangle generate the dimension of Space of Multi-variable Polynomials of degree n or less in two variables.

Each dimension here can be generated using the formula $\binom{n+2}{2}$ where n is the degree

Polynomials of degree n or less in three variables

Just like the previous example we are going to consider polynomial in three variables while vary the degree. then we will study the Basis and dimension and see how it relates to pascal triangle.

- Polynomial of Degree one on three variables
  A Basis is $\{1,\ x_2,\ x_1,\ x_0\}$
  Dimension=4
- Polynomial of Degree two or less in three variables
  A Basis is $\{1,\ x_2,\ x_1,\ x_0,\ x_2^2,\ x_1^2,\ x_0^2,\ x_1x_2,\ x_0x_2,\ x_0x_1\}$
  Dimension=10
- Polynomial of Degree three or less in three variables
  $$\textit{A Basis is} \left\{\begin{array}{l} 1,\ x_2,\ x_1,\ x_0,\ x_2^2,\ x_1^2,\ x_0^2,\ x_2^3,\ x_1^3,\ x_0^3,\ x_1x_2,\ x_0x_2, \\ x_0x_1,\ x_1x_2^2,\ x_1^2x_2,\ x_0x_2^2,\ x_0x_1^2,\ x_0^2x_2,\ x_0^2x_1,\ x_0x_1x_2 \end{array}\right\}$$

Contd.

Dimension=20

- Polynomial of Degree four or less in three variables

$$A\ Basis\ is \left\{ \begin{array}{c} 1,\ x_2,\ x_1,\ x_0,\ x_2^2,\ x_1^2,\ x_0^2,\ x_2^3,\ x_1^3,\ x_0^3,\ x_2^4,\ x_1^4,\ x_0^4, \\ x_1x_2,\ x_0x_2,\ x_0x_1,\ x_1x_2^2\ x_1^2x_2,\ x_0x_2^2,\ x_0x_1^2,\ x_0^2x_2, \\ x_0^2x_1,\ x_1x_2^3,\ x_1^3x_2,\ x_0x_2^3\ x_0x_1^3,\ x_0^3x_2,\ x_0^3x_1,\ x_1^2x_2^2, \\ x_0^2x_2^2,\ x_0^2x_1^2,\ x_0x_1x_2,\ x_0x_1x_2^2,\ x_0x_1^2x_2,\ x_0^2x_1x_2 \end{array} \right\}$$

Dimension=35

Discovery

If we continue with this and Study the sequence the dimension of the vector space above,we will see that it can be generated using the formula $\binom{n+3}{3}$ where n is the degree this follows the pattern of fourth diagonal of a Pascal triangle.

Polynomials of degree n or less in Four variables

- Polynomial of Degree one or less in four variables
  A Basis is $\{1,\ x_3,\ x_2,\ x_1,\ x_0\}$
  Dimension=5
- Polynomial of Degree two or less in four variables
  A Basis is $\left\{\begin{array}{c} 1,\ x_3,\ x_2,\ x_1,\ x_0,\ x_3^2,\ x_2^2,\ x_1^2,\ x_0^2, \\ x_2x_3,\ x_1x_3,\ x_1x_2,\ x_0x_3,\ x_0x_2,\ x_0x_1 \end{array}\right\}$
  Dimension=15
- Polynomial of Degree three or less in four variables

$$A\ Basis\ is \left\{\begin{array}{c} 1,\ x_3,\ x_2,\ x_1,\ x_0,\ x_3^2,\ x_2^2,\ x_1^2,\ x_0^2,\ x_3^3,\ x_2^3,\ x_1^3,\ x_0^3, \\ x_2x_3,\ x_1x_3,\ x_1x_2,\ x_0x_3,\ x_0x_2,\ x_0x_1,\ x_2x_3^2,\ x_2^2x_3, \\ x_1x_3^2,\ x_1x_2^2,\ x_1^2x_3,\ x_1^2x_2,\ x_0x_3^2,\ x_0x_2^2,\ x_0x_1^2,\ x_0^2x_3, \\ x_0^2x_2,\ x_0^2x_1,\ x_1x_2x_3,\ x_0x_2x_3,\ x_0x_1x_3,\ x_0x_1x_2 \end{array}\right\}$$

  Dimension=35

CONTD.

- Polynomial of Degree four or less in four variables

$$
\textit{A Basis is} \left\{ \begin{array}{c}
1,\ x_3,\ x_2,\ x_1,\ x_0,\ x_3^2,\ x_2^2,\ x_1^2,\ x_0^2,\ x_3^3,\ x_2^3,\ x_1^3, \\
x_0^3,\ x_3^4,\ x_2^4,\ x_1^4,\ x_0^4,\ x_2x_3,\ x_1x_3,\ x_1x_2,\ x_0x_3, \\
x_0x_2,\ x_0x_1,\ x_2x_3^2,\ x_2^2x_3,\ x_1x_3^2,\ x_1x_2^2,\ x_1^2x_3,\ x_1^2x_2, \\
x_0x_3^2,\ x_0x_2^2,\ x_0x_1^2,\ x_0^2x_3,\ x_0^2x_2,\ x_0^2x_1,\ x_2x_3^3, \\
x_2^3x_3,\ x_1x_3^3,\ x_1x_2^3,\ x_1^3x_3,\ x_1^3x_2,\ x_0x_3^3,\ x_0x_2^3,\ x_0x_1^3, \\
x_0^3x_3,\ x_0^3x_2,\ x_0^3x_1,\ x_2^2x_3^2,\ x_1^2x_3^2,\ x_1^2x_2^2,\ x_0^2x_3^2, \\
x_0^2x_2^2,\ x_0^2x_1^2,\ x_1x_2x_3,\ x_0x_2x_3,\ x_0x_1x_3,\ x_0x_1x_2, \\
x_1x_2x_3^2,\ x_1x_2^2x_3,\ x_1^2x_2x_3,\ x_0x_2x_3^2, x_0x_2^2x_3,\ x_0x_1x_3^2, \\
x_0x_1x_2^2,\ x_0x_1^2x_3, x_0x_1^2x_2,\ x_0^2x_2x_3,\ x_0^2x_1x_3,\ x_0^2x_1x_2, \\
x_0x_1x_2x_3
\end{array} \right\}
$$

Dimension=70

Discovery

If we study the sequence the dimension of the vector space above we will see that it follows the sequence of the third diagonal of a pascal triangle. So, we can say that the third diagonal of a Pascal triangle generate the dimension of Space of Multi-variable Polynomials of degree n or less in two variables.
Each dimension here can be generated using the formula $\binom{n+4}{4}$ where n is the degree

Space of Homogeneous polynomial of degree n in k variables

A homogeneous polynomial is a polynomial whose nonzero terms all have the same degree. Space of Homogeneous polynomial of degree n in k variables is denoted by $S^n[x_k]$

Examples of space of Homogeneous polynomial are:

- Space of Homogeneous polynomial of degree two in k variable
- Space of Homogeneous polynomial of degree three in k variable
- Space of Homogeneous polynomial of degree Four in k variable
- Space of Homogeneous polynomial of degree Five in k variable

HOMOGENEOUS POLYNOMIAL OF DEGREE TWO IN K VARIABLES

- Space of Homogeneous polynomial of Degree two in 2 variables
  $V = \left\{ F(x) : F(x) = a_0x_1^2 + a_1x_0^2 + a_2x_0x_1 \;\; a_i \in \mathbb{R} \;\; \forall_{i\geq 0} \right\}$
  A Basis is $\left\{ x_1^2 \quad x_0^2 \quad x_0x_1 \right\}$
  Dimension$=3$
- Space of Homogeneous polynomial of Degree two in 3 variables
  $V = \left\{ \begin{matrix} F(x) : F(x) = a_0x_2^2 + a_1x_1^2 + a_2x_0^2 + a_3x_1x_2 + a_4x_0x_2 + a_5x_0x_1 \\ a_i \in \mathbb{R} \;\; \forall_{i\geq 0} \end{matrix} \right\}$
  A Basis is $\left\{ x_2^2 \quad x_1^2 \quad x_0^2 \quad x_1x_2 \quad x_0x_2 \quad x_0x_1 \right\}$
  Dimension$=6$
- Space of Homogeneous polynomial of Degree two in 4 variables
  $V = \left\{ \begin{matrix} g(x) : g(x) = a_0x_3^2 + a_1x_2^2 + a_2x_1^2 + a_3x_0^2 + a_4x_2x_3 + \\ a_5x_1x_3 + a_6x_1x_2 + a_7x_0x_3 + a_8x_0x_2 + a_9x_0x_1 \end{matrix} \right\}$

CONTD.

A Basis is $\left\{x_3^2 \quad x_2^2 \quad x_1^2 \quad x_0^2 \quad x_2x_3 \quad x_1x_3 \quad x_1x_2 \quad x_0x_3 \quad x_0x_2 \quad x_0x_1\right\}$

Dimension= 10

- Space of Homogeneous polynomial of Degree two in 5 variables

$V = \{g(x) : g(x) = a_0x_4^2 + a_1x_3^2 + a_2x_2^2 + a_3x_1^2 + a_4x_0^2 + a_5x_3x_4 + a_6x_2x_4 \ldots + a_{14}x_0x_1\}$

$$A \text{ Basis is} \left\{\begin{matrix} x_4^2 & x_3^2 & x_2^2 & x_1^2 & x_0^2 & x_3x_4 & x_2x_4 & x_2x_3 & x_1x_4 & x_1x_3 \\ x_1x_2 & x_0x_4 & x_0x_3 & x_0x_2 & x_0x_1 \end{matrix}\right\}$$

Dimension= 15

CONTD.

Discovery

If we study the sequence the dimension of the vector space above we will see that it follows the sequence of the third diagonal of a pascal triangle. So, we can say that the third diagonal of a Pascal triangle generate the dimension of Space of Homogeneous polynomial of Degree two in k variables.

Each dimension here can be generated using the formula $\binom{k+1}{2}$ where k is the number of variables

SPACE OF HOMOGENEOUS POLYNOMIAL OF DEGREE THREE IN K VARIABLES

- Homogeneous Polynomial of Degree three in 2 variables
  $a_0x_1^3 + a_1x_0^3 + a_2x_0x_1^2 + a_3x_0^2x_1$
  A Basis is
  $\left\{x_1^3 \quad x_0^3 \quad x_0x_1^2 \quad x_0^2x_1\right\}$
  Dimension$= 4$
- Homogeneous Polynomial of Degree three in 3 variables
  $V = \left\{g(x) : g(x) = a_0x_2^3 + a_1x_1^3 + a_2x_0^3 + \ldots + a_8x_0^2x_1 + a_9x_0x_1x_2\right\}$
  $$A \text{ Basis is } \left\{\begin{matrix} x_2^3 & x_1^3 & x_0^3 & x_1x_2^2 & x_1^2x_2 & x_0x_2^2 \\ x_0x_1^2 & x_0^2x_2 & x_0^2x_1 & x_0x_1x_2 & & \end{matrix}\right\}$$
  Dimension$= 10$

Contd.

- Homogeneous Polynomial of Degree three in 4 variables
  $V = \{g(x) : g(x) = a_0x_3^3 + a_1x_2^3 + a_2x_1^3 + a_3x_0^3 + \ldots + a_{19}x_0x_1x_2\}$
  $$A\ Basis\ is \left\{\begin{array}{cccccc} x_3^3 & x_2^3 & x_1^3 & x_0^3 & x_2x_3^2 & x_2^2x_3 \\ x_1x_3^2 & x_1x_2^2x_1^2x_3 & x_1^2x_2 & x_0x_3^2x_0x_2^2 & x_0x_1^2 & x_0^2x_3 \\ x_0^2x_2 & x_0^2x_1 & x_1x_2x_3 & x_0x_2x_3 & x_0x_1x_3 & x_0x_1x_2 \end{array}\right\}$$
  Dimension$=$ 20
- Homogeneous Polynomial of Degree three in 5 variables

$$A\ Basis\ is \left\{\begin{array}{ccccccc} x_4^3 & x_3^3 & x_2^3 & x_1^3 & x_0^3 & x_3x_4^2 & x_3^2x_4 \\ x_2x_4^2 & x_2x_3^2 & x_2^2x_4 & x_2^2x_3 & x_1x_4^2 & x_1x_3^2 & x_1x_2^2 \\ x_1^2x_4 & x_1^2x_3 & x_1^2x_2 & x_0x_4^2 & x_0x_3^2 & x_0x_2^2 & x_0x_1^2 \\ x_0^2x_4 & x_0^2x_3 & x_0^2x_2 & x_0^2x_1 & x_2x_3x_4 & x_1x_3x_4 & x_1x_2x_4 \\ x_1x_2x_3 & x_0x_3x_4 & x_0x_2x_4 & x_0x_2x_3 & x_0x_1x_4 & x_0x_1x_3 & x_0x_1x_2 \end{array}\right\}$$

Dimension$=$ 35

Homogeneous Polynomial of Degree four in k variables

- Homogeneous Polynomial of Degree four in 2 variables
  $$V = \{g(x) : g(x) = a_0x_1^4 + a_1x_0^4 + a_2x_0x_1^3 + a_3x_0^3x_1 + a_4x_0^2x_1^2\}$$
  A Basis is $\{x_1^4 \quad x_0^4 \quad x_0x_1^3 \quad x_0^3x_1 \quad x_0^2x_1^2\}$
  Dimension$=5$
- Homogeneous Polynomial of Degree four in 3 variables
  $$V = \{g(x) : g(x) = a_0x_2^4 + a_1x_1^4 + a_2x_0^4 + a_3x_1x_2^3 + \ldots + a_{14}x_0^2x_1x_2\}$$
  $$A\ Basis\ is \begin{Bmatrix} x_2^4 & x_1^4 & x_0^4 & x_1x_2^3 & x_1^3x_2 & x_0x_2^3 & x_0x_1^3 & x_0^3x_2 \\ x_0^3x_1 & x_1^2x_2^2 & x_0^2x_2^2 & x_0^2x_1^2 & x_0x_1x_2^2 & x_0x_1^2x_2 & x_0^2x_1x_2 & \end{Bmatrix}$$
  Dimension$=15$

CONTD.

- Homogeneous Polynomial of Degree four in 4 variables
  $V = \{g(x) : g(x) = a_0x_3^4 + a_1x_2^4 + a_2x_1^4 + a_3x_0^4 + ... + a_{34}x_0x_1x_2x_3\}$

$$A\ Basis\ is \left\{ \begin{array}{ccccccc} x_3^4 & x_2^4 & x_1^4 & x_0^4 & x_2x_3^3 & x_2^3x_3 & x_1x_3^3 \\ x_1x_2^3 & x_1^3x_3 & x_1^3x_2 & x_0x_3^3 & x_0x_2^3 & x_0x_1^3 & x_0^3x_3 \\ x_0^3x_2 & x_0^3x_1 & x_2^2x_3^2 & x_1^2x_3^2 & x_1^2x_2^2 & x_0^2x_3^2 & x_0^2x_2^2 \\ x_0^2x_1^2 & x_1x_2x_3^2 & x_1x_2^2x_3 & x_1^2x_2x_3 & x_0x_2x_3^2 & x_0x_2^2x_3 & x_0x_1x_3^2 \\ x_0x_1x_2^2 & x_0x_1^2x_3 & x_0x_1^2x_2 & x_0^2x_2x_3 & x_0^2x_1x_3 & x_0^2x_1x_2 & x_0x_1x_2x_3 \end{array} \right\}$$

Dimension= 35

SYMMETRIC TENSOR OF ORDER 3 DEFINED ON $\mathbb{R}^n$ $[S^3(\mathbb{R}^n)]$

- Symmetric Tensor of order 3 defined on $\mathbb{R}^2$ written as $S^3(\mathbb{R}^2)$

$$V = \left\{ A : A = \left[\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \right] a_i \in \mathbb{R} \right\}$$

$$e.g, \left[\begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 7 \end{pmatrix} \right] \in V$$

A Basis is below

$$\left\{ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \right\}$$

Dimension= 4

CONTD.

- Symmetric Tensor of order 3 defined on $\mathbb{R}^3$ written as $S^3(\mathbb{R}^3)$
  V=$\left\{ A : A = \left[\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \end{pmatrix}, \begin{pmatrix} a_1 & a_3 & a_4 \\ a_3 & a_6 & a_7 \\ a_4 & a_7 & a_8 \end{pmatrix}, \begin{pmatrix} a_2 & a_4 & a_5 \\ a_4 & a_7 & a_8 \\ a_5 & a_8 & a_9 \end{pmatrix} \right] a_i \in \mathbb{R} \right\}$
  e.g $\left[\begin{pmatrix} 0 & 2 & 41 \\ 2 & 3 & 17 \\ 41 & 17 & 33 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 17 \\ 3 & 7 & 11 \\ 17 & 11 & 12 \end{pmatrix}, \begin{pmatrix} 41 & 17 & 33 \\ 17 & 11 & 12 \\ 33 & 12 & 28 \end{pmatrix} \right] \in V$
  Dimension is 10

CONTD.

- Symmetric Tensor of order 3 defined on $\mathbb{R}^4$ written as $S^3(\mathbb{R}^4)$

$$V = \left\{ A : A = \left[\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_4 & a_5 & a_6 \\ a_2 & a_5 & a_7 & a_8 \\ a_3 & a_6 & a_8 & a_9 \end{pmatrix}, \begin{pmatrix} a_1 & a_4 & a_5 & a_6 \\ a_4 & a_{10} & a_1 & a_{12} \\ a_5 & a_1 & a_{13} & a_{14} \\ a_6 & a_{12} & a_{14} & a_{15} \end{pmatrix}, \begin{pmatrix} a_2 & a_5 & a_7 & a_8 \\ a_5 & a_1 & a_{13} & a_{14} \\ a_7 & a_{13} & a_{16} & a_{17} \\ a_8 & a_{14} & a_{17} & a_{18} \end{pmatrix}, \begin{pmatrix} a_3 & a_6 & a_8 & a_9 \\ a_6 & a_{12} & a_{14} & a_{15} \\ a_8 & a_{14} & a_{17} & a_{18} \\ a_9 & a_{15} & a_{18} & a_{19} \end{pmatrix} \right] \quad \textit{where} \quad a_i \in \mathbb{R} \right\}$$

Dimension is 20

Programming

Implementing Python codes to deal with the vector space listed above as many as possible. As we know computer can go far than we can in a short time. So,You can see the raw codes here.
Copy the codes once the link is open and make use of it in Python or Sage.

Usage of Homogeneous Polynomial

create a homogeneous polynomial using Homogeneous function with two parameters.e.g Homogeneous(2,4) where the first number is the degree and the second is the variables. After creating a vector space then you can check for the following properties.

1. **Function:** this generate the general function of the polynomial.
2. **Basis:** this get you the basis of the vector space defined
3. **Dimension:** generate the dimension of the vector space.

Programming

Usage of Polynomial

create a polynomial of degree n or less in k variable using Polynomial function with two parameters.e.g Polynomials(2,4) where the first number is the degree and the second is the variables. After creating a vector space then you can check for the following properties.

1. **Function:** this generate the general function of the polynomial.
2. **Basis:** this get you the basis of the vector space defined
3. **Dimension:** generate the dimension of the vector space.

you can also use my calculator it is going to generate everything you need without any codes.

Programming

Usage of Symmetric tensor or cubic Matrices

create the space the same way as previous and make use of the following attributes

1. **Tensor:** this generate the matrices
2. **Dimension:** generate the dimension of the vector space.

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Questions And Answers

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