Linear Algebra

Vector Space

Group 1

A project providing solution to different problems on vector Space.

Mathematics (BSc. and B.Ed)
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Project 1

Questions

1. Prove that

   $$Y=\left\{\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}\in\mathbb{R}^4:\ a-b+3d=0\right\}$$

   is a subspace of $\mathbb{R}^4$

2. Prove that $X=\left\{A\in M_n(\mathbb{R}):\ A=A^T\right\}$ is a subspace of $M_n(\mathbb{R})$
3. Prove that $\mathscr{C}\subset M_n(\mathbb{R})\ of\ skewed\ symetric$
   $\mathscr{C}=\left\{B\in M_n(\mathbb{R}):\ B=-B^T\right\}$ is a subspace of $M_n(\mathbb{R})$
4. Show that the set $\mathscr{B}\subset M_n(\mathbb{F})$ of upper triangular matrices is a subspace of $M_n(\mathbb{F})$ i.e $\mathscr{B}=\{D\in M_n(\mathbb{R}): a_{ij}=0 \text{ if } i>j\}$

Solutions

1. Since 0-0+3*0=0, then Y has zero element i.e

   $$there\ exist\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\in Y$$

   Hence Y is not empty.

   choose $x,y\in Y$ then

   $$x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\ and\ y=\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

   $where\ x_1-x_2+3x_4=0\ and\ y_1-y_2+3y_4=0$

choose $\alpha \in \mathbb{F}$ where $\mathbb{F}$ is a Field then,

$$\text{x+}\alpha y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \alpha \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} \alpha y_1 \\ \alpha y_2 \\ \alpha y_3 \\ \alpha y_4 \end{pmatrix}$$

$$x + \alpha y = \begin{pmatrix} x_1 + \alpha y_1 \\ x_2 + \alpha y_2 \\ x_3 + \alpha y_3 \\ x_4 + \alpha y_4 \end{pmatrix}$$

Now we need to check whether $(x + \alpha y)$ is an element in Y or not. To do that we will check for the condition necessary for element in Y.

$(x_1 + \alpha y_1) - (x_2 + \alpha y_2) + 3(x_4 + \alpha y_4)$

$x_1 + \alpha y_1 - x_2 - \alpha y_2 + 3x_4 + 3\alpha y_4$

Rearrange

$x_1 - x_2 + 3x_4 + \alpha y_1 - \alpha y_2 + 3\alpha y_4$
$x_1 - x_2 + 3x_4 + \alpha(y_1 - y_2 + 3y_4) = 0 + 3(0) = 0$

This satisfied the condition necessary for element in Y.
So, $(x + \alpha y) \in Y$ Therefore, Y is a subspace of $\mathbb{R}^4$

2. Since all zero matrix of $M_n(\mathbb{R})$ is symmetric, then Y has zero element i.e

$$there\ exist \begin{pmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \end{pmatrix} \in X$$

Hence X is not empty.

choose $A, B \in X$, since A and B are symmetric matrix ‘then

$a_{i,j} = a_{j,i}$ *and* $b_{i,j} = b_{j,i}$ $(where\ 1 \le i \le n\ and\ 1 \le j \le n)$
a_{ij} *and* $b_{i,j}$ denote entry in row-i column-j in A and B respectively let say,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & . & . & . & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & . & . & . & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & . & . & . & a_{3,n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n} \end{pmatrix}, B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & . & . & . & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & . & . & . & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & . & . & . & b_{3,n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ b_{n,1} & b_{n,2} & b_{n,3} & . & . & . & b_{n,n} \end{pmatrix}$$

Now, let us simplify $A + \alpha B$ (where $\alpha \in \mathbb{F}$ and $\mathbb{F}$ is a Field of Real number)

$$A + \alpha B = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & . & . & . & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & . & . & . & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & . & . & . & a_{3,n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n} \end{pmatrix} + \alpha \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & . & . & . & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & . & . & . & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & . & . & . & b_{3,n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ b_{n,1} & b_{n,2} & b_{n,3} & . & . & . & b_{n,n} \end{pmatrix}$$

$$A+\alpha B=\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & . & . & . & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & . & . & . & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & . & . & . & a_{3,n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n} \end{pmatrix} + \begin{pmatrix} \alpha b_{1,1} & \alpha b_{1,2} & \alpha b_{1,3} & . & . & . & \alpha b_{1,n} \\ \alpha b_{2,1} & \alpha b_{2,2} & \alpha b_{2,3} & . & . & . & \alpha b_{2,n} \\ \alpha b_{3,1} & \alpha b_{3,2} & \alpha b_{3,3} & . & . & . & \alpha b_{3,n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \alpha b_{n,1} & \alpha b_{n,2} & \alpha b_{n,3} & . & . & . & \alpha b_{n,n} \end{pmatrix}$$

$$A+\alpha B=\begin{pmatrix} a_{1,1}\alpha+b_{1,1} & a_{1,2}+\alpha b_{1,2} & a_{1,3}+\alpha b_{1,3} & . & . & . & a_{1,n}+\alpha b_{1,n} \\ a_{2,1}\alpha+b_{2,1} & a_{2,2}+\alpha b_{2,2} & a_{2,3}+\alpha b_{2,3} & . & . & . & a_{2,n}+\alpha b_{2,n} \\ a_{3,1}\alpha+b_{3,1} & a_{3,2}+\alpha b_{3,2} & a_{3,3}+\alpha b_{3,3} & . & . & . & a_{3,n}+\alpha b_{3,n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n,1}+\alpha b_{n,1} & a_{n,2}+\alpha b_{n,2} & a_{n,3}+\alpha b_{n,3} & . & . & . & a_{n,n}+\alpha b_{n,n} \end{pmatrix}$$

Let C be a matrix such such that $C = A+\alpha B$
then we see that $c_{i,j} = a_{i,j}+\alpha b_{i,j}$
($c_{i,j}, a_{i,j}$ *and* $b_{i,j}$ is an entry in row i column j of matrix C, A and B respectively)

Now let's check if C is symmetric or not
$c_{i,j} = a_{i,j}+\alpha b_{i,j}$
By symmetric property of A and B
$b_{i,j} = b_{j,i}$ and $a_{i,j} = a_{j,i}$
So, $c_{i,j} = a_{i,j}+\alpha b_{i,j} = a_{j,i}+\alpha b_{j,i}$
But $c_{j,i} = a_{j,i}+\alpha b_{j,i}$
This implies that $c_{i,j} = c_{j,i}$
showing that $C = C^T$ *i.e* $A+\alpha B$ is a symmetric matrix
Therefore $A+\alpha B \in X$
Hence X is a subspace of M_n

3. Since all zero matrix of $M_n(\mathbb{R})$ are symmetrically skewed, then $\mathscr{C}$ has zero element
Hence $\mathscr{C}$ is not empty.

choose $A, B \in \mathscr{C}$, since A and B are symmetrically skewed matrix then $a_{i,j} = -a_{j,i}$ *and* $b_{i,j} = -b_{j,i}$ (*where* $1 \le i \le n$ *and* $1 \le j \le n$)

a_{ij} and $b_{i,j}$ denote entry in row-i column-j in A and B respectively

Just as you have seen the addition of matrix in Question 2
Let C be a matrix such such that $C = A + \alpha B$ then we see that
$c_{i,j} = a_{i,j} + \alpha b_{i,j}$ ($c_{i,j}, a_{i,j}$ *and* $b_{i,j}$,is an entry in row i column j of matrix C, A and B respectively)

Now let's check if C is skew-symmetric or not
$c_{i,j} = a_{i,j} + \alpha b_{i,j}$
from the property of A and B, we know that
$b_{i,j} = -b_{j,i}$ and $a_{i,j} = -a_{j,i}$
So, $c_{i,j} = a_{i,j} + \alpha b_{i,j} = -a_{j,i} + \alpha * -b_{j,i}$
$c_{i,j} = -a_{j,i} - \alpha * b_{j,i} = -(a_{j,i} + \alpha b_{j,i})$
But $c_{j,i} = a_{j,i} + \alpha b_{j,i}$
So, $c_{i,j} = -c_{j,i}$
showing that $C = -C^T$ $i.e$ $(A + \alpha B)$ is a skew-symmetric matrix
Therefore $(A + \alpha B) \in \mathscr{C}$
Hence $\mathscr{C}$ is a subspace of $M_n(\mathbb{R})$

4. There exist Zero matrix of $M_n(\mathbb{F})$ that is upper triangular matrix therefore $\mathscr{B}$ is not empty

   choose $A, B \in \mathscr{B}$, since A and B are upper triangular matrix then
   $a_{i,j} = 0$ *and* $b_{i,j} = 0\ \forall_{i>j}$ (*where* $1 \leq i \leq n$ *and* $1 \leq j \leq n$)
   a_{ij} *and* $b_{i,j}$ denote entry in row-i column-j in A and B respectively
   Let C be a matrix such such that
   $C = A + \alpha B$ then we see that
   $c_{i,j} = a_{i,j} + \alpha b_{i,j}$
   ($c_{i,j}, a_{i,j}$ *and* $b_{i,j}$ is an entry in row-i column-j of matrix C, A and B respectively)

   Now let's check if C is a upper triangular matrix or not
   $c_{i,j} = a_{i,j} + \alpha b_{i,j}$
   from the property of A and B,
   we know that $b_{i,j} = 0$ and $a_{i,j} = 0$ (*when ever* $i > j$)
   So, $c_{i,j} = 0 + \alpha * 0$ (*whenever* $i > j$)
   $c_{i,j} = 0 + 0$ (*whenever* $i > j$)
   $c_{i,j} = 0$ (*whenever* $i > j$)

This shows that matrix C is an upper triangular matrix
Therefore, $C \in \mathscr{C}$ i.e $(A+\alpha B) \in \mathscr{C}$

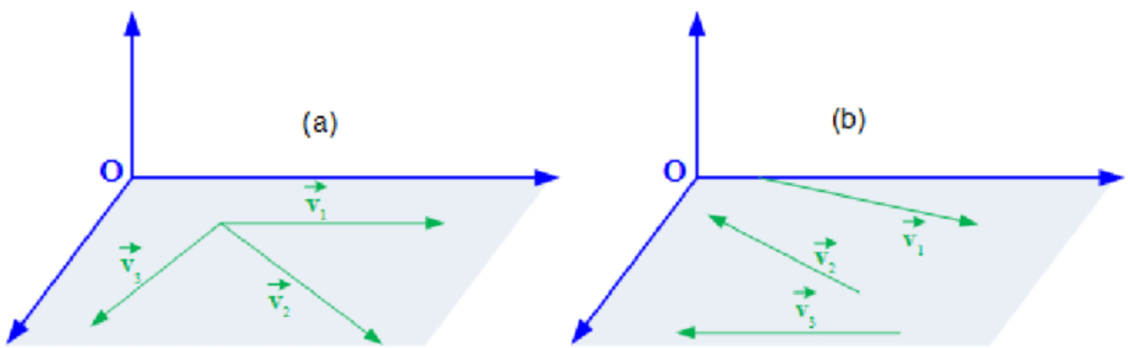
Hence $\mathscr{C}$ is a subspace of $M_n(\mathbb{F})$

Project 2

Questions
Give the geometric interpretation of three linearly dependent vectors.

Solution

vectors v_1, v_2 and v_3 are linearly dependent means that one of them is in a plane generated by the other two vectors.



Project 3

Question
Explore Pascal triangle, Give the meaning/interpretation of the number in the 4th ,5th and 6th diagonal in terms of dimension of vector space.

Solution
Firstly, Note that the space of all symmetric tensors of order k defined on V which often denoted by $S^k(V)$ or $Sym^k(V)$. It is itself a vector space, and if V has dimension N then the dimension of $Sym^k(V)$ is the binomial coefficient.
$dim\ Sym^k(V) = \binom{N+k-1}{k}$.

Now studying the 4th diagonal of a pascal triangle w get tetrahedral numbers. 1, 4, 10, 20 3v ,vvi ...
the sequence follows the formula
$\frac{(n+2)(n+1)n}{3!} = \binom{n+2}{3} = \binom{n+3-1}{3}$
The formula above look similar to the general formula for dimension of the space of all symmetric tensors of order k defined on N-dimensional vector V. so, comparing the formula. n is the dimension of V and k=3.
Hence, the 4th diagonal of a pascal triangle generate the dimension of the space of all symmetric tensors of order 3 defined on n-dimensional vector V

Studying the 5th diagonal of a pascal triangle we get pentatopes numbers. 1, , 10, 20 35 ,5vi ...
the sequence follows the formula
$\frac{(n+3)(n+2)(n+1)}{4!} = \binom{n+3}{4} = \binom{n+4-1}{4}$
comparing the formula also. n is the dimension of v and k=4.
Hence, the 5th diagonal of a pascal triangle generate the dimension of the space of symmetric tensors of order 4 defined on n-dimensional vector V

Project 4

Question
Let $T = O \in M_n(\mathbb{R}) : Tr(O) = 0$ be a vector space of Trace-less nxn matrices. compute a basis and dimension of T.

Solutions
A Basis of Traceless matrices is below
let $A_{i,n} = \{e_{i,1},\ e_{i,2},\ e_{i,3}...\ e_{i,n}\}$

$$then\ the\ Basis\ of\ traceess\ matrix\ is\ (\bigcup_{i=1}^{n} A_i) \setminus \{e_{n,n}\}$$

$$e_{i,j} = \begin{bmatrix} 0 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & . & . & 0 & k \end{bmatrix}$$

where the **1** is at i^{th} row, j^{th} column
k=-1 whenever $j = i$ and k=0 whenever $i \neq j$
Since $\bigcup_{i=1}^{n} A_i$ has n*n element then the Dimension of the space of n by n traceless matrix is $n^2 - 1$

Project 5

Questions

1. Prove that the set
$$\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\-1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\}$$
is a linear independent set of vectors in $\mathbb{R}^4$

2. Prove that $\begin{pmatrix}1\\2\\1\\1\end{pmatrix}$ can be expressed as linear combinations

3. Let $\{V_1,V_2,V_3,V_4\}$ be a linear independent family of vectors in $\mathbb{R}^4$ prove that $\{V_1+V_2,\ V_2+V_3,\ V_3+V_4,\ V_4+V_1\}$ is not linearly independent but $\{V_1+V_2,V_2,V_3,V_4\}$ is linearly independent.

4. Let f,g,h belong to the space of infinitely continuously differentiable real value function $C^\infty(\mathbb{R})$ such that $f(x)=e^x,\ g(x)=e^{2x}\ and\ h(x)=e^{3x}$ show that f,g,h are linearly independent.

Solutions

1. To prove that $\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\-1\\1\end{pmatrix},\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\}$ is a set of linearly dependent vectors we need to show that there exist real numbers $\alpha_1,\ \alpha_2,\ \alpha_3\ and\ \alpha_4$, not all of them are zero, such that
$$\alpha_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+\alpha_2\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}+\alpha_3\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}+\alpha_4\begin{pmatrix}1\\1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Proof:

$$\alpha_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+\alpha_2\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}+\alpha_3\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}+\alpha_4\begin{pmatrix}1\\1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1\\\alpha_1\\\alpha_1\\\alpha_1\end{pmatrix}+\begin{pmatrix}\alpha_2\\\alpha_2\\-\alpha_2\\\alpha_2\end{pmatrix}+\begin{pmatrix}\alpha_3\\-\alpha_3\\\alpha_3\\-\alpha_3\end{pmatrix}+\begin{pmatrix}\alpha_4\\\alpha_4\\0\\\alpha_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1+\alpha_2+\alpha_3+\alpha_4\\\alpha_1+\alpha_2-\alpha_3+\alpha_4\\\alpha_1-\alpha_2+\alpha_3\\\alpha_1+\alpha_2-\alpha_3+\alpha_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$\alpha_1+\alpha_2+\alpha_3+\alpha_4=0$
$\alpha_1+\alpha_2-\alpha_3+\alpha_4=0$
$\alpha_1-\alpha_2+\alpha_3=0$
$\alpha_1+\alpha_2-\alpha_3+\alpha_4=0$

Let us solve the equation.

$\alpha_1+\alpha_2+\alpha_3+\alpha_4=0...eq(1)$
$\alpha_1+\alpha_2-\alpha_3+\alpha_4=0...eq(2)$
$\alpha_1-\alpha_2+\alpha_3=0...eq(3)$
$\alpha_1+\alpha_2-\alpha_3+\alpha_4=0...eq(4)$

From eq(iii), $\alpha_2=\alpha_1+\alpha_3$...eq(5)

substitute $\alpha_2=\alpha_1+\alpha_3$ in to equation (1),(2) and (4).

$\alpha_1+\alpha_1+\alpha_3+\alpha_3+\alpha_4=0$
$\alpha_1+\alpha_1+\alpha_3-\alpha_3+\alpha_4=0$
$\alpha_1+\alpha_1+\alpha_3-\alpha_3+\alpha_4=0$

$2\alpha_1+2\alpha_3+\alpha_4=0...eq(6)$
$2\alpha_1+\alpha_4=0...eq(7)$
$2\alpha_1+\alpha_4=0...eq(8)$

subtract eq(7) from eq(6)

$2\alpha_3 = 0 \implies \alpha_3 = 0$

Note: eq(7) and eq(8) are the same this shows that there will be infinitely many solutions.

So, choose $\alpha_1 = 1$, then

$2(1) + \alpha_4 = 0 \implies \alpha_4 = -2$

But Recall that $\alpha_2 = \alpha_1 + \alpha 3$

so, $\alpha_2 = 1 + 0 = 1$,

Therefore, $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0 \; and \; \alpha_4 = -2$

Since We find some real numbers ($\alpha_1, \alpha_2, \alpha_3 \; and \; \alpha_4$) that not all of them equal to Zero therefore the given set of vectors are **linearly dependent**.

2. 1 If set of vector V_1, V_2, V_3, V_4 are linearly independent then

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ only if } \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

Now let us consider set of vector $V_1 + V_2, V_2, V_3, V_4$ we need to show that,

$$\beta_1(V_1 + V_2) + \beta_2 V_2 + \beta_3 V_3 + \beta_4 V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ only if } \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$$

Proof:

$$\beta_1(V_1 + V_2) + \beta_2 V_2 + \beta_3 V_3 + \beta_4 V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\beta_1 V_1 + \beta_1 V_2 + \beta_2 V_2 + \beta_3 V_3 + \beta_4 V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\beta_1 V_1 + (\beta_1+\beta_2)V_2 + \beta_3 V_3 + \beta_4 V_4 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

let $\beta_v = \beta_1 + \beta_2$ then,

$$\beta_1 V_1 + \beta_v V_2 + \beta_3 V_3 + \beta_4 V_4 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Since we know that V_1, V_2, V_3, V_4 are linearly independent then
$\beta_1 = \beta_v = \beta_3 = \beta_4 = 0$
Recall that $\beta_3 = \beta_1 + \beta_2$
so, $\beta_1 + \beta_2 = 0$
since $\beta_1 = 0$ then, $0 + \beta_2$=0
therefore, $\beta_2 = 0$
Hence, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$
Since , $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$
then the set of vector $V_1 + V_2, V_2, V_3, V_4$ are linearly independent then

2 we need to show that there exist some real numbers such that not all are zero that satisfied below equation

$$\beta_1(V_1+V_2) + \beta_2(V_2+V_3) + \beta_3(V_3+V_4) + \beta_4(V_4+V_1) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Proof:

$$\beta_1(V_1+V_2) + \beta_2(V_2+V_3) + \beta_3(V_3+V_4) + \beta_4(V_4+V_1) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\beta_1 V_1 + \beta_1 V_2 + \beta_2 V_2 + \beta_2 V_3 + \beta_3 V_3 + \beta_3 V_4 + \beta_4 V_4 + \beta_4 V_1 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\beta_1 V_1 + \beta_4 V_1 + \beta_2 V_2 + \beta_1 V_2 + \beta_3 V_3 + \beta_2 V_3 + \beta_4 V_4 + \beta_3 V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(\beta_1 + \beta_4)V_1 + (\beta_1 + \beta_2)V_2 + (\beta_2 + \beta_3)V_3 + (\beta_3 + \beta_4)V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

let $\beta_5 = \beta_1 + \beta_4,\ \beta_6 = \beta_1 + \beta_2$
$\beta_7 = \beta_2 + \beta_3$ *and* $\beta_8 = \beta_3 + \beta_4$, then,

$$\beta_5 V_1 + \beta_6 V_2 + \beta_7 V_3 + \beta_8 V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since we know that V_1, V_2, V_3, V_4 are linearly independent then
$\beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$
Recall that, $\beta_v = \beta_1 + \beta_2,\ \beta_5 = \beta_1 + \beta_4,\ \beta_{vi} = \beta_1 + \beta_2$
$\beta_7 = \beta_2 + \beta_3$ *and* $\beta_8 = \beta_3 + \beta_4$

so, $\beta_1 + \beta_4 = 0 \quad eq(i)$
$\beta_1 + \beta_2 = 0 \quad eq(ii)$
$\beta_2 + \beta_3 = 0 \quad eq(iii)$
$\beta_3 + \beta_4 = 0 \quad eq(iv)$
Subtract eq(ii) from eq(i) and eq(iii) from eq(iv)
then the set of vector $V_1 + V_2, V_2, V_3, V_4$ are linearly independent then
$\beta_4 - \beta_2 = 0$
$\beta_4 - \beta_2 = 0$
Both equation gotten is the same. So, choose $\beta_2 = 1$
then, $\beta_4 - 1 = 0 \implies \beta_4 = 1$
From eq(iii)$\beta_2 + \beta_3 = 0 \implies \beta_1 = -1$ Therefore, $\beta_1 = -1, \beta_2 = 1, \beta_3 = -1$ *and* $beta_4 = 1$
Hence, the vector $(V_1 + V_2), (V_2 + V_3), (V_3 + V_4)$ and $(V_4 + V_1)$ are **linearly dependent**.