

Sampling Rate Conversions: Upsampling (Interpolation)

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Motivation for Upsampling

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Motivation and Definitions

Examples

Summary

Why increase the sampling rate?

- Interface between systems running at different sample rates (audio, telecom, sensors).
- Enable finer time resolution for subsequent processing (filtering, D/A conversion).

Goal: Given samples $x[n] = x_c(nT)$, produce samples

$$x_i[n] = x_c(nT_i), \quad T_i = \frac{T}{L}$$

so that the new sampling rate is L times larger.

Upsampling: Definition and Approach

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$$x_i[n] = x\left[\frac{n}{L}\right] = x_c\left(\frac{nT}{L}\right), \quad n = 0, \pm L, \pm 2L, \dots$$

At integer multiples of L , we have the original samples. Elsewhere, we need intermediate values.

Problem: Need the “missing” samples while preserving bandlimited structure.

Solution Outline:

- 1 **Expand:** Insert $L - 1$ ‘null’ samples between each original sample (creates $x_e[n]$).
- 2 **Interpolate:** Lowpass filter with gain L and cutoff π/L to reconstruct intermediate samples.

This process preserves the bandlimited nature of the signal.

Expander and Interpolator Structure

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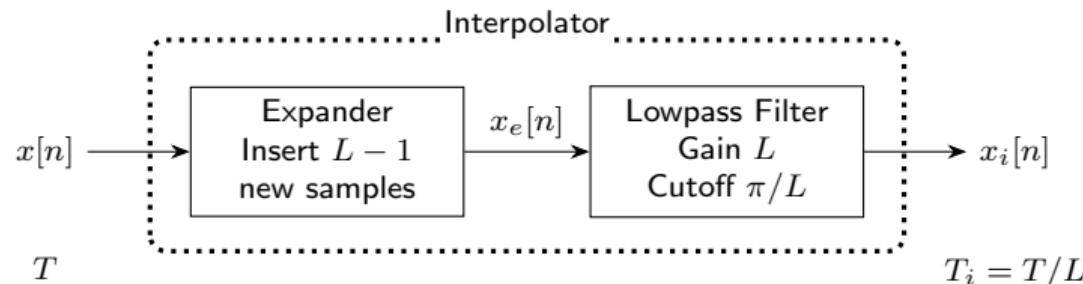
Expander (Sampling Rate Expander):

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases} \iff x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL]$$

Interpolator: Lowpass filter $h_i[n]$ (gain L , cutoff π/L) applied to $x_e[n]$:

$$x_i[n] = (h_i * x_e)[n]$$

System Diagram:



Terminology: Expander + Antialias (Interpolation) Filter = **Interpolator**.

Frequency-Domain Effect of Expansion: DTFT of $x_e[n]$

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Expander definition:

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases} \iff x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL]$$

Compute the DTFT of $x_e[n]$:

$$\begin{aligned} X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_e[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k] \delta[n - kL] \right) e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega(kL)} \\ &= \sum_{k=-\infty}^{\infty} x[k] e^{-j(\omega L)k} \\ &= X(e^{j\omega L}) \end{aligned}$$

Frequency-Domain Effect of Expansion: Interpretation and Filter

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Result:

$$X_e(e^{j\omega}) = X(e^{j\omega L})$$

Interpretation:

- Baseband spectrum is **compressed** by factor L .
- Multiple frequency-scaled images appear within $|\omega| \leq \pi$ due to DTFT periodicity.
- An interpolation lowpass (cutoff π/L) removes images and rescales amplitude.

Interpolator output:

$$X_i(e^{j\omega}) = H_i(e^{j\omega}) X_e(e^{j\omega}) \approx X(e^{j\omega}) \quad (\text{ideal})$$

Ideal interpolation filter:

$$H_i(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \pi/L \\ 0, & \pi/L < |\omega| \leq \pi \end{cases}$$

Gain justification: Scale by L so spectral amplitude matches new sampling rate:

$$L \cdot \frac{1}{T} = \frac{1}{T/L} = \frac{1}{T_i}.$$

Interpolation - Frequency Domain

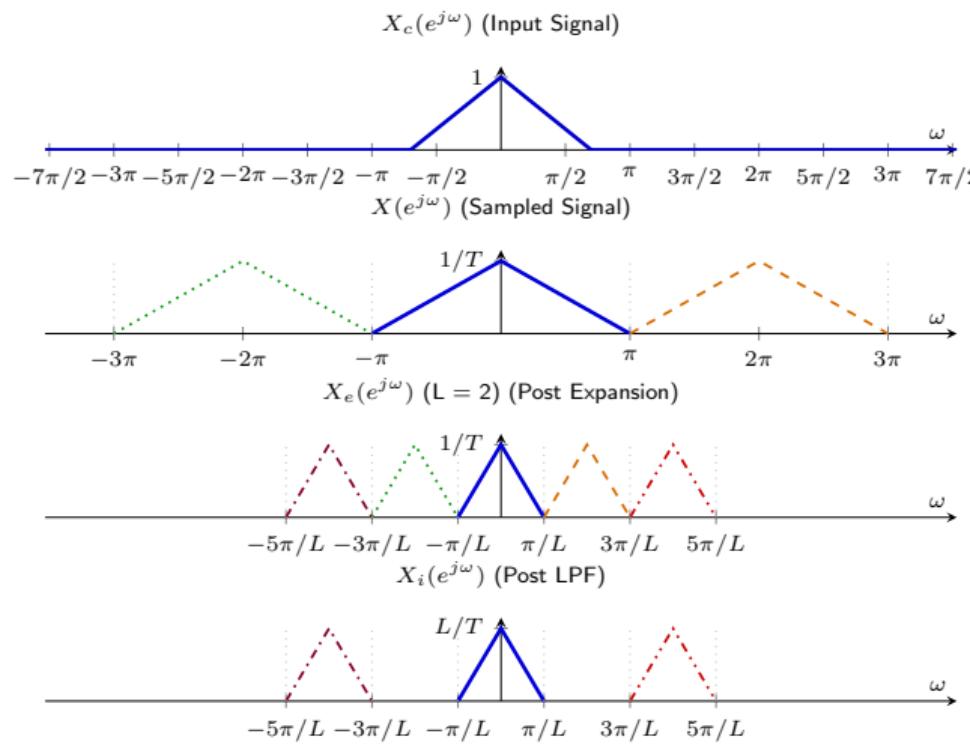
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Example: Upsampling with $L = 3$

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Assume original $X(e^{j\omega})$ occupies $|\omega| \leq \pi/2$.

After expansion by $L = 3$:

$$X_e(e^{j\omega}) = X(e^{j3\omega})$$

Nonzero only for $|\omega| \leq \pi/6$ in principal lobe; other compressed images appear.

Interpolator H_i :

$$H_i(e^{j\omega}) = \begin{cases} 3, & |\omega| \leq \pi/3 \\ 0, & \pi/3 < |\omega| \leq \pi \end{cases}$$

Result:

$$X_i(e^{j\omega}) = H_i(e^{j\omega})X_e(e^{j\omega}) \approx X(e^{j\omega})$$

Bandwidth Relation: Needed cutoff $\pi/3$ ensures only baseband compressed copy passes. Requires gain correction (Gain = 3).

Example: $L = 3$ (Frequency Domain Visualization)

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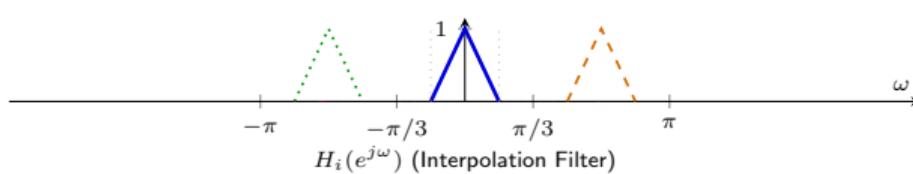
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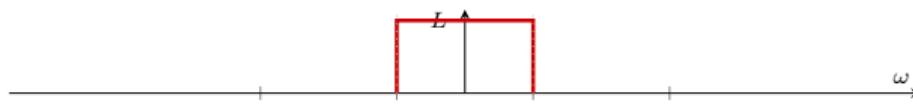
$X(e^{j\omega})$ (Original, $|\omega| \leq \pi/2$)



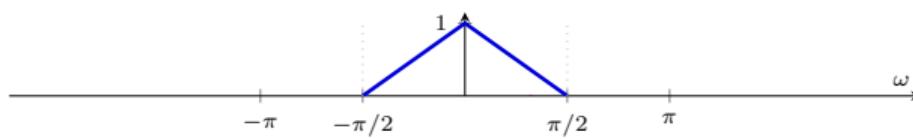
$X_e(e^{j\omega}) = X(e^{j3\omega})$ (Expanded, compressed bandwidth)



$H_i(e^{j\omega})$ (Interpolation Filter)



$X_i(e^{j\omega}) = H_i(e^{j\omega})X_e(e^{j\omega})$ (Interpolated Output)



Summary of Upsampling Concepts

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Core Steps:

$$x[n] \xrightarrow{\text{Insert } L-1 \text{ 'null' samples}} x_e[n] \xrightarrow{h_i[n]} x_i[n]$$

Key Formulas:

$$X_e(e^{j\omega}) = X(e^{j\omega L})$$

$$H_i(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \pi/L \\ 0, & \text{else} \end{cases}$$

Conditions:

- Original sequence must be bandlimited (no alias in initial sampling).
- Interpolation filter approximates ideal lowpass.