

# EE 264: Digital Signal Processing

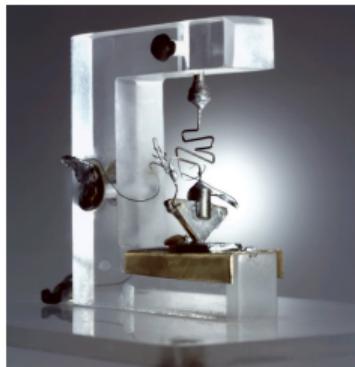
July 31, 2025

# Why Digital Signal Processing?

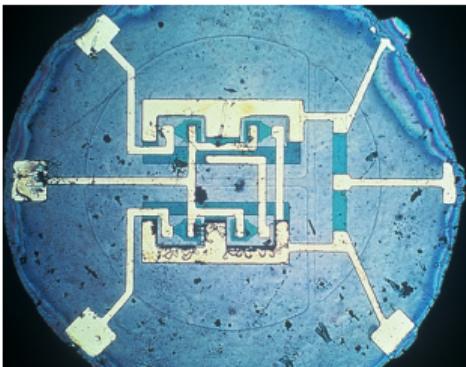
- ▶ Flexibility
- ▶ Accuracy
- ▶ Multi-purpose hardware
- ▶ Easy to implement sophisticated operations
- ▶ Today we have tremendous computer power

# Why Digital Signal Processing?

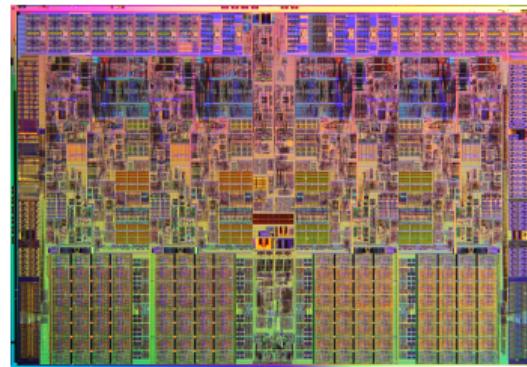
The development of low-cost and high-speed digital electronics paved the way for digital signal processing



First point contact  
transistor (1947)



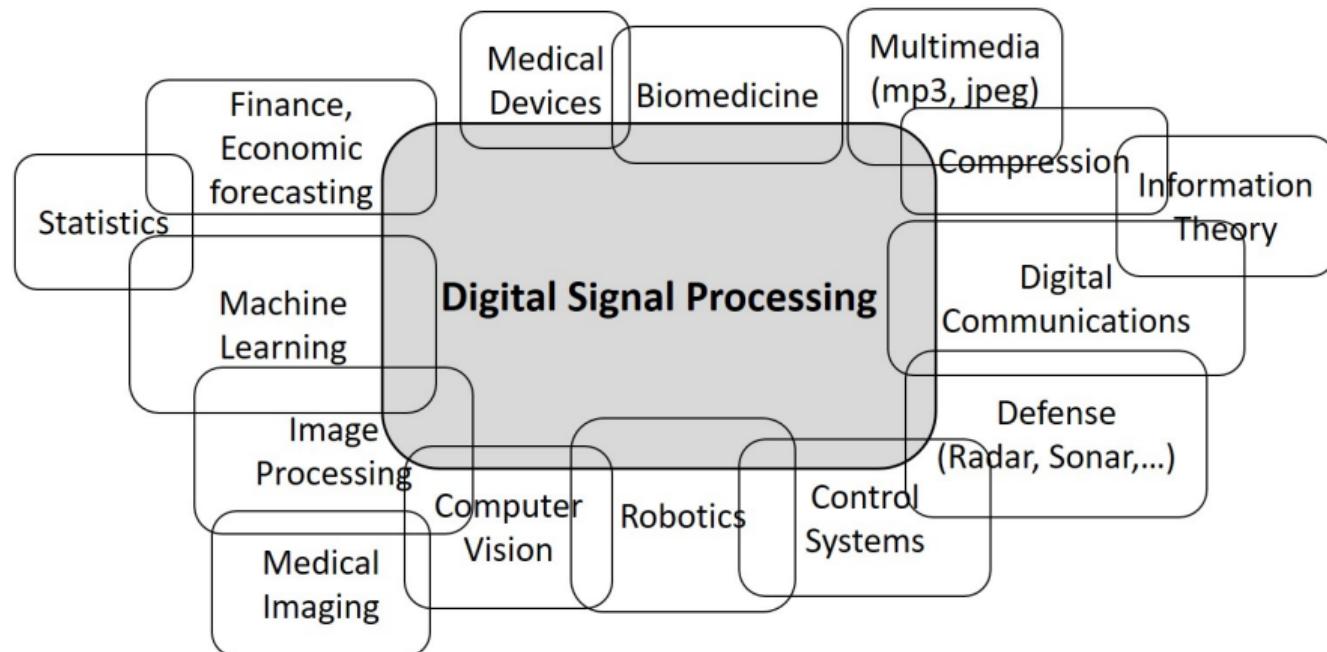
First integrated  
circuit (1961)



Modern processor (200X)

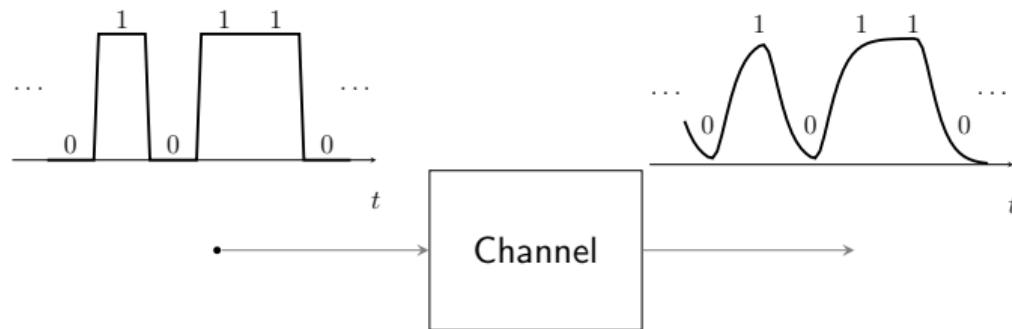
# Why learn digital signal processing?

- ▶ Present in essentially all fields of modern EE
- ▶ Countless applications



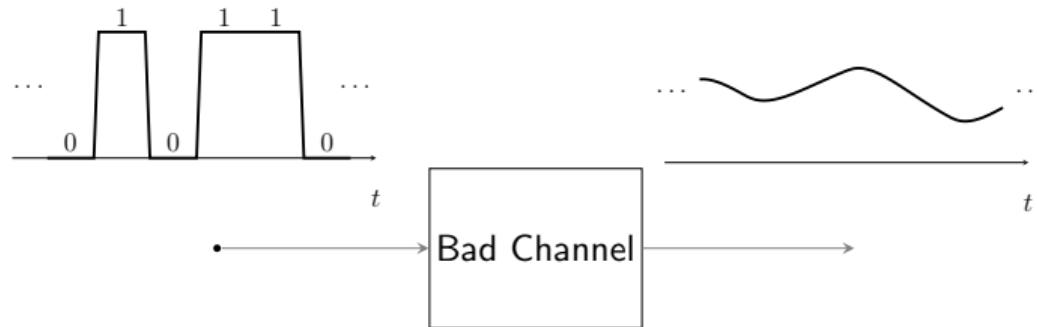
## Example: digital communication

Problem:

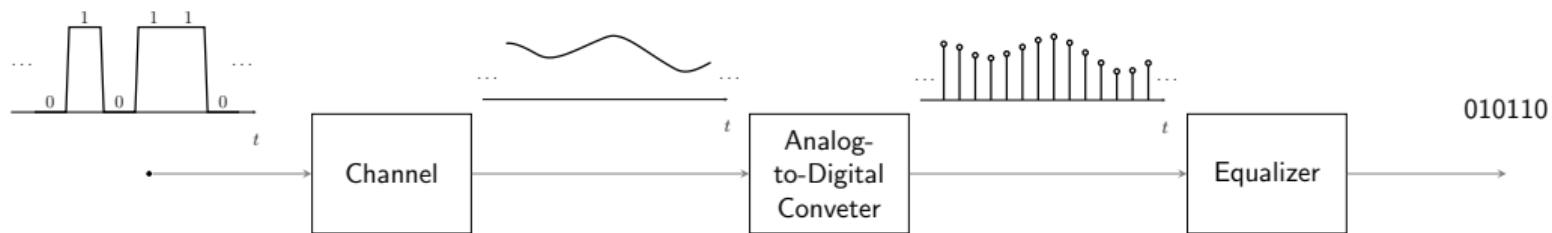


# Example: digital communication

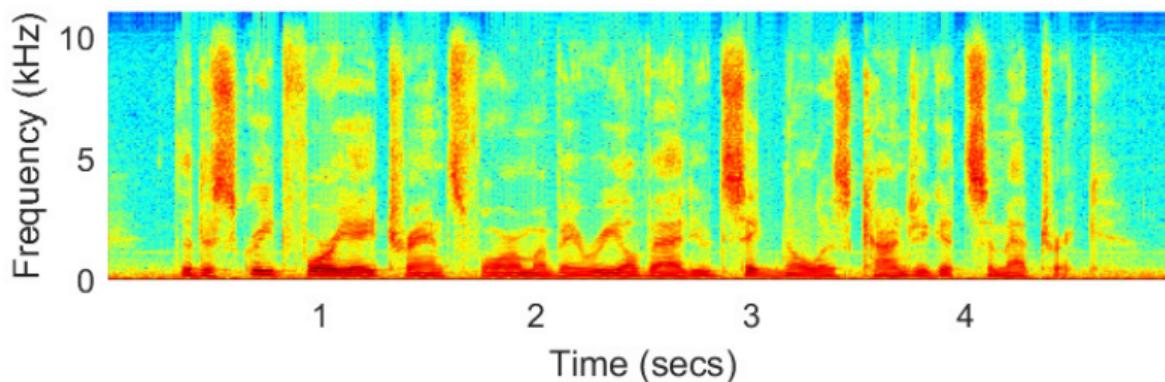
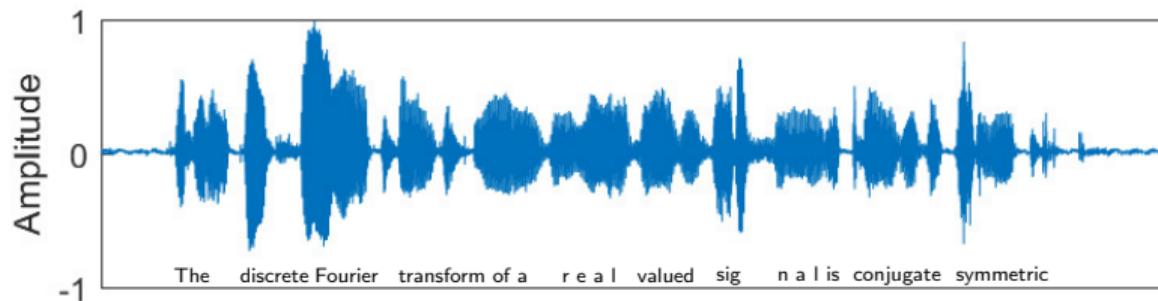
Problem:



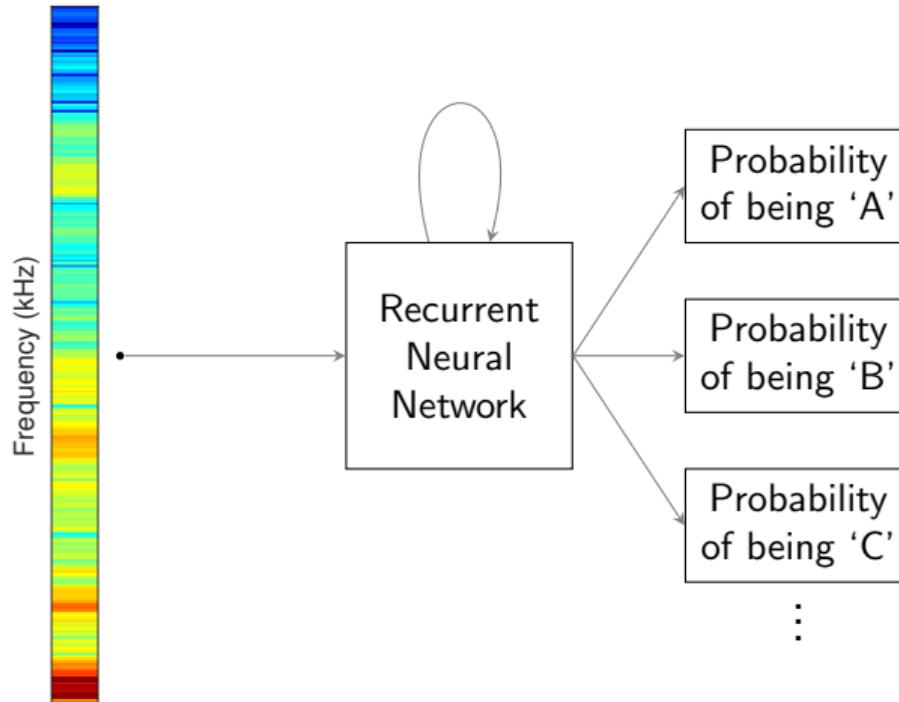
Solution:



## Example: speech recognition



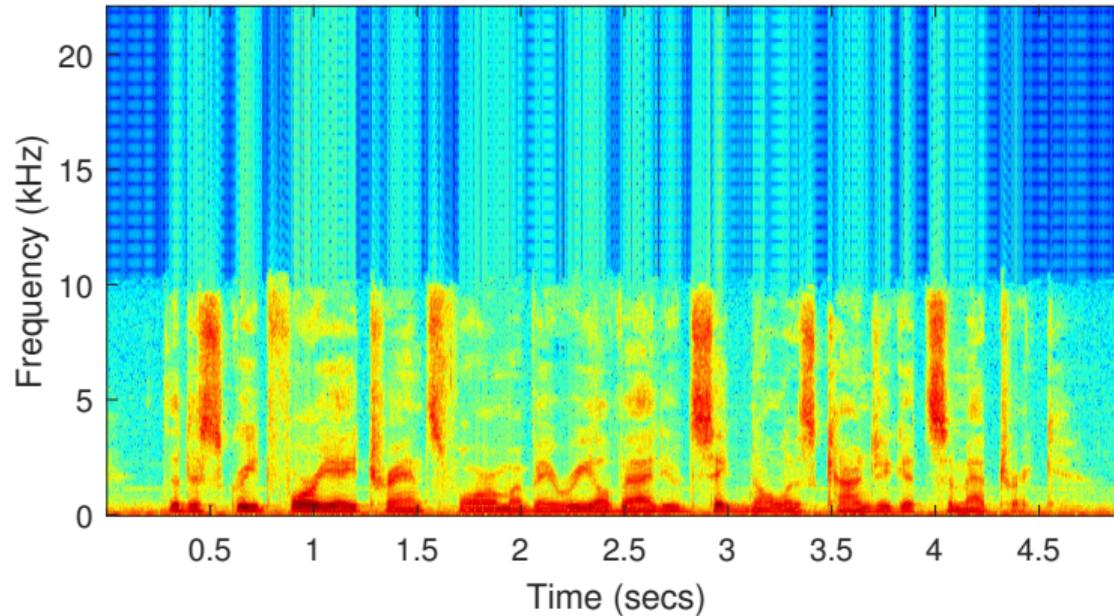
## Example: speech recognition



Slice of the  
spectrogram

## Example: speech recognition

Spectrogram of the same speech signal, now recorded with sampling rate of 44.1 kHz



More on spectrograms and short-time Fourier transform on lecture 11.

# Digital processing of analog signals



## Analog-to-digital converter (ADC)

- ▶ Performs filtering, sampling, and quantization
- ▶ Sampling rate may be of tens of kHz (audio processing), or it may be of tens of GHz (optical communications)

## Digital signal processor

- ▶ Performs some operation e.g., filtering, FFT, etc
- ▶ May be implemented on PCs with 64-bit floating-point precision, or on ASICs with limited arithmetic precision (e.g., 6 bits).

## Digital-to-analog converter (DAC)

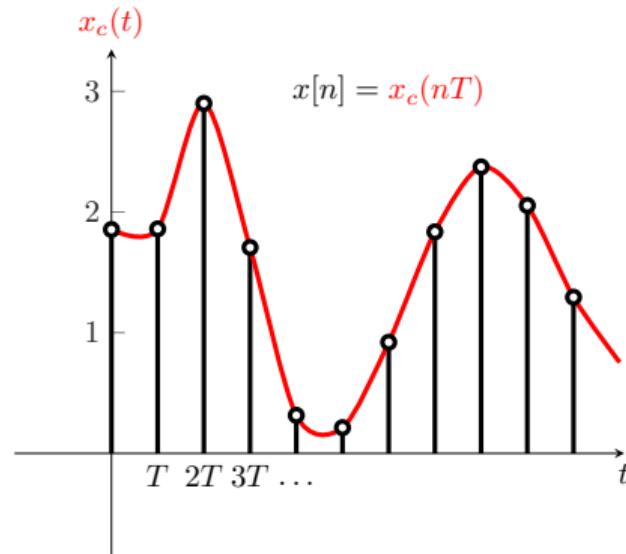
- ▶ Performs quantization and reconstruction (filtering)
- ▶ Sampling rate could be similar to ADC

# Review

- ▶ Discrete-time signals and systems
- ▶ Discrete-time Fourier transform (DTFT)
- ▶ The  $z$ -transform
- ▶ Difference equations

# Discrete-time signals

Discrete-time signals (or simply *sequences*) may be inherently discrete or they may be obtained by sampling a continuous-time signal. More about sampling on lectures 4 and 5.

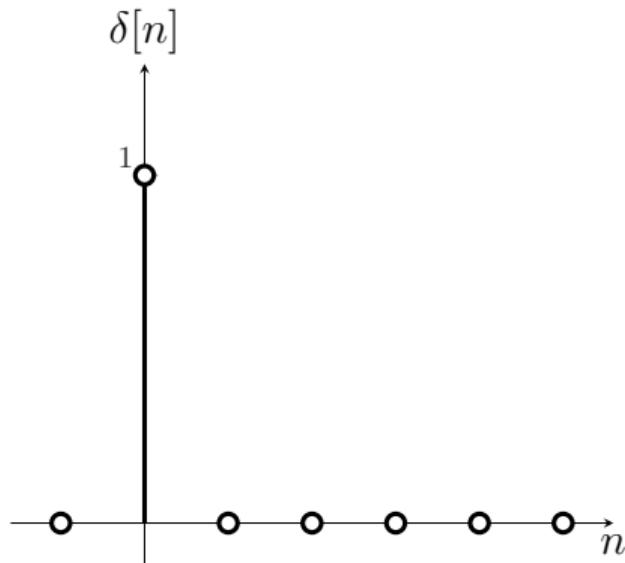


$x[n]$  is only defined for  $n \in \mathbb{Z}$

# Basic sequences

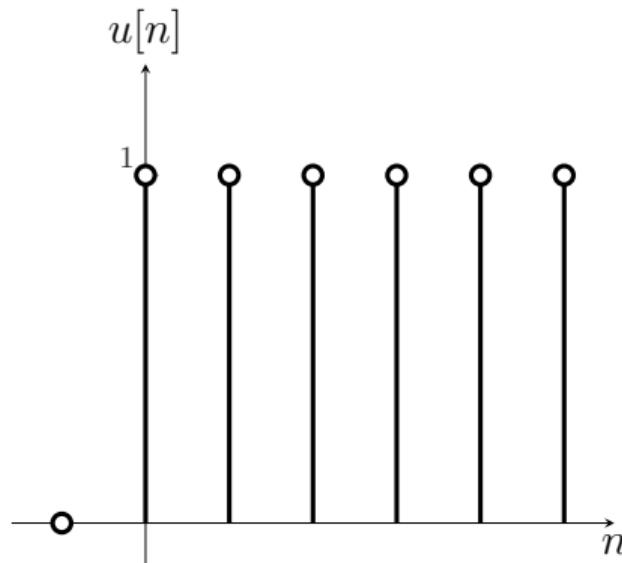
**Unit impulse:**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$



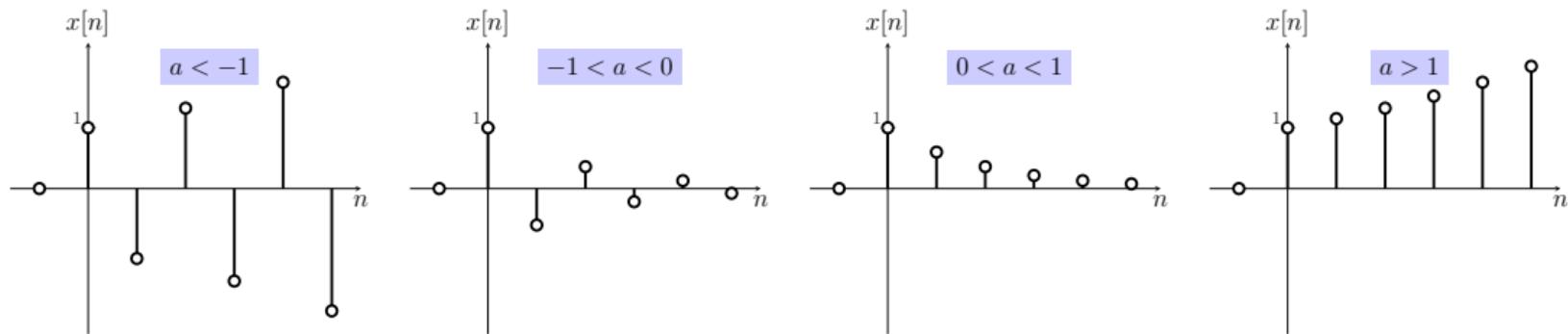
**Unit step:**

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



# Basic sequences

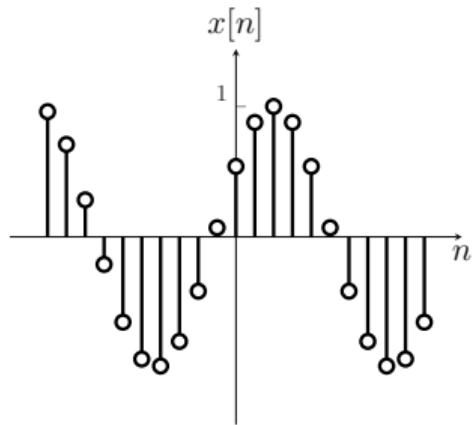
**Right-sided exponential:**  $x[n] = a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$



- ▶ Exponential exhibits oscillatory behavior for  $a < 0$ .
- ▶ Exponential is **unbounded** if  $|a| > 1$ .

## Basic sequences

**Sinusoids:**  $x[n] = \cos(\omega_0 n + \phi)$



Differently from continuous-time sinusoids, discrete-time sinusoids...

- ▶ are periodic only if  $\omega_0/\pi$  is rational.
- ▶ have maximum frequency  $\omega = \pi$ .

**Complex exponentials:**

$$e^{j(\omega_0 n + \phi)} = \cos(\omega_0 n + \phi) + j \sin(\omega_0 n + \phi)$$

(from Euler's equation)

# Classification of discrete-time signals

Energy and power signals

**Energy signals** have finite energy

$$E \equiv \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

**Power signals** have infinite energy (e.g., periodic signals), but they have finite average power

$$P \equiv \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2 < \infty$$

# Classification of discrete-time signals

## Symmetry

$$x[n] = x[-n] \quad (\text{even symmetry})$$

$$x[n] = -x[-n] \quad (\text{odd symmetry})$$

Any signal  $x[n]$  can be decomposed as a sum of an even ( $x_e[n]$ ) and an odd ( $x_o[n]$ ) component:

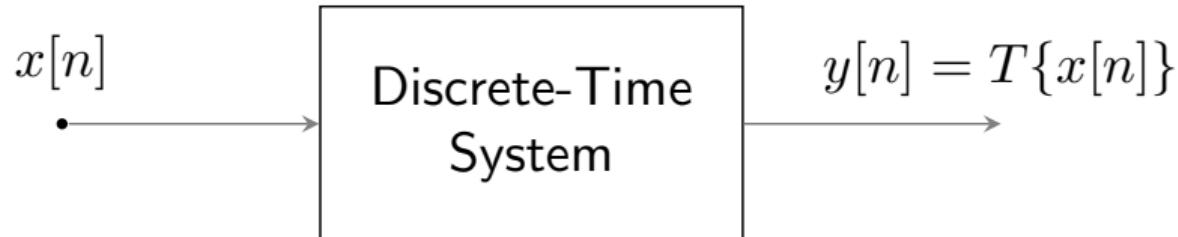
$$x[n] = x_e[n] + x_o[n]$$

where

$$x_e[n] = \frac{1}{2}(x[n] + x[-n]) \quad (\text{even component})$$

$$x_o[n] = \frac{1}{2}(x[n] - x[-n]) \quad (\text{odd component})$$

# Discrete-time systems



Some important properties

**Linearity or superposition**

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

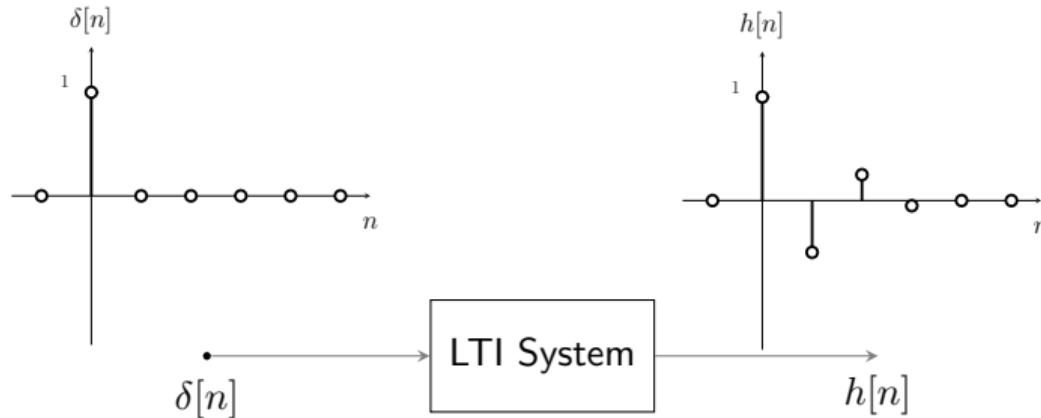
**Time invariance or shift invariance**

$$\text{If } d[n] = x[n - n_d], \text{ then } T\{d[n]\} = y[n - n_d]$$

A time shift of the input causes an equal time shift of the output

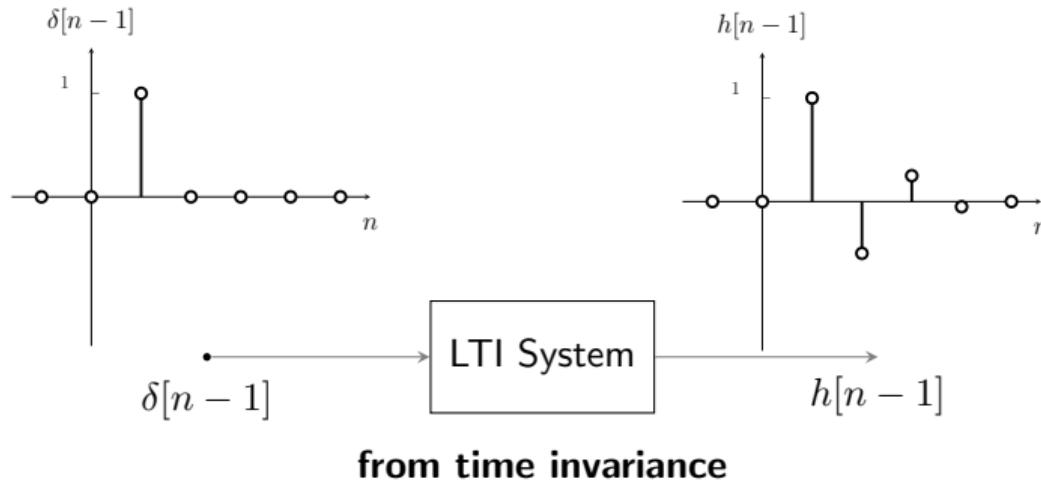
# Linear time-invariant (LTI) systems

LTI systems are **completely characterized** by their impulse response.



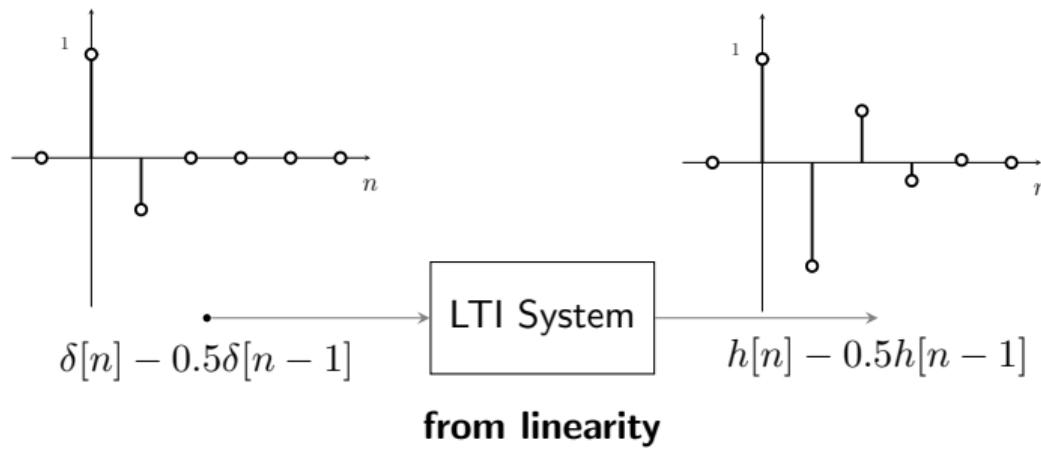
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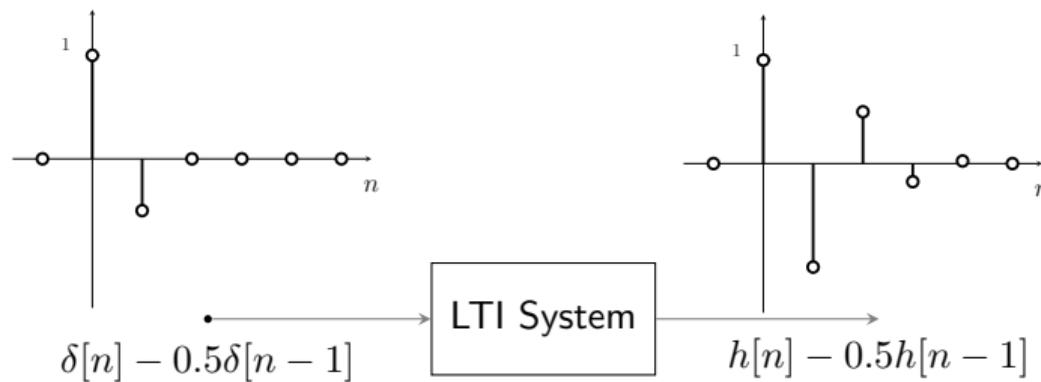
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# Linear time-invariant (LTI) systems

LTI systems are **completely characterized** by their impulse response.



The convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Shorthand notation:  $y[n] = x[n] * h[n]$  or  $y[n] = (x * h)[n]$

# Other properties of discrete-time systems

## Memorylessness

A system is **memoryless** if its output at time  $n$ ,  $y[n]$ , depends only on the present input  $x[n]$ .

## Causality

A system is **causal** if its output at time  $n$ ,  $y[n]$ , depends only on the present and past samples of the input  $\{x[n], x[n - 1], x[n - 2], \dots\}$ . All physical systems are causal.

For **LTI systems**, causality implies  $h[n] = 0, n < 0$

$$y[n] = \sum_{k=0}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^n x[k]h[n-k] \quad (\text{Convolution sum for causal systems})$$

# Other properties of discrete-time systems

## Stability

A system is **BIBO stable** if every bounded input ( $|x[n]| < B_x < \infty$ ) produces a bounded output ( $|y[n]| < B_y < \infty$ )

For **LTI systems**,

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \quad (\text{Convolution sum})$$

$$\leq \sum_{k=-\infty}^{\infty} |x[n-k]| |h[k]| \quad (\text{Triangle inequality})$$

$$\leq B_x \sum_{k=-\infty}^{\infty} |h[k]| \quad (\text{Bounded input } |x[n]| < B_x < \infty)$$

$$< B_y < \infty \quad (\text{only if } \sum_{k=-\infty}^{\infty} |h[k]| < \infty)$$

Therefore, an LTI system is BIBO stable only if its impulse response  $h[k]$  is **absolute summable**.

## Classify the following discrete-time systems

System	Linear	Time invariant	Memoryless	Causal	BIBO stable
Constant offset $y[n] = x[n] + C, C \neq 0$					
Time shift $y[n] = x[n - n_d]$					
Squaring $y[n] = x^2[n]$					
Accumulator $y[n] = \sum_{k=-\infty}^n x[k]$					
Compressor $y[n] = x[Mn], M > 1$					
Differentiator $y[n] = x[n] - x[n - 1]$					
A difference equation $y[n] = x[n] + y[n - 1]$					

# Frequency-domain representation of LTI systems

Let's use the convolution sum to compute the output of an LTI system to a complex exponential:

$$\begin{array}{ccc} x[n] = e^{j\omega n} & \xrightarrow{\text{LTI System}} & y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\ & & = \underbrace{\left( \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \right)}_{H(e^{j\omega})} e^{j\omega n} = H(e^{j\omega})e^{j\omega n} \end{array}$$

- ▶  $H(e^{j\omega})$  is a complex number that depends only on the impulse response of the LTI system
- ▶ The output of an LTI system to a complex exponential is also a complex exponential of same frequency, but with possibly different amplitude and phase

## Conclusions

- ▶ Complex exponentials are **eigenfunctions** of LTI systems, and  $H(e^{j\omega})$  are the corresponding eigenvalues.
- ▶ By representing signals as a sum of complex exponentials, we can readily calculate their output to an LTI system:  $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

# Discrete-time Fourier transform (DTFT)

## Definition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (\text{Direct transform})$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \quad (\text{Inverse transform})$$

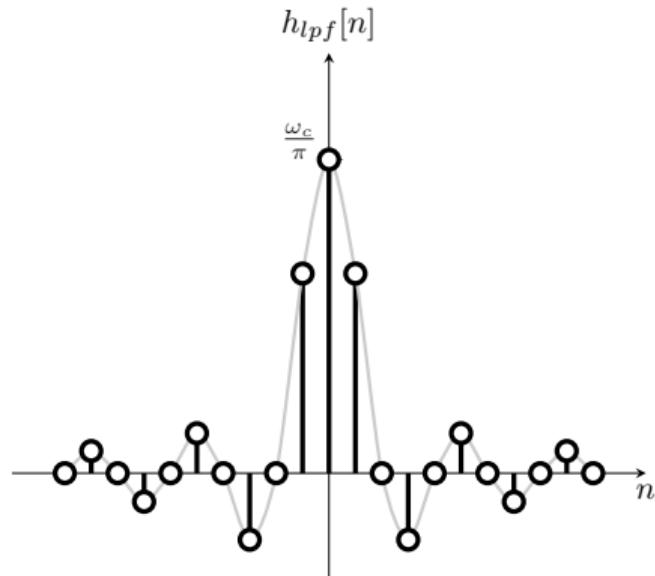
## Important:

- ▶ The DTFT is periodic with period  $2\pi$ :  $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$
- ▶ If  $x[n]$  is **absolute summable** i.e.,  $\sum_{k=-\infty}^{\infty} |x[k]| < \infty$ , then the DTFT exists. This is a *sufficient* condition, but it is *not a necessary* condition.
- ▶ Absolute summability also implies that the DTFT **converges uniformly** to a continuous function of  $\omega$ .

## Example: the ideal lowpass filter

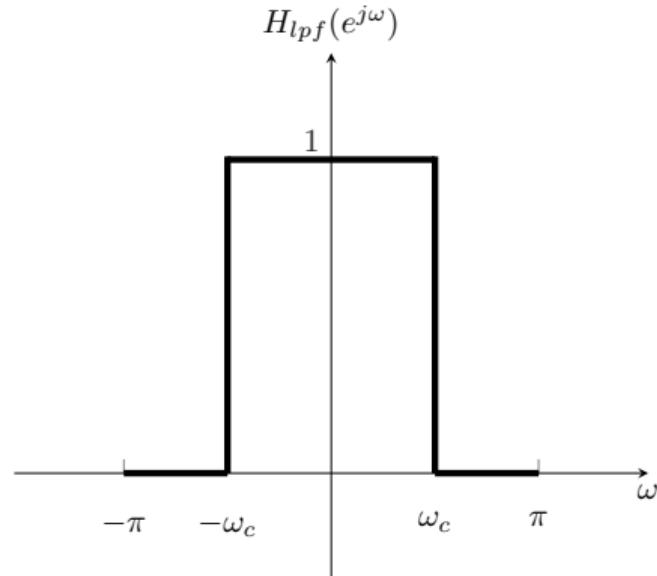
### Time domain

$$h_{lpf}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c}{\pi} n\right)$$



### Frequency domain

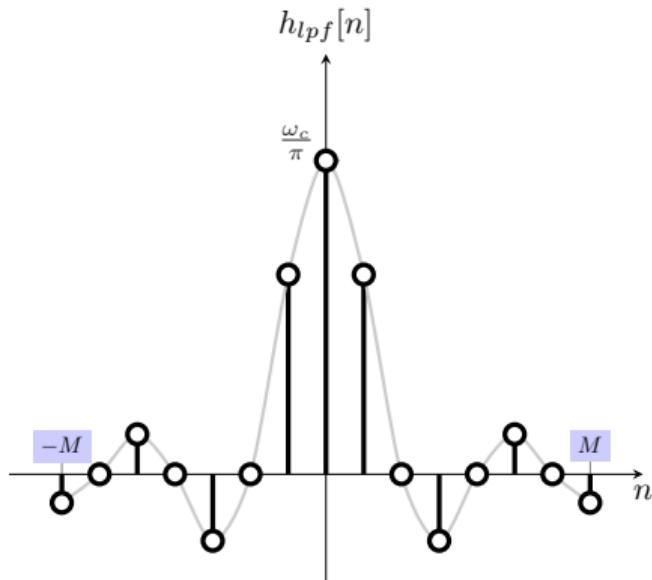
$$H_{lpf}(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$



# Example: the ideal lowpass filter

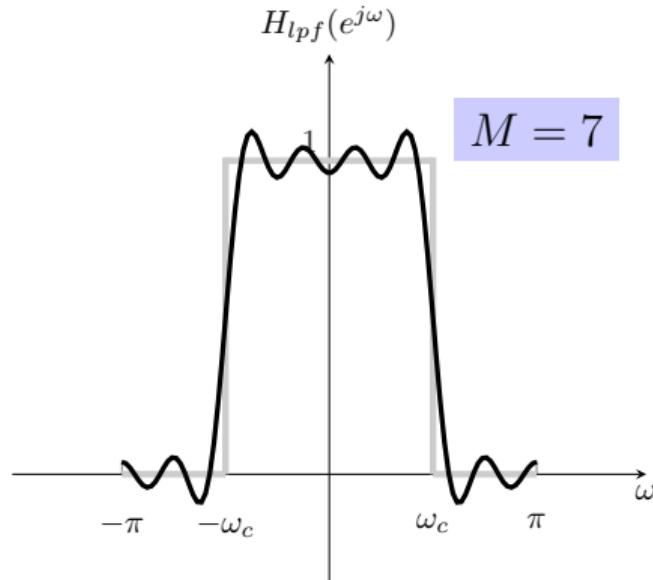
## Time domain

$$h_{lpf}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c}{\pi} n\right)$$



## Frequency domain

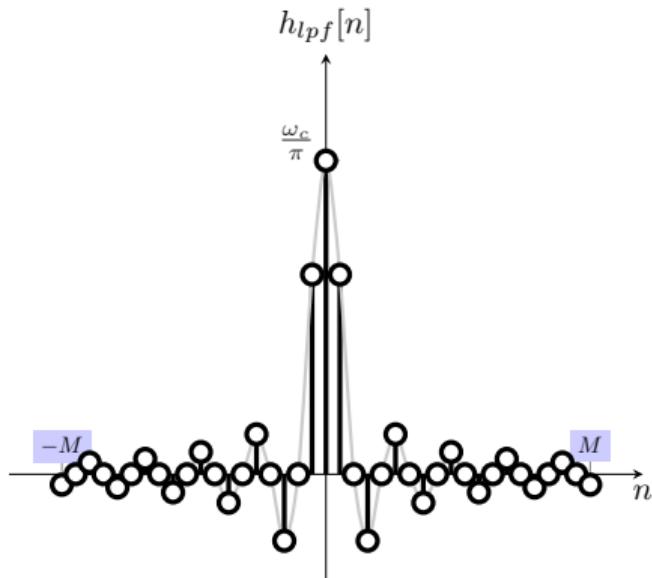
$$H_M(e^{j\omega}) = \sum_{n=-M}^M \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$



## Example: the ideal lowpass filter

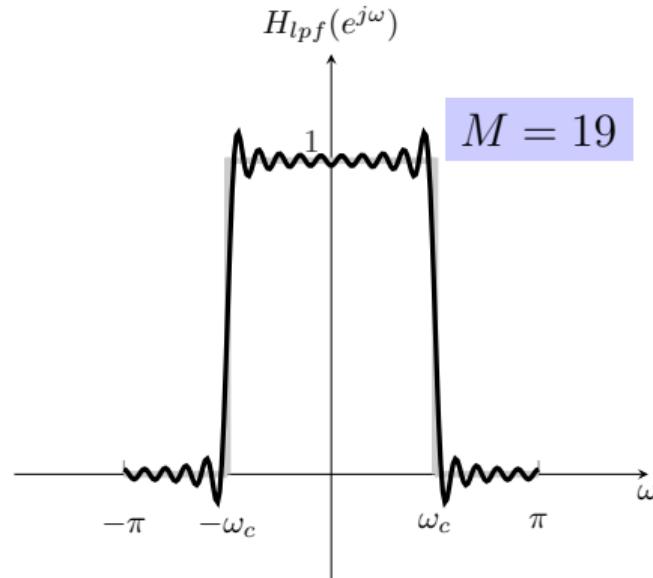
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### Frequency domain

$$H_M(e^{j\omega}) = \sum_{n=-M}^M \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$



## Convergence of the DTFT

- ▶ The sinc function is not absolute summable, but it is **square summable**

$$\sum_{n=-\infty}^{\infty} |\text{sinc}(n)|^2 < \infty$$

- ▶ In this case, the DTFT does not converge uniformly; it converges in the mean-square sense
- ▶ The oscillations are known as **Gibbs phenomenon**, and they occur whenever there is a discontinuity in the frequency domain
- ▶ Interestingly, as  $M \rightarrow \infty$  the oscillations become more rapid, but the size of the ripples does not decrease
- ▶ The DTFT may exist even when sequences are neither absolute summable nor square summable. Examples: a constant, unit step, complex exponentials
- ▶ Although all this may seem like a mathematical curiosity, it has important implications in filter design (lecture 9) and in spectrum analysis (lecture 11)

# DTFT properties

- ▶ Linearity

$$ax_1[n] + bx_2[n] \iff aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- ▶ Time shift, delay for  $n_d > 0$ , advance for  $n_d < 0$

$$x[n - n_d] \iff e^{-j\omega n_d} X(e^{j\omega})$$

- ▶ Frequency shift (modulation)

$$e^{j\omega_0 n} x[n] \iff X(e^{j(\omega - \omega_0)})$$

- ▶ Convolution

$$y[n] = x[n] * h[n] \iff Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- ▶ Multiplication of sequences (windowing)

$$x[n]w[n] \iff \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega - \theta)}) d\theta$$

# DTFT properties

- ▶ Linear weighting

$$nx[n] \iff j \frac{dX(e^{j\omega})}{d\omega}$$

- ▶ Time reversal

$$x[-n] \iff X(e^{-j\omega})$$

- ▶ Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega}) d\omega$$

if  $x[n] = y[n]$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \quad (\text{signal energy})$$

- ▶ Deterministic autocorrelation function

$$c_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m]x[n+m] \iff X(e^{j\omega})X^*(e^{j\omega}) = |X(e^{j\omega})|^2$$

# The $z$ -Transform

The  $z$ -transform is a generalization of the DTFT and it is applicable to a broader class of signals and systems.

## Definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (\text{Direct transform})$$

$$x[n] = \frac{1}{2j\pi} \oint_{\mathcal{C}} X(z)z^{n-1}dz \quad (\text{Inverse transform})$$

- ▶ The  $z$ -transform has a **region of convergence (ROC)**, which is the values of  $z$  for which the infinite sum in the direct transform is finite. Without the ROC, the  $z$ -transform is an **ambiguous representation** of a signal.
- ▶ The inverse transform is given by the contour integral over some complex path  $\mathcal{C}$ . This is generally laborious, so we'll obtain the inverse  $z$ -transform through indirect methods such as **partial fraction expansion**.

# The $z$ -Transform

$x[n] = z^n, z \in \mathbb{C}$  is another **eigenfunction** of LTI systems

*Proof:*

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]z^{(n-k)} && \text{(from convolution sum)} \\ &= \left( \sum_{k=-\infty}^{\infty} h[k]z^{-k} \right) z^n && \text{(rearranging)} \\ &= H(z)z^n && \text{(By definition } H(z) \equiv \sum_{k=-\infty}^{\infty} h[k]z^{-k}) \end{aligned}$$

- ▶ We can always write  $z^n = r^n e^{j\omega n}$  (polar coordinates)
- ▶ The eigenfunction  $e^{j\omega n}$ , used in the DTFT, is just a particular case ( $r = 1$ ) of the eigenfunction used in the  $z$ -transform
- ▶ The factor  $r^n$  helps the  $z$ -transform sum converge to a broader class of signals

# Relation between DTFT and the $z$ -Transform

DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

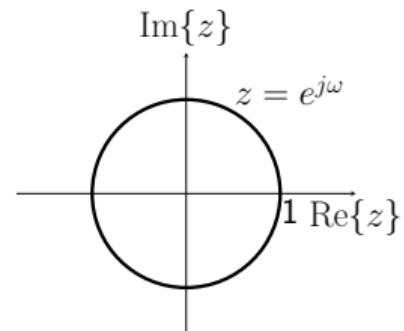
(Direct transform)

$z$ -Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(Direct transform)

- ▶ The DTFT is equal to the  $z$ -transform evaluated on the unit circle
- ▶ DTFT is periodic with period  $2\pi$ .
- ▶ **Question:** Does the DTFT exist if the ROC of the  $z$ -transform does not include the unit circle?



## For what will we use the $z$ -transform?

1. Representing LTI systems
2. Determining stability of LTI systems
3. Solving difference equations

## Rational $z$ -transforms

Rational  $z$ -transforms are a ratio of two polynomials in  $z^{-1}$  (or  $z$ )

$$\begin{aligned} X(z) &= \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} \\ &= \frac{b_0}{a_0} z^{N-M} \frac{(z - z_1)(z - z_2) \dots (z - z_M)}{(z - p_1)(z - p_2) \dots (z - p_N)} \end{aligned}$$

- ▶ The **zeros** of a  $z$ -transform are the values of  $z$  for which  $X(z) = 0$
- ▶ The **poles** of a  $z$ -transform are the values of  $z$  for which  $X(z) = \infty$ . By definition, the ROC cannot contain any pole
- ▶  $X(z)$  has  $M$  zeros  $\{z_1, z_2, \dots, z_M\}$ , which are the roots of the numerator polynomial  $B(z)$
- ▶  $X(z)$  has  $N$  poles  $\{p_1, p_2, \dots, p_N\}$ , which are the roots of the denominator polynomial  $A(z)$
- ▶ If  $N - M > 0$ ,  $X(z)$  has more  $N - M$  zeros at  $z = 0$
- ▶ If  $N - M < 0$ ,  $X(z)$  has more  $M - N$  poles at  $z = 0$
- ▶ If the coefficients  $\{b_0, \dots, b_M\}, \{a_0, \dots, a_N\}$  are real, the poles and zeros are either real or they appear in complex conjugate pairs

## Examples

1. Right-sided exponential  $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

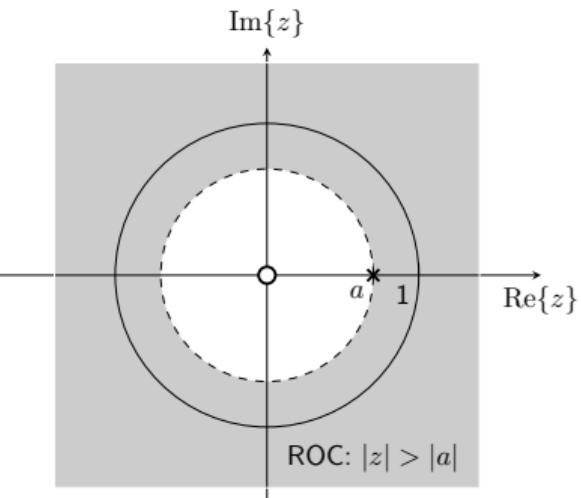
This sum converges only if  $|az^{-1}| < 1$ . Hence, ROC:  $|z| > |a|$ .

From the sum of an infinite geometric progression:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Pole at  $z = a$  and zero at  $z = 0$ .

The ROC of a causal signal is the **exterior of a circle** whose radius is the magnitude of the outermost pole  $|a|$ .



## Examples

2. Left-sided exponential  $x[n] = -a^n u[-n - 1]$

$$X(z) = - \sum_{n=-\infty}^{-1} a^n z^{-n} = - \sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

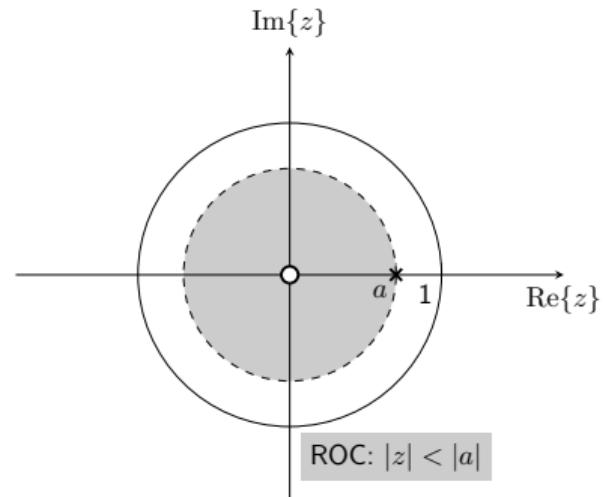
This sum converges only if  $|a^{-1}z| < 1$ . Hence, ROC:  $|z| < |a|$ .

Once again from the sum of an infinite geometric progression:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad (\text{Same as before!})$$

Without the ROC, the  $z$ -transform is an **ambiguous representation** of a signal.

The ROC of an anti-causal signal is the **interior of a circle** whose radius is the magnitude of the innermost pole  $|a|$ .



## Examples

3. Two-sided exponential  $x[n] = -b^n u[-n-1] + a^n u[n]$

We can use the previous results and the linearity property of the  $z$ -transform:

$$X(z) = - \sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-bz^{-1}} + \frac{1}{1-az^{-1}}$$

The ROC is the intersection of the two previous ROCs, i.e.,  $\text{ROC} = |b| < |z| < |a|$ .

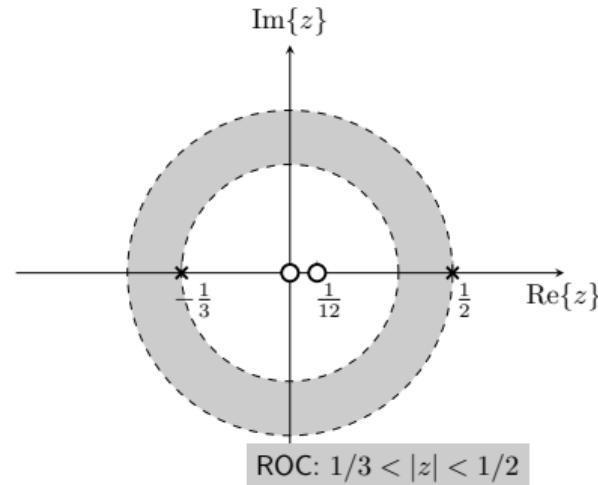
Assuming  $a = -1/3$  and  $b = 1/2$

$$X(z) = \frac{2z(z - 1/12)}{(z + 1/3)(z - 1/2)}$$

Poles at  $z = -1/3, 1/2$

Zeros at  $z = 0, 1/12$

The ROC of a two-sided signal is an **annulus**  
(ring-shaped region).



# Properties of the region of convergence

The ROC tells a lot about a signal or system

1. If the ROC is the exterior of a circle ( $\text{ROC} = \{|z| > |a|\}$ ), the system/signal is **causal**, where  $a$  is the outermost pole. The ROC cannot contain any pole
2. If a **LTI system is BIBO stable** (i.e.,  $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ ), the ROC must contain the unit circle.

*Proof:*

$$\begin{aligned}|H(z)| &= \left| \sum_{n=-\infty}^{\infty} h[n]z^{-n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |h[n]| |z|^{-n} && \text{(From triangle inequality)} \\ &= \sum_{n=-\infty}^{\infty} |h[n]| < \infty && (z = e^{j\omega} \implies |z| = 1)\end{aligned}$$

3. A **causal and stable LTI system** has all poles inside the unit circle. This follows from the first two properties.

# Properties of the $z$ -transform

Property	Time-Domain	$z$ -Domain	ROC
Notation	$x[n]$	$X(z)$	$\text{ROC} = \{r_2 <  z  < r_1\}$
	$x_1[n]$	$X_1(z)$	$\text{ROC}_1$
	$x_2[n]$	$X_2(z)$	$\text{ROC}_2$
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	at least $\text{ROC}_1 \cap \text{ROC}_2$
Time shifting	$x[n - k]$	$X(z)z^{-k}$	$\text{ROC} - \begin{cases} \{0\}, k > 0 \\ \{\infty\}, k < 0 \end{cases}$
Scaling in the $z$ -domain	$a^n x[n]$	$X(\frac{z}{a})$	$ ar_1  <  z  <  ar_2 $
Time reversal	$x[-n]$	$X(z^{-1})$	$ \frac{1}{r_1}  <  z  <  \frac{1}{r_2} $
Conjugation	$x[n]^*$	$X(z^*)^*$	$\text{ROC}$
Real part	$\text{Re}\{x[n]\}$	$\frac{X(z) + X(z^*)^*}{2}$	includes $\text{ROC}$
Imaginary part	$\text{Im}\{x[n]\}$	$\frac{X(z) - X(z^*)^*}{2j}$	includes $\text{ROC}$
Differentiation in the $z$ -domain	$nx[n]$	$-z \frac{dX(z)}{dz}$	$\text{ROC}$
Convolution	$x_1[n] * x_2[n]$	$X_1(z) \cdot X_2(z)$	at least $\text{ROC}_1 \cap \text{ROC}_2$
Correlation	$x_1[n] * x_2[-n]$	$X_1(z) \cdot X_2(z^{-1})$	at least $\text{ROC}_1 \cap \text{ROC}_2(z^{-1})$

# Linear constant-coefficient difference equation

Many practical problems appear in the form of difference equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

This difference equation defines an LTI system only if:

1. All coefficients  $a_k, b_k$  are constant
2. The initial conditions (or *rest* conditions) are zero  $y[-N] = y[-N+1] = \dots = y[-1] = 0$

Applying the linearity and time-shift properties of the  $z$ -transform:

$$\sum_{k=0}^N a_k Y(z) z^{-k} = \sum_{k=0}^M b_k X(z) z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-jk}}$$

$H(z)$  is a **rational  $z$ -transform**.

## Example: first-order system

**First-order system:**  $y[n] - ay[n - 1] = x[n]$

Calculating the  $z$ -transform:

$$\begin{aligned}Y(z) - aY(z)z^{-1} &= X(z) \\ Y(z)(1 - az^{-1}) &= X(z) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}\end{aligned}$$

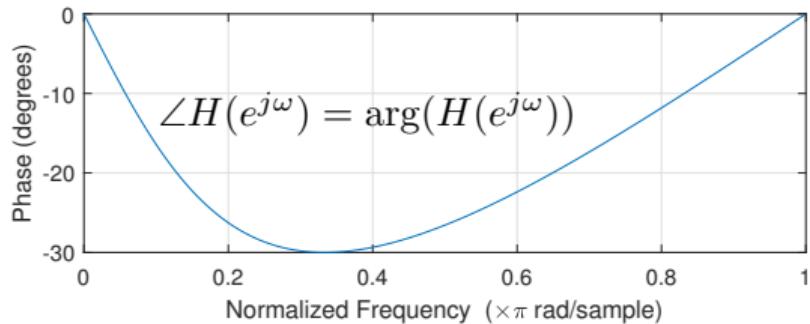
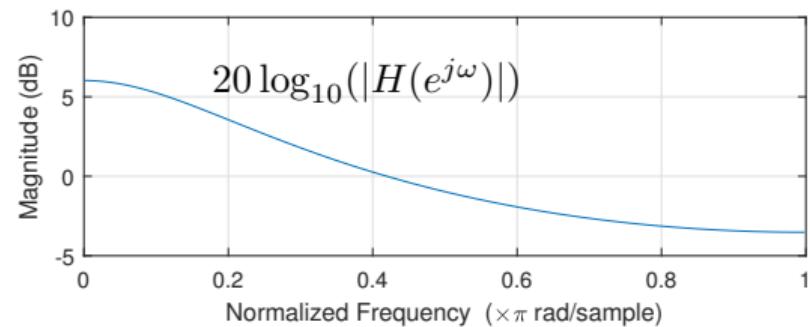
- ▶ This corresponds to the exponential:  $h[n] = a^n u[n]$ .
- Questions:** Why the causal exponential? For what values of  $a$  is this system stable?
- ▶ This system is **autoregressive** i.e., the present output depends on previous outputs
- ▶ Autoregressive systems have **infinite impulse response (IIR)**
- ▶ Systems with rational  $z$ -transforms with non-zero poles are IIR

Let's assume  $a = 0.5$ , then  $y[n] - 0.5y[n - 1] = x[n]$

$$H(z) = \frac{1}{1 - 0.5z^{-1}} \xrightarrow{z=e^{j\omega}} H(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

In Matlab:

```
> freqz(1, [1, -0.5])
```



## Example: moving average system

**Moving average:**  $y[n] = \frac{1}{M}(x[n] + x[n - 1] + x[n - 2] + \dots + x[n - M + 1])$

Calculating the  $z$ -transform:

$$Y(z) = \frac{1}{M}X(z)(1 + z^{-1} + \dots + z^{-(n-M+1)})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M}(1 + z^{-1} + \dots + z^{-(n-M+1)})$$

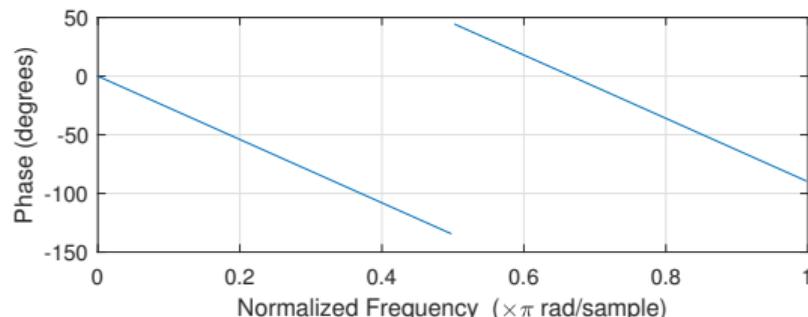
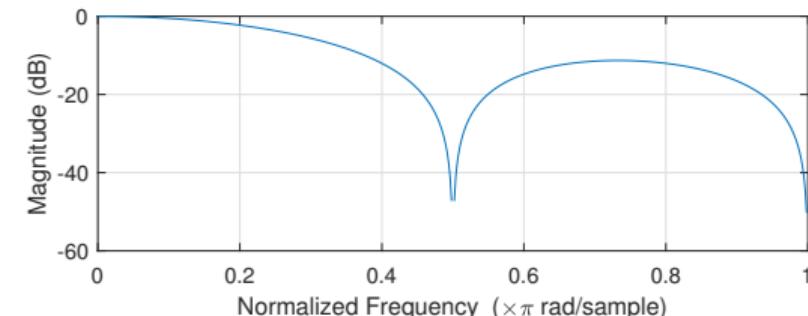
- ▶ This system has impulse response  $h[n] = 1/M(\delta[n] + \delta[n - 1] + \dots + \delta[n - M + 1])$
- ▶ The impulse response only depends on a finite number of previous inputs. Hence, this system has a **finite impulse response (FIR)**

Let's assume  $M = 4$ , then  $y[n] = \frac{1}{4}(x[n] + x[n - 1] + x[n - 2] + x[n - 3])$

$$H(z) = \frac{1}{4}(1 + z^{-1} + \dots + z^{-3}) \xrightarrow{z=e^{j\omega}} H(e^{j\omega}) = \frac{1}{4}(1 + e^{-j\omega} + \dots + e^{-3j\omega})$$

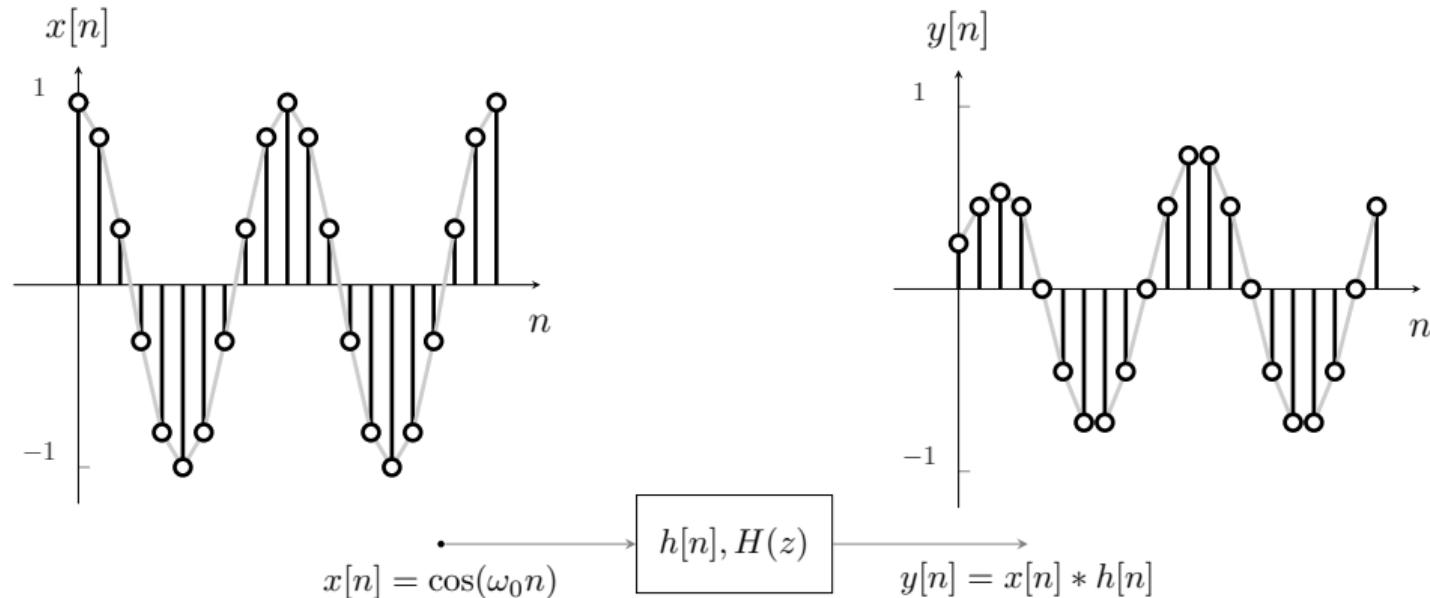
In Matlab:

```
> freqz([1, 1, 1, 1]/4, 1)
```



## Example: output of a moving average filter

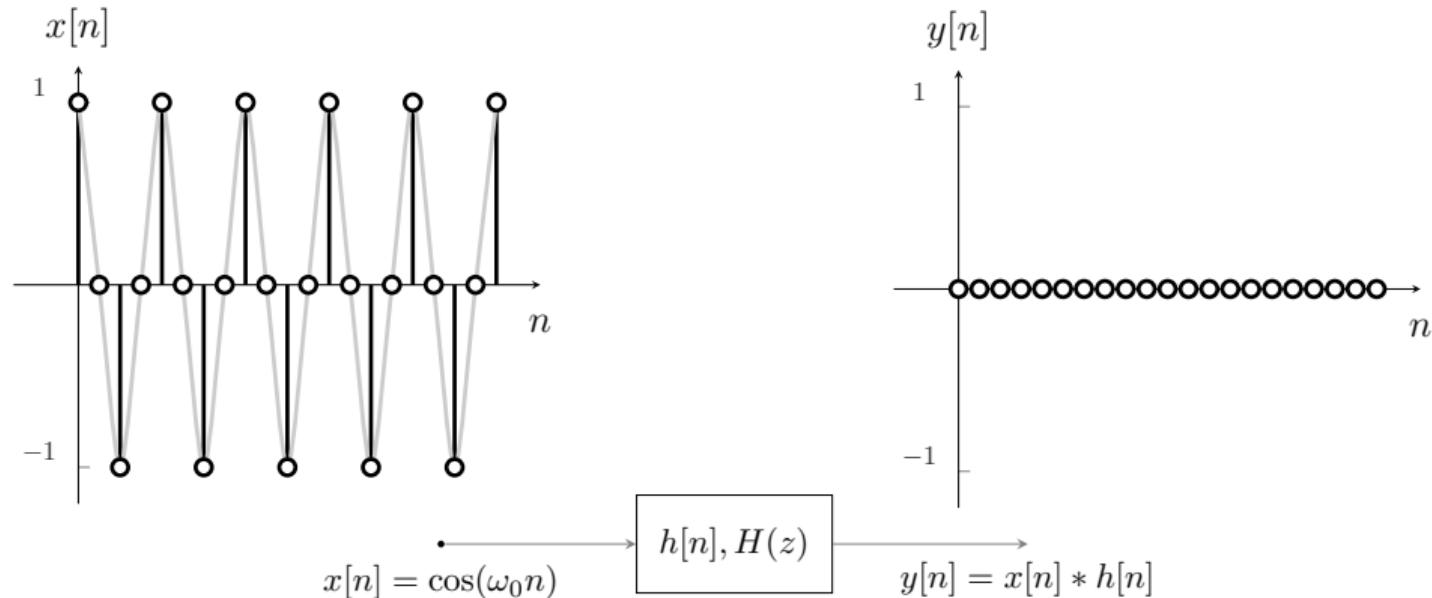
Suppose the input signal frequency is  $\omega_0 = 0.2\pi$



$$h[n] = \frac{1}{4}(\delta[n] + \dots + \delta[n-3]) \iff H(z) = \frac{1}{4}(1 + z^{-1} + \dots + z^{-3})$$

## Example: output of a moving average filter

What would happen if  $\omega_0 = 0.5\pi$ ?



$$h[n] = \frac{1}{4}(\delta[n] + \dots + \delta[n-3]) \iff H(z) = \frac{1}{4}(1 + z^{-1} + \dots + z^{-3})$$

## Summary

- ▶ Systems can be linear, time-invariant, memoryless, causal, and stable
- ▶ Linear and time-invariant (LTI) systems are completely characterized by their impulse response
- ▶ We can use the convolution sum to calculate the output of an LTI system to any signal
- ▶ The complex exponential  $e^{j\omega n}$ , and more generally  $z^n$ , are eigenfunctions of LTI systems
- ▶ Frequency-domain representation of signals and systems
  - ▶ Discrete-time Fourier transform (DTFT)
  - ▶  $z$ -transform and ROC. Without specifying the ROC the  $z$ -transform is ambiguous
- ▶ The DTFT is equivalent to the  $z$ -transform evaluated on the unit circle:  
 $H(e^{j\omega}) = H(z = e^{j\omega})$ , provided that the unit circle is in the ROC of the  $z$ -transform