

Sampling Rate Conversions: Downsampling

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Downsampling: Motivation

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Frequency
Domain Analysis

Decimation

Summary

Why downsample?

- Reduce computation, memory, bandwidth, and power.
- Match device/IO rate constraints (sensors, radios, codecs).
- Enable multirate algorithms, filter banks.

Definition (Compressor / Downampler):

$$x_d[n] = x[nM] = x_c(nMT) \quad \text{with} \quad T_d = MT$$

Equivalent to sampling the bandlimited reconstruction $x_c(t)$ with period T_d .

Downsampling: Aliasing Considerations

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When is it alias-free?

- If $X_c(j\Omega) = 0$ for $|\Omega| \geq \Omega_N$, then $x_d[n]$ exactly represents $x_c(t)$ if

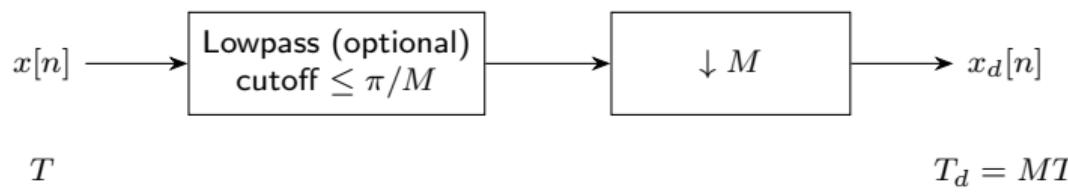
$$\frac{\pi}{T_d} = \frac{\pi}{MT} \geq \Omega_N \iff \omega_N \leq \frac{\pi}{M}.$$

- Interpretations:

- Original sampling rate is at least M times the Nyquist rate, or
- Prefilter to reduce bandwidth by factor M before downsampling (decimation).

Terminology:

- Compressor: just $\downarrow M$ (rate reduction).
- Decimator: antialiasing lowpass + $\downarrow M$.



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DTFT Relationship:

DTFT of $x[n] = x_c(nT)$:

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$

Downsampling sequence DTFT (with $T_d = MT$):

$$X_d(e^{j\omega}) = \frac{1}{T_d} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T_d} - j\frac{2\pi r}{T_d}\right)$$

Key Result – Relationship Between DTFTs:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

Interpretation:

- $X_d(e^{j\omega})$ is sum of M scaled, shifted copies of $X(e^{j\omega})$.
- Frequency axis compressed by factor M ; shifts at $2\pi i/M$.
- Overall amplitude scaling $1/M$.

Derivation of DTFT Relationship

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Starting point (sampling with $T_d = MT$):

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi r}{MT}\right)$$

Let $r = i + kM$ with $i = 0, 1, \dots, M-1$:

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{i=0}^{M-1} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi(i + kM)}{MT}\right)$$

Group terms:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi i}{MT} - j\frac{2\pi k}{T}\right)$$

Recognize inner sum as $X(e^{j(\omega/M - 2\pi i/M)})$. Therefore:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

Downsampling Example: $M = 2$ (No Aliasing)

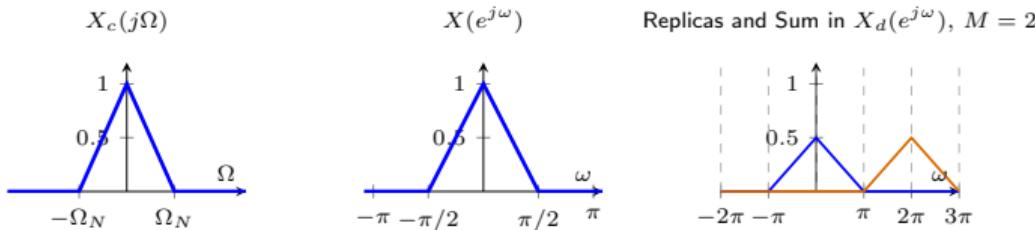
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Setup: Original sampling rate is twice the minimum Nyquist rate; bandwidth fits after compression.



Result: No aliasing because the compressed baseband fits in $[-\pi, \pi]$. For $M = 2$ with $\omega_N = \pi/2$, the baseband and the shifted replica (centered at 2π) do not overlap within $[-\pi, \pi]$.

Downsampling with Aliasing: $M = 3$

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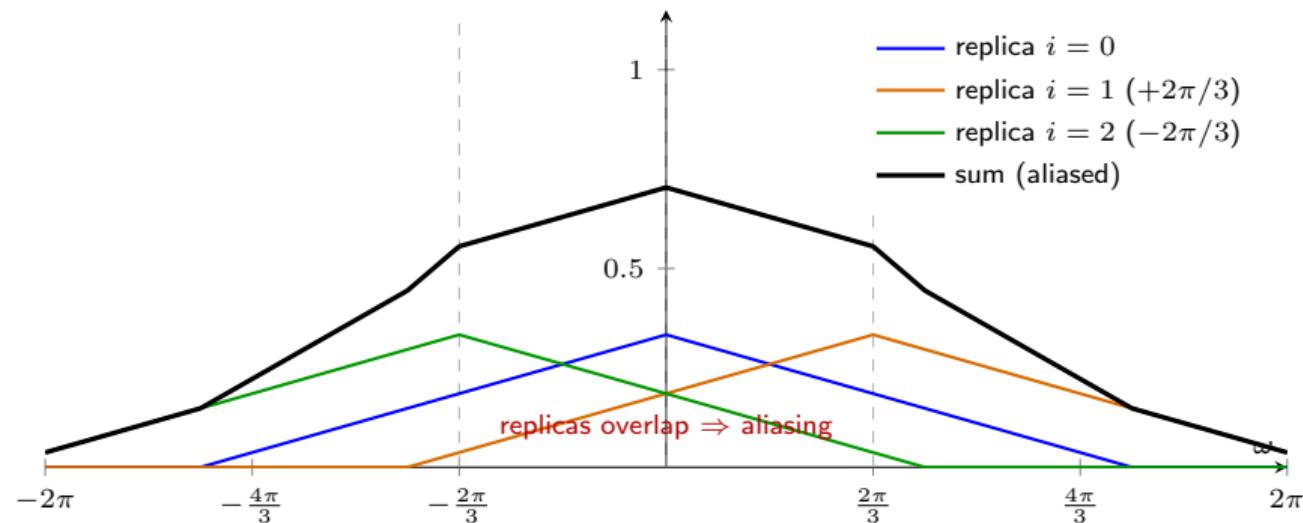
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Setup: Compression factor too large, spectral copies overlap.



Downsampling with Aliasing - Frequency Domain

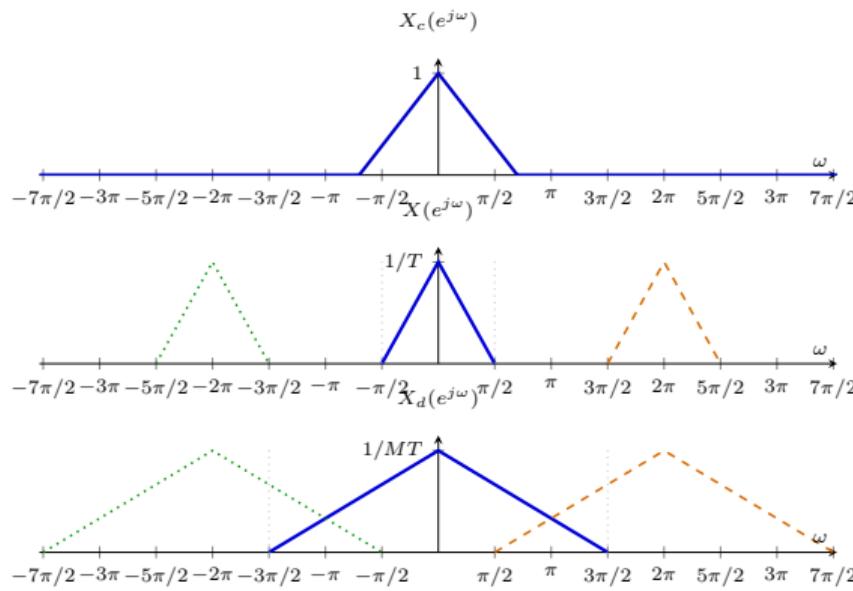
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Decimation: Prefiltering Before Downsampling

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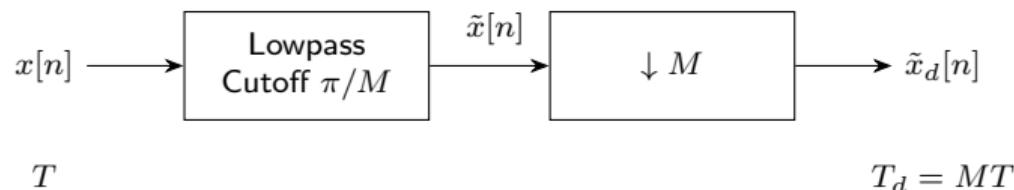
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Problem: If original bandwidth exceeds π/M , downsampling aliases.

Solution: Apply a lowpass (antialiasing) filter of cutoff π/M before compression.



Ideal filter:

$$\tilde{H}_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \pi/M \\ 0, & \pi/M < |\omega| \leq \pi \end{cases}$$

After filtering, $\tilde{x}[n]$ has bandwidth π/M , so $\tilde{x}_d[n] = \tilde{x}[nM]$ is alias-free.

Downsampling with Anti Aliasing - Frequency Domain

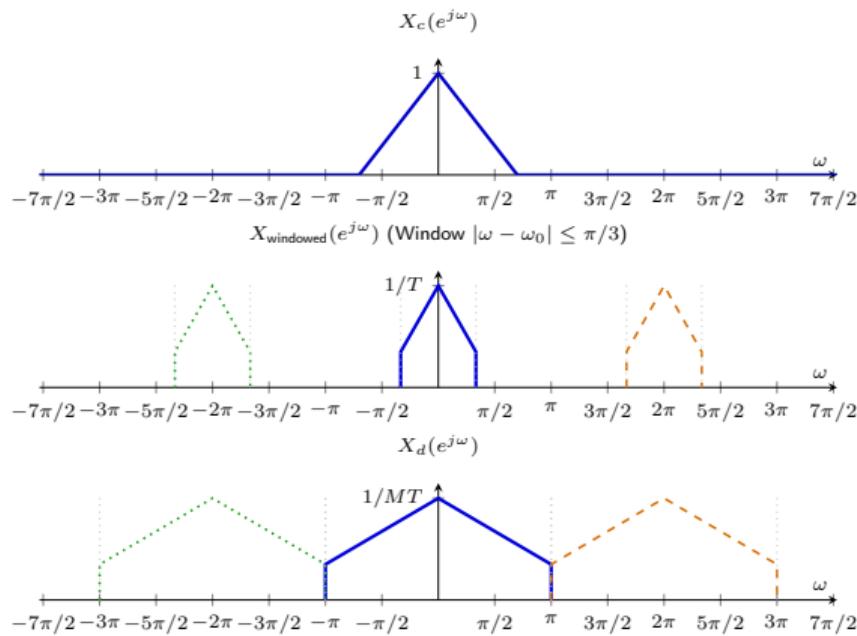
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Key Concepts:

- Resampling changes sampling rate using discrete-time operations.
- Downsampling by M : $x_d[n] = x[nM]$.
- Compressor vs. Decimator: Decimator adds antialias filter.

Frequency-Domain Relationship:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

Aliasing Condition:

$$\omega_N \leq \frac{\pi}{M} \quad (\text{required for alias-free direct downsampling})$$

If violated: prefilter to bandwidth π/M before rate reduction.