Sampling of Continuous-Time Signals

Maxx Seminario

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Periodic Sampling: Basic Definition

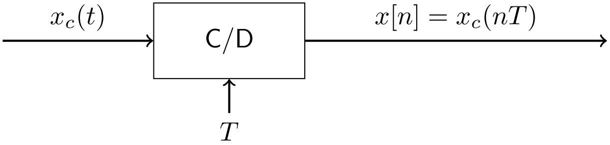
Fundamental Sampling Equation:

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

Key Parameters:

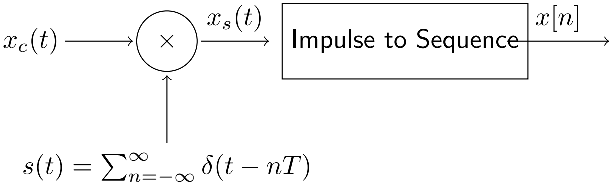
- T = Sampling period (seconds)
- $f_s = \frac{1}{T}$ = Sampling frequency (samples/second or Hz)
- $\Omega_s = \frac{2\pi}{T}$ = Sampling frequency (radians/second)

Ideal C/D Converter:



Mathematical Model: Impulse Train Sampling

Two-Stage Process:



Impulse Train Modulation:

$$x_s(t) = x_c(t) \cdot s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Using Sifting Property $x(t)\delta(t) = x(0)\delta(t)$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

Time-Domain Visualization: Sampling Process

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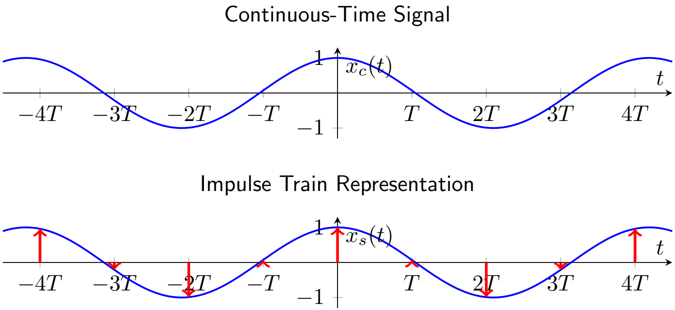
Introduction

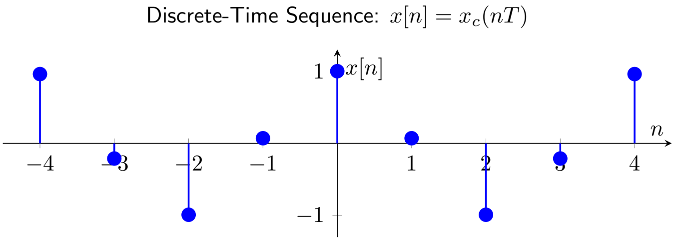
Frequency Domain

Nyquist Theorem

Examples



Time-Domain: Discrete-Time Sequence



Key Observation: Time normalization

- $x_s(t)$: Spacing $=$ T seconds
- $x[n]$: Spacing $=$ 1 'sample index' (dimensionless)

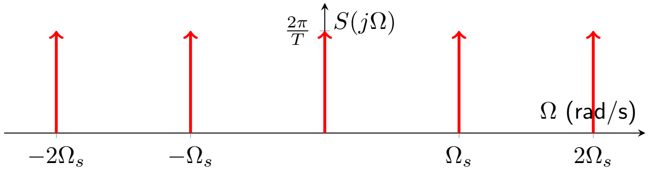
Frequency-Domain: Fourier Transform of Impulse Train

Impulse Train in Time Domain:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Fourier Transform (Impulse Train in Frequency):

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s), \quad \Omega_s = \frac{2\pi}{T}$$



Frequency-Domain Representation of Sampling

Convolution Property: $x_s(t) = x_c(t) \cdot s(t)$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

Result - Periodic Replication:

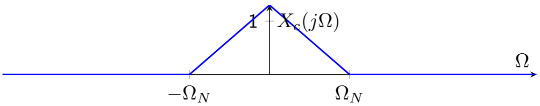
$$\boxed{X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))}$$

Physical Interpretation:

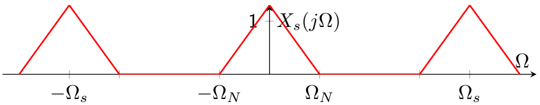
- Original spectrum $X_c(j\Omega)$ is **replicated** at intervals of Ω_s
- Scaled by factor $\frac{1}{T}$
- Copies centered at $\Omega = 0, \pm\Omega_s, \pm 2\Omega_s, \ldots$

Case 1: No Aliasing ($\Omega_s > 2\Omega_N$)

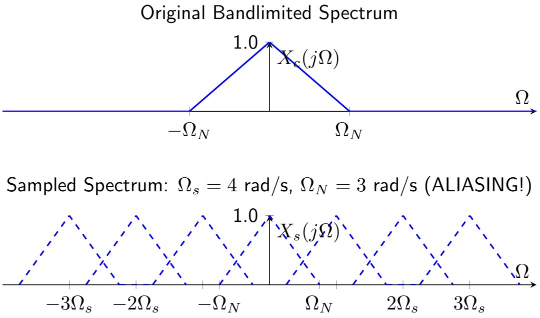
Original Bandlimited Spectrum



Sampled Spectrum: $\Omega_s = 8$ rad/s, $\Omega_N = 3$ rad/s (No Overlap)



Case 2: Aliasing ($\Omega_s < 2\Omega_N$)



Relation Between $X_s(j\Omega)$ and $X(e^{j\omega})$

From Impulse Train:

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega nT}$$

DTFT of Sequence:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Relationship - Frequency Scaling:

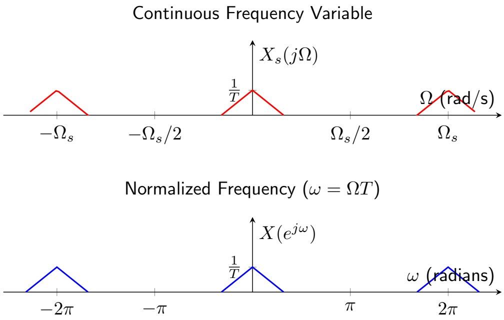
$$\boxed{X(e^{j\omega}) = X_s(j\Omega)\Big|_{\Omega=\omega/T} = X_s(j\omega/T)}$$

or equivalently:

$$\boxed{X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)}$$

Normalization: $\Omega = \Omega_s$ maps to $\omega = 2\pi$

Frequency Axis Normalization



Key Points:

- $\Omega_s = 2\pi/T$ maps to $\omega = 2\pi$
- $X(e^{j\omega})$ is always 2π-periodic

Nyquist-Shannon Sampling Theorem

Statement:
Let $x_c(t)$ be a bandlimited signal with:

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

Then $x_c(t)$ is **uniquely determined** by its samples $x[n] = x_c(nT)$ if:

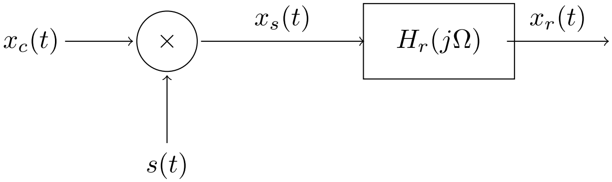
$$\boxed{\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N}$$

Terminology:

- Ω_N = **Nyquist frequency** (highest frequency in signal)
- $2\Omega_N$ = **Nyquist rate** (minimum sampling frequency)
- $\Omega_s/2$ = **Folding frequency**

Reconstruction from Samples

System:



Ideal Reconstruction Filter:

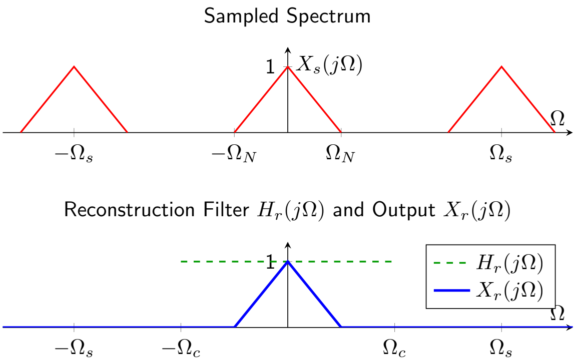
$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

where $\Omega_N \leq \Omega_c \leq (\Omega_s - \Omega_N)$

Output:

$$X_r(j\Omega) = H_r(j\Omega) \cdot X_s(j\Omega) = X_c(j\Omega) \quad \text{if } \Omega_s \geq 2\Omega_N$$

Ideal Lowpass Reconstruction Filter



Result: $X_r(j\Omega) = X_c(j\Omega) \Rightarrow x_r(t) = x_c(t)$

Example 1: Sampling a Sinusoid (No Aliasing)

Signal: $x_c(t) = \cos(\Omega_0 t)$ with $\Omega_0 = 4000\pi$ rad/s
Sampling: $T = 1/6000$ s $\Rightarrow \Omega_s = 12000\pi$ rad/s

Check Nyquist: $\Omega_s = 12000\pi > 2\Omega_0 = 8000\pi$

Sampled Sequence:

$$x[n] = \cos(4000\pi \cdot n/6000) = \cos[2\pi n/3]$$

Normalized frequency: $\omega_0 = \Omega_0 T = 2\pi/3$

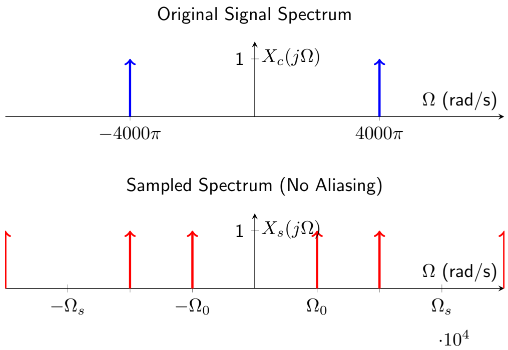
Fourier Transform:

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$

After Sampling:

$$X(e^{j\omega}) = \pi\delta(\omega - 2\pi/3) + \pi\delta(\omega + 2\pi/3) + \text{periodic}$$

Example 1: Frequency Domain Visualization



Reconstruction: Lowpass filter extracts center copy $\rightarrow x_r(t) = x_c(t)$

Example 2: Sampling a Sinusoid (With Aliasing)

Signal: $x_c(t) = \cos(\Omega_0 t)$ with $\Omega_0 = 16000\pi$ rad/s
Sampling: $T = 1/6000$ s $\Rightarrow \Omega_s = 12000\pi$ rad/s

Check Nyquist: $\Omega_s = 12000\pi < 2\Omega_0 = 32000\pi$ $\times$

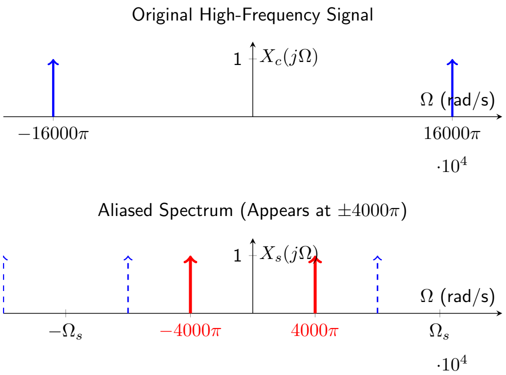
Sampled Sequence:

$$x[n] = \cos(16000\pi \cdot n/6000) = \cos(8\pi n/3)$$

But $\cos(8\pi n/3) = \cos(8\pi n/3 - 2\pi n) = \cos(2\pi n/3)$

Same samples as Example 1!
Alias frequency: $\Omega_{\text{alias}} = \Omega_s - \Omega_0 = 12000\pi - 16000\pi = -4000\pi$
Or equivalently: $|\Omega_{\text{alias}}| = 4000\pi$

Example 2: Aliasing Visualization



Reconstruction: Filter extracts $4000\pi \to x_r(t) = \cos(4000\pi t) \neq x_c(t)$

Aliasing: Multiple Signals $\rightarrow$ Same Samples

Key Insight: For any integer k:

$$\cos[(\Omega_s k + \Omega_0)nT] = \cos(\Omega_0 nT)$$

Family of Alias Frequencies:

$$\Omega_{\text{alias},k} = \Omega_0 + k\Omega_s, \quad k = 0, \pm 1, \pm 2, \ldots$$

All produce the same sequence when sampled at rate Ω_s!

Example: $\Omega_s = 12000\pi$, sample values $x[n] = \cos(2\pi n/3)$

Frequency Ω_0 (rad/s)	Normalized ω_0
4000π	$2\pi/3$
16000π	$8\pi/3 = 2\pi/3 + 2\pi$
-8000π	$-4\pi/3 = 2\pi/3 - 2\pi$
28000π	$14\pi/3 = 2\pi/3 + 4\pi$

Resolution: Restrict to $|\Omega_0| \leq \Omega_s/2$ (Nyquist criterion)

Summary: Sampling in Frequency Domain

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Introduction

Frequency Domain

Nyquist Theorem

Examples

Key Equation:

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

Three Cases:

1. $\Omega_s > 2\Omega_N$: No overlap $\rightarrow$ Perfect reconstruction possible
2. $\Omega_s = 2\Omega_N$: Critical sampling (Nyquist rate)
3. $\Omega_s < 2\Omega_N$: Aliasing $\rightarrow$ Information loss

Practical Considerations:

- Real signals are never perfectly bandlimited (noise, harmonics)
- Oversampling provides guard band
- Trade-off: sampling rate vs. computation/storage

Reconstruction Formula (Time Domain)

Ideal Lowpass Filter Impulse Response:

$$h_r(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

Reconstruction Formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT) \frac{\sin[\Omega_c(t - nT)]}{\pi(t - nT)}$$

For $\Omega_c = \pi/T$:

$$\boxed{x_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}}$$

Interpolation: Each sample weighted by sinc function
Shannon Interpolation Formula: Exact for bandlimited signals

Conclusion

Main Results:

1. Sampling creates **periodic replication** in frequency domain
2. **Nyquist criterion**: $\Omega_s \geq 2\Omega_N$ prevents aliasing
3. **Perfect reconstruction** possible if Nyquist satisfied
4. Frequency normalization: $\omega = \Omega T$

Practical Applications:

- Digital audio: $f_s = 44.1$ kHz (covers 0-20 kHz hearing range)
- Speech: $f_s = 8$ kHz (telephone quality)

Next Topics:

- Practical A/D conversion, quantization effects
- Multirate signal processing, decimation, interpolation
- Filter design for anti-aliasing