

Changing the Sampling Rate: Upsampling (Interpolation)

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Motivation for Upsampling

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Summary

Why increase the sampling rate?

- Interface between systems running at different sample rates (audio, telecom, sensors).
- Enable finer time resolution for subsequent processing (filtering, D/A conversion).

Goal: Given samples $x[n] = x_c(nT)$, produce samples

$$x_i[n] = x_c(nT_i), \quad T_i = \frac{T}{L}$$

so that the new sampling rate is L times larger.

Definition (Upsampling / Interpolation):

$$x_i[n] = x\left[\frac{n}{L}\right] = x_c\left(\frac{nT}{L}\right), \quad n = 0, \pm L, \pm 2L, \dots$$

Problem: Need the “missing” samples while preserving bandlimited structure.

Solution Outline:

- 1 Insert “new samples” (expander) within the sequence.
- 2 Lowpass filter with appropriate gain and cutoff to reconstruct intermediate samples.

Expander and Interpolator Structure

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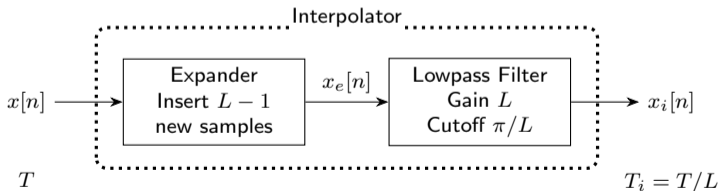
Expander (Sampling Rate Expander):

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases} \iff x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL]$$

Interpolator: Lowpass filter $h_i[n]$ (gain L , cutoff π/L) applied to $x_e[n]$:

$$x_i[n] = (h_i * x_e)[n]$$

System Diagram:



Terminology: Expander + Antialias (Interpolation) Filter = **Interpolator**.

Frequency-Domain Effect of Expansion: DTFT of $x_e[n]$

Expander definition:

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases} \iff x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL]$$

Compute the DTFT of $x_e[n]$:

$$\begin{aligned} X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_e[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k] \delta[n - kL] \right) e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega(kL)} \\ &= \sum_{k=-\infty}^{\infty} x[k] e^{-j(\omega L)k} \\ &= X(e^{j\omega L}) \end{aligned}$$

Frequency-Domain Effect of Expansion: Interpretation and Filter

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Result:

$$X_e(e^{j\omega}) = X(e^{j\omega L})$$

Interpretation:

- Baseband spectrum is **compressed** by factor L .
- Multiple frequency-scaled images appear within $|\omega| \leq \pi$ due to DTFT periodicity.
- An interpolation lowpass (cutoff π/L) removes images and rescales amplitude.

Interpolator output:

$$X_i(e^{j\omega}) = H_i(e^{j\omega}) X_e(e^{j\omega}) \approx X(e^{j\omega}) \quad (\text{ideal})$$

Ideal interpolation filter:

$$H_i(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \pi/L \\ 0, & \pi/L < |\omega| \leq \pi \end{cases}$$

Gain justification: Scale by L so spectral amplitude matches new sampling rate:

$$L \cdot \frac{1}{T} = \frac{1}{T/L} = \frac{1}{T_i}.$$

Downsampling with Aliasing - Frequency Domain

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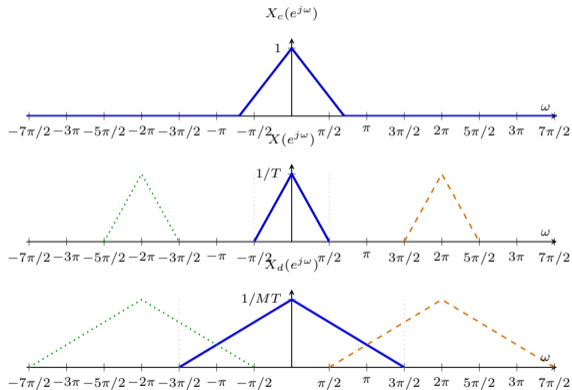
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Downsampling with Anti Aliasing - Frequency Domain

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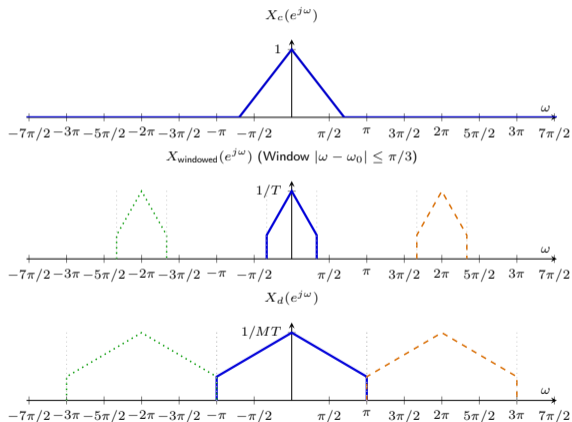
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Interpolation Filter Impulse Response

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Ideal lowpass interpolation filter (L -fold upsampling):

$$H_i(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \pi/L \\ 0, & \pi/L < |\omega| \leq \pi \end{cases}$$

Impulse response (sinc form):

$$h_i[n] = \frac{\sin(\pi n/L)}{\pi n/L}$$

Key Properties:

- $h_i[0] = 1$.
- $h_i[n] = 0$ for $n = \pm L, \pm 2L, \dots$ (zeros at integer multiples of L).
- Acts like ideal reconstruction between the nonzero locations of $x_e[n]$.

Output Samples:

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n - kL)/L]}{\pi(n - kL)/L}$$

For $n = mL$ (multiples of L):

$$x_i[mL] = x[m]$$

(Perfect retention of existing samples)

Derivation Recap

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Starting with expander output:

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL]$$

Filtering:

$$x_i[n] = (h_i * x_e)[n] = \sum_{m=-\infty}^{\infty} h_i[n - m] x_e[m] = \sum_{k=-\infty}^{\infty} h_i[n - kL] x[k]$$

Using $h_i[n]$ definition:

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin(\pi(n - kL)/L)}{\pi(n - kL)/L}$$

Matches continuous-time sampling at $T_i = T/L$:

$$x_i[n] = x_c(nT_i) = x_c\left(\frac{nT}{L}\right)$$

Conclusion: Ideal L -fold interpolation exactly reconstructs bandlimited signal at higher rate.

Example: $L = 3$ (Spectral View)

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Assume original $X(e^{j\omega})$ occupies $|\omega| \leq \pi/2$.

After expansion by $L = 3$:

$$X_e(e^{j\omega}) = X(e^{j3\omega})$$

Nonzero only for $|\omega| \leq \pi/6$ in principal lobe; other compressed images appear.

Interpolator H_i :

$$H_i(e^{j\omega}) = \begin{cases} 3, & |\omega| \leq \pi/3 \\ 0, & \pi/3 < |\omega| \leq \pi \end{cases}$$

Result:

$$X_i(e^{j\omega}) = H_i(e^{j\omega})X_e(e^{j\omega}) \approx X(e^{j\omega})$$

Bandwidth Relation: Needed cutoff $\pi/3$ ensures only baseband compressed copy passes plus gain correction.

Numerical Example

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Let $x[n] = \cos(0.25\pi n)$, upsample by $L = 4$.

Zero-stuff:

$$x_e[n] = \begin{cases} \cos(0.25\pi(n/4)), & n \equiv 0 \pmod{4} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \cos(0.0625\pi n), & n = 0, \pm 4, \pm 8, \dots \\ 0, & \text{else} \end{cases}$$

Filter with ideal interpolation $h_i[n]$ (cutoff $\pi/4$, gain 4) gives:

$$x_i[n] = \cos(0.25\pi n) \quad (\text{now evaluated at finer grid})$$

Energy / Amplitude: Gain 4 compensates for zero insertion dilution.

Practical: Use finite-length FIR approximation of $h_i[n]$ (windowed sinc) and polyphase decomposition for efficiency.

Practical Interpolation Filter Design

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Ideal Filter Not Implementable: Infinite-length sinc, noncausal.

Design Approaches:

- Windowed-sinc FIR (Kaiser, Hamming) with cutoff π/L .
- Parks–McClellan (equiripple) lowpass design specifying passband ripple and stopband attenuation.
- IIR filters (less common due to phase distortion concerns).

Gain Handling:

- Include factor L inside filter taps.
- Or multiply output after filtering (less efficient).

Performance Metrics:

- Passband ripple (affects amplitude accuracy of retained frequencies).
- Transition width (larger L often permits narrower filter).
- Stopband attenuation (controls interpolation imaging).

Alias and Imaging Considerations

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Assumption: Original $x[n]$ from alias-free sampling of bandlimited $x_c(t)$.

If Not Bandlimited:

- Interpolation filter will pass baseband AND distort edges.
- Imaging components (scaled copies) may leak if filter insufficient.

Quality Factors:

- Stopband attenuation sets rejection of compressed images.
- Passband flatness ensures accurate amplitude of restored spectrum.
- Group delay linearity affects waveform shape (phase-sensitive applications).

Preconditioning: Sometimes mild lowpass applied before interpolation to suppress high-frequency noise.

Upsampling vs. Downsampling (Comparison)

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- **Downsampling:** Bandwidth must shrink (filter then decimate). Aliasing risk if not filtered.
- **Upsampling:** Bandwidth must remain limited; expansion creates *images* requiring lowpass removal.
- **Filters:** Downsampling uses antialias filter (cutoff π/M). Upsampling uses interpolation filter (cutoff π/L).
- **Scaling:** Interpolation filter has gain L ; decimation filter usually unity gain in passband.
- **Artifacts:** Downsampling risk: aliasing; Upsampling risk: imaging.

Summary of Upsampling Concepts

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Core Steps:

$$x[n] \xrightarrow{\text{Insert } L-1 \text{ zeros}} x_e[n] \xrightarrow{h_i[n]} x_i[n]$$

Key Formula:

$$X_e(e^{j\omega}) = X(e^{j\omega L}), \quad H_i(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \pi/L \\ 0, & \text{else} \end{cases}$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n - kL)/L]}{\pi(n - kL)/L}$$

Conditions:

- Original sequence must be bandlimited (no alias in initial sampling).
- Interpolation filter approximates ideal lowpass.

Design Notes:

- Use finite-length FIR (windowed-sinc) or polyphase structure.
- Gain normalization critical (factor L).

Artifacts to Control:

- Imaging (spectral copies).

Practical Interpolation Checklist

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Given: Target upsampling factor L .

Checklist:

- 1 Confirm input bandlimit $|\omega| \leq \omega_B \leq \pi$.
- 2 Determine required cutoff: $\omega_c = \min(\pi/L, \omega_B)$.
- 3 Specify passband ripple (e.g. ± 0.1 dB) and stopband attenuation (e.g. ≥ 60 dB).
- 4 Design FIR filter taps; include scaling by L .
- 5 Implement via polyphase (split taps into L subfilters).
- 6 Validate amplitude and spectral suppression (FFT of output).
- 7 (Optional) Multistage interpolation if L factor large (factorization).

Performance Tips:

- Larger L shrinks required passband, enabling shorter filters.
- Multistage: reduce overall tap count vs single sharp filter.
- Fixed-point: ensure sufficient headroom after gain L .

Closing Remarks

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Upsampling (Interpolation) is the discrete-time dual of:

- D/C reconstruction (conceptually).
- Downsampling with antialias filter (role reversal of images vs aliases).

Takeaways:

- Zero insertion alone distorts spectrum (images).
- Proper lowpass filter restores original bandwidth and amplitude.
- Exact reconstruction relies on original bandlimited assumption.

Next Lecture: *Rational sampling-rate conversion* (cascade interpolate + decimate).