Sampling Rate Conversions: Downsampling

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Downsampling: Motivation

Why downsample?

- Reduce computation, memory, bandwidth, and power.
- Match device/IO rate constraints (sensors, radios, codecs).
- Enable multirate algorithms, filter banks.

Definition (Compressor / Downsampler):

$$x_d[n] = x[nM] = x_c(nMT) \quad \text{with} \quad T_d = MT$$

Equivalent to sampling the bandlimited reconstruction $x_c(t)$ with period T_d.

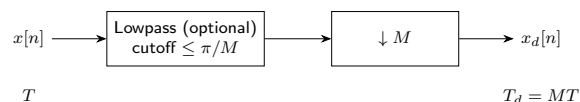# Downsampling: Aliasing Considerations

When is it alias-free?

- If $X_c(j\Omega) = 0$ for $|\Omega| \geq \Omega_N$, then $x_d[n]$ exactly represents $x_c(t)$ if

$$\frac{\pi}{T_d} = \frac{\pi}{MT} \geq \Omega_N \iff \omega_N \leq \frac{\pi}{M}.$$

- Interpretations:
  - Original sampling rate is at least M times the Nyquist rate, or
  - Prefilter to reduce bandwidth by factor M before downsampling (decimation).

Terminology:

- Compressor: just $\downarrow M$ (rate reduction).
- Decimator: antialiasing lowpass + $\downarrow M$.

$$x[n] \longrightarrow \boxed{\begin{array}{c}\text{Lowpass (optional)}\\ \text{cutoff} \leq \pi/M\end{array}} \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$$

$$T \qquad\qquad\qquad\qquad\qquad\qquad T_d = MT$$

Frequency-Domain Analysis

DTFT Relationship:
DTFT of $x[n] = x_c(nT)$:

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$

Downsampled sequence DTFT (with $T_d = MT$):

$$X_d(e^{j\omega}) = \frac{1}{T_d} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T_d} - j\frac{2\pi r}{T_d}\right)$$

Key Result – Relationship Between DTFTs:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

Interpretation:

- $X_d(e^{j\omega})$ is sum of M scaled, shifted copies of $X(e^{j\omega})$.
- Frequency axis compressed by factor M; shifts at $2\pi i/M$.
- Overall amplitude scaling $1/M$.

Derivation of DTFT Relationship

Starting point (sampling with $T_d = MT$):

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi r}{MT}\right)$$

Let $r = i + kM$ with $i = 0, 1, \ldots, M-1$:

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{i=0}^{M-1} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi(i+kM)}{MT}\right)$$
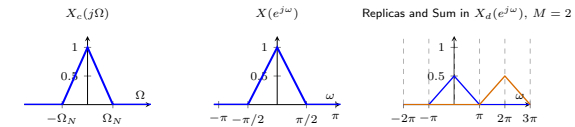
Group terms:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi i}{MT} - j\frac{2\pi k}{T}\right)$$

Recognize inner sum as $X\left(e^{j(\omega/M - 2\pi i/M)}\right)$. Therefore:

$$\boxed{X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)}$$
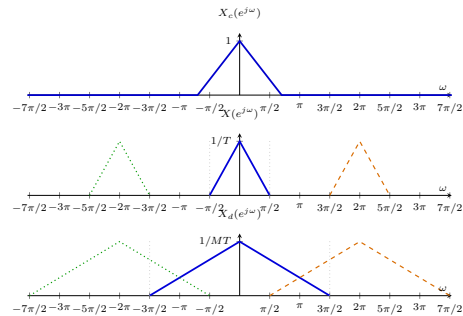
Downsampling Example: $M = 2$ (No Aliasing)

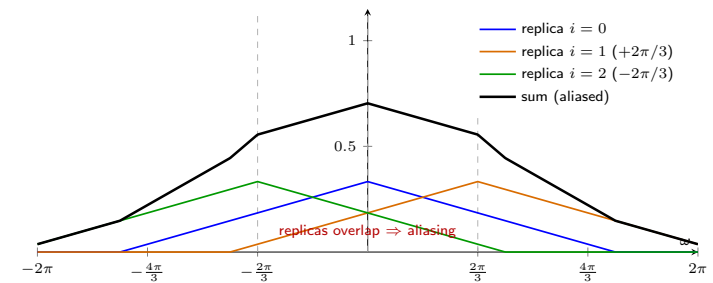Setup: Original sampling rate is twice the minimum Nyquist rate; bandwidth fits after compression.

Result: No aliasing because the compressed baseband fits in $[-\pi, \pi]$. For $M = 2$ with $\omega_N = \pi/2$, the baseband and the shifted replica (centered at 2π) do not overlap within $[-\pi, \pi]$.
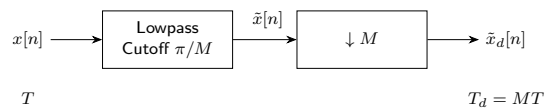
Downsampling with Aliasing - Frequency Domain

Downsampling with Aliasing: $M = 3$

Setup: Compression factor too large, spectral copies overlap.

replicas overlap $\Rightarrow$ aliasing

legend:
replica $i = 0$
replica $i = 1$ $(+2\pi/3)$
replica $i = 2$ $(-2\pi/3)$
sum (aliased)

Decimation: Prefiltering Before Downsampling

Problem: If original bandwidth exceeds π/M, downsampling aliases.
Solution: Apply a lowpass (antialiasing) filter of cutoff π/M before compression.

$$x[n] \longrightarrow \boxed{\begin{array}{c} \text{Lowpass} \\ \text{Cutoff } \pi/M \end{array}} \xrightarrow{\ \tilde{x}[n]\ } \boxed{\downarrow M} \longrightarrow \tilde{x}_d[n]$$

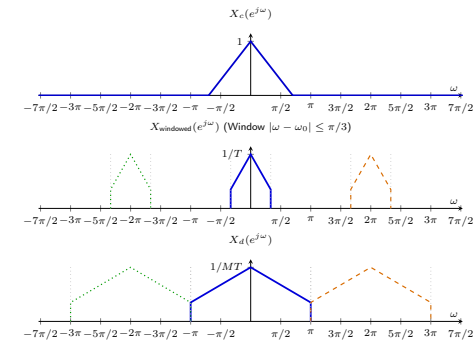$$T \qquad\qquad\qquad\qquad\qquad\qquad T_d = MT$$

Ideal filter:

$$\tilde{H}_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \pi/M \\ 0, & \pi/M < |\omega| \leq \pi \end{cases}$$

After filtering, $\tilde{x}[n]$ has bandwidth π/M, so $\tilde{x}_d[n] = \tilde{x}[nM]$ is alias-free.

Downsampling with Anti Aliasing - Frequency Domain

Summary

Key Concepts:

- Resampling changes sampling rate using discrete-time operations.
- Downsampling by M: $x_d[n] = x[nM]$.
- Compressor vs. Decimator: Decimator adds antialias filter.

Frequency-Domain Relationship:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

Aliasing Condition:

$$\omega_N \leq \frac{\pi}{M} \quad \text{(required for alias-free direct downsampling)}$$

If violated: prefilter to bandwidth π/M before rate reduction.

Sampling Rate Conversions: Upsampling (Interpolation)

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Motivation for Upsampling

Why increase the sampling rate?

- Interface between systems running at different sample rates (audio, telecom, sensors).
- Enable finer time resolution for subsequent processing (filtering, D/A conversion).

Goal: Given samples $x[n] = x_c(nT)$, produce samples

$$x_i[n] = x_c(nT_i), \qquad T_i = \frac{T}{L}$$

so that the new sampling rate is L times larger.

Upsampling: Definition and Approach

Definition (Upsampling / Interpolation):

$$x_i[n] = x\left[\frac{n}{L}\right] = x_c\left(\frac{nT}{L}\right), \quad n = 0, \pm L, \pm 2L, \dots$$

At integer multiples of L, we have the original samples. Elsewhere, we need intermediate values.

Problem: Need the "missing" samples while preserving bandlimited structure.

Solution Outline:

1. **Expand:** Insert $L-1$ 'null' samples between each original sample (creates $x_e[n]$).
2. **Interpolate:** Lowpass filter with gain L and cutoff π/L to reconstruct intermediate samples.

This process preserves the bandlimited nature of the signal.

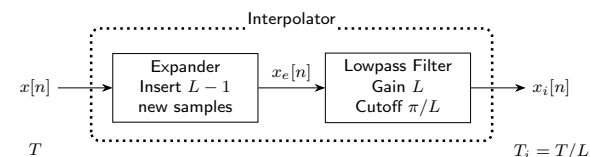## Expander and Interpolator Structure

Expander (Sampling Rate Expander):

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases} \quad \Longleftrightarrow \quad x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n - kL]$$

Interpolator: Lowpass filter $h_i[n]$ (gain L, cutoff π/L) applied to $x_e[n]$:

$$x_i[n] = (h_i * x_e)[n]$$

System Diagram:

Terminology: Expander + Antialias (Interpolation) Filter = **Interpolator**.

Frequency-Domain Effect of Expansion: DTFT of $x_e[n]$

Expander definition:

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases} \iff x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

Compute the DTFT of $x_e[n]$:

$$
\begin{aligned}
X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_e[n]\,e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL] \right) e^{-j\omega n} \\
&= \sum_{k=-\infty}^{\infty} x[k]\,e^{-j\omega(kL)} \\
&= \sum_{k=-\infty}^{\infty} x[k]\,e^{-j(\omega L)k} \\
&= X(e^{j\,\omega L})
\end{aligned}
$$

Frequency-Domain Effect of Expansion: Interpretation and Filter

Result:

$$X_e(e^{j\omega}) = X(e^{j\omega L})$$

Interpretation:

- Baseband spectrum is **compressed** by factor L.
- Multiple frequency-scaled images appear within $|\omega| \leq \pi$ due to DTFT periodicity.
- An interpolation lowpass (cutoff π/L) removes images and rescales amplitude.

Interpolator output:

$$X_i(e^{j\omega}) = H_i(e^{j\omega})\,X_e(e^{j\omega}) \approx X(e^{j\omega}) \quad \text{(ideal)}$$
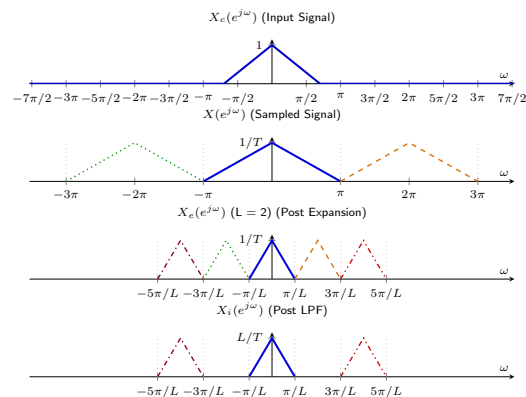
Ideal interpolation filter:

$$H_i(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \pi/L \\ 0, & \pi/L < |\omega| \leq \pi \end{cases}$$

Gain justification: Scale by L so spectral amplitude matches new sampling rate:

$$L \cdot \frac{1}{T} = \frac{1}{T/L} = \frac{1}{T_i}.$$

Interpolation - Frequency Domain

Example: Upsampling with $L = 3$

Assume original $X(e^{j\omega})$ occupies $|\omega| \leq \pi/2$.
After expansion by $L = 3$:

$$X_e(e^{j\omega}) = X(e^{j3\omega})$$

Nonzero only for $|\omega| \leq \pi/6$ in principal lobe; other compressed images appear.
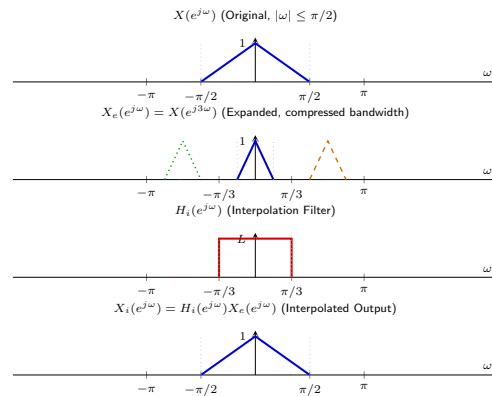Interpolator H_i:

$$H_i(e^{j\omega}) = \begin{cases} 3, & |\omega| \leq \pi/3 \\ 0, & \pi/3 < |\omega| \leq \pi \end{cases}$$

Result:

$$X_i(e^{j\omega}) = H_i(e^{j\omega})X_e(e^{j\omega}) \approx X(e^{j\omega})$$

Bandwidth Relation: Needed cutoff $\pi/3$ ensures only baseband compressed copy passes. Requires gain correction (Gain $= 3$).

Example: $L = 3$ (Frequency Domain Visualization)

$X(e^{j\omega})$ (Original, $|\omega| \leq \pi/2$)

$X_e(e^{j\omega}) = X(e^{j3\omega})$ (Expanded, compressed bandwidth)

$H_i(e^{j\omega})$ (Interpolation Filter)

$X_i(e^{j\omega}) = H_i(e^{j\omega})X_e(e^{j\omega})$ (Interpolated Output)

Summary of Upsampling Concepts

Core Steps:

$$x[n] \xrightarrow{\text{Insert } L-1 \text{ 'null' samples}} x_e[n] \xrightarrow{h_i[n]} x_i[n]$$

Key Formulas:

$$X_e(e^{j\omega}) = X(e^{j\omega L})$$

$$H_i(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \pi/L \\ 0, & \text{else} \end{cases}$$

Conditions:

- Original sequence must be bandlimited (no alias in initial sampling).
- Interpolation filter approximates ideal lowpass.