

# Linear Constant-Coefficient Difference Equations

Maxx Seminario

University of Nebraska-Lincoln

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# Introduction to Difference Equations

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Introduction

Accumulator  
System

Moving Average  
System

General Solution

Particular  
Solution

Summary

- **Important Class of LTI Systems:** Systems where input  $x[n]$  and output  $y[n]$  satisfy a difference equation
- **General Form:**

$$\sum_{k=0}^N a_k y[n - k] = \sum_{m=0}^M b_m x[n - m]$$

- **Parameters:**
  - $N$ : Order of the system (highest delay in output)
  - $a_k$ : Output coefficients (constant)
  - $b_m$ : Input coefficients (constant)
  - $M$ : Highest delay in input terms
- **Why Important?:**
  - Provides computational algorithms for LTI systems
  - Foundation for digital filter implementation
  - Connects time-domain and system analysis

# Example 1: The Accumulator System

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**Problem:** Find the difference equation for the accumulator system.

**System Definition:**

$$y[n] = \sum_{k=-\infty}^n x[k]$$

**Approach:**

- Rewrite the sum to separate current and past inputs
- Use the relationship between  $y[n]$  and  $y[n - 1]$

# Example 1: Accumulator Solution

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$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

**Step 2:** Recognize that the sum is  $y[n - 1]$ :

$$y[n - 1] = \sum_{k=-\infty}^{n-1} x[k]$$

**Step 3:** Substitute to get the difference equation:

$$y[n] = x[n] + y[n - 1]$$

**Standard Form:**

$$y[n] - y[n - 1] = x[n]$$

# Block Diagram: Recursive Accumulator

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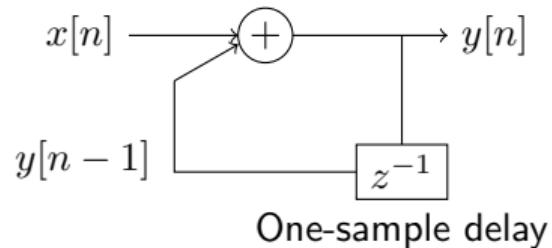
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## Recursive Implementation:

- Each output value computed using previously computed values
- $y[n] = x[n] + y[n - 1]$ : Add current input to previous output
- Requires initial condition (e.g.,  $y[-1] = 0$ )

# Example 2: Moving Average System

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**Problem:** Find difference equation for causal moving average system.

**System Definition** (with  $M_1 = 0$ , so the system is causal):

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n - k]$$

**Two Approaches:**

- 1 **Direct (Non-recursive):** Use convolution form directly
- 2 **Recursive:** Express as cascade of simpler systems

# Moving Average: Direct Implementation

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## Direct Form:

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n - k]$$

## Standard Form:

$$y[n] = \sum_{k=0}^{M_2} \frac{1}{M_2 + 1} x[n - k]$$

## Disadvantages of Direct Implementation:

- Requires  $(M_2 + 1)$  multiplications per output sample
- Must store  $(M_2 + 1)$  input samples in memory
- $O(M_2)$  : Computational cost grows linearly with window size  $M_2$

# Block Diagram: Recursive Moving Average

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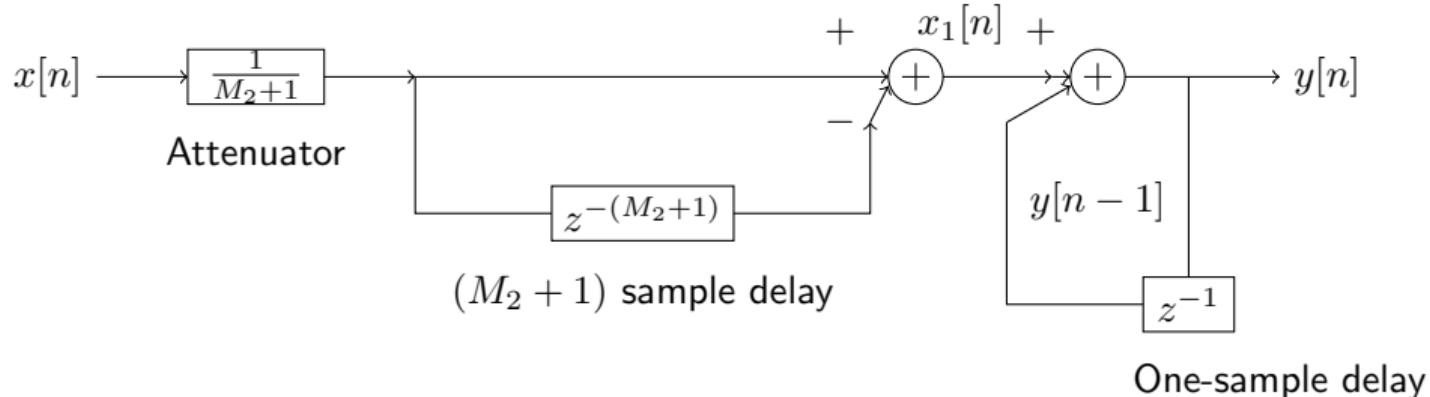
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## Signal Flow:

- Input  $x[n] \rightarrow$  Attenuator  $\frac{1}{M_2+1}$
- Attenuated signal splits: direct path and  $(M_2 + 1)$  sample delay
- Sum:  $x_1[n] = \frac{1}{M_2+1}[x[n] - x[n - (M_2 + 1)]]$
- $x_1[n] \rightarrow$  Accumulator  $\rightarrow$  Output  $y[n]$

# Moving Average as Difference Equation

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**Intermediate signal:**

$$x_1[n] = \frac{1}{M_2 + 1} [x[n] - x[n - M_2 - 1]]$$

**Accumulator relation:**  $y[n] = x_1[n] + y[n - 1]$

**Final recursive form:**

$$y[n] - y[n - 1] = \frac{1}{M_2 + 1} [x[n] - x[n - M_2 - 1]]$$

Note: There is an unlimited number of distinct difference equations to represent an LTI I/O relation.

# General Solution of Difference Equations

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**Key Issue:** Difference equation alone does not uniquely specify output!

**General Solution Structure:**

$$y[n] = y_p[n] + y_h[n]$$

- $y_p[n]$ : Particular solution (satisfies original equation)
- $y_h[n]$ : Homogeneous solution (satisfies homogeneous equation)

**Homogeneous Equation ( $x[n] = 0$ ):**

$$\sum_{k=0}^N a_k y_h[n - k] = 0$$

**Homogeneous Solution Form:**

$$y_h[n] = \sum_{m=1}^N A_m z_m^n$$

where  $z_m$  are roots of characteristic polynomial  $A(z) = \sum_{k=0}^N a_k z^{-k} = 0$

# Auxiliary Conditions

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## Need for Auxiliary Conditions:

- $N$  undetermined coefficients  $A_m$  in homogeneous solution
- Need  $N$  auxiliary (boundary) conditions for unique solution

## Types of Auxiliary Conditions:

- 1 **Fixed Values:** Specify  $y[-1], y[-2], \dots, y[-N]$
- 2 **Initial Rest:** If  $x[n] = 0$  for  $n < n_0$ , then  $y[n] = 0$  for  $n < n_0$

# Forward and Backward Computation for Specific Class of Difference Equations

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## Specific Class of Difference Equations:

- Inputs  $x[n]$  and Outputs  $y[n]$  satisfy an  $N$ th-order linear constant coefficient difference equation:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

## Recursive Computation (forward):

$$y[n] = -\sum_{k=1}^N \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^M \frac{b_k}{a_0} x[n-k]$$

## Recursive Computation (backward):

$$y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^M \frac{b_k}{a_N} x[n-k]$$

# Example 3: First-Order Difference Equation

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**Problem:** Solve the difference equation with given input and initial condition.

**Difference Equation:**

$$y[n] = ay[n - 1] + x[n]$$

**Input:**

$$x[n] = K\delta[n]$$

**Auxiliary Condition:**

$$y[-1] = c$$

**Find:** The complete solution  $y[n]$  for all  $n$ .

# Summary: Linear Constant-Coefficient Difference Equations

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## ■ Definition:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

Describes the relationship between input  $x[n]$  and output  $y[n]$  in LTI systems.

## ■ Key Components:

- $N$  and  $M$ : Order of the system
- $a_k, b_m$ : Constant coefficients
- Recursive and direct computation methods

## ■ General Solution:

$$y[n] = y_p[n] + y_h[n]$$

- $y_p[n]$ : Particular solution (due to input  $x[n]$ )
- $y_h[n]$ : Homogeneous solution (due to initial conditions)
- Auxiliary conditions: Necessary to determine a unique / particular solutions.

- **Applications:** Digital filters, computational algorithms for LTI systems, and time-domain analysis.