Frequency Domain Representation of Discrete Systems

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Introduction to Frequency-Domain Analysis

- **Multiple Signal Representations**:
  - Time-domain: Weighted sum of delayed impulses
  - Frequency-domain: Weighted sum of sinusoids/complex exponentials
- **Why Complex Exponentials?**
  - Complex exponentials are *eigenfunctions* of LTI systems
  - Input: $e^{j\omega n}$ $\rightarrow$ Output: $H(e^{j\omega})e^{j\omega n}$
  - Sinusoidal input $\rightarrow$ Sinusoidal output (same frequency)
- **Fundamental Property**:
  - LTI systems preserve frequency of sinusoidal inputs
  - Only amplitude and phase change
  - Changes determined by system's frequency response

Review: Eigenfunctions and Eigenvalues

Definition: For a linear operator $\mathcal{T}$, a function $\phi(t)$ is an **eigenfunction** if:

$$\mathcal{T}\{\phi(t)\} = \lambda\phi(t)$$

where λ is the corresponding **eigenvalue**.

Key Properties:

- Operator transforms eigenfunction into a scaled version of itself
- Shape is preserved, only amplitude changes
- Eigenvalue λ determines the scaling factor

Why This Matters:

- Transform complicated operations into simple scaling
- Solving differential/difference equations becomes algebraic

Eigenfunctions for LTI Systems

Key Concept: Complex exponentials are eigenfunctions of LTI systems.

Input-Output Relation:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = H(e^{j\omega})e^{j\omega n}$$

Interpretation:

- $e^{j\omega n}$ is an **eigenfunction**
- $H(e^{j\omega})$ is the corresponding **eigenvalue**
- $H(e^{j\omega})$ describes amplitude/phase change as function of frequency

Complex Representation:

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$

Example 1: Ideal Delay System

System Definition:

$$y[n] = x[n - n_d]$$

where n_d is a fixed integer delay.

Method 1 - Direct Substitution: For input $x[n] = e^{j\omega n}$:

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d} \cdot e^{j\omega n}$$

Therefore: $H(e^{j\omega}) = e^{-j\omega n_d}$

Method 2 - Using Impulse Response: Impulse response: $h[n] = \delta[n - n_d]$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}$$

Example 1: Ideal Delay Analysis

Frequency Response: $H(e^{j\omega}) = e^{-j\omega n_d} = \cos(\omega n_d) - j\sin(\omega n_d)$

Real and Imaginary Parts:

$$H_R(e^{j\omega}) = \cos(\omega n_d) \tag{1}$$

$$H_I(e^{j\omega}) = -\sin(\omega n_d) \tag{2}$$

Magnitude and Phase:

$$|H(e^{j\omega})| = 1 \tag{3}$$

$$\angle H(e^{j\omega}) = -\omega n_d \tag{4}$$

Interpretation:

- **Magnitude = 1**: No amplitude change at any frequency
- **Phase =** $-\omega n_d$: Linear phase (pure delay)

Superposition Principle

Signal Representation: If we can represent a signal as:

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}$$

System Output: By linearity and the eigenfunction property:

$$y[n] = \sum_k \alpha_k H(e^{j\omega_k}) e^{j\omega_k n}$$

Key Insight:

- Each frequency component is processed independently
- System acts as a "filter" for different frequencies
- Only need to know $H(e^{j\omega})$ at frequencies ω_k

This is the foundation for:

- Fourier analysis
- Filter design
- Spectral analysis

Example 2: Sinusoidal Response

Input: $x[n] = A\cos(\omega_0 n + \phi)$

Step 1 - Express using complex exponentials:

$$x[n] = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

Step 2 - Apply superposition:

$$y_1[n] = H(e^{j\omega_0})\frac{A}{2}e^{j\phi}e^{j\omega_0 n} \tag{5}$$

$$y_2[n] = H(e^{-j\omega_0})\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n} \tag{6}$$

Step 3 - Total response:

$$y[n] = \frac{A}{2}\left[H(e^{j\omega_0})e^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})e^{-j\phi}e^{-j\omega_0 n}\right]$$

Example 2: Final Result

For real impulse response: $H(e^{-j\omega_0}) = H^*(e^{j\omega_0})$

Simplified output:

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta)$$

where $\theta = \angle H(e^{j\omega_0})$

Key Results:

- **Frequency preserved**: Output has same frequency ω_0
- **Amplitude scaled**: By factor $|H(e^{j\omega_0})|$
- **Phase shifted**: By angle $\theta = \angle H(e^{j\omega_0})$

Periodicity of Frequency Response

Fundamental Property: $H(e^{j\omega})$ is periodic with period 2π.

Proof:

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n}$$

Since $e^{-j2\pi n} = 1$ for integer n:

$$e^{-j(\omega+2\pi)n} = e^{-j\omega n}e^{-j2\pi n} = e^{-j\omega n}$$

Therefore: $H(e^{j(\omega+2\pi)}) = H(e^{j\omega})$

Why this occurs:

- Sequences $\{e^{j\omega n}\}$ and $\{e^{j(\omega+2\pi)n}\}$ are identical (in discrete time!)
- System must respond identically to identical inputs
- Frequencies ω and $\omega + 2\pi$ are indistinguishable

Frequency Response Specification

Key Consequence: Only need to specify $H(e^{j\omega})$ over one period!

Common Choices:

- $0 \leq \omega \leq 2\pi$
- $-\pi < \omega \leq \pi$ (most common)

Frequency Interpretation:

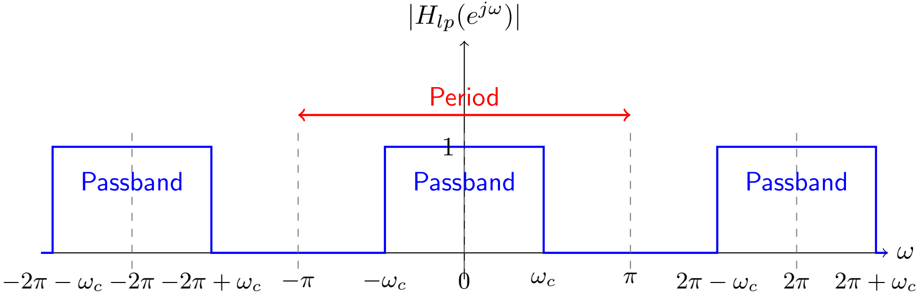
- **Low frequencies**: Close to $\omega = 0$ (or even multiples of π)
- **High frequencies**: Close to $\omega = \pm\pi$ (or odd multiples of π)

Important Note:

- This is different from continuous-time systems
- Continuous-time: $H(j\Omega)$ defined for all Ω
- Discrete-time: $H(e^{j\omega})$ periodic with period 2π
- Digital frequency ω is normalized (radians per sample)

Ideal Frequency-Selective Filters

Ideal Lowpass Filter:

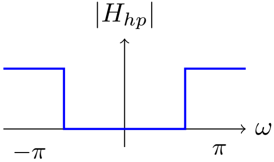


Properties:

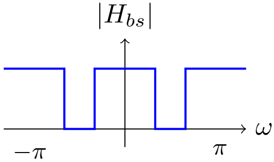
- Passbands: $|\omega - 2\pi k| \leq \omega_c$ for integer k
- **Periodic with period** 2π: $H(e^{j(\omega+2\pi)}) = H(e^{j\omega})$
- Each period has identical rectangular passband around multiples of 2π

Other Ideal Filters

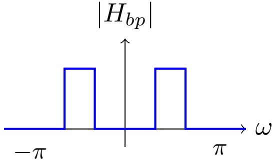
Highpass



Bandstop



Bandpass



Filter Types:

- **Highpass**: Passes high frequencies, rejects low frequencies
- **Bandstop (Notch)**: Rejects frequencies in a band
- **Bandpass**: Passes frequencies in a band, rejects others

Note: All responses are periodic with period 2π

Example 3: Moving Average System

System Definition (causal, $M_1 = 0$):

$$h[n] = \begin{cases} \frac{1}{M_2+1}, & 0 \leq n \leq M_2 \\ 0, & \text{otherwise} \end{cases}$$

Frequency Response:

$$H(e^{j\omega}) = \frac{1}{M_2+1} \sum_{n=0}^{M_2} e^{-j\omega n}$$

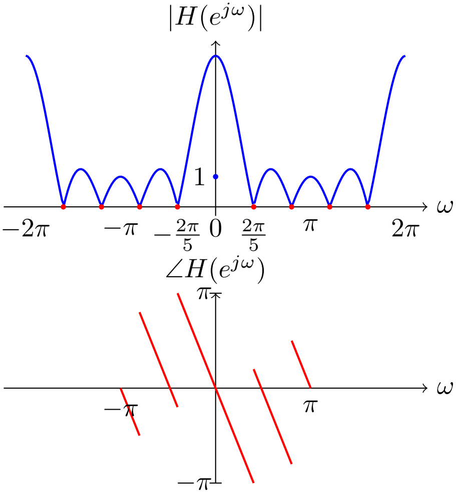
Using Geometric Series Formula:

$$H(e^{j\omega}) = \frac{1}{M_2+1} \frac{1 - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}}$$

Simplified Form:

$$H(e^{j\omega}) = \frac{1}{M_2+1} \frac{\sin[\omega(M_2+1)/2]}{\sin(\omega/2)} e^{-j\omega M_2/2}$$

Moving Average: Frequency Response Analysis



For $M_1 = 0, M_2 = 4$:

Key Observations:

- **Lowpass character**: Attenuates high frequencies
- **Linear phase**: $\angle H(e^{j\omega}) = -2\omega \bmod \frac{2\pi}{5}$
- **Phase jumps**: At frequencies where $\sin(5\omega/2) = 0$

Moving Average: Physical Interpretation

Why does it act like a lowpass filter?

Time Domain View:

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

- Averages $(M_2 + 1)$ consecutive samples
- Rapid variations (high frequencies) tend to cancel out
- Slow variations (low frequencies) are preserved
- Smoothing effect on the input signal

Frequency Domain Confirmation:

- $|H(e^{j\omega})|$ maximum at $\omega = 0$ (DC)
- $|H(e^{j\omega})|$ decreases as ω increases
- First zero at $\omega = \frac{2\pi}{M_2+1}$
- Higher frequencies are attenuated

Summary: Key Concepts

- **Eigenfunction Property**:

$$e^{j\omega n} \rightarrow H(e^{j\omega})e^{j\omega n}$$

- **Frequency Response**:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- **Periodicity**: $H(e^{j(\omega+2\pi)}) = H(e^{j\omega})$
- **Sinusoidal Response**:

$$A\cos(\omega_0 n + \phi) \rightarrow A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \angle H(e^{j\omega_0}))$$

- **Filter Design**: Use $H(e^{j\omega})$ to shape frequency content
- **System Analysis**: Frequency response reveals system behavior

Next Topics: Discrete-Time Fourier Transform (DTFT), Z-Transform