The Inverse z-Transform

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Overview: The Inverse z-Transform

- **Motivation**:
  - Need to move between time-domain and z-domain representations
  - Analysis often involves finding z-transform, manipulating, then inverting
  - Essential for discrete-time signal and system analysis
- **Formal Definition**:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

  where C is a closed contour within the ROC
- **Practical Methods**:
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

Inspection Method

Concept: Recognize common transform pairs "by inspection"

Some Transform Pairs:

- $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|$
- $-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| < |a|$

Example: $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

- If $|z| > \frac{1}{2}$: $x[n] = \left(\frac{1}{2}\right)^n u[n]$
- If $|z| < \frac{1}{2}$: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

Key Point: ROC determines which sequence! Same $X(z)$ can represent different sequences.

Partial Fraction Expansion: Overview

For Rational z-Transforms:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

General Procedure:

1. Factor denominator to find poles d_k
2. Determine expansion form:
   - $M < N$: $X(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$
   - $M \geq N$: $X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$
3. Calculate coefficients: $A_k = (1 - d_k z^{-1})X(z)\Big|_{z=d_k}$
4. Use ROC to determine sequence type:
   - Poles inside inner ROC boundary $\rightarrow$ right-sided
   - Poles outside outer ROC boundary $\rightarrow$ left-sided

Partial Fractions: Simple Poles Example

Example:
$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z| > \frac{1}{2}$

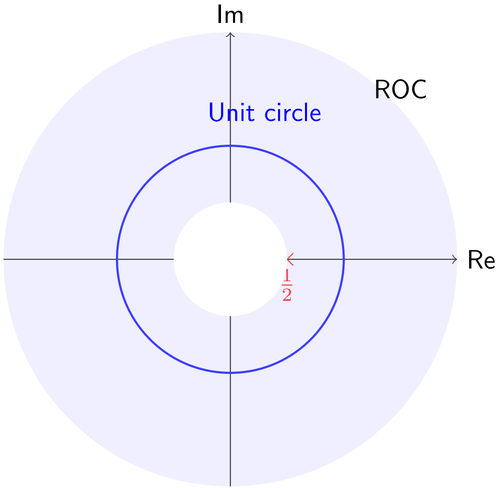
Partial fraction expansion:

$$X(z) = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

Since ROC is $|z| > \frac{1}{2}$: Both poles inside ROC $\rightarrow$ right-sided sequences

Result:

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$



Partial Fractions: Case $M \geq N$

Example: $X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}, \quad |z| > 1$

Long division yields: $B_0 = 2$, remainder $= 5z^{-1} - 1$

Expansion:

$$X(z) = 2 + \frac{-9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

Since ROC is $|z| > 1$: All sequences are right-sided

Result:

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Power Series Expansion

Direct from Definition: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

Example 1:

- $X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1}) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$
- $x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$

Example: Complete Inverse Transform

Given:

$$X(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})}$$

ROC: $\frac{1}{4} < |z| < \frac{1}{2}$

Partial fractions:

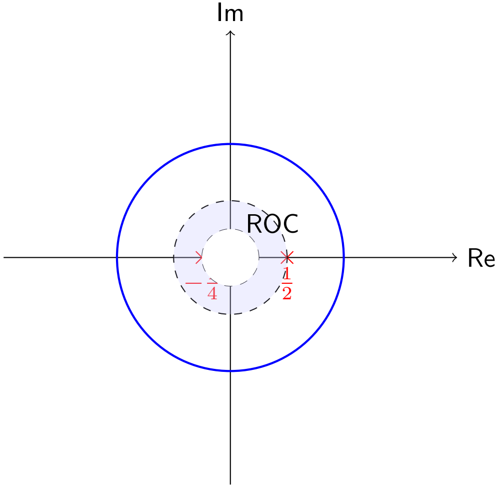
$$X(z) = \frac{3}{1 - \frac{1}{2}z^{-1}} + \frac{-2}{1 + \frac{1}{4}z^{-1}}$$

ROC analysis:

- Pole at $\frac{1}{2}$ outside ROC $\rightarrow$ left-sided
- Pole at $-\frac{1}{4}$ inside ROC $\rightarrow$ right-sided

Result:

$$x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] - 2\left(-\frac{1}{4}\right)^n u[n]$$



Summary of Inverse z-Transform Methods

1. **Inspection Method**:
   - Best for simple, recognizable forms
2. **Partial Fraction Expansion**:
   - Most useful for rational functions
   - Systematic procedure for simple and multiple poles
   - ROC critical for determining sequence type
3. **Power Series Expansion**:
   - Direct from definition

Key Principle: ROC determines sequence type

- Poles inside inner boundary $\rightarrow$ right-sided sequences
- Poles outside outer boundary $\rightarrow$ left-sided sequences