Discrete Time Signals

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What is a Signal?

- A **signal** conveys information about the state or behavior of a physical system.
- Signals can be synthesized for communication between humans or between humans and machines.
- Mathematically, signals are functions of one or more independent variables (e.g., time, space).
- Example: **Speech signal** is a function of time.
  Image is a function of two spatial variables (brightness).

Signals: Continuous vs. Discrete

- The independent variable of a signal can be:
  - **Continuous** (Continuous-time or analog signals)
  - **Discrete** (Discrete-time signals)
- Amplitude can also be:
  - **Continuous**
  - **Discrete**
- **Digital signals:** Both time and amplitude are discrete.

Types of Signal Processing Systems

- **Continuous-time systems:** Input and output are continuous-time signals.
- **Discrete-time systems:** Input and output are discrete-time signals.
- **Digital systems:** Input and output are digital signals (discrete in both time and amplitude).
- **Digital signal processing (DSP):** Concerned with transforming signals that are discrete in both amplitude and time.

Discrete-Time Signals as Sequences

- Discrete-time signals are sequences of numbers:

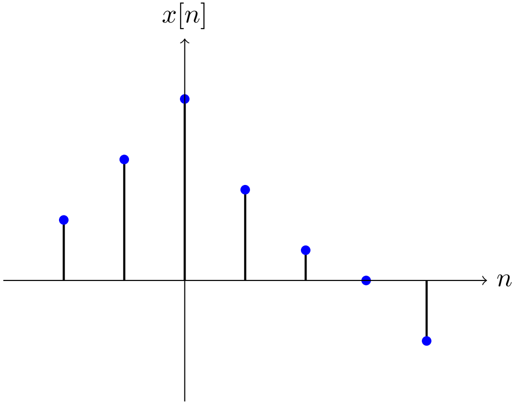
$$x = \{x[n]\}, \quad -\infty < n < \infty$$

- n is an integer; $x[n]$ is the nth sample.
- Often arise from sampling a continuous-time signal $x_a(t)$:

$$x[n] = x_a(nT)$$

- T is the sampling period; its reciprocal is the sampling frequency.

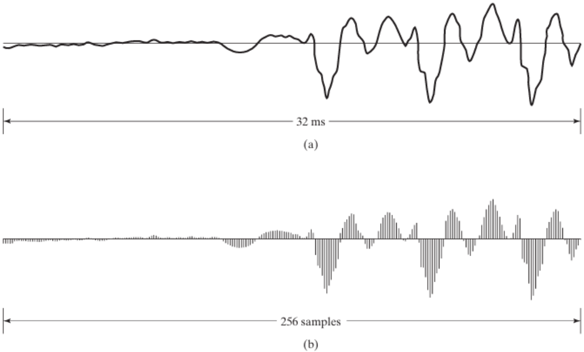
Graphical Representation of a Sequence



- $x[n]$ is defined only for integer values of n.
- Not defined for non-integer values.

Sampling: From Analog to Discrete

Continuous-time (Analog) Signal

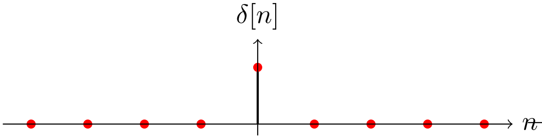


- Sampling theorem: original analog signal can be reconstructed if sampled frequently enough.

Basic Discrete-Time Sequences: Impulse and Step

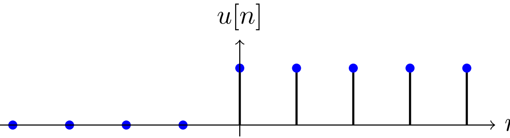
Unit Sample (Impulse) Sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Unit Step Sequence

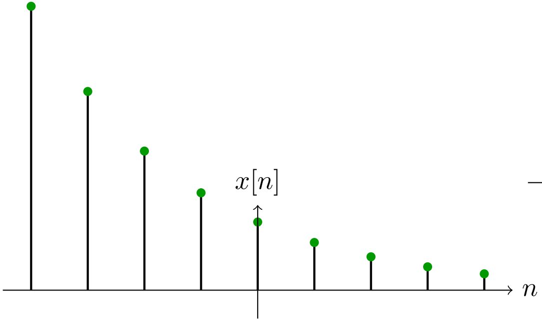
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Basic Discrete-Time Sequences: Exponential and Sinusoidal

Exponential Sequence

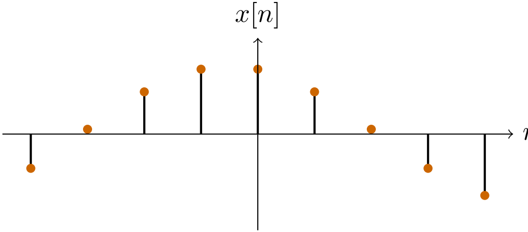
$$x[n] = A\alpha^n$$



Example: $A = 1.2$, $\alpha = 0.7$

Sinusoidal Sequence

$$x[n] = A\cos(\omega_0 n + \phi)$$



Example: $A = 1.2$, $\omega_0 = 0.6$, $\phi = 0.3$

Impulse Representation of Sequences

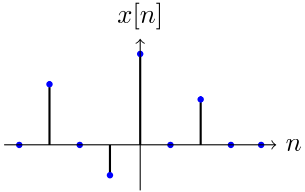
- Any sequence can be written as a sum of scaled, delayed impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

- Example with a random sequence:

$$x[n] = 2\delta[n+3] - \delta[n+1] + 3\delta[n] + 1.5\delta[n-2]$$

n	-4	-3	-2	-1	0	1	2	3	4
$x[n]$	0	2	0	-1	3	0	1.5	0	0



Relations Between Impulse and Step Sequences

- The unit step is the sum of all impulses up to n:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

- Or, as sum of delayed impulses:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

- The impulse is the first backward difference of the unit step:

$$\delta[n] = u[n] - u[n-1]$$

Exponential and Sinusoidal Sequences

- **Real exponential:** $x[n] = A\alpha^n$ (growth/decay depends on $|\alpha|$)
- **Complex exponential:**

$$x[n] = |A||\alpha|^n e^{j(\omega_0 n + \phi)}$$

- **Sinusoidal:**

$$x[n] = |A| \cos(\omega_0 n + \phi)$$

- Key property: In discrete-time, $e^{j(\omega_0 + 2\pi)n} = e^{j\omega_0 n}$
- Need only consider frequencies in $-\pi < \omega_0 \leq \pi$ (or $0 \leq \omega_0 < 2\pi$)

Periodicity in Discrete-Time Sinusoids

- A discrete-time sequence is periodic if $x[n] = x[n+N]$ for all n, where N is a positive integer (**the period**).
- For the sinusoid $x[n] = A\cos(\omega_0 n + \phi)$, periodicity requires that:

$$\omega_0 N = 2\pi k, \quad \text{where } k \text{ is an integer}$$

- A similar statement holds for the complex exponential sequence $x[n] = Ce^{j\omega_0 n}$:

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

- **Conclusion:** Not all discrete-time sinusoids or complex exponentials are periodic, and their periodicity depends on whether the frequency ω_0 satisfies the above condition for some integer N.

Example 1: Discrete-Time Sinusoid $x_1[n]$

- Consider the sequence, what is the period?

$$x_1[n] = \cos\left(\frac{\pi n}{4}\right)$$

Example 1 Result: Discrete-Time Sinusoid $x_1[n]$

- $x_1[n]$ is periodic with period $N = 8$.
- Proof:

$$x_1[n+8] = \cos\left(\frac{\pi(n+8)}{4}\right) = \cos\left(\frac{\pi n}{4} + 2\pi\right) = \cos\left(\frac{\pi n}{4}\right) = x_1[n]$$

Example 2: Discrete-Time Sinusoid $x_2[n]$

- Now consider the sequence, what is the period?

$$x_2[n] = \cos\left(\frac{3\pi n}{8}\right)$$

Example 2 Result: Discrete-Time Sinusoid $x_2[n]$

- $x_2[n]$ has a higher frequency but is *not* periodic with period 8.
- $$x_2[n+8] = \cos\left(\frac{3\pi(n+8)}{8}\right) = \cos\left(\frac{3\pi n}{8} + 3\pi\right) = -x_2[n]$$
- $x_2[n]$ has period $N = 16$.

Non-Periodic Discrete-Time Sinusoids

- **Some discrete-time sinusoids are not periodic at all.**
- **Example:**

$$x_3[n] = \cos(n)$$

- For periodicity, we require that for some integer N: $x_3[n] = x_3[n+N]$ for all n, which implies

$$\cos(n) = \cos(n+N) \implies N = 2\pi k \text{ for some integer } k$$

- However, 2π is not a rational multiple of 1, so there is **no integer** N that satisfies this condition.
- **Conclusion:** $x_3[n]$ never exactly repeats itself. This aperiodicity is unique to discrete-time sinusoids with "irrational" frequencies (relative to 2π).
- **Key Point:** In discrete time, only certain frequencies yield periodic sinusoids; others are aperiodic because the period would have to be a non-integer.

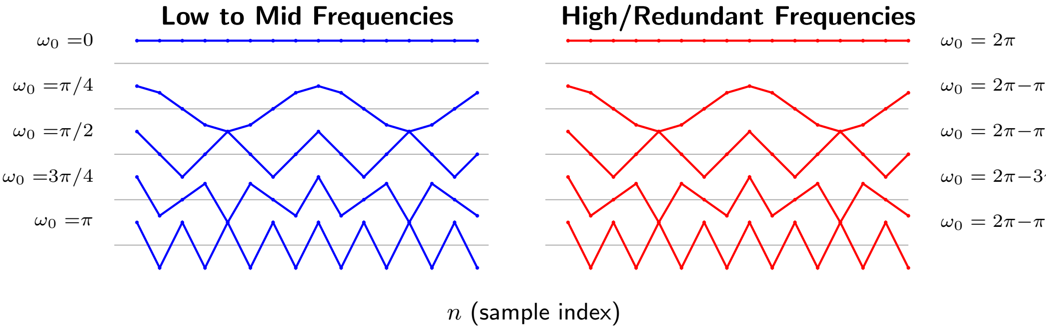
Distinct Frequencies and Periodicity in Discrete-Time

- ω_0 and $(\omega_0 + 2\pi r)$ are indistinguishable for integer r.
- There are exactly N distinguishable frequencies for sequences periodic with period N:

$$\omega_k = \frac{2\pi k}{N}, \quad k = 0, 1, \ldots, N-1$$

- These properties are fundamental for discrete-time Fourier analysis.

Indistinguishable Discrete-Time Frequencies



Each row compares signals with frequencies ω_0 and $2\pi - \omega_0$. Blue and red lines connect the points of each sinusoid for better visibility.

Oscillation Rate vs. Frequency in Discrete-Time

- As ω_0 increases from 0 toward π, oscillations become more rapid.
- As ω_0 increases from π toward 2π, oscillations slow down.
- Due to periodicity, $\omega_0 = 2\pi$ is identical to $\omega_0 = 0$.

Summary: Discrete-Time Signals & Systems

- **Signals:** Functions that convey information; can be continuous or discrete in time and amplitude.
- **Discrete-Time Signals:** Sequences $x[n]$, typically obtained by sampling a continuous signal.
- **Basic Sequences:**
  - Impulse ($\delta[n]$), step ($u[n]$), exponential, and sinusoidal sequences.
  - Any sequence can be expressed as a sum of scaled, shifted impulses.
- **Impulse and Step Relations:**
  - $u[n]$ is the running sum of $\delta[n]$; $\delta[n]$ is the difference of $u[n]$.
- **Sinusoids in Discrete Time:**
  - Periodicity depends on frequency: only some discrete-time sinusoids are periodic.
  - Frequencies separated by 2π are indistinguishable (aliasing effect).
  - Only N unique frequencies for period-N sequences.
- **Oscillation Rate:** Oscillation increases from 0 to π; slows from π to 2π.