Sampling Rate Conversions: Downsampling

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Fall 2025

Downsampling: Motivation

Why downsample?

- Reduce computation, memory, bandwidth, and power.
- Match device/IO rate constraints (sensors, radios, codecs).
- Enable multirate algorithms, filter banks.

Definition (Compressor / Downsampler):

$$x_d[n] = x[nM] = x_c(nMT) \quad \text{with} \quad T_d = MT$$

Equivalent to sampling the bandlimited reconstruction $x_c(t)$ with period T_d.

Downsampling: Aliasing Considerations

When is it alias-free?

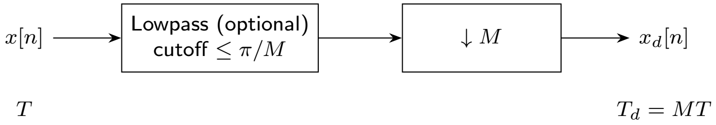
- If $X_c(j\Omega) = 0$ for $|\Omega| \geq \Omega_N$, then $x_d[n]$ exactly represents $x_c(t)$ if

$$\frac{\pi}{T_d} = \frac{\pi}{MT} \geq \Omega_N \iff \omega_N \leq \frac{\pi}{M}.$$

- Interpretations:
  - Original sampling rate is at least M times the Nyquist rate, or
  - Prefilter to reduce bandwidth by factor M before downsampling (decimation).

Terminology:

- Compressor: just $\downarrow M$ (rate reduction).
- Decimator: antialiasing lowpass $+ \downarrow M$.



Frequency-Domain Analysis

DTFT Relationship:
DTFT of $x[n] = x_c(nT)$:

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$

Downsampled sequence DTFT (with $T_d = MT$):

$$X_d(e^{j\omega}) = \frac{1}{T_d} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T_d} - j\frac{2\pi r}{T_d}\right)$$

Key Result – Relationship Between DTFTs:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

Interpretation:

- $X_d(e^{j\omega})$ is sum of M scaled, shifted copies of $X(e^{j\omega})$.
- Frequency axis compressed by factor M; shifts at $2\pi i/M$.
- Overall amplitude scaling $1/M$.

Derivation of DTFT Relationship

Starting point (sampling with $T_d = MT$):

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi r}{MT}\right)$$

Let $r = i + kM$ with $i = 0, 1, \ldots, M-1$:

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{i=0}^{M-1} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi(i+kM)}{MT}\right)$$

Group terms:

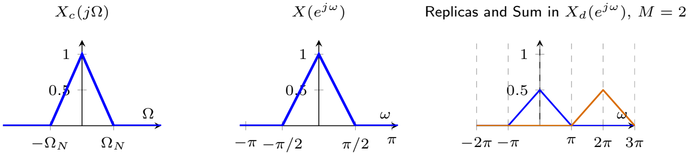
$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi i}{MT} - j\frac{2\pi k}{T}\right)$$

Recognize inner sum as $X\left(e^{j(\omega/M - 2\pi i/M)}\right)$. Therefore:

$$\boxed{X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)}$$

Downsampling Example: $M = 2$ (No Aliasing)

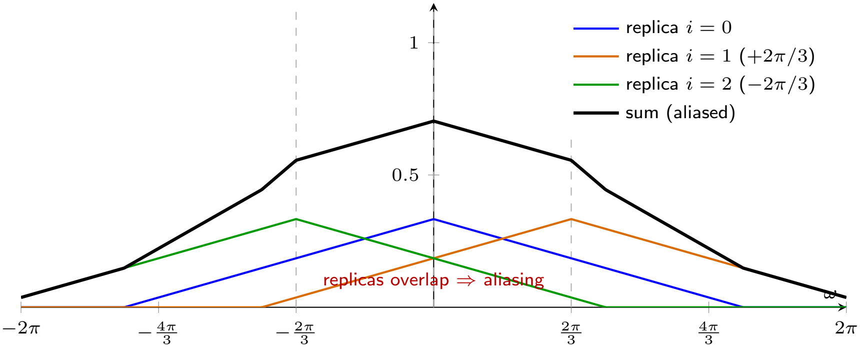
Setup: Original sampling rate is twice the minimum Nyquist rate; bandwidth fits after compression.



Result: No aliasing because the compressed baseband fits in $[-\pi, \pi]$. For $M = 2$ with $\omega_N = \pi/2$, the baseband and the shifted replica (centered at 2π) do not overlap within $[-\pi, \pi]$.

Downsampling with Aliasing: $M = 3$

Setup: Compression factor too large, spectral copies overlap.



Decimation: Prefiltering Before Downsampling

Problem: If original bandwidth exceeds π/M, downsampling aliases.
Solution: Apply a lowpass (antialiasing) filter of cutoff π/M before compression.

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{Lowpass}\\ \text{Cutoff } \pi/M\end{matrix}} \xrightarrow{\tilde{x}[n]} \boxed{\downarrow M} \longrightarrow \tilde{x}_d[n]$$

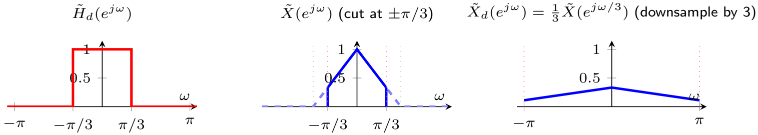
T $\qquad\qquad$ $T_d = MT$

Ideal filter:

$$\tilde{H}_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \pi/M \\ 0, & \pi/M < |\omega| \leq \pi \end{cases}$$

After filtering, $\tilde{x}[n]$ has bandwidth π/M, so $\tilde{x}_d[n] = \tilde{x}[nM]$ is alias-free.

Decimation Example: $M = 3$ With Prefiltering



Notes:

- Original $X(e^{j\omega})$ (triangular to $\pm\pi/2$) is truncated by the ideal LPF to $\pm\pi/3$, producing vertical cutoffs and a triangular top peaking at 1.
- Downsampling by $M = 3$ yields $\tilde{X}_d(e^{j\omega}) = \frac{1}{3}\tilde{X}(e^{j\omega/3})$: bandwidth expands to $\pm\pi$, apex is $1/3$.

Summary

Key Concepts:

- Resampling changes sampling rate using discrete-time operations.
- Downsampling by M: $x_d[n] = x[nM]$.
- Compressor vs. Decimator: Decimator adds antialias filter.

Frequency-Domain Relationship:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

Aliasing Condition:

$$\omega_N \leq \frac{\pi}{M} \quad \text{(required for alias-free direct downsampling)}$$

If violated: prefilter to bandwidth π/M before rate reduction.