DSP HW2 Part 2: Digital Parametric Resonators

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1 Introduction

A parametric resonator is a second-order digital filter designed to produce sustained oscillations at a specific frequency when excited by a brief input signal. These filters are used as blocks in digital audio synthesis and can simulate the resonant behavior of physical instruments.

2 Mathematical Derivation

2.1 Time Domain Difference Equation

A second-order digital resonator can be described by the following difference equation:

$$y[n] = b_0 x[n] - a_1 y[n-1] - a_2 y[n-2] \tag{1}$$

where:

- $x[n]$ is the input signal
- $y[n]$ is the output signal
- b_0 is the feedforward coefficient
- a_1, a_2 are the feedback coefficients

The difference equation $y[n] = Gx[n] - a_1 y[n-1] - a_2 y[n-2]$ creates oscillations because it implements a **digital memory system with delayed feedback**. When you input a brief pulse, the system “remembers” its recent outputs through the $y[n-1]$ and $y[n-2]$ terms, and the carefully chosen coefficients a_1 and a_2 cause these delayed versions to reinforce each other in a periodic pattern that naturally ’rings’ at the designed frequency.

2.2 Z-Transform and Transfer Function

Taking the z-transform of equation (1):

$$Y(z) = b_0 X(z) - a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) \tag{2}$$

Rearranging to solve for the transfer function:

$$Y(z)[1 + a_1 z^{-1} + a_2 z^{-2}] = b_0 X(z) \tag{3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{4}$$

2.3 Pole Placement for Resonance

To create a resonator at frequency ω_0 with decay factor R, we place complex conjugate poles at:

$$p_{1,2} = Re^{\pm j\omega_0} \tag{5}$$

The denominator polynomial with these poles is:

$$\begin{aligned} D(z) &= (1 - Re^{j\omega_0}z^{-1})(1 - Re^{-j\omega_0}z^{-1}) && (6)\\ &= 1 - R(e^{j\omega_0} + e^{-j\omega_0})z^{-1} + R^2z^{-2} && (7)\\ &= 1 - 2R\cos(\omega_0)z^{-1} + R^2z^{-2} && (8) \end{aligned}$$

Therefore, the coefficients are:

$$\begin{aligned} a_1 &= -2R\cos(\omega_0) && (9)\\ a_2 &= R^2 && (10) \end{aligned}$$

2.4 Frequency Domain Analysis

The frequency response is obtained by substituting $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{b_0}{1 + a_1e^{-j\omega} + a_2e^{-j2\omega}} \tag{11}$$

Substituting our coefficients:

$$H(e^{j\omega}) = \frac{b_0}{1 - 2R\cos(\omega_0)e^{-j\omega} + R^2e^{-j2\omega}} \tag{12}$$

2.5 Gain Normalization

To ensure unity gain at the resonant frequency ω_0, we set:

$$|H(e^{j\omega_0})| = 1 \tag{13}$$

The magnitude at resonance is:

$$|H(e^{j\omega_0})| = \frac{b_0}{|1 - 2R\cos(\omega_0)e^{-j\omega_0} + R^2e^{-j2\omega_0}|} \tag{14}$$

After algebraic manipulation, the normalization factor is:

$$b_0 = G = (1 - R)\sqrt{1 - 2R\cos(2\omega_0) + R^2} \tag{15}$$

2.6 Digital Frequency Conversion

For a desired analog frequency f Hz and sampling rate F_s:

$$\omega_0 = \frac{2\pi f}{F_s} \tag{16}$$

3 MATLAB Implementation

3.1 Resonator Coefficient Function

Listing 1: res_coeffs.m - Resonator coefficient generator

```matlab
function [a, b] = res_coeffs(freq, R, Fs)
    % RES_COEFFS Generate coefficients for parametric resonator
    %
    % Inputs:
    %   freq - Desired resonant frequency in Hz
    %   R    - Pole radius (0 < R < 1), controls decay rate
    %   Fs   - Sampling frequency in Hz
    %
    % Outputs:
    %   a - Denominator coefficients [1, a1, a2]
    %   b - Numerator coefficient [G]

    % Convert analog frequency to digital frequency
    omega_0 = (2 * pi * freq) / Fs;

    % Calculate denominator coefficients (feedback)
    a1 = -2 * R * cos(omega_0);
    a2 = R^2;

    % Calculate gain normalization factor
    G = (1 - R) * sqrt(1 - 2*R*cos(2*omega_0) + R^2);

    % Return filter coefficients
    b = [G];            % Numerator (feedforward)
    a = [1, a1, a2];    % Denominator (feedback)
end
```

4 Example Figures

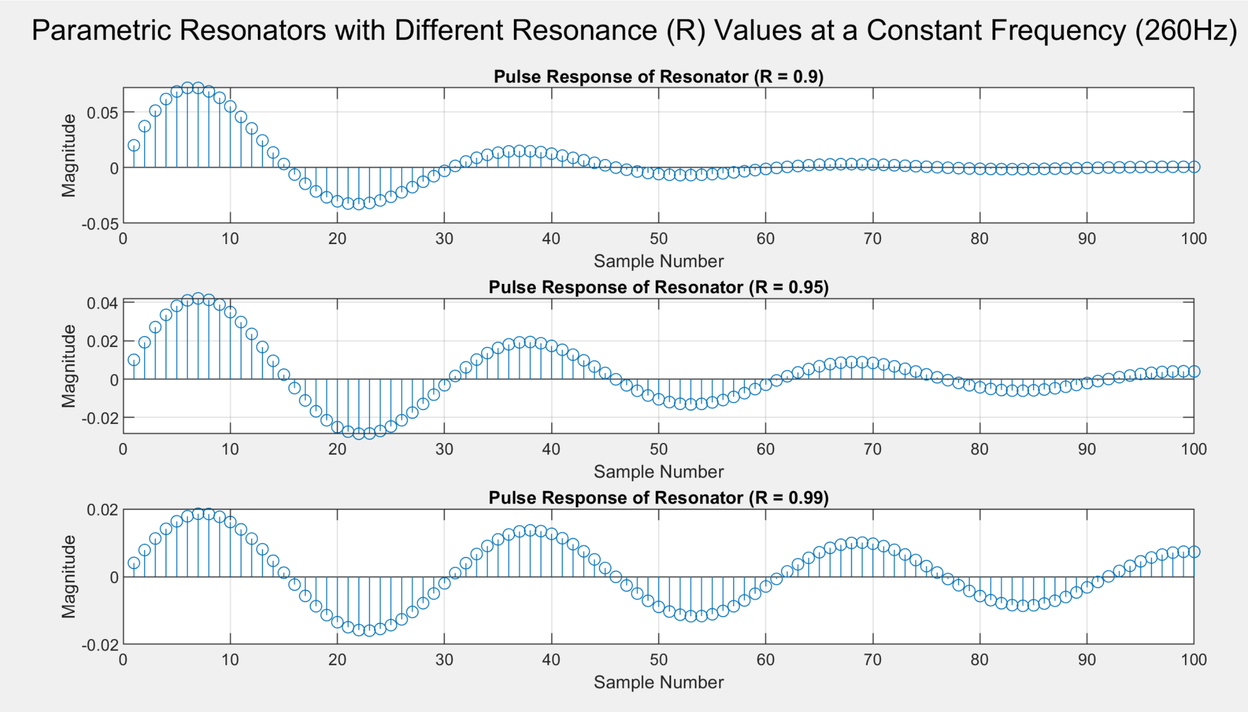

Figure 1: Resonance Sweep in time domain.

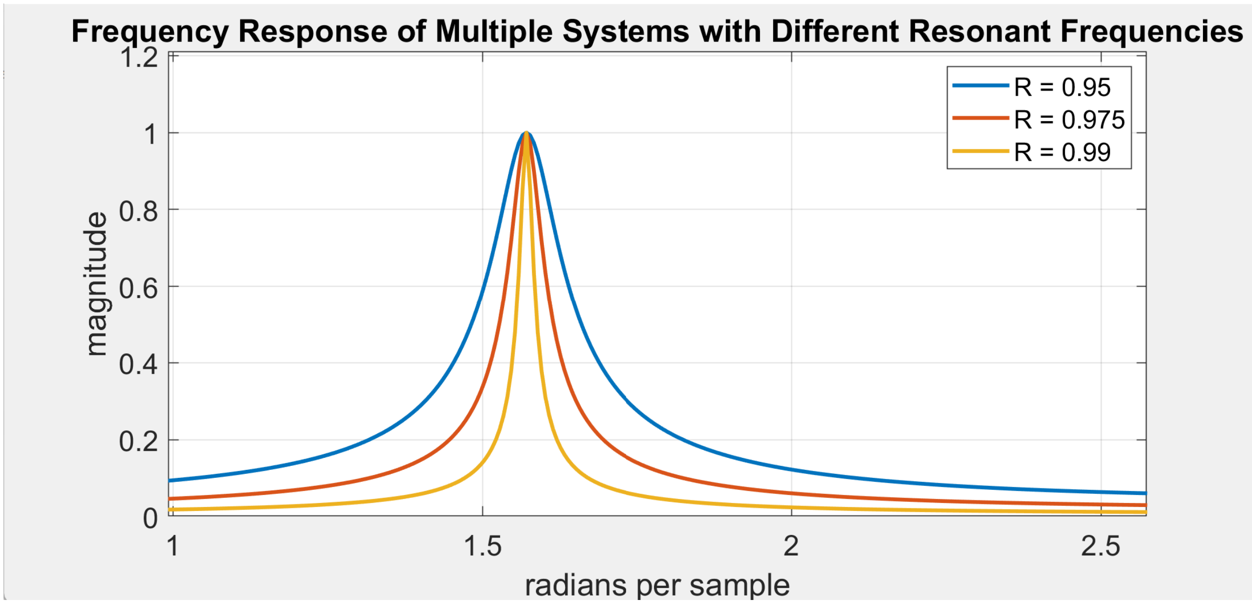

Figure 2: Resonance Sweep in frequency domain.

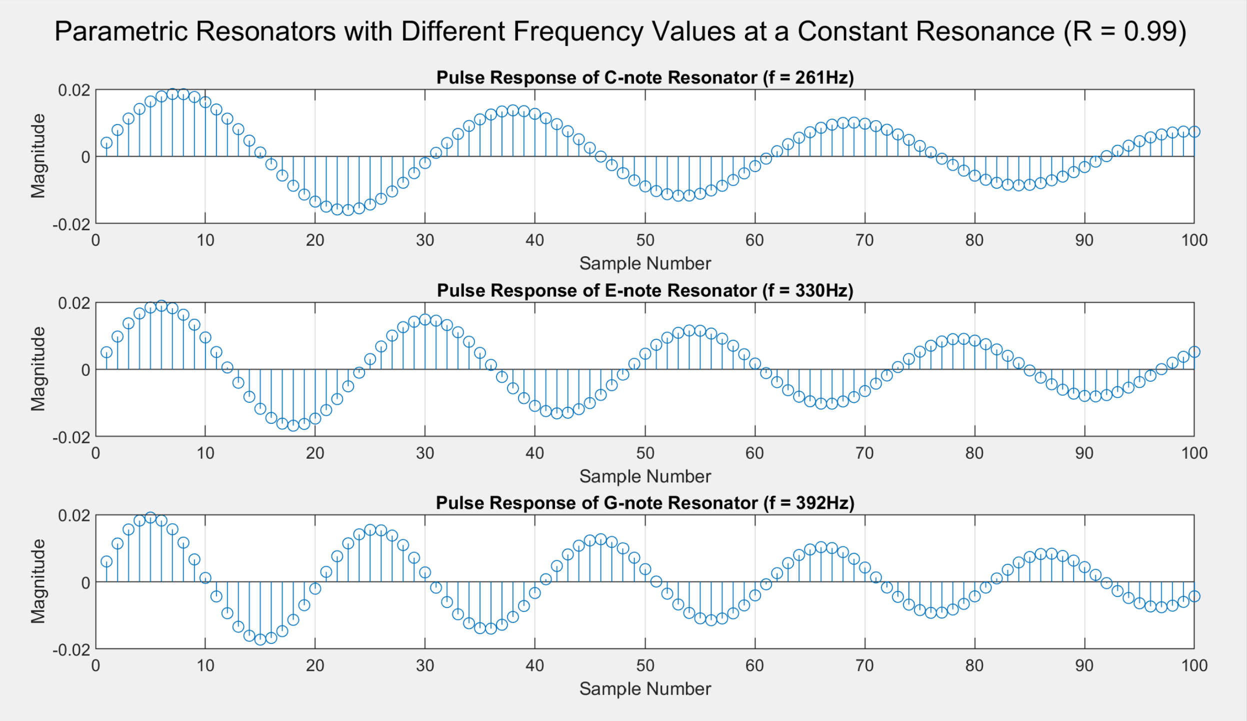

Figure 3: Resonance Frequency Sweep in time domain.

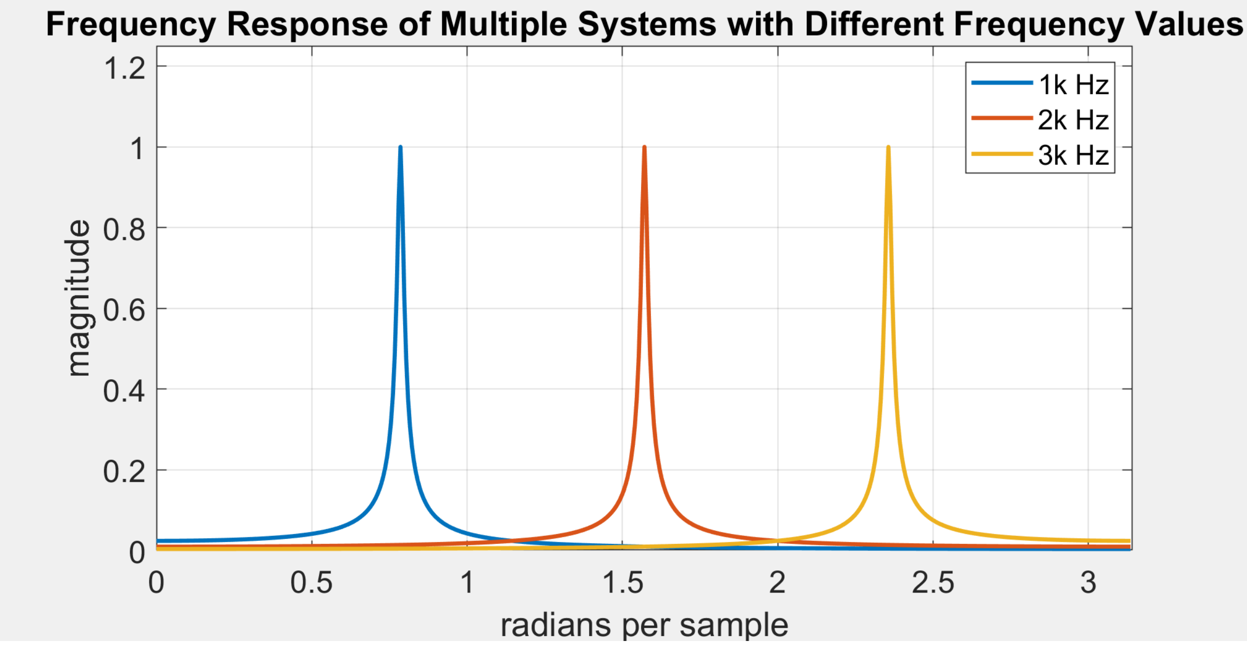

Figure 4: Resonance Frequency Sweep in frequency domain.