

Linear Constant-Coefficient Difference Equations

Maxx Seminario

University of Nebraska-Lincoln

November 9, 2025

Introduction to Difference Equations

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

- **Important Class of LTI Systems:** Systems where input $x[n]$ and output $y[n]$ satisfy a difference equation

- **General Form:**

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

- **Parameters:**

- N : Order of the system (highest delay in output)
- a_k : Output coefficients (constant)
- b_m : Input coefficients (constant)
- M : Highest delay in input terms

- **Why Important?:**

- Provides computational algorithms for LTI systems
- Foundation for digital filter implementation
- Connects time-domain and system analysis

Example 1: The Accumulator System

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Problem: Find the difference equation for the accumulator system.

System Definition:

$$y[n] = \sum_{k=-\infty}^n x[k]$$

Approach:

- Rewrite the sum to separate current and past inputs
- Use the relationship between $y[n]$ and $y[n-1]$

Example 1: Accumulator Solution

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Step 1: Rewrite the accumulator equation:

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

Step 2: Recognize that the sum is $y[n-1]$:

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

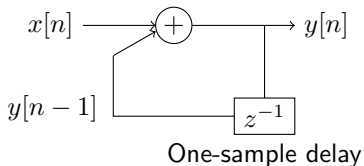
Step 3: Substitute to get the difference equation:

$$y[n] = x[n] + y[n-1]$$

Standard Form:

$$y[n] - y[n-1] = x[n]$$

Block Diagram: Recursive Accumulator



Recursive Implementation:

- Each output value computed using previously computed values
- $y[n] = x[n] + y[n-1]$: Add current input to previous output
- Requires initial condition (e.g., $y[-1] = 0$)

Example 2: Moving Average System

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Problem: Find difference equation for causal moving average system.

System Definition (with $M_1 = 0$, so the system is causal):

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n - k]$$

Two Approaches:

- 1 Direct (Non-recursive):** Use convolution form directly
- 2 Recursive:** Express as cascade of simpler systems

Moving Average: Direct Implementation

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Direct Form:

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n - k]$$

Standard Form:

$$y[n] = \sum_{k=0}^{M_2} \frac{1}{M_2 + 1} x[n - k]$$

Disadvantages of Direct Implementation:

- Requires $(M_2 + 1)$ multiplications per output sample
- Must store $(M_2 + 1)$ input samples in memory
- $O(M_2)$: Computational cost grows linearly with window size M_2

Block Diagram: Recursive Moving Average

ECEN 463/863

Maxx Seminario

Introduction

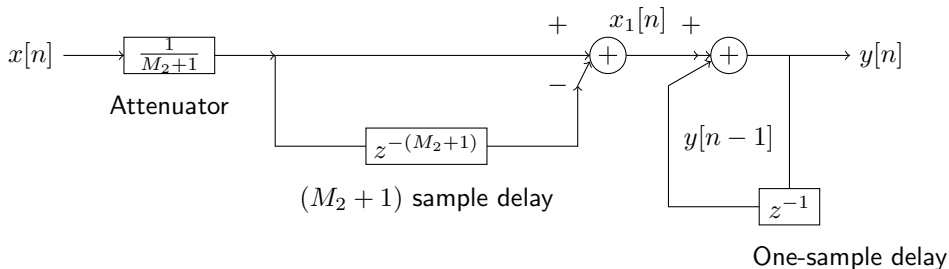
Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary



Signal Flow:

- Input $x[n] \rightarrow$ Attenuator $\frac{1}{M_2+1}$
- Attenuated signal splits: direct path and $(M_2 + 1)$ sample delay
- Sum: $x_1[n] = \frac{1}{M_2+1}[x[n] - x[n - (M_2 + 1)]]$
- $x_1[n] \rightarrow$ Accumulator \rightarrow Output $y[n]$

Moving Average as Difference Equation

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Intermediate signal:

$$x_1[n] = \frac{1}{M_2 + 1} [x[n] - x[n - M_2 - 1]]$$

Accumulator relation: $y[n] = x_1[n] + y[n - 1]$

Final recursive form:

$$y[n] - y[n - 1] = \frac{1}{M_2 + 1} [x[n] - x[n - M_2 - 1]]$$

Note: There is an unlimited number of distinct difference equations to represent an LTI I/O relation.

General Solution of Difference Equations

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Key Issue: Difference equation alone does not uniquely specify output!

General Solution Structure:

$$y[n] = y_p[n] + y_h[n]$$

- $y_p[n]$: Particular solution (satisfies original equation)
- $y_h[n]$: Homogeneous solution (satisfies homogeneous equation)

Homogeneous Equation ($x[n] = 0$):

$$\sum_{k=0}^N a_k y_h[n - k] = 0$$

Homogeneous Solution Form:

$$y_h[n] = \sum_{m=1}^N A_m z_m^n$$

where z_m are roots of characteristic polynomial $A(z) = \sum_{k=0}^N a_k z^{-k} = 0$

Auxiliary Conditions

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Need for Auxiliary Conditions:

- N undetermined coefficients A_m in homogeneous solution
- Need N auxiliary (boundary) conditions for unique solution

Types of Auxiliary Conditions:

- 1 **Fixed Values:** Specify $y[-1], y[-2], \dots, y[-N]$
- 2 **Initial Rest:** If $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$

Forward and Backward Computation for Specific Class of Difference Equations

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Specific Class of Difference Equations:

- Inputs $x[n]$ and Outputs $y[n]$ satisfy an N th-order linear constant coefficient difference equation:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Recursive Computation (forward):

$$y[n] = - \sum_{k=1}^N \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^M \frac{b_k}{a_0} x[n-k]$$

Recursive Computation (backward):

$$y[n-N] = - \sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^M \frac{b_k}{a_N} x[n-k]$$

Example 3: First-Order Difference Equation

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

Problem: Solve the difference equation with given input and initial condition.

Difference Equation:

$$y[n] = ay[n-1] + x[n]$$

Input:

$$x[n] = K\delta[n]$$

Auxiliary Condition:

$$y[-1] = c$$

Find: The complete solution $y[n]$ for all n .

Summary: Linear Constant-Coefficient Difference Equations

ECEN 463/863

Maxx Seminario

Introduction

Accumulator
System

Moving Average
System

General Solution

Particular
Solution

Summary

■ Definition:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

Describes the relationship between input $x[n]$ and output $y[n]$ in LTI systems.

■ Key Components:

- N and M : Order of the system
- a_k, b_m : Constant coefficients
- Recursive and direct computation methods

■ General Solution:

$$y[n] = y_p[n] + y_h[n]$$

- $y_p[n]$: Particular solution (due to input $x[n]$)
- $y_h[n]$: Homogeneous solution (due to initial conditions)
- Auxiliary conditions: Necessary to determine a unique / particular solutions.

- ## ■ Applications:
- Digital filters, computational algorithms for LTI systems, and time-domain analysis.