

The Discrete Fourier Transform

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October 3, 2025

Overview: Representation of Sequences

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Summary

- **Motivation:** How to represent arbitrary input sequences in frequency domain?
 - We know LTI system response to $e^{j\omega n}$ is $H(e^{j\omega})e^{j\omega n}$
 - Need to decompose arbitrary signals into complex exponentials
- **Key Questions:**
 - Can we represent any sequence as a sum of complex exponentials?
 - What are the conditions for such representations to exist?
 - How do we compute the representation?
- **Today's Topics:**
 - Fourier Transform pair for discrete-time signals
 - Convergence conditions
 - Symmetry properties
 - Fourier Transform theorems

The Discrete-Time Fourier Transform (DTFT)

Fourier Transform Pair:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad (\text{Synthesis}) \quad (1)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad (\text{Analysis}) \quad (2)$$

Key Interpretations:

- **Synthesis:** Represents $x[n]$ as superposition of complex sinusoids
- **Analysis:** Determines "how much" of each frequency is in $x[n]$
- $X(e^{j\omega})$ is generally complex-valued
- Integration can be over any interval of length 2π

Connection to LTI Systems:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Frequency response = DTFT of impulse response

Complex Representation of DTFT

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Rectangular Form:

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

Polar Form:

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

Phase Considerations:

- Phase is not unique (can add multiples of 2π)
- **Principal value:** $\text{ARG}[X(e^{j\omega})] \in [-\pi, \pi]$
- **Continuous phase:** $\arg[X(e^{j\omega})]$ (unwrapped phase)

Periodicity:

- $X(e^{j\omega})$ is periodic in ω with period 2π
- Similar to Fourier series for periodic functions
- Eq. (2.131) is Fourier series of periodic function $X(e^{j\omega})$

Convergence of the DTFT

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Question: When does $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ converge?

Absolute Summability (Sufficient Condition):

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \quad (3)$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega n}| \quad (4)$$

$$= \sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad (5)$$

Implications:

- If $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, then $X(e^{j\omega})$ exists
- Series converges uniformly to a continuous function
- All stable sequences have Fourier transforms
- All FIR systems have finite, continuous frequency responses

Example: Suddenly-Applied Exponential

Sequence: $x[n] = a^n u[n]$

DTFT Calculation:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} \quad (6)$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n \quad (7)$$

$$= \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |ae^{-j\omega}| < 1 \quad (8)$$

Convergence Condition: $|a| < 1$

Absolute Summability Check:

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty \quad \text{if } |a| < 1$$

This confirms that absolute summability guarantees convergence.

Square Summability

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Alternative Condition: Some sequences are not absolutely summable but are square summable:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Mean-Square Convergence:

$$\lim_{M \rightarrow \infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 d\omega = 0$$

where $X_M(e^{j\omega}) = \sum_{n=-M}^M x[n]e^{-j\omega n}$

Important Distinction:

- Error may not approach zero at each ω
- Total "energy" in error approaches zero
- Leads to Gibbs phenomenon at discontinuities

Example: Ideal Lowpass Filter

Frequency Response:

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

Impulse Response:

$$h_{lp}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \quad (9)$$

$$= \frac{1}{2\pi j n} [e^{j\omega_c n} - e^{-j\omega_c n}] \quad (10)$$

$$= \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty \quad (11)$$

Properties:

- **Noncausal:** $h_{lp}[n] \neq 0$ for $n < 0$
- **Not absolutely summable:** Decays as $1/n$
- **Square summable:** Mean-square convergence

Gibbs Phenomenon

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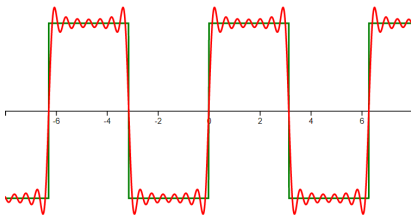
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Finite Sum Approximation:

$$H_M(e^{j\omega}) = \sum_{n=-M}^M \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$$



Key Points:

- Oscillations near discontinuity persist as $M \rightarrow \infty$
- Maximum overshoot $\approx 9\%$ (doesn't decrease)
- Oscillations become more rapid but localized

Fourier Transform of Special Sequences

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Case 1: Constant Sequence

- Sequence: $x[n] = 1$ for all n
- Neither absolutely nor square summable
- DTFT (using impulse functions):

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

Case 2: Complex Exponential

- Sequence: $x[n] = e^{j\omega_0 n}$, $-\pi < \omega_0 \leq \pi$
- DTFT:

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r)$$

- Verification: Substituting into synthesis equation yields $x[n] = e^{j\omega_0 n}$

Unit Step Sequence

Sequence: $u[n]$

DTFT:

$$U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{r=-\infty}^{\infty} \pi \delta(\omega + 2\pi r)$$

Interpretation:

- First term: Response for $\omega \neq 0$
- Second term: DC component (impulses at $\omega = 2\pi r$)
- Neither absolutely nor square summable
- Requires generalized function theory

General Form: For sequences with discrete frequency components:

$$x[n] = \sum_k a_k e^{j\omega_k n} \Rightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_k 2\pi a_k \delta(\omega - \omega_k + 2\pi r)$$

Conjugate Symmetry Definitions

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For Sequences:

- **Conjugate-symmetric:** $x_e[n] = x_e^*[-n]$
- **Conjugate-antisymmetric:** $x_o[n] = -x_o^*[-n]$

Decomposition: Any sequence can be written as:

$$x[n] = x_e[n] + x_o[n] \quad (12)$$

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) \quad (13)$$

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) \quad (14)$$

For Real Sequences:

- **Even:** $x_e[n] = x_e[-n]$ (conjugate-symmetric when real)
- **Odd:** $x_o[n] = -x_o[-n]$ (conjugate-antisymmetric when real)

Symmetry Properties of DTFT

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General Complex Sequences:

Sequence	Fourier Transform
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\text{Re}\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part)
$j\text{Im}\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part)
$x_e[n]$	$X_R(e^{j\omega}) = \text{Re}\{X(e^{j\omega})\}$
$x_o[n]$	$jX_I(e^{j\omega}) = j\text{Im}\{X(e^{j\omega})\}$

Key Insight:

- Conjugate-symmetric sequences \leftrightarrow Real transforms
- Conjugate-antisymmetric sequences \leftrightarrow Imaginary transforms

Symmetry Properties for Real Sequences

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If $x[n]$ is real:

Property	Result
DTFT	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (conjugate symmetric)
Real part	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (even function)
Imaginary part	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (odd function)
Magnitude	$ X(e^{j\omega}) = X(e^{-j\omega}) $ (even function)
Phase	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (odd function)
Even part of $x[n]$	Transforms to $X_R(e^{j\omega})$
Odd part of $x[n]$	Transforms to $jX_I(e^{j\omega})$

Practical Implication: For real signals, we only need to compute DTFT for $\omega \in [0, \pi]$

Example: Symmetry Properties

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Sequence: $x[n] = a^n u[n]$, $|a| < 1$ (real)

DTFT: $X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$

Components:

$$X_R(e^{j\omega}) = \frac{1 - a \cos \omega}{1 + a^2 - 2a \cos \omega} \quad (\text{even}) \quad (15)$$

$$X_I(e^{j\omega}) = \frac{-a \sin \omega}{1 + a^2 - 2a \cos \omega} \quad (\text{odd}) \quad (16)$$

$$|X(e^{j\omega})| = \frac{1}{(1 + a^2 - 2a \cos \omega)^{1/2}} \quad (\text{even}) \quad (17)$$

$$\angle X(e^{j\omega}) = \tan^{-1} \left(\frac{-a \sin \omega}{1 - a \cos \omega} \right) \quad (\text{odd}) \quad (18)$$

Verification: Each property satisfies the symmetry conditions for real sequences

Overview of DTFT Theorems

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Key Theorems:

- 1 **Linearity:** $ax[n] + by[n] \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$
- 2 **Time Shifting:** $x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$
- 3 **Frequency Shifting:** $e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)})$
- 4 **Time Reversal:** $x[-n] \leftrightarrow X(e^{-j\omega})$
- 5 **Differentiation:** $nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega}$
- 6 **Convolution:** $x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$
- 7 **Multiplication:** $x[n]y[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$

Why Important?

- Simplify analysis of complex signals
- Transform difficult operations into simple ones
- Foundation for filter design and signal processing

Linearity and Time Shifting

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1. Linearity:

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{F}} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

2. Time Shifting:

$$x[n - n_d] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_d} X(e^{j\omega})$$

Proof of Time Shifting:

$$\mathcal{F}\{x[n - n_d]\} = \sum_{n=-\infty}^{\infty} x[n - n_d] e^{-j\omega n} \quad (19)$$

$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+n_d)} \quad (m = n - n_d) \quad (20)$$

$$= e^{-j\omega n_d} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} \quad (21)$$

$$= e^{-j\omega n_d} X(e^{j\omega}) \quad (22)$$

Frequency Shifting and Time Reversal

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3. Frequency Shifting (Modulation):

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega - \omega_0)})$$

Application: Modulation shifts spectrum by ω_0

4. Time Reversal:

$$x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$$

For real sequences: $x[-n] \xleftrightarrow{\mathcal{F}} X^*(e^{j\omega})$

Example Application:

- Signal $x[n] = \cos(\omega_0 n) = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n})$
- Using frequency shifting: spectrum has impulses at $\pm\omega_0$

Differentiation in Frequency

5. Differentiation Theorem:

$$nx[n] \xleftrightarrow{\mathcal{F}} j \frac{dX(e^{j\omega})}{d\omega}$$

Proof Sketch:

$$\frac{dX(e^{j\omega})}{d\omega} = \frac{d}{d\omega} \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (23)$$

$$= \sum_{n=-\infty}^{\infty} x[n](-jn)e^{-j\omega n} \quad (24)$$

$$= -j \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n} \quad (25)$$

Applications:

- Finding moments of sequences
- Analyzing group delay of filters
- Computing derivatives of frequency response

Parseval's Theorem

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Energy Conservation:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Interpretation:

- Total energy in time domain = Total energy in frequency domain
- $|X(e^{j\omega})|^2$ is the **energy density spectrum**
- Shows how energy is distributed across frequencies

General Form:

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Application: Signal power calculation, filter energy analysis

The Convolution Theorem

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Time Domain Convolution \leftrightarrow Frequency Domain Multiplication:

$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Why This Works:

- Complex exponentials are eigenfunctions of LTI systems
- Input $e^{j\omega n}$ produces output $H(e^{j\omega})e^{j\omega n}$
- By superposition, each frequency component is scaled by $H(e^{j\omega})$

Practical Impact:

- Convolution (complex operation) \rightarrow Multiplication (simple)
- Foundation for frequency-domain filtering
- Basis for fast convolution algorithms

The Modulation (Windowing) Theorem

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Time Domain Multiplication \leftrightarrow Frequency Domain Convolution:

$$y[n] = x[n]w[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

Key Points:

- This is **periodic convolution** (circular convolution)
- Integration over one period only
- Dual to the convolution theorem

Applications:

- **Windowing:** Multiplying by window function in time domain
- **Modulation:** Shifting spectrum by multiplication with $e^{j\omega_0 n}$
- **Spectral analysis:** Effect of finite-length observation

Example: Using Tables and Theorems

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Problem: Find DTFT of $x[n] = a^n u[n - 5]$

Example: Inverse Transform Using Partial Fractions

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Given: $X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$

Example: Highpass Filter Design

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Desired Frequency Response:

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_c < |\omega| < \pi \\ 0, & |\omega| < \omega_c \end{cases}$$

Approach: Express as $H(e^{j\omega}) = e^{-j\omega n_d}(1 - H_{lp}(e^{j\omega}))$

Example: Solving Difference Equations

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System: $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$

Finding Frequency Response:

- 1 Apply DTFT to both sides:
- 2 Factor and solve:
- 3 Partial fractions:
- 4 Inverse Transform:

Summary of DTFT

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Key Concepts:

1 DTFT Pair:

- Analysis: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
- Synthesis: $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$

2 Convergence:

- Absolute summability \rightarrow uniform convergence
- Square summability \rightarrow mean-square convergence

3 Symmetry Properties:

- Real sequences have conjugate-symmetric DTFTs
- Even/odd decomposition in both domains

4 Transform Theorems:

- Convolution \leftrightarrow Multiplication
- Time shift \leftrightarrow Linear phase
- Differentiation, modulation, Parseval's theorem

Important Transform Pairs

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Sequence	DTFT
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$a^n u[n], a < 1$	$\frac{1}{1 - ae^{-j\omega}}$
$e^{j\omega_0 n}$	$2\pi \sum_r \delta(\omega - \omega_0 + 2\pi r)$
$\cos(\omega_0 n)$	$\pi \sum_r [\delta(\omega - \omega_0 + 2\pi r) + \delta(\omega + \omega_0 + 2\pi r)]$
$\frac{\sin(\omega_c n)}{\pi n}$	Ideal lowpass filter
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \pi \sum_r \delta(\omega + 2\pi r)$

Remember: DTFT is periodic with period 2π , so we only need to consider $\omega \in [-\pi, \pi]$