

# Sampling Rate Conversions: Downsampling

Maxx Seminario

University of Nebraska-Lincoln

Fall 2025

# Downsampling: Motivation and Key Ideas

ECEN 463/863

Maxx Seminario

Frequency  
Domain Analysis

Decimation

Summary

## Why downsample?

- Reduce computation, memory, bandwidth, and power.
- Match device/IO rate constraints (sensors, radios, codecs).
- Enable multirate algorithms, filter banks, and pyramids.
- Coarser-time analysis or stage-wise processing at lower rates.

## Definition (Compressor / Downampler):

$$x_d[n] = x[nM] = x_c(nMT) \quad \text{with} \quad T_d = MT$$

Equivalent to sampling the bandlimited reconstruction  $x_c(t)$  with period  $T_d$ .

# Downsampling: When is it Alias-Free?

ECEN 463/863

Maxx Seminario

Frequency  
Domain Analysis

Decimation

Summary

## When is it alias-free?

- If  $X_c(j\Omega) = 0$  for  $|\Omega| \geq \Omega_N$ , then  $x_d[n]$  exactly represents  $x_c(t)$  if

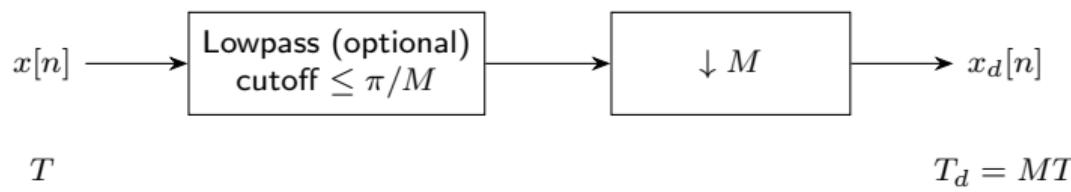
$$\frac{\pi}{T_d} = \frac{\pi}{MT} \geq \Omega_N \iff \omega_N \leq \frac{\pi}{M}.$$

- Interpretations:

- Original sampling rate is at least  $M$  times the Nyquist rate, or
- Prefilter to reduce bandwidth by factor  $M$  before downsampling (decimation).

## Terminology:

- Compressor: just  $\downarrow M$  (rate reduction).
- Decimator: antialiasing lowpass +  $\downarrow M$ .



# Frequency-Domain Analysis

ECEN 463/863

Maxx Seminario

Frequency-  
Domain Analysis

Decimation  
Summary

## DTFT Relationship:

DTFT of  $x[n] = x_c(nT)$ :

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$

Downsampling sequence DTFT (with  $T_d = MT$ ):

$$X_d(e^{j\omega}) = \frac{1}{T_d} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T_d} - j\frac{2\pi r}{T_d}\right)$$

## Key Result – Relationship Between DTFTs:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

## Interpretation:

- $X_d(e^{j\omega})$  is sum of  $M$  scaled, shifted copies of  $X(e^{j\omega})$ .
- Frequency axis compressed by factor  $M$ ; shifts at  $2\pi i/M$ .
- Overall amplitude scaling  $1/M$ .

# Derivation of DTFT Relationship

ECEN 463/863

Maxx Seminario

Frequency-  
Domain Analysis

Decimation

Summary

Starting point (sampling with  $T_d = MT$ ):

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi r}{MT}\right)$$

Let  $r = i + kM$  with  $i = 0, 1, \dots, M-1$ :

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{i=0}^{M-1} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi(i + kM)}{MT}\right)$$

Group terms:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi i}{MT} - j\frac{2\pi k}{T}\right)$$

Recognize inner sum as  $X(e^{j(\omega/M - 2\pi i/M)})$ . Therefore:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

# Downsampling Example: $M = 2$ (No Aliasing)

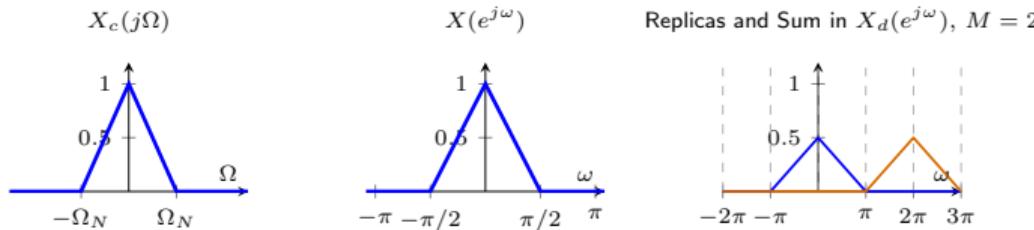
ECEN 463/863

Maxx Seminario

Frequency-  
Domain Analysis

Decimation  
Summary

Setup: Original sampling rate is twice the minimum Nyquist rate; bandwidth fits after compression.



Result: No aliasing because the compressed baseband fits in  $[-\pi, \pi]$ . For  $M = 2$  with  $\omega_N = \pi/2$ , the baseband and the shifted replica (centered at  $2\pi$ ) do not overlap within  $[-\pi, \pi]$ .

# Downsampling with Aliasing: $M = 3$

ECEN 463/863

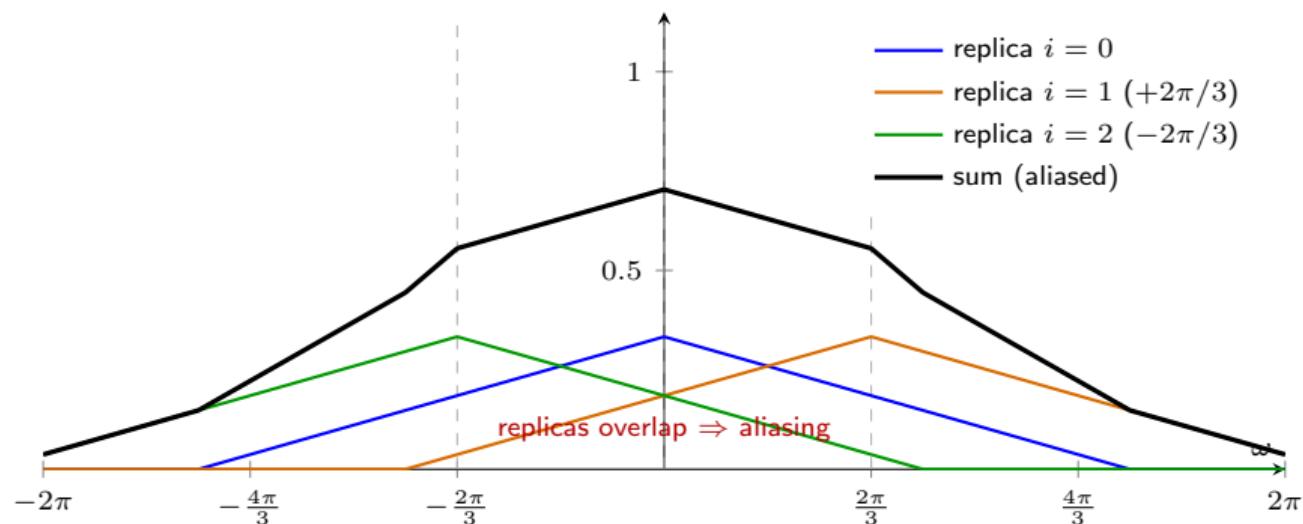
Maxx Seminario

Frequency-  
Domain Analysis

Decimation

Summary

Setup: Compression factor too large, spectral copies overlap.



# Decimation: Prefiltering Before Downsampling

ECEN 463/863

Maxx Seminario

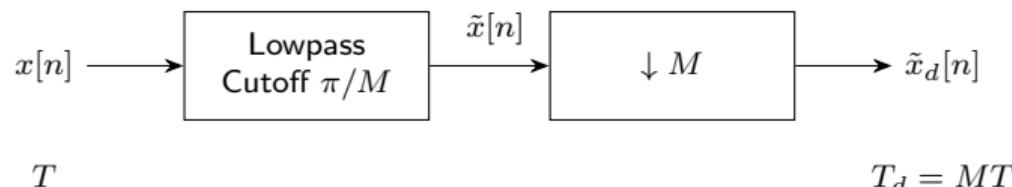
Frequency  
Domain Analysis

Decimation

Summary

Problem: If original bandwidth exceeds  $\pi/M$ , downsampling aliases.

Solution: Apply a lowpass (antialiasing) filter of cutoff  $\pi/M$  before compression.



Ideal filter:

$$\tilde{H}_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \pi/M \\ 0, & \pi/M < |\omega| \leq \pi \end{cases}$$

After filtering,  $\tilde{x}[n]$  has bandwidth  $\pi/M$ , so  $\tilde{x}_d[n] = \tilde{x}[nM]$  is alias-free.

# Decimation Example: $M = 3$ With Prefiltering

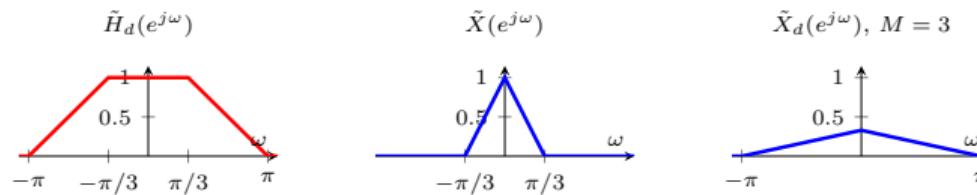
ECEN 463/863

Maxx Seminario

Frequency  
Domain Analysis

Decimation

Summary



Notes:

- $\tilde{x}_d[n] \neq x[nM]$ ; it is from the filtered version.
- Trade-off: reduced bandwidth for alias-free rate reduction.

# Summary

ECEN 463/863

Maxx Seminario

Frequency  
Domain Analysis

Decimation  
Summary

## Key Concepts:

- Resampling changes sampling rate using discrete-time operations.
- Downsampling by  $M$ :  $x_d[n] = x[nM]$ .
- Compressor vs. Decimator: Decimator adds antialias filter.

## Frequency-Domain Relationship:

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

## Aliasing Condition:

$$\omega_N \leq \frac{\pi}{M} \quad (\text{required for alias-free direct downsampling})$$

If violated: prefilter to bandwidth  $\pi/M$  before rate reduction.

## Applications:

- Audio/video sample-rate conversion
- Multirate filter banks
- Computational load reduction

Decimation