The Discrete Fourier Transform

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Overview: Representation of Sequences

- **Motivation**: How to represent arbitrary input sequences in frequency domain?
  - We know LTI system response to $e^{j\omega n}$ is $H(e^{j\omega})e^{j\omega n}$
  - Need to decompose arbitrary signals into complex exponentials
- **Key Questions**:
  - Can we represent any sequence as a sum of complex exponentials?
  - What are the conditions for such representations to exist?
  - How do we compute the representation?
- **Today's Topics**:
  - Fourier Transform pair for discrete-time signals
  - Convergence conditions
  - Symmetry properties
  - Fourier Transform theorems

The Discrete-Time Fourier Transform (DTFT)

Fourier Transform Pair:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{(Synthesis)} \tag{1}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \text{(Analysis)} \tag{2}$$

Key Interpretations:

- **Synthesis**: Represents $x[n]$ as superposition of complex sinusoids
- **Analysis**: Determines "how much" of each frequency is in $x[n]$
- $X(e^{j\omega})$ is generally complex-valued
- Integration can be over any interval of length 2π

Connection to LTI Systems:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Frequency response = DTFT of impulse response

Complex Representation of DTFT

Rectangular Form:

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

Polar Form:

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

Phase Considerations:

- Phase is not unique (can add multiples of 2π)
- **Principal value**: ARG$[X(e^{j\omega})] \in [-\pi, \pi]$
- **Continuous phase**: $\arg[X(e^{j\omega})]$ (unwrapped phase)

Periodicity:

- $X(e^{j\omega})$ is periodic in ω with period 2π
- Similar to Fourier series for periodic functions
- Eq. (2.131) is Fourier series of periodic function $X(e^{j\omega})$

Convergence of the DTFT

Question: When does $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ converge?
Absolute Summability (Sufficient Condition):

$$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \tag{3}$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \tag{4}$$

$$= \sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{5}$$

Implications:

- If $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, then $X(e^{j\omega})$ exists
- Series converges uniformly to a continuous function
- All stable sequences have Fourier transforms
- All FIR systems have finite, continuous frequency responses

Example: Suddenly-Applied Exponential

Sequence: $x[n] = a^n u[n]$
DTFT Calculation:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} \tag{6}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n \tag{7}$$

$$= \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |ae^{-j\omega}| < 1 \tag{8}$$

Convergence Condition: $|a| < 1$
Absolute Summability Check:

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty \quad \text{if } |a| < 1$$

This confirms that absolute summability guarantees convergence.

Square Summability

Alternative Condition: Some sequences are not absolutely summable but are square summable:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Mean-Square Convergence:

$$\lim_{M \to \infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 d\omega = 0$$

where $X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$

Important Distinction:

- Error may not approach zero at each ω
- Total "energy" in error approaches zero
- Leads to Gibbs phenomenon at discontinuities

Example: Ideal Lowpass Filter

Frequency Response:

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

Impulse Response:

$$h_{lp}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \tag{9}$$

$$= \frac{1}{2\pi j n} [e^{j\omega_c n} - e^{-j\omega_c n}] \tag{10}$$

$$= \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty \tag{11}$$

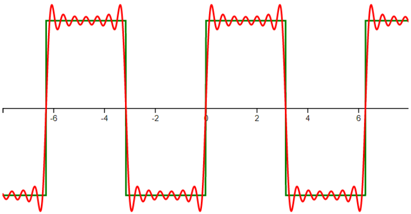
Properties:

- **Noncausal**: $h_{lp}[n] \neq 0$ for $n < 0$
- **Not absolutely summable**: Decays as $1/n$
- **Square summable**: Mean-square convergence

Gibbs Phenomenon

Finite Sum Approximation:

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$$



Key Points:

- Oscillations near discontinuity persist as $M \to \infty$
- Maximum overshoot $\approx 9\%$ (doesn't decrease)
- Oscillations become more rapid but localized

Fourier Transform of Special Sequences

Case 1: Constant Sequence

- Sequence: $x[n] = 1$ for all n
- Neither absolutely nor square summable
- DTFT (using impulse functions):

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

Case 2: Complex Exponential

- Sequence: $x[n] = e^{j\omega_0 n}$, $-\pi < \omega_0 \leq \pi$
- DTFT:

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r)$$

- Verification: Substituting into synthesis equation yields $x[n] = e^{j\omega_0 n}$

Unit Step Sequence

Sequence: $u[n]$
DTFT:

$$U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{r=-\infty}^{\infty} \pi\delta(\omega + 2\pi r)$$

Interpretation:

- First term: Response for $\omega \neq 0$
- Second term: DC component (impulses at $\omega = 2\pi r$)
- Neither absolutely nor square summable
- Requires generalized function theory

General Form: For sequences with discrete frequency components:

$$x[n] = \sum_k a_k e^{j\omega_k n} \Rightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_k 2\pi a_k \delta(\omega - \omega_k + 2\pi r)$$

Conjugate Symmetry Definitions

For Sequences:

- **Conjugate-symmetric**: $x_e[n] = x_e^*[-n]$
- **Conjugate-antisymmetric**: $x_o[n] = -x_o^*[-n]$

Decomposition: Any sequence can be written as:

$$x[n] = x_e[n] + x_o[n] \tag{12}$$

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) \tag{13}$$

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) \tag{14}$$

For Real Sequences:

- **Even**: $x_e[n] = x_e[-n]$ (conjugate-symmetric when real)
- **Odd**: $x_o[n] = -x_o[-n]$ (conjugate-antisymmetric when real)

Symmetry Properties of DTFT

General Complex Sequences:

Sequence	Fourier Transform
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\text{Re}\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part)
$j\text{Im}\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part)
$x_e[n]$	$X_R(e^{j\omega}) = \text{Re}\{X(e^{j\omega})\}$
$x_o[n]$	$jX_I(e^{j\omega}) = j\text{Im}\{X(e^{j\omega})\}$

Key Insight:

- Conjugate-symmetric sequences $\leftrightarrow$ Real transforms
- Conjugate-antisymmetric sequences $\leftrightarrow$ Imaginary transforms

Symmetry Properties for Real Sequences

If $x[n]$ is real:

Property	**Result**
DTFT	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (conjugate symmetric)
Real part	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (even function)
Imaginary part	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (odd function)
Magnitude	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (even function)
Phase	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (odd function)
Even part of $x[n]$	Transforms to $X_R(e^{j\omega})$
Odd part of $x[n]$	Transforms to $jX_I(e^{j\omega})$

Practical Implication: For real signals, we only need to compute DTFT for $\omega \in [0, \pi]$

Example: Symmetry Properties

Sequence: $x[n] = a^n u[n]$, $|a| < 1$ (real)
DTFT: $X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$
Components:

$$X_R(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} \quad \text{(even)} \tag{15}$$

$$X_I(e^{j\omega}) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} \quad \text{(odd)} \tag{16}$$

$$|X(e^{j\omega})| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} \quad \text{(even)} \tag{17}$$

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) \quad \text{(odd)} \tag{18}$$

Verification: Each property satisfies the symmetry conditions for real sequences

Overview of DTFT Theorems

Key Theorems:

1. **Linearity**: $ax[n] + by[n] \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$
2. **Time Shifting**: $x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$
3. **Frequency Shifting**: $e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
4. **Time Reversal**: $x[-n] \leftrightarrow X(e^{-j\omega})$
5. **Differentiation**: $nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$
6. **Convolution**: $x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$
7. **Multiplication**: $x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Why Important?

- Simplify analysis of complex signals
- Transform difficult operations into simple ones
- Foundation for filter design and signal processing

Linearity and Time Shifting

1. **Linearity**:

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{F}} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

2. **Time Shifting**:

$$x[n - n_d] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_d} X(e^{j\omega})$$

Proof of Time Shifting:

$$\mathcal{F}\{x[n - n_d]\} = \sum_{n=-\infty}^{\infty} x[n - n_d] e^{-j\omega n} \tag{19}$$

$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+n_d)} \quad (m = n - n_d) \tag{20}$$

$$= e^{-j\omega n_d} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} \tag{21}$$

$$= e^{-j\omega n_d} X(e^{j\omega}) \tag{22}$$

Frequency Shifting and Time Reversal

3. Frequency Shifting (Modulation):

$$e^{j\omega_0 n}x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

Application: Modulation shifts spectrum by ω_0

4. Time Reversal:

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

For real sequences: $x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(e^{j\omega})$

Example Application:

- Signal $x[n] = \cos(\omega_0 n) = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n})$
- Using frequency shifting: spectrum has impulses at $\pm\omega_0$

Differentiation in Frequency

5. Differentiation Theorem:

$$nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Proof Sketch:

$$\frac{dX(e^{j\omega})}{d\omega} = \frac{d}{d\omega}\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{23}$$

$$= \sum_{n=-\infty}^{\infty} x[n](-jn)e^{-j\omega n} \tag{24}$$

$$= -j\sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n} \tag{25}$$

Applications:

- Finding moments of sequences
- Analyzing group delay of filters
- Computing derivatives of frequency response

Parseval's Theorem

Energy Conservation:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Interpretation:

- Total energy in time domain $=$ Total energy in frequency domain
- $|X(e^{j\omega})|^2$ is the **energy density spectrum**
- Shows how energy is distributed across frequencies

General Form:

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Application: Signal power calculation, filter energy analysis

The Convolution Theorem

Time Domain Convolution $\leftrightarrow$ **Frequency Domain Multiplication**:

$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Why This Works:

- Complex exponentials are eigenfunctions of LTI systems
- Input $e^{j\omega n}$ produces output $H(e^{j\omega})e^{j\omega n}$
- By superposition, each frequency component is scaled by $H(e^{j\omega})$

Practical Impact:

- Convolution (complex operation) $\rightarrow$ Multiplication (simple)
- Foundation for frequency-domain filtering
- Basis for fast convolution algorithms

The Modulation (Windowing) Theorem

Time Domain Multiplication $\leftrightarrow$ **Frequency Domain Convolution**:

$$y[n] = x[n]w[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

Key Points:

- This is **periodic convolution** (circular convolution)
- Integration over one period only
- Dual to the convolution theorem

Applications:

- **Windowing**: Multiplying by window function in time domain
- **Modulation**: Shifting spectrum by multiplication with $e^{j\omega_0 n}$
- **Spectral analysis**: Effect of finite-length observation

Example: Using Tables and Theorems

Problem: Find DTFT of $x[n] = a^n u[n-5]$

Example: Inverse Transform Using Partial Fractions

Given: $X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$

Example: Highpass Filter Design

Desired Frequency Response:

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_c < |\omega| < \pi \\ 0, & |\omega| < \omega_c \end{cases}$$

Approach: Express as $H(e^{j\omega}) = e^{-j\omega n_d}(1 - H_{lp}(e^{j\omega}))$

Example: Solving Difference Equations

System: $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$

Finding Frequency Response:

1. Apply DTFT to both sides:
2. Factor and solve:
3. Partial fractions:
4. Inverse Transform:

Summary of DTFT

Key Concepts:

1. **DTFT Pair**:
   - Analysis: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
   - Synthesis: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
2. **Convergence**:
   - Absolute summability $\rightarrow$ uniform convergence
   - Square summability $\rightarrow$ mean-square convergence
3. **Symmetry Properties**:
   - Real sequences have conjugate-symmetric DTFTs
   - Even/odd decomposition in both domains
4. **Transform Theorems**:
   - Convolution $\leftrightarrow$ Multiplication
   - Time shift $\leftrightarrow$ Linear phase
   - Differentiation, modulation, Parseval's theorem

Important Transform Pairs

Sequence	**DTFT**
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$a^n u[n],\ \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$
$e^{j\omega_0 n}$	$2\pi \sum_r \delta(\omega-\omega_0+2\pi r)$
$\cos(\omega_0 n)$	$\pi \sum_r [\delta(\omega-\omega_0+2\pi r)+\delta(\omega+\omega_0+2\pi r)]$
$\frac{\sin(\omega_c n)}{\pi n}$	Ideal lowpass filter
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\pi \sum_r \delta(\omega+2\pi r)$

Remember: DTFT is periodic with period 2π, so we only need to consider $\omega \in [-\pi, \pi]$