Frequency Domain Analysis of Circuits

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Why Frequency Domain Analysis?

Limitations of Time Domain:

- Differential equations for AC circuits
- Complex trig math
- Difficult for sinusoidal steady-state

Frequency Domain Advantages:

- Converts differential equations to algebra
- Easy handling of sinusoidal signals
- Simplifies AC circuit analysis

Applications:

- AC power systems (60 Hz)
- Audio systems (20 Hz - 20 kHz)
- Radio frequency circuits (MHz - GHz)
- Signal processing and filtering

Domain Transformation Tool

Phasor transform converts time-domain sinusoids to frequency-domain complex numbers

Goal for this lecture

Review frequency domain (phasor) analysis for AC circuits

Sinusoidal Signals: The Foundation

General Sinusoidal Signal:

$$v(t) = V_m \cos(\omega t + \phi)$$

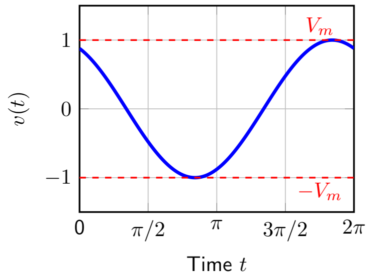
where:

- V_m = amplitude (peak value)
- ω = angular frequency (rad/s)
- ϕ = phase angle (radians or degrees)

Related Parameters:

- Frequency: $f = \omega/(2\pi)$ (Hz)
- Period: $T = 1/f = 2\pi/\omega$ (s)
- RMS value: $V_{rms} = V_m/\sqrt{2}$

Sinusoidal Waveform:



Phasor Concept: From Time to Frequency Domain

Euler's Identity:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Sinusoid as Complex Exponential:

$$v(t) = V_m \cos(\omega t + \phi)$$

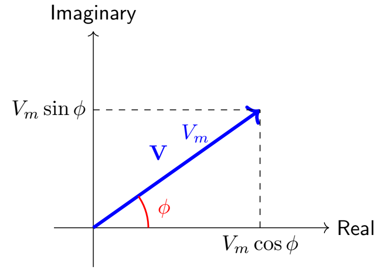
$$v(t) = \text{Re}\{V_m e^{j(\omega t + \phi)}\}$$

$$v(t) = \text{Re}\{V_m e^{j\phi} e^{j\omega t}\}$$

Phasor Definition

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi$$

Phasor Diagram:



Rectangular Form:

$$\mathbf{V} = V_m \cos\phi + jV_m \sin\phi$$

Phasor Transform: Summary

Time Domain	**Phasor Domain**	**Operation**
$V_m \cos(\omega t + \phi)$	$\mathbf{V} = V_m \angle \phi$	Domain transformation
$\frac{d}{dt}$	$j\omega$	Differentiation → multiplication
$\int dt$	$\frac{1}{j\omega}$	Integration → division
Addition	Addition	Same (LTI Systems)

Key Advantage

- ☺ **Differentiation** in time domain → **Multiplication** by $j\omega$ in phasor domain.
- ☹ Phasor analysis only works for **linear circuits** with **sinusoidal sources** at the **same frequency** in **steady-state**

Electrical Impedance

Definition:
Impedance is the ratio of phasor voltage to phasor current:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

Polar Form:

$$\mathbf{Z} = |\mathbf{Z}| \angle \theta$$

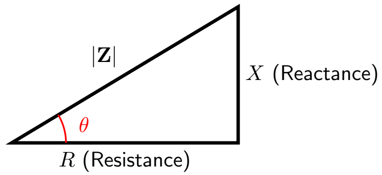
Rectangular Form:

$$\mathbf{Z} = R + jX$$

where:

- R = resistance (real part)
- X = reactance (imaginary part)

Impedance in Complex Plane:



Relationships:

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}$$

$$\theta = \tan^{-1}\left(\frac{X}{R}\right)$$

Impedance of R, L, and C

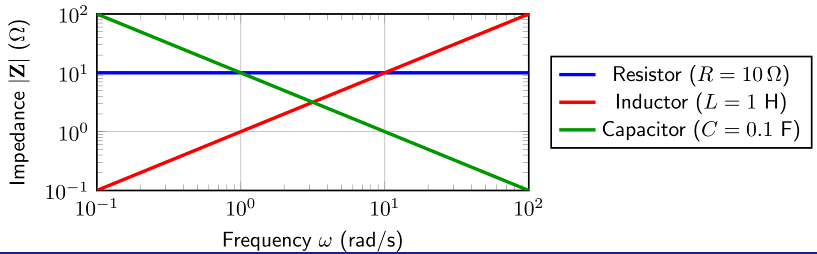
Element	Time Domain	Impedance	Phase
Resistor	$v = iR$	$\mathbf{Z}_R = R$	0
Inductor	$v = L\dfrac{di}{dt}$	$\mathbf{Z}_L = j\omega L$	$+90$
Capacitor	$i = C\dfrac{dv}{dt}$	$\mathbf{Z}_C = \dfrac{1}{j\omega C} = \dfrac{-j}{\omega C}$	-90

Resistor:
- Real impedance
- V and I in phase
- Frequency independent

Inductor:
- Imaginary impedance
- V leads I by 90°
- $|\mathbf{Z}_L| = \omega L$ increases with ω

Capacitor:
- Imaginary impedance
- I leads V by 90°
- $|\mathbf{Z}_C| = 1/(\omega C)$ decreases with ω

Frequency Behavior of Impedance



Frequency Behavior
- **Resistor**: Constant impedance (frequency independent)
- **Inductor**: High impedance at high frequencies (blocks AC, passes DC)
- **Capacitor**: Low impedance at high frequencies (blocks DC, passes AC)

Phasor Analysis: Circuit Laws

All DC circuit analysis techniques apply to phasors

Kirchhoff's Voltage Law (KVL):

$$\sum \mathbf{V}_k = 0$$

Kirchhoff's Current Law (KCL):

$$\sum \mathbf{I}_k = 0$$

Ohm's Law:

$$\mathbf{V} = \mathbf{IZ}$$

Series Impedances:

$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n$$

Parallel Impedances:

$$\mathbf{Z}_{eq}^{-1} = \mathbf{Z}_1^{-1} + \mathbf{Z}_2^{-1} + \cdots + \mathbf{Z}_n^{-1}$$

Voltage Divider:

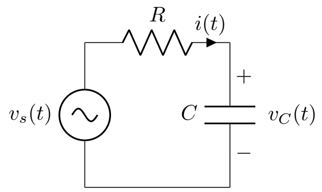
$$\mathbf{V}_k = \mathbf{V}_s\mathbf{Z}_k(\mathbf{Z}_1 + \mathbf{Z}_2)^{-1}$$

Key Point

Replace resistances with impedances, and voltages/currents with phasors. Then use the standard DC techniques

Example: Series RC Circuit

Circuit:



Given:

- $v_s(t) = V_m \cos(\omega t)$
- $R = 100\,\Omega$
- $C = 10\,\mu$F
- $\omega = 1000$ rad/s

Phasor Analysis:
Source phasor: $\mathbf{V}_s = V_m\angle 0$
Impedances:

$$\mathbf{Z}_R = 100\,\Omega$$

$$\mathbf{Z}_C = \frac{-j}{\omega C} = \frac{-j}{0.01} = -j100\,\Omega$$

Total impedance:

$$\mathbf{Z}_{eq} = R - jX_C = 100 - j100$$

$$= 141.4\angle -45\,\Omega$$

Current:

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}_{eq}} = \frac{V_m\angle 0}{141.4\angle -45} = \frac{V_m}{141.4}\angle 45$$

Example: Phasor Diagram

Voltage Divider:
Capacitor voltage:

$$\mathbf{V}_C = \mathbf{V}_s \frac{\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C}$$
$$= \mathbf{V}_s \frac{-j100}{100 - j100}$$
$$= \mathbf{V}_s \frac{100\angle -90}{141.4\angle -45}$$
$$= 0.707 V_m \angle -45$$

Resistor voltage:

$$\mathbf{V}_R = \mathbf{I}R = 0.707 V_m \angle 45$$

Phasor Diagram:

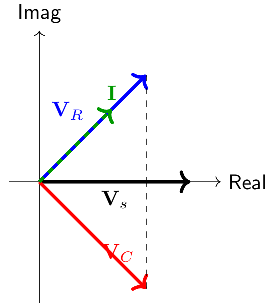
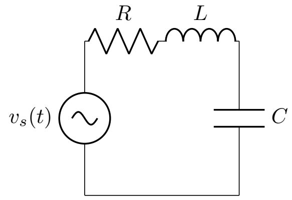


Figure 1: $\mathbf{V}_R + \mathbf{V}_C = \mathbf{V}_s$ (KVL)

Example: Series RLC Circuit

Circuit:



Total Impedance:

$$\mathbf{Z} = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$
$$= R + j(X_L - X_C)$$

Three Cases:

1. **Inductive** $(X_L > X_C)$:
   - Net reactance is positive
   - Voltage leads current
   - Behaves like RL circuit
2. **Capacitive** $(X_L < X_C)$:
   - Net reactance is negative
   - Current leads voltage
   - Behaves like RC circuit
3. **Resonant** $(X_L = X_C)$:
   - Net reactance is zero
   - $\mathbf{Z} = R$ (purely resistive)
   - V and I in phase

Resonance in RLC Circuits

Resonance Condition:
At resonance: $X_L = X_C$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

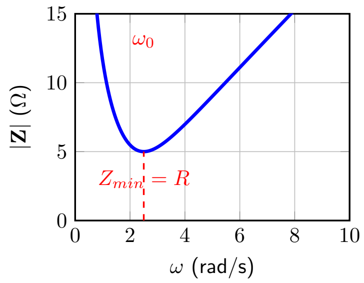
Resonant Frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

At Resonance:

- $\mathbf{Z} = R$ (minimum impedance)
- Maximum current
- Zero phase angle

Impedance vs. Frequency:



AC Power: Instantaneous and Average

Instantaneous Power:
For $v(t) = V_m \cos(\omega t)$ and $i(t) = I_m \cos(\omega t - \theta)$:

$$p(t) = v(t) \cdot i(t)$$

$$= V_m I_m \cos(\omega t) \cos(\omega t - \theta)$$

Using trig identity:

$$p(t) = \frac{V_m I_m}{2} \cos\theta + \frac{V_m I_m}{2} \cos(2\omega t - \theta)$$

Average Power:

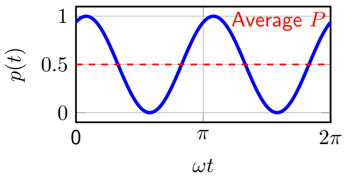
$$P = \frac{1}{T} \int_0^T p(t)\, dt = \frac{V_m I_m}{2} \cos\theta$$

Using RMS Values:

$$V_{rms} = \frac{V_m}{\sqrt{2}}, \quad I_{rms} = \frac{I_m}{\sqrt{2}}$$

Average (Real) Power

$$P = V_{rms} I_{rms} \cos\theta$$



Reactive and Apparent Power

Power Components:

1. **Real (Average) Power**:

$$P = V_{rms} I_{rms} \cos\theta \quad \text{(W)}$$

- Power dissipated (resistors)

2. **Reactive Power**:

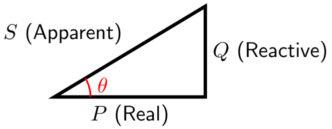
$$Q = V_{rms} I_{rms} \sin\theta \quad \text{(VAR)}$$

- Power stored/returned (L/C)

3. **Apparent Power**:

$$S = V_{rms} I_{rms} \quad \text{(VA)}$$

Power Triangle:



$$S = \sqrt{P^2 + Q^2}$$

$$P = S\cos\theta, \quad Q = S\sin\theta$$

Power Factor:

$$\text{pf} = \cos\theta = \frac{P}{S}$$

Power Factor and Its Importance

Power Factor Definition:

$$\text{pf} = \cos\theta = \frac{P}{S}$$

Range: $0 \leq \text{pf} \leq 1$

Special Cases:

- ☺ **pf = 1** (unity): purely resistive, $\theta = 0$
- ☹ **pf = 0**: purely reactive, $\theta = \pm 90$

Leading vs. Lagging:

- Lagging pf: inductive load (current lags voltage)
- Leading pf: capacitive load (current leads voltage)

Low Power Factor Problems

- ☹ Higher current required
- ☹ Larger conductor sizes needed
- ☹ More I^2R losses in transmission

Power Factor Correction:
Add capacitors in parallel with inductive loads to:

- ☺ Increase power factor
- ☺ Reduce reactive power
- ☺ Lower current draw

Power in Circuit Elements

Element	Phase	Real Power P	Reactive Power Q	pf
Resistor	$\theta = 0$	I^2R	0	1
Inductor	$\theta = 90$	0	I^2X_L (positive)	0
Capacitor	$\theta = -90$	0	$-I^2X_C$ (negative)	0

Key Observations

- Only **resistors** dissipate real power (convert to heat · or light if you mess up)
- **Inductors** and **capacitors** store and return energy (reactive power)
- Reactive power from L and C have opposite signs (can cancel to form resonant networks)

Summary: Frequency Domain Analysis

Phasor Analysis:

- Transform: $V_m \cos(\omega t + \phi) \leftrightarrow V_m\angle\phi$
- ☺ Differential equations → algebra
- $d/dt \to j\omega$, $\int dt \to 1/(j\omega)$

Impedance:

- $\mathbf{Z} = R + jX$
- Resistor: $\mathbf{Z}_R = R$
- Inductor: $\mathbf{Z}_L = j\omega L$
- Capacitor: $\mathbf{Z}_C = 1/(j\omega C)$

Circuit Analysis:

- ☺ All DC techniques apply
- KVL, KCL, voltage/current dividers
- Series/parallel combinations

AC Power:

- Real power: $P = V_{rms}I_{rms}\cos\theta$
- Reactive power: $Q = V_{rms}I_{rms}\sin\theta$
- Apparent power: $S = V_{rms}I_{rms}$

Power Factor:

- pf $= \cos\theta = P/S$
- Lagging pf: inductive
- Leading pf: capacitive
- ☹ Low pf → higher losses

Resonance:

- Occurs when $X_L = X_C$
- $\omega_0 = 1/\sqrt{LC}$
- ☺ Minimum Z, maximum I

Comparison: Time vs. Frequency Domain

Aspect	**Time Domain**	**Frequency Domain**
Signals	$v(t)$, $i(t)$ (real functions)	$\mathbf{V}$, $\mathbf{I}$ (complex phasors)
Math	Differential equations	Algebraic equations
Circuit elements	R, L, C (time relations)	$\mathbf{Z}_R$, $\mathbf{Z}_L$, $\mathbf{Z}_C$ (impedances)
Analysis	Initial conditions, transients	Steady-state, magnitude/phase
Advantages	Shows time evolution	Simplifies sinusoidal analysis
Limitations	Complex for AC steady-state	Only sinusoidal steady-state

When to Use Each

Time Domain: Transients, switching, initial conditions, non-sinusoidal signals
Frequency Domain: AC steady-state, sinusoidal sources, impedance analysis