

Linear Time-Invariant Systems

Maxx Seminario

University of Nebraska-Lincoln

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Introduction to LTI Systems

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Introduction

Convolution

LTI System

Properties

Stability of LTI

Systems

- In discrete-time signal processing, a particularly important class of systems consists of those that are both **linear** and **time invariant**.
- These two properties in combination lead to especially convenient representations for such systems.
- Most important, this class of systems has significant signal-processing applications.
- LTI systems can be completely characterized by their impulse response.

Linear Systems and Superposition

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- The class of linear systems is defined by the **principle of superposition**.
- If linearity is combined with the representation of a general sequence as a linear combination of delayed impulses, a linear system can be completely characterized by its impulse response.
- Let $h_k[n]$ be the response of the system to the input $\delta[n - k]$, an impulse occurring at $n = k$.

System Response using Superposition

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- Using the impulse representation of the input:

$$y[n] = T \left\{ \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \right\}$$

- Applying the principle of superposition:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n - k]\} = \sum_{k=-\infty}^{\infty} x[k] h_k[n]$$

- If only linearity is imposed, then $h_k[n]$ depends on both n and k .

Time Invariance Property

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- The property of **time invariance** implies that if $h[n]$ is the response to $\delta[n]$, then the response to $\delta[n - k]$ is $h[n - k]$.
- With this additional constraint, the system response becomes:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k], \quad \text{for all } n$$

- An LTI system is completely characterized by its impulse response $h[n]$.

LTI Output as Responses to Individual Input Samples

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- Let's consider a concrete example with:

- **Input sequence:**

$$x[n] = \begin{cases} 1, & n = 0 \\ -1, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

- **Impulse response:**

$$h[n] = \begin{cases} 1, & n = 0 \\ 0.5, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

LTI Output as Responses to Individual Input Samples: Visualization

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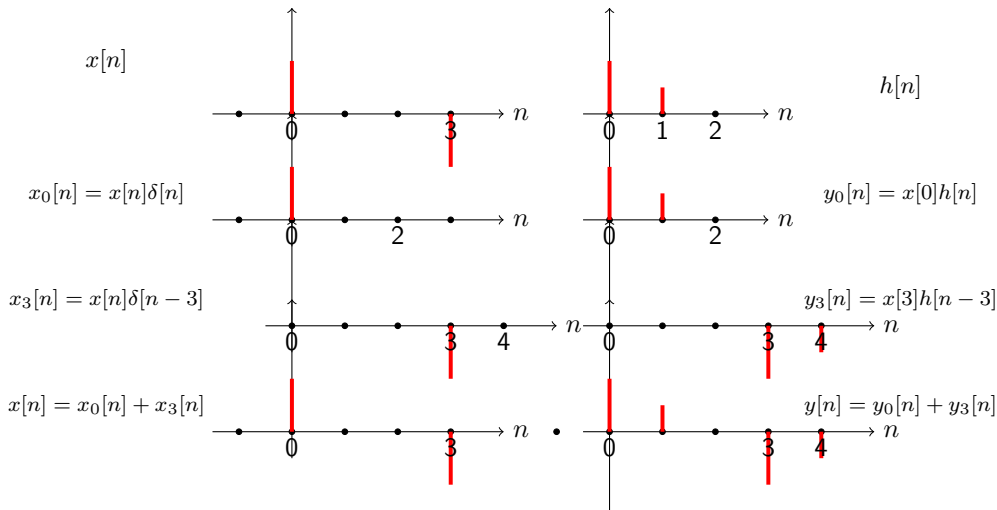
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The Convolution Sum

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- The equation:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

is referred to as the **convolution sum**.

- We represent this by the operator notation:

$$y[n] = x[n] * h[n]$$

- The operation of discrete-time convolution takes two sequences $x[n]$ and $h[n]$ and produces a third sequence $y[n]$.

Transforming $h[k]$ into $h[n - k]$

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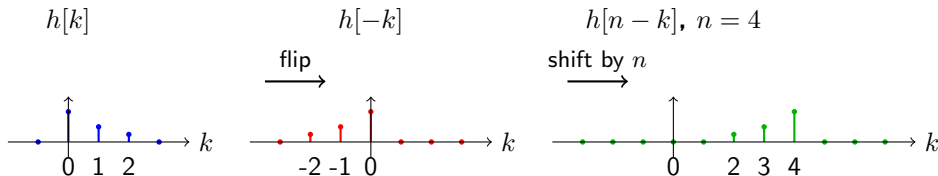
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$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

- 1 The original sequence $h[k]$.
- 2 Flipping around the y-axis to get $h[-k]$.
- 3 Shifting by n to get $h[n - k]$ (or equivalently $h[-(k - n)]$).



- Note: $h[n - k] = h[-(k - n)]$ represents $h[-k]$ shifted right by n samples.

Computing Convolution: Key Steps

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- To compute $y[n]$, we need to form the sequence $h[n - k]$ for $-\infty < k < \infty$.
- Key insight: $h[n - k] = h[-(k - n)]$
- Steps to form $h[n - k]$ from $h[k]$:
 - 1 Reverse $h[k]$ in time about $k = 0$ to get $h[-k]$
 - 2 Delay the time-reversed signal by n samples to get $h[n - k]$
- Then multiply $x[k]$ and $h[n - k]$ sample by sample and sum all products.

Example 1: Convolution of Two Sequences

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- Consider a system with impulse response:

$$h[n] = u[n] - u[n - N] = \begin{cases} 1, & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

- Input signal:

$$x[n] = a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Example 1 : Convolution of Two Sequences (Case 1: $n < 0$)

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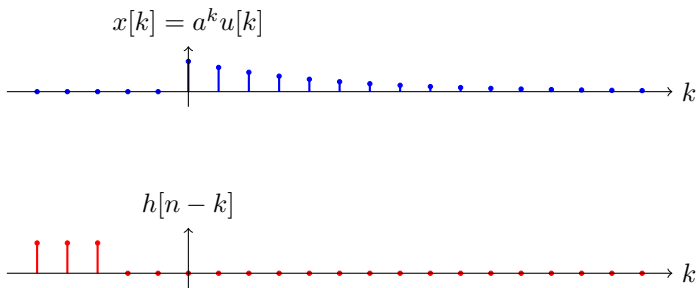
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■ Case 1: $n < 0$

$$y[n] = 0$$

- For $n < 0$, there is no overlap between $x[k]$ and $h[n - k]$, so the output is zero.

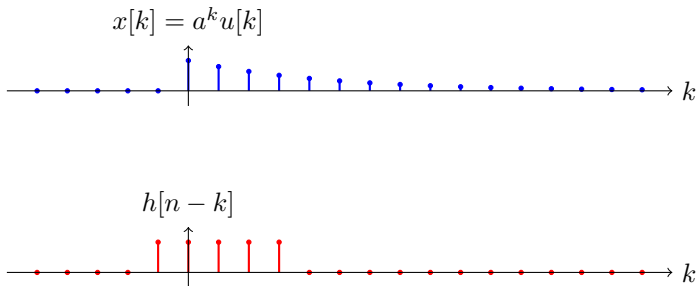


Example 1 : Convolution of Two Sequences (Case 2: $0 \leq n \leq N - 1$)

- **Case 2:** $0 \leq n \leq N - 1$

$$y[n] = \sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a}$$

- For $0 \leq n \leq N - 1$, there is partial overlap between $x[k]$ and $h[n - k]$.

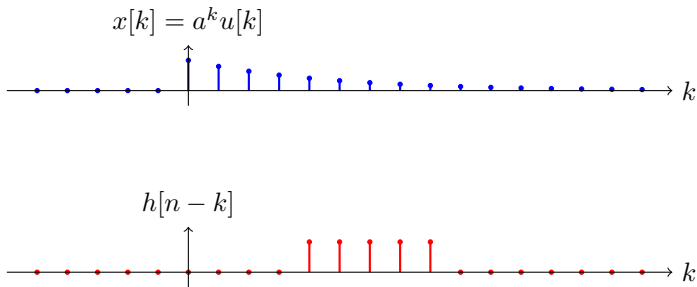


Example 1 : Convolution of Two Sequences (Case 3: $n > N - 1$)

■ Case 3: $n > N - 1$

$$y[n] = \sum_{k=n-N+1}^n a^k = a^{n-N+1} \frac{1 - a^N}{1 - a}$$

- For $n > N - 1$, there is full overlap between $x[k]$ and $h[n - k]$.



Example 1: Complete Solution

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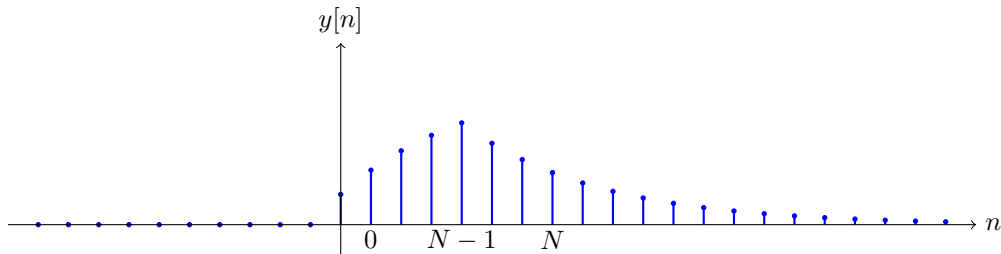
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- The complete closed-form expression for $y[n]$ is:

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-a^{n+1}}{1-a}, & 0 \leq n \leq N-1 \\ a^{n-N+1} \frac{1-a^N}{1-a}, & n > N-1 \end{cases}$$

- Below is a visualization of $y[n]$ for $a = 0.8$ and $N = 5$.



Example 2: Convolution of Two Sequences

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- Consider a system with impulse response:

$$h[n] = u[n] - u[n - 4] = \begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

- Input signal:

$$x[n] = u[n - 2] - u[n - 6] = \begin{cases} 1, & 2 \leq n \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

Visualizing the Solution to Example 2

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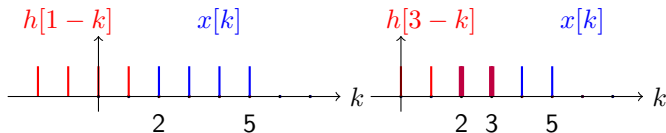
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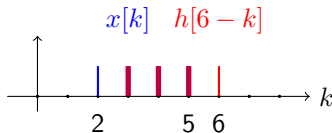
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Case 1: (no overlap)

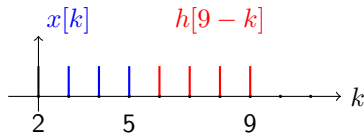


Case 2: (increasing overlap)

Case 3: (full overlap)



Case 4: (decreasing overlap)



Solution to Example 2

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1 Case 1: $n < 2$

$$y[n] = 0 \quad (\text{no overlap between } x[k] \text{ and } h[n - k])$$

2 Case 2: $2 \leq n \leq 5$

$$y[n] = \sum_{k=2}^n h[n - k] = \min(n - 2 + 1, 4)$$

3 Case 3: $6 \leq n \leq 8$

$$y[n] = \sum_{k=n-3}^5 h[n - k] = 4$$

4 Case 4: $n \geq 9$

$$y[n] = \sum_{k=n-3}^5 h[n - k] = 6 - n$$

Complete Output for Example 2

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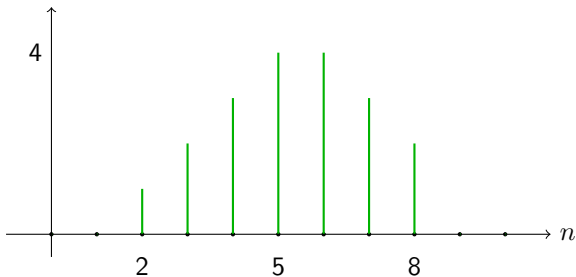
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Complete Output $y[n]$



Properties of Linear Time-Invariant (LTI) Systems

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- LTI systems are defined by the convolution operation:

$$y[n] = x[n] * h[n]$$

- The impulse response $h[n]$ fully characterizes an LTI system.
- Key properties derive from the convolution operation:
 - Commutativity
 - Distributivity over addition
 - Associativity

Commutativity of LTI Systems

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- The convolution operation is commutative:

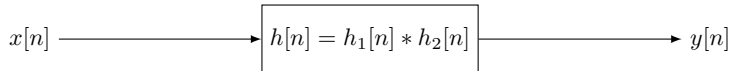
$$x[n] * h[n] = h[n] * x[n]$$

- This means the order of the input signal and impulse response does not matter.
- Proof:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \quad (\text{substitute } m = n - k, \text{ so } k = n - m) \\ &= h[n] * x[n] \end{aligned}$$

- Implication:
 - A system with input $x[n]$ and impulse response $h[n]$ produces the same output as a system with input $h[n]$ and impulse response $x[n]$.

Commutativity of LTI Systems: Cascade Connection



Distributivity of LTI Systems

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- The convolution operation distributes over addition:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- This property allows parallel systems to be simplified into a single equivalent system.
- Example (Parallel Combination):
 - Two systems with impulse responses $h_1[n]$ and $h_2[n]$.
 - Equivalent system impulse response:

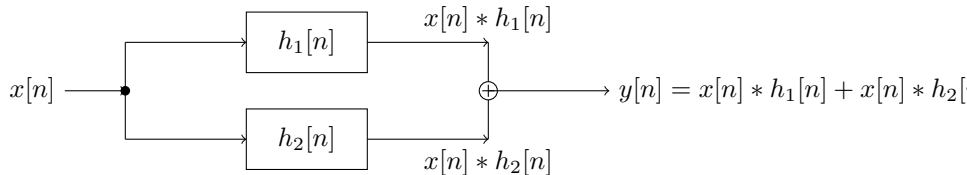
$$h[n] = h_1[n] + h_2[n]$$

Block Diagram: Distributivity of LTI Systems

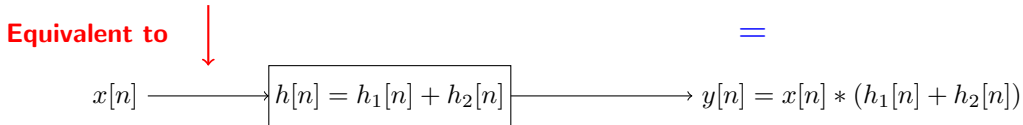
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Parallel Systems



Single Equivalent System



Key Point:
Both systems produce
identical outputs

Stability and Causality of LTI Systems

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■ Stability:

- An LTI system is stable if its impulse response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

■ Causality:

- An LTI system is causal if $h[n] = 0$ for $n < 0$.

■ Example:

- The accumulator with $h[n] = u[n]$ is causal but not stable.
- A system with $h[n] = a^n u[n]$, $|a| < 1$, is both causal and stable.

Simplifications Using Convolution Properties

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- Delay System:

$$x[n] * \delta[n - n_d] = x[n - n_d]$$

- Example: Cascading Systems

- Forward Difference:

$$h_1[n] = \delta[n + 1] - \delta[n]$$

- Delay:

$$h_2[n] = \delta[n - 1]$$

- Equivalent system:

$$h[n] = h_1[n] * h_2[n] = \delta[n] - \delta[n - 1]$$

Stability of LTI Systems

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- **Definition:** A stable system produces bounded output for every bounded input (BIBO)
- **LTI Stability Condition:**

$$\text{System is stable} \iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- **Proof:**

- *Sufficient:* If $\sum |h[n]| < \infty$ and $|x[n]| \leq B_x$, then:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|$$

- *Necessary:* If $\sum |h[n]| = \infty$, we can construct a bounded input that produces unbounded output
- **Key Insight:** For LTI systems, stability depends only on the impulse response!

Causality of LTI Systems

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- **Definition:** A causal system's output at time n_0 depends only on inputs at times $n \leq n_0$

- **LTI Causality Condition:**

$$\text{System is causal} \iff h[n] = 0 \text{ for } n < 0$$

- **Justification:**

- From convolution: $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
- For causality: $y[n]$ should not depend on $x[m]$ for $m > n$
- This requires $h[k] = 0$ when $n - k > n$, i.e., when $k < 0$

- **Terminology:**

- A sequence that is zero for $n < 0$ is called a *causal sequence*
- Such sequences could be impulse responses of causal systems

Example 1: Ideal Delay System

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Problem: Analyze the stability and causality of the ideal delay system.

System Definition:

$$y[n] = x[n - n_d]$$

where n_d is a fixed integer (positive, negative, or zero).

Find:

- 1 The impulse response $h[n]$
- 2 Determine if the system is stable
- 3 Determine if the system is causal

Example 1: Ideal Delay System - Solution

Solution:

1. Impulse Response:

$$h[n] = \delta[n - n_d]$$

2. Stability Analysis:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n - n_d]| = 1 < \infty$$

⇒ **System is STABLE**

3. Causality Analysis:

- If $n_d \geq 0$: $h[n] = 0$ for $n < 0 \Rightarrow$ **CAUSAL**
- If $n_d < 0$: $h[n_d] = 1$ with $n_d < 0 \Rightarrow$ **NON-CAUSAL**

Conclusion: All delay systems are stable. Causality depends on the sign of the delay.

Example 2: Moving Average System

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Problem: Analyze the stability and causality of the moving average system.

System Definition:

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n - k]$$

where M_1 and M_2 are non-negative integers.

Find:

- 1 The impulse response $h[n]$
- 2 Determine if the system is stable
- 3 Determine conditions for causality

Example 2: Moving Average System - Solution

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1. Impulse Response:

$$h[n] = \begin{cases} \frac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2 \\ 0, & \text{otherwise} \end{cases}$$

2. Stability Analysis:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-M_1}^{M_2} \frac{1}{M_1 + M_2 + 1} = \frac{M_1 + M_2 + 1}{M_1 + M_2 + 1} = 1 < \infty$$

⇒ **System is STABLE** (FIR system with finite values)

3. Causality Analysis:

- For causality: $h[n] = 0$ for $n < 0$
- This requires $-M_1 \geq 0$, so $M_1 = 0$
- **Causal moving average:** $M_1 = 0$, $M_2 \geq 0$

Example 3: Accumulator System

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Problem: Analyze the stability and causality of the accumulator system.

System Definition:

$$y[n] = \sum_{k=-\infty}^n x[k]$$

Find:

- 1 The impulse response $h[n]$
- 2 Determine if the system is stable
- 3 Determine if the system is causal

Example 3: Accumulator System - Solution

Solution:

1. Impulse Response:

$$h[n] = \sum_{k=-\infty}^n \delta[k] = u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

2. Stability Analysis:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |u[n]| = \sum_{n=0}^{\infty} 1 = \infty$$

⇒ **System is UNSTABLE**

3. Causality Analysis:

$$h[n] = 0 \text{ for } n < 0$$

⇒ **System is CAUSAL**

Conclusion: The accumulator is causal but unstable (IIR system).

Example 4: Exponential System

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Problem: Analyze the stability and causality of the exponential system.

System Definition:

$$h[n] = a^n u[n]$$

where a is a real constant.

Find:

- 1 Determine conditions for stability
- 2 Determine if the system is causal
- 3 What happens for different values of $|a|$?

Example 4: Exponential System - Solution

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Solution:

1. Stability Analysis:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |a|^n$$

- If $|a| < 1$: $\sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty \Rightarrow$ **STABLE**
- If $|a| \geq 1$: $\sum_{n=0}^{\infty} |a|^n = \infty \Rightarrow$ **UNSTABLE**

2. Causality Analysis:

$$h[n] = a^n u[n] = 0 \text{ for } n < 0$$

\Rightarrow **System is CAUSAL**

Conclusion: The system is always causal. Stability depends on $|a| < 1$.

Summary: Stability and Causality Examples

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System	Impulse Response	Stable?	Causal?
Ideal Delay	$\delta[n - n_d]$	Yes	Yes if $n_d \geq 0$ No if $n_d < 0$
Moving Average	$\frac{1}{M_1 + M_2 + 1}$	Yes	Yes if $M_1 = 0$ No if $M_1 > 0$
Accumulator	$u[n]$	No	Yes
Exponential	$a^n u[n]$	Yes if $ a < 1$ No if $ a \geq 1$	Yes

Key Observations:

- FIR systems (finite impulse response) are always stable
- IIR systems (infinite impulse response) may or may not be stable
- Causality is determined by whether $h[n] = 0$ for $n < 0$