ECEN 463/863

Maxx Seminario

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Accumulator
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Moving Average
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General Solution

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Solution

Summary

Linear Constant-Coefficient Difference Equations

Maxx Seminario

University of Nebraska-Lincoln

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Introduction to Difference Equations

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- **Important Class of LTI Systems**: Systems where input $x[n]$ and output $y[n]$ satisfy a difference equation
- **General Form**:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- **Parameters**:
  - N: Order of the system (highest delay in output)
  - a_k: Output coefficients (constant)
  - b_m: Input coefficients (constant)
  - M: Highest delay in input terms
- **Why Important?**:
  - Provides computational algorithms for LTI systems
  - Foundation for digital filter implementation
  - Connects time-domain and system analysis

Example 1: The Accumulator System

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Problem: Find the difference equation for the accumulator system.
System Definition:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Approach:

- Rewrite the sum to separate current and past inputs
- Use the relationship between $y[n]$ and $y[n-1]$

Example 1: Accumulator Solution

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Step 1: Rewrite the accumulator equation:

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

Step 2: Recognize that the sum is $y[n-1]$:

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

Step 3: Substitute to get the difference equation:

$$y[n] = x[n] + y[n-1]$$

Standard Form:

$$y[n] - y[n-1] = x[n]$$

Block Diagram: Recursive Accumulator

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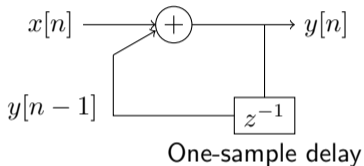
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One-sample delay

Recursive Implementation:

- Each output value computed using previously computed values
- $y[n] = x[n] + y[n-1]$: Add current input to previous output
- Requires initial condition (e.g., $y[-1] = 0$)

Example 2: Moving Average System

Problem: Find difference equation for causal moving average system.

System Definition (with $M_1 = 0$, so the system is causal):

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n - k]$$

Two Approaches:

1. **Direct (Non-recursive)**: Use convolution form directly
2. **Recursive**: Express as cascade of simpler systems

Moving Average: Direct Implementation

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Direct Form:

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Standard Form:

$$y[n] = \sum_{k=0}^{M_2} \frac{1}{M_2 + 1} x[n-k]$$

Disadvantages of Direct Implementation:

- Requires $(M_2 + 1)$ multiplications per output sample
- Must store $(M_2 + 1)$ input samples in memory
- $O(M_2)$: Computational cost grows linearly with window size M_2

Block Diagram: Recursive Moving Average

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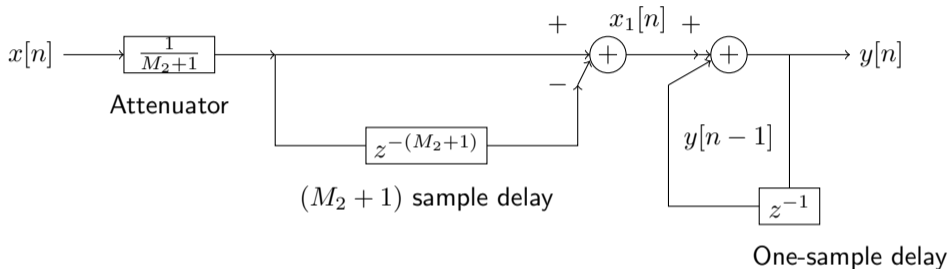
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Signal Flow:
- Input $x[n] \to$ Attenuator $\frac{1}{M_2+1}$
- Attenuated signal splits: direct path and (M_2+1) sample delay
- Sum: $x_1[n] = \frac{1}{M_2+1}[x[n] - x[n-(M_2+1)]]$
- $x_1[n] \to$ Accumulator $\to$ Output $y[n]$

Moving Average as Difference Equation

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Intermediate signal:

$$x_1[n] = \frac{1}{M_2 + 1}[x[n] - x[n - M_2 - 1]]$$

Accumulator relation: $y[n] = x_1[n] + y[n - 1]$

Final recursive form:

$$y[n] - y[n - 1] = \frac{1}{M_2 + 1}[x[n] - x[n - M_2 - 1]]$$

Note: There is an unlimited number of distinct difference equations to represent an LTI I/O relation.

General Solution of Difference Equations

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Key Issue: Difference equation alone does not uniquely specify output!

General Solution Structure:

$$y[n] = y_p[n] + y_h[n]$$

- $y_p[n]$: Particular solution (satisfies original equation)
- $y_h[n]$: Homogeneous solution (satisfies homogeneous equation)

Homogeneous Equation (x[n] = 0):

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

Homogeneous Solution Form:

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n$$

where z_m are roots of characteristic polynomial $A(z) = \sum_{k=0}^{N} a_k z^{-k} = 0$

Auxiliary Conditions

Need for Auxiliary Conditions:

- N undetermined coefficients A_m in homogeneous solution
- Need N auxiliary (boundary) conditions for unique solution

Types of Auxiliary Conditions:

1. **Fixed Values**: Specify $y[-1], y[-2], \ldots, y[-N]$
2. **Initial Rest**: If $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$

Forward and Backward Computation for Specific Class of Difference Equations

Specific Class of Difference Equations:

- Inputs $x[n]$ and Outputs $y[n]$ satisfy an Nth-order linear constant coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Recursive Computation (forward):

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

Recursive Computation (backward):

$$y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_N} x[n-k]$$

Example 3: First-Order Difference Equation

Problem: Solve the difference equation with given input and initial condition.

Difference Equation:
$$y[n] = ay[n-1] + x[n]$$

Input:
$$x[n] = K\delta[n]$$

Auxiliary Condition:
$$y[-1] = c$$

Find: The complete solution $y[n]$ for all n.

Example 3: Solution for $n > 0$

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Method: Use recursive computation starting from $n = 0$.

For $n = 0$:

$$y[0] = ay[-1] + x[0] \tag{1}$$
$$= ay[-1] + K\delta[0] \tag{2}$$
$$= ac + K \cdot 1 \tag{3}$$
$$= ac + K \tag{4}$$

For $n = 1$:

$$y[1] = ay[0] + x[1] \tag{5}$$
$$= a(ac + K) + K\delta[1] \tag{6}$$
$$= a^2c + aK + K \cdot 0 \tag{7}$$
$$= a^2c + aK \tag{8}$$

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Example 3: Solution for $n > 0$

For $n = 2$:

$$y[2] = ay[1] + x[2] \tag{9}$$
$$= a(a^2 c + aK) + K\delta[2] \tag{10}$$
$$= a^3 c + a^2 K + 0 \tag{11}$$
$$= a^3 c + a^2 K \tag{12}$$

For $n = 3$:

$$y[3] = ay[2] + x[3] \tag{13}$$
$$= a(a^3 c + a^2 K) + 0 \tag{14}$$
$$= a^4 c + a^3 K \tag{15}$$

Pattern Recognition: For $n \geq 0$: $y[n] = a^{n+1} c + a^n K$

Example 3: Solution for $n < 0$

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For $n < 0$: Use backward recursion.
Rearrange difference equation:

$$y[n-1] = \frac{y[n] - x[n]}{a}$$

For $n = 0$ (find $y[-1]$):

$$y[-1] = \frac{y[0] - x[0]}{a} = \frac{(ac + K) - K}{a} = \frac{ac}{a} = c$$

This confirms our auxiliary condition
For $n = -1$ (find $y[-2]$):

$$y[-2] = \frac{y[-1] - x[-1]}{a} = \frac{c - 0}{a} = \frac{c}{a}$$

General pattern for $n < 0$:

$$y[n] = ca^{n+1} = \frac{c}{a^{n+1}}$$

Example 3: Complete Solution

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Final Answer:

$$y[n] = \begin{cases} ca^{n+1}, & n < 0 \\ a^n(ac + K), & n \geq 0 \end{cases}$$

Alternative Compact Form:

$$y[n] = ca^{n+1} + Ka^n u[n]$$

where $u[n]$ is the unit step function.

Physical Interpretation:

- **Homogeneous part**: ca^{n+1} (due to initial condition)
- **Forced response**: $Ka^n u[n]$ (due to impulse input)
- **System behavior**: Exponential with base a
- **Stability**: System stable if $|a| < 1$

Summary: Linear Constant-Coefficient Difference Equations

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- **Definition**:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

  Describes the relationship between input $x[n]$ and output $y[n]$ in LTI systems.
- **Key Components**:
    - N and M: Order of the system
    - a_k, b_m: Constant coefficients
    - Recursive and direct computation methods
- **General Solution**:

$$y[n] = y_p[n] + y_h[n]$$

    - $y_p[n]$: Particular solution (due to input $x[n]$)
    - $y_h[n]$: Homogeneous solution (due to initial conditions)
    - Auxiliary conditions: Necessary to determine a unique / particular solutions.
- **Applications**: Digital filters, computational algorithms for LTI systems, and time-domain analysis.