The z-Transform

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Overview: The z-Transform

- **Motivation**:
  - Fourier transform doesn't converge for all sequences
  - Need a more general transform that encompasses broader class of signals
  - z-transform notation often more convenient for analysis
- **Key Relationships**:
  - z-transform for discrete-time $\leftrightarrow$ Laplace transform for continuous-time
  - Similar relationship to corresponding Fourier transforms
  - Fourier transform is special case: $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$
- **Today's Topics**:
  - Definition and convergence of z-transform
  - Region of Convergence (ROC) properties
  - Examples of common z-transform pairs
  - Properties of rational z-transforms

The z-Transform

Definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \text{(bilateral z-transform)} \tag{1}$$

where z is a complex variable.

z-Transform Operator:

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)$$

Notation: $x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$

One-sided z-Transform:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \quad \text{(unilateral z-transform)}$$

Relationship to Fourier Transform

Complex Variable in Polar Form:

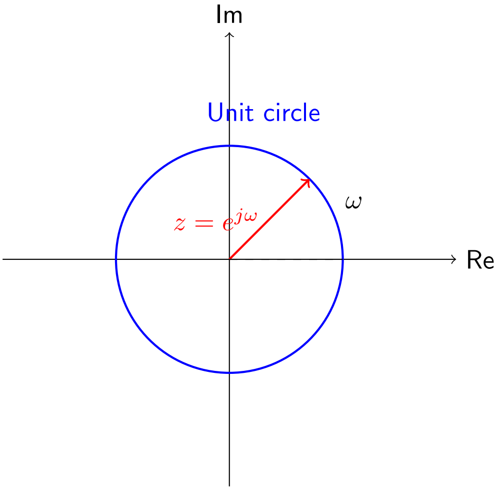
$$z = re^{j\omega} \tag{2}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} \tag{3}$$

$$= \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n} \tag{4}$$

Interpretation:

- z-transform = Fourier transform of $x[n]r^{-n}$
- For $r = 1$ (unit circle): $X(e^{j\omega}) =$ Fourier transform
- $|z| = 1$ defines the unit circle in z-plane



Region of Convergence (ROC)

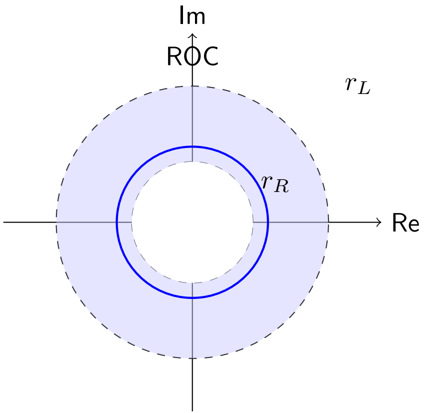
Definition: Set of values of z for which the z-transform converges

Convergence Condition:

$$|X(re^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

Key Properties:

- Convergence depends only on $|z| = r$
- ROC consists of a ring in z-plane: $r_R < |z| < r_L$
- If ROC includes unit circle $\Rightarrow$ Fourier transform exists

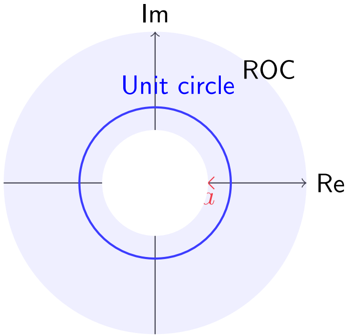


Example: Right-Sided Exponential

Signal: $x[n] = a^n u[n]$
z-Transform:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} \tag{5}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n \tag{6}$$



Convergence:

- Requires $|az^{-1}| < 1$
- Therefore: $|z| > |a|$

Closed Form:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad \text{for } |z| > |a|$$

Example: Left-Sided Exponential

Signal: $x[n] = -a^n u[-n-1]$

z-Transform:

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} \tag{7}$$

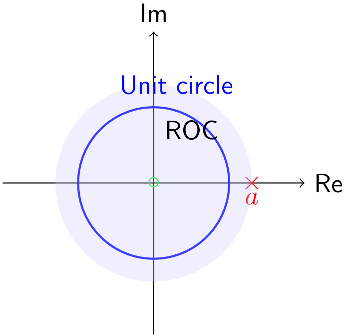
$$= 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n \tag{8}$$

Convergence:

- Requires $|a^{-1} z| < 1$
- Therefore: $|z| < |a|$

Closed Form:

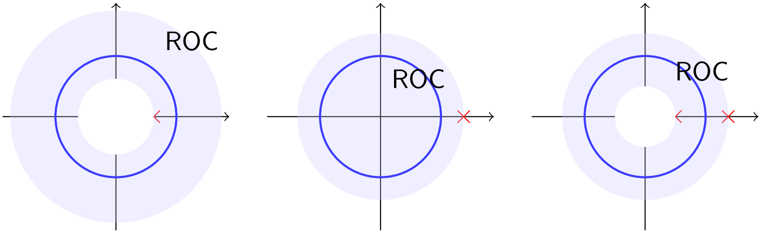
$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad \text{for } |z| < |a|$$



Properties of the ROC

1. **General Form**: $0 \leq r_R < |z| < r_L \leq \infty$ (annulus)
2. **Fourier Transform**: Exists iff ROC includes unit circle
3. **Poles**: ROC cannot contain any poles
4. **Finite-Duration**: ROC is entire z-plane except possibly $z = 0$ or $z = \infty$
5. **Right-Sided**: ROC extends outward from outermost pole
6. **Left-Sided**: ROC extends inward from innermost pole
7. **Two-Sided**: ROC is a ring bounded by poles
8. **Connected Region**: ROC must be connected

ROC for Different Sequence Types



Right-sided Left-sided Two-sided

Example: Sum of Two Exponentials

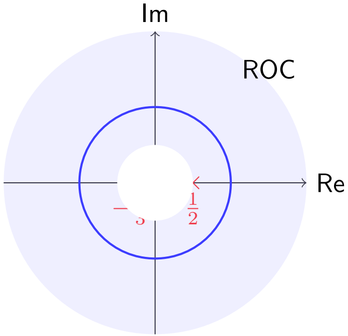
Signal:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

z-Transform (by linearity):

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} \tag{9}$$

$$= \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})} \tag{10}$$



ROC: Intersection of individual ROCs

- First term: $|z| > \frac{1}{2}$
- Second term: $|z| > \frac{1}{3}$
- Combined: $|z| > \frac{1}{2}$

Example: Two-Sided Exponential

Signal:

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

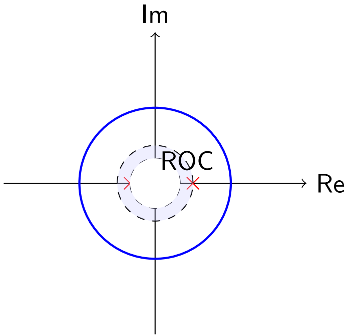
z-Transform:

$$X(z) = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

ROC:

- First term (right-sided): $|z| > \frac{1}{3}$
- Second term (left-sided): $|z| < \frac{1}{2}$
- Combined: $\frac{1}{3} < |z| < \frac{1}{2}$

Note: ROC doesn't include unit circle $\Rightarrow$ no Fourier transform



Finite-Length Sequences

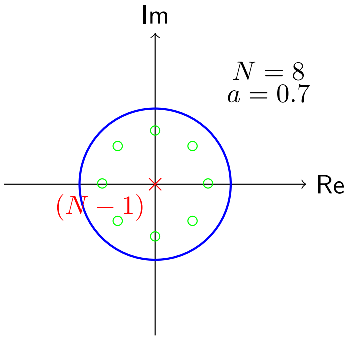
Example: $x[n] = a^n, \quad 0 \leq n \leq N-1$

z-Transform:

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} \tag{11}$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} \tag{12}$$

$$= \frac{z^N - a^N}{z^{N-1}(z-a)} \tag{13}$$



Zeros:

$$z_k = ae^{j2\pi k/N}, \quad k = 0, 1, ..., N-1$$

ROC: Entire z-plane except $z = 0$ (assuming $|a| < \infty$)

Common z-Transform Pairs

Sequence	**Transform**	**ROC**
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Rational z-Transforms

General Form:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

Key Points:

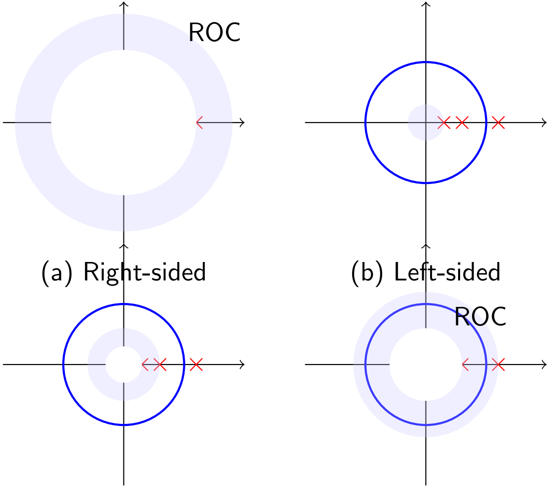
- Zeros: roots of $P(z) = 0$
- Poles: roots of $Q(z) = 0$
- ROC determined by pole locations
- Any sum of exponentials $\Rightarrow$ rational z-transform

Pole-Zero Plot:

- Poles: marked with $\times$
- Zeros: marked with $\circ$
- Must specify ROC to uniquely determine sequence

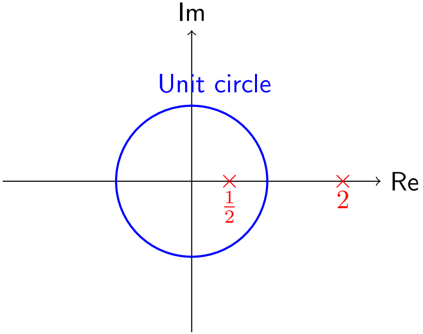
Example: Multiple ROCs for Same Pole-Zero Pattern

Given: Poles at $z = a, b, c$ with $|a| < |b| < |c|$



Stability and Causality

Example: System with poles at $z = \frac{1}{2}$ and $z = 2$



Three possible ROCs:

1. $|z| < \frac{1}{2}$: Left-sided, not stable
2. $\frac{1}{2} < |z| < 2$: Two-sided, stable, not causal
3. $|z| > 2$: Right-sided, causal, not stable

Summary

- **z-Transform Definition**:
  - Generalization of Fourier transform
  - Power series in complex variable z
  - Reduces to Fourier transform on unit circle
- **Region of Convergence**:
  - Critical for uniquely specifying sequence
  - Depends on sequence type (right/left/two-sided)
  - Cannot contain poles
  - Must be connected annular region
- **Rational z-Transforms**:
  - Result from sums of exponentials
  - Characterized by poles and zeros
  - ROC determines sequence properties
- **Next Time**: z-Transform properties and inverse z-transform