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Maxx Seminario

Introduction

Frequency
Domain Analysis

Example:
Nonstopic Data

Example: Moving
Average

Summary

Continuous-Time Processing of Discrete-Time Signals

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University of Nebraska-Lincoln

Fall 2025

Overview: System Architecture
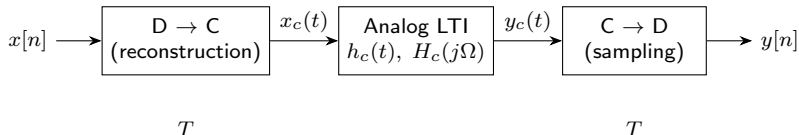
Complementary to Previous Topic:

- Previously: Discrete-time processing of continuous-time signals
- Now: Continuous-time processing of discrete-time signals

General System Configuration:

$$x[n] \longrightarrow \boxed{\begin{array}{c} \text{D} \rightarrow \text{C} \\ \text{(reconstruction)} \end{array}} \xrightarrow{x_c(t)} \boxed{\begin{array}{c} \text{Analog LTI} \\ h_c(t), \ H_c(j\Omega) \end{array}} \xrightarrow{y_c(t)} \boxed{\begin{array}{c} \text{C} \rightarrow \text{D} \\ \text{(sampling)} \end{array}} \longrightarrow y[n]$$

$$T \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T$$

Key Characteristics:

- Input and output: discrete-time sequences
- Intermediate processing: continuous-time domain
- Provides useful interpretation of certain discrete-time systems
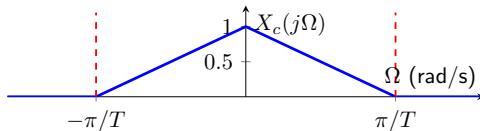- Not typically implemented

Bandlimited Signal Constraint

Fundamental Property:
The ideal D/C converter produces a bandlimited signal:

$$X_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T$$

Consequence: No aliasing in C/D conversion

- $y_c(t)$ is also bandlimited: $Y_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$
- Sampling $y_c(t)$ at rate $1/T$ satisfies Nyquist criterion
- Perfect reconstruction of $y[n]$ from $y_c(nT)$

Time-Domain Representations

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Input Signal - Bandlimited Interpolation:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

Output Signal - After Continuous-Time Processing:

$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

where $y[n] = y_c(nT)$

Key Relationships:

- $x[n] = x_c(nT)$ - samples of reconstructed signal
- $y[n] = y_c(nT)$ - samples of processed signal
- Both sequences connected through continuous-time system

Frequency Domain Representations

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Three Key Equations:

1. D/C Conversion:
$$X_c(j\Omega) = TX(e^{j\Omega T}), \quad |\Omega| < \pi/T$$

2. Continuous-Time Processing:
$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

3. C/D Conversion:
$$Y(e^{j\omega}) = \frac{1}{T}Y_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Overall Discrete-Time System Response

Combining All Relationships:

Substitute Eq. (1) and (2) into Eq. (3):

$$Y(e^{j\omega}) = \frac{1}{T} Y_c \left(j\frac{\omega}{T} \right)$$

$$= \frac{1}{T} H_c \left(j\frac{\omega}{T} \right) X_c \left(j\frac{\omega}{T} \right)$$

$$= \frac{1}{T} H_c \left(j\frac{\omega}{T} \right) \cdot T X(e^{j\omega})$$

Result - Effective Discrete-Time Frequency Response:

$$\boxed{H(e^{j\omega}) = H_c \left(j\frac{\omega}{T} \right), \quad |\omega| < \pi}$$

Therefore the overall system behaves as a discrete-time system with frequency response $H(e^{j\omega})$

Design Relationship

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Forward Design: Given desired $H(e^{j\omega})$, find $H_c(j\Omega)$

Solution:

$$H_c(j\Omega) = H(e^{j\Omega T}), \quad |\Omega| < \pi/T$$

Arbitrary Extension:

- Since $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, we can choose $H_c(j\Omega)$ arbitrarily above π/T
- Typically (out of convenience): $H_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$
- Makes $H_c(j\Omega)$ bandlimited

Key Notes:

- This is the **inverse** of impulse invariance
- Impulse invariance: $H(e^{j\omega}) = H_c(j\omega/T)$
- This method: $H_c(j\Omega) = H(e^{j\Omega T})$

Frequency Domain Illustration
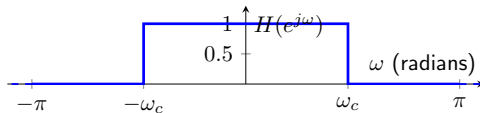
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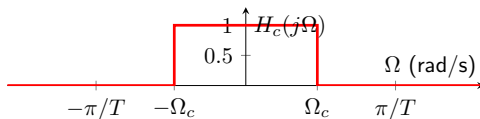
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Discrete-Time Frequency Response $H(e^{j\omega})$:

Continuous-Time Frequency Response $H_c(j\Omega)$:

Relationship: $H_c(j\Omega) = H(e^{j\Omega T})$ for $|\Omega| < \pi/T$

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Example: Noninteger Delay System

Discrete-Time Frequency Response:

$$H(e^{j\omega}) = e^{-j\omega\Delta}, \quad |\omega| < \pi$$

Case 1 - **Integer Delay** ($\Delta = n_0$, integer):

$$y[n] = x[n - n_0]$$

Straightforward interpretation: shift sequence by n_0 samples

Case 2 - **Noninteger Delay** (Δ not integer):

- Expression $y[n] = x[n - \Delta]$ has no direct meaning
- Cannot shift discrete sequence by fractional samples
- Need continuous-time interpretation

Continuous-Time Interpretation

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Apply Design Relationship:

$$H_c(j\Omega) = H(e^{j\Omega T}) = e^{-j\Omega \Delta T}$$

We recognize this is an ideal time delay

$$y_c(t) = x_c(t - \Delta T)$$

Physical Interpretation:

1. Start with discrete sequence $x[n]$
2. Reconstruct bandlimited $x_c(t)$ via D/C converter
3. Delay $x_c(t)$ by ΔT seconds
4. Sample delayed signal to get $y[n] = y_c(nT)$

Therefore the noninteger delay operates on the **interpolated** continuous signal

Noninteger Delay: Time Domain Visualization

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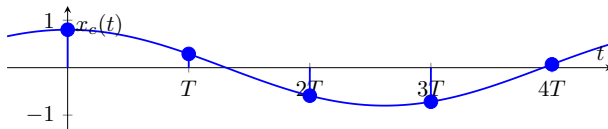Introduction

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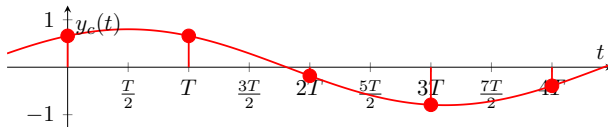
Example:
Noninteger Delay

Example: Moving
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Example: $\Delta = 0.5$ (half-sample delay)

Input: Interpolated Signal with Samples $x[n]$

Output: Delayed by $T/2$, Sampled at $y[n]$

Example: Moving-Average System

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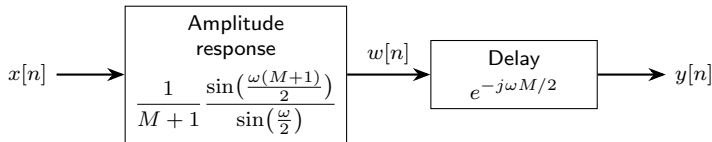
Example:
Noise-type Data

Example: Moving
Average

Summary

General $(M+1)$-**Point Moving Average**:

$$y[n] = \frac{1}{M+1} \sum_{k=0}^{M} x[n-k]$$

Frequency Response (from DTFT Lecture):

$$H(e^{j\omega}) = \frac{1}{M+1} \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$$

Decomposition:

Moving Average: Integer vs. Noninteger Delay

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Case 1 - **Odd Number of Points** (M even):
Example: $M = 4$ (5-point average)

$$\text{Delay} = \frac{M}{2} = 2 \text{ samples (integer)}$$

$$y[n] = w[n - 2]$$

Simple interpretation: 2-sample shift

Case 2 - **Even Number of Points** (M odd):
Example: $M = 5$ (6-point average)

$$\text{Delay} = \frac{M}{2} = 2.5 \text{ samples (noninteger)}$$

Must use continuous-time interpretation:
- Bandlimited interpolation of $w[n]$
- Continuous delay of $MT/2 = 2.5T$ seconds
- Resampling to get $y[n]$

Moving Average: Numerical Example

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Example:
Multistage Delay

Example: Moving
Average

Summary

Input: $x[n] = \cos(0.25\pi n)$
System: 6-point moving average ($M = 5$)

Frequency Response at Input Frequency:

$$H(e^{j0.25\pi}) = \frac{1}{6} \frac{\sin[3(0.25\pi)]}{\sin(0.125\pi)} e^{-j(0.25\pi)(2.5)}$$

Calculate magnitude:

$$|H(e^{j0.25\pi})| = \frac{1}{6} \frac{\sin(0.75\pi)}{\sin(0.125\pi)} \approx 0.308$$

Calculate phase:

$$\angle H(e^{j0.25\pi}) = -0.25\pi \times 2.5 = -0.625\pi$$
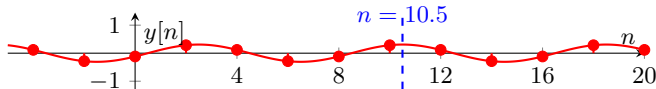
Moving Average: Visualization

Input Signal: $x[n] = \cos(0.25\pi n)$

Input: Sampled Cosine
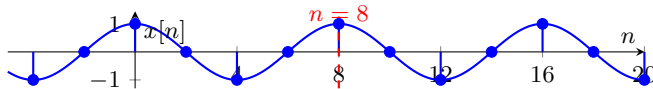
Output Signal: $y[n] = 0.308 \cos[0.25\pi(n - 2.5)]$

Output: Amplitude 0.308, Delay 2.5 samples

Summary: Concepts

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System Architecture:

$$x[n] \xrightarrow{\text{D/C}} x_c(t) \xrightarrow{H_c(j\Omega)} y_c(t) \xrightarrow{\text{C/D}} y[n]$$

Fundamental Relationships:

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right) \quad \Leftrightarrow \quad H_c(j\Omega) = H(e^{j\Omega T})$$

Limitations:

- Not typically used for actual implementation
- Requires bandlimited signals for exact analysis
- Mainly a conceptual/analytical tool

Summary: Mathematical Framework

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Time Domain:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

$$h[n] = \frac{\sin[\pi(n - \Delta)]}{\pi(n - \Delta)} \text{ for delay } \Delta$$

Frequency Domain:

$$X_c(j\Omega) = T X(e^{j\Omega T}), \quad Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T} Y_c\left(j\frac{\omega}{T}\right) = H_c\left(j\frac{\omega}{T}\right) X(e^{j\omega})$$

Design Equation:

$$\boxed{H_c(j\Omega) = H(e^{j\Omega T}), \quad |\Omega| < \pi/T}$$