

## Sampling of Continuous-Time Signals

Maxx Seminario

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## Periodic Sampling: Basic Definition

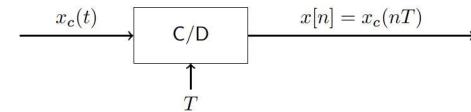
### Fundamental Sampling Equation:

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

### Key Parameters:

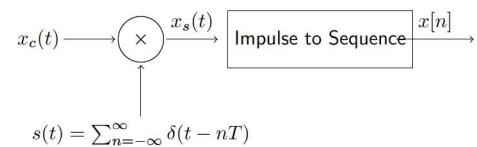
- $T$  = Sampling period (seconds)
- $f_s = \frac{1}{T}$  = Sampling frequency (samples/second or Hz)
- $\Omega_s = \frac{2\pi}{T}$  = Sampling frequency (radians/second)

### Ideal C/D Converter:



## Mathematical Model: Impulse Train Sampling

### Two-Stage Process:



### Impulse Train Modulation:

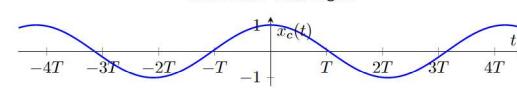
$$x_s(t) = x_c(t) \cdot s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

**Using Sifting Property**  $x(t)\delta(t) = x(0)\delta(t)$ :

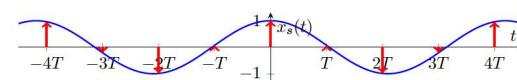
$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

## Time-Domain Visualization: Sampling Process

Continuous-Time Signal



Impulse Train Representation



## Time-Domain: Discrete-Time Sequence

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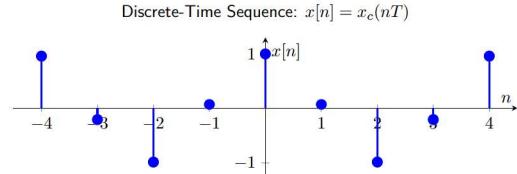
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Windowed Sinusoids



**Key Observation:** Time normalization

- $x_s(t)$ : Spacing =  $T$  seconds
- $x[n]$ : Spacing = 1 'sample index' (dimensionless)

## Frequency-Domain: Fourier Transform of Impulse Train

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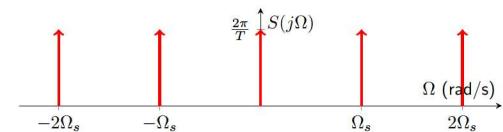
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**Impulse Train in Time Domain:**

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

**Fourier Transform (Impulse Train in Frequency):**

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s), \quad \Omega_s = \frac{2\pi}{T}$$



## Frequency-Domain Representation of Sampling

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**Convolution Property:**  $x_s(t) = x_c(t) \cdot s(t)$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

**Result - Periodic Replication:**

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

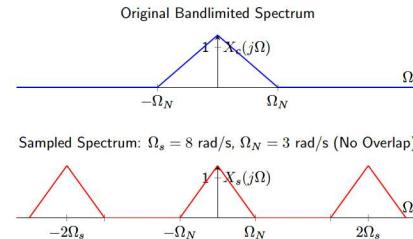
**Physical Interpretation:**

- Original spectrum  $X_c(j\Omega)$  is **replicated** at intervals of  $\Omega_s$
- Scaled by factor  $\frac{1}{T}$
- Copies centered at  $\Omega = 0, \pm\Omega_s, \pm 2\Omega_s, \dots$

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## Case 1: No Aliasing ( $\Omega_s > 2\Omega_N$ )



## Case 2: Aliasing ( $\Omega_s < 2\Omega_N$ )

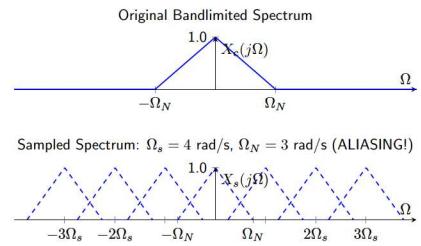
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## Relation Between $X_s(j\Omega)$ and $X(e^{j\omega})$

**From Impulse Train:**

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega nT}$$

**DTFT of Sequence:**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

**Relationship - Frequency Scaling:**

$$X(e^{j\omega}) = X_s(j\Omega) \Big|_{\Omega=\omega/T} = X_s(j\omega/T)$$

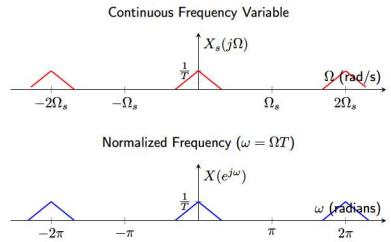
or equivalently:

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)$$

**Normalization:**  $\Omega = \Omega_s$  maps to  $\omega = 2\pi$

## Frequency Axis Normalization

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### Key Points:

- $\Omega_s = 2\pi/T$  maps to  $\omega = 2\pi$
- $X(e^{j\omega})$  is always  $2\pi$ -periodic

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## Nyquist-Shannon Sampling Theorem

### Statement:

Let  $x_c(t)$  be a bandlimited signal with:

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

Then  $x_c(t)$  is **uniquely determined** by its samples  $x[n] = x_c(nT)$  if:

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$$

### Terminology:

- $\Omega_N$  = **Nyquist frequency** (highest frequency in signal)
- $2\Omega_N$  = **Nyquist rate** (minimum sampling frequency)
- $\Omega_s/2$  = **Folding frequency**

## Reconstruction from Samples

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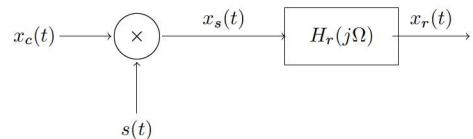
Convolution

Filtering

Hilbert

Theorem

### System:



### Ideal Reconstruction Filter:

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

where  $\Omega_N \leq \Omega_c \leq (\Omega_s - \Omega_N)$

### Output:

$$X_r(j\Omega) = H_r(j\Omega) \cdot X_s(j\Omega) = X_c(j\Omega) \quad \text{if } \Omega_s \geq 2\Omega_N$$

## Ideal Lowpass Reconstruction Filter

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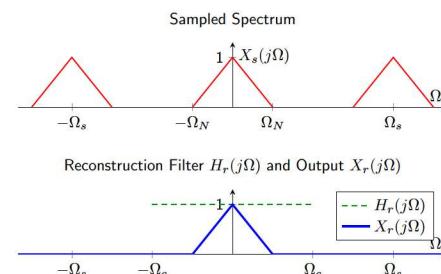
Introduction

Convolution

Filtering

Hilbert

Theorem



**Result:**  $X_r(j\Omega) = X_c(j\Omega) \Rightarrow x_r(t) = x_c(t)$

## Example 1: Sampling a Sinusoid (No Aliasing)

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**Signal:**  $x_c(t) = \cos(\Omega_0 t)$  with  $\Omega_0 = 4000\pi$  rad/s  
**Sampling:**  $T = 1/6000$  s  $\Rightarrow \Omega_s = 12000\pi$  rad/s

**Check Nyquist:**  $\Omega_s = 12000\pi > 2\Omega_0 = 8000\pi$  ✓

**Sampled Sequence:**

$$x[n] = \cos(4000\pi \cdot n/6000) = \cos(2\pi n/3)$$

Normalized frequency:  $\omega_0 = \Omega_0, T = 2\pi/3$

**Fourier Transform:**

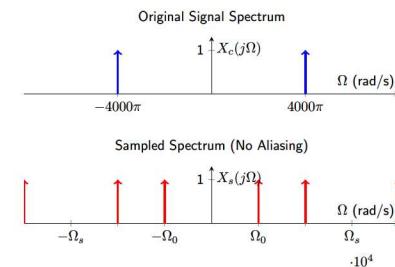
$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$

**After Sampling:**

$$X(e^{j\omega}) = \pi\delta(\omega - 2\pi/3) + \pi\delta(\omega + 2\pi/3) + \text{periodic}$$

## Example 1: Frequency Domain Visualization

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**Reconstruction:** Lowpass filter extracts center copy  $\rightarrow x_r(t) = x_c(t)$

## Example 2: Sampling a Sinusoid (With Aliasing)

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**Signal:**  $x_c(t) = \cos(\Omega_0 t)$  with  $\Omega_0 = 16000\pi$  rad/s  
**Sampling:**  $T = 1/6000$  s  $\Rightarrow \Omega_s = 12000\pi$  rad/s

**Check Nyquist:**  $\Omega_s = 12000\pi < 2\Omega_0 = 32000\pi$   $\times$

**Sampled Sequence:**

$$x[n] = \cos(16000\pi \cdot n/6000) = \cos(8\pi n/3)$$

But  $\cos(8\pi n/3) = \cos(8\pi n/3 - 2\pi n) = \cos(2\pi n/3)$

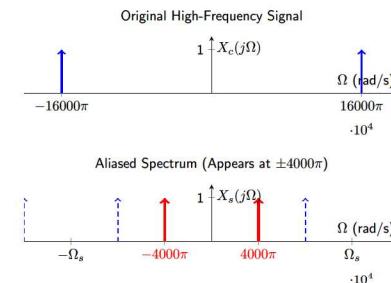
**Same samples as Example 1!**

**Alias frequency:**  $\Omega_{\text{alias}} = \Omega_s - \Omega_0 = 12000\pi - 16000\pi = -4000\pi$   
Or equivalently:  $|\Omega_{\text{alias}}| = 4000\pi$

## Example 2: Aliasing Visualization

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**Reconstruction:** Filter extracts  $4000\pi \rightarrow x_r(t) = \cos(4000\pi t) \neq x_c(t)$

## Aliasing: Multiple Signals → Same Samples

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**Key Insight:** For any integer  $k$ :

$$\cos[(\Omega_s k + \Omega_0)nT] = \cos(\Omega_0 nT)$$

**Family of Alias Frequencies:**

$$\Omega_{\text{alias},k} = \Omega_0 + k\Omega_s, \quad k = 0, \pm 1, \pm 2, \dots$$

All produce the same sequence when sampled at rate  $\Omega_s$ !

**Example:**  $\Omega_s = 12000\pi$ , sample values  $x[n] = \cos(2\pi n/3)$

Frequency $\Omega_0$ (rad/s)	Normalized $\omega_0$
$4000\pi$	$2\pi/3$
$16000\pi$	$8\pi/3 = 2\pi/3 + 2\pi$
$-8000\pi$	$-4\pi/3 = 2\pi/3 - 2\pi$
$28000\pi$	$14\pi/3 = 2\pi/3 + 4\pi$

**Resolution:** Restrict to  $|\Omega_0| \leq \Omega_s/2$  (Nyquist criterion)

## Summary: Sampling in Frequency Domain

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**Key Equation:**

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

**Three Cases:**

- 1  $\Omega_s > 2\Omega_N$ : No overlap → Perfect reconstruction possible
- 2  $\Omega_s = 2\Omega_N$ : Critical sampling (Nyquist rate)
- 3  $\Omega_s < 2\Omega_N$ : Aliasing → Information loss

**Practical Considerations:**

- Real signals are never perfectly bandlimited
- Anti-aliasing filters used before sampling
- Oversampling provides guard band
- Trade-off: sampling rate vs. computation/storage

## Reconstruction Formula (Time Domain)

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Perfect Reconstruction

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### Ideal Lowpass Filter Impulse Response:

$$h_r(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

### Reconstruction Formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT) = \sum_{n=-\infty}^{\infty} x_c(nT) \frac{\sin[\Omega_c(t-nT)]}{\pi(t-nT)}$$

For  $\Omega_c = \pi/T$ :

$$x_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

**Interpolation:** Each sample weighted by sinc function

**Shannon Interpolation Formula:** Exact for bandlimited signals

## Conclusion

### Main Results:

- 1 Sampling creates **periodic replication** in frequency domain
- 2 **Nyquist criterion:**  $\Omega_s \geq 2\Omega_N$  prevents aliasing
- 3 **Perfect reconstruction** possible if Nyquist satisfied
- 4 Frequency normalization:  $\omega = \Omega T$

### Practical Applications:

- Digital audio:  $f_s = 44.1$  kHz (covers 0-20 kHz hearing range)
- Speech:  $f_s = 8$  kHz (telephone quality)
- Video: Various rates depending on bandwidth
- Software-defined radio, data acquisition, control systems

### Next Topics:

- Practical A/D conversion, quantization effects
- Multirate signal processing, decimation, interpolation
- Filter design for anti-aliasing