Linear Constant-Coefficient Difference Equations

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Introduction to Difference Equations

- **Important Class of LTI Systems**: Systems where input $x[n]$ and output $y[n]$ satisfy a difference equation
- **General Form**:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- **Parameters**:
  - N: Order of the system (highest delay in output)
  - a_k: Output coefficients (constant)
  - b_m: Input coefficients (constant)
  - M: Highest delay in input terms
- **Why Important?**:
  - Provides computational algorithms for LTI systems
  - Foundation for digital filter implementation
  - Connects time-domain and system analysis

Example 1: The Accumulator System

Problem: Find the difference equation for the accumulator system.
System Definition:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Approach:

- Rewrite the sum to separate current and past inputs
- Use the relationship between $y[n]$ and $y[n-1]$

Example 1: Accumulator Solution

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Step 1: Rewrite the accumulator equation:

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

Step 2: Recognize that the sum is $y[n-1]$:

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

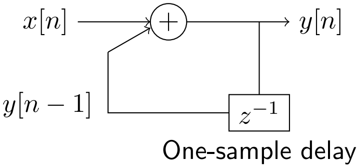
Step 3: Substitute to get the difference equation:

$$y[n] = x[n] + y[n-1]$$

Standard Form:

$$y[n] - y[n-1] = x[n]$$

Block Diagram: Recursive Accumulator



Recursive Implementation:

- Each output value computed using previously computed values
- $y[n] = x[n] + y[n-1]$: Add current input to previous output
- Requires initial condition (e.g., $y[-1] = 0$)

Example 2: Moving Average System

Problem: Find difference equation for causal moving average system.

System Definition (with $M_1 = 0$, so the system is causal):

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Two Approaches:

1. **Direct (Non-recursive)**: Use convolution form directly
2. **Recursive**: Express as cascade of simpler systems

Moving Average: Direct Implementation

Direct Form:

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

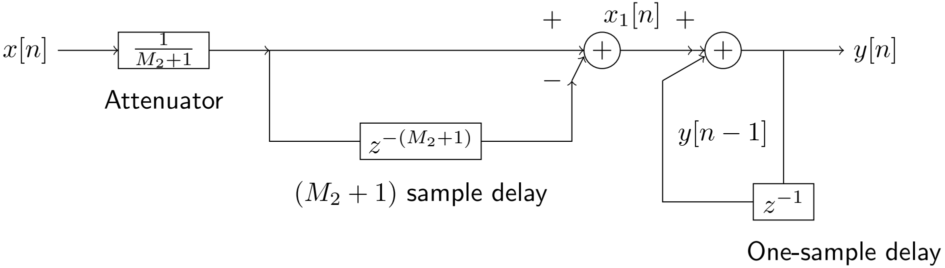
Standard Form:

$$y[n] = \sum_{k=0}^{M_2} \frac{1}{M_2 + 1} x[n-k]$$

Disadvantages of Direct Implementation:

- Requires $(M_2 + 1)$ multiplications per output sample
- Must store $(M_2 + 1)$ input samples in memory
- $O(M_2)$: Computational cost grows linearly with window size M_2

Block Diagram: Recursive Moving Average



Signal Flow:

- Input $x[n]$ → Attenuator $\frac{1}{M_2+1}$
- Attenuated signal splits: direct path and $(M_2 + 1)$ sample delay
- Sum: $x_1[n] = \frac{1}{M_2+1}[x[n] - x[n - (M_2 + 1)]]$
- $x_1[n]$ → Accumulator → Output $y[n]$

Moving Average as Difference Equation

Intermediate signal:

$$x_1[n] = \frac{1}{M_2 + 1}[x[n] - x[n - M_2 - 1]]$$

Accumulator relation: $y[n] = x_1[n] + y[n-1]$

Final recursive form:

$$y[n] - y[n-1] = \frac{1}{M_2 + 1}[x[n] - x[n - M_2 - 1]]$$

Note: There is an unlimited number of distinct difference equations to represent an LTI I/O relation.

General Solution of Difference Equations

Key Issue: Difference equation alone does not uniquely specify output!

General Solution Structure:

$$y[n] = y_p[n] + y_h[n]$$

- $y_p[n]$: Particular solution (satisfies original equation)
- $y_h[n]$: Homogeneous solution (satisfies homogeneous equation)

Homogeneous Equation (x[n] = 0):

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

Homogeneous Solution Form:

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n$$

where z_m are roots of characteristic polynomial $A(z) = \sum_{k=0}^{N} a_k z^{-k} = 0$

Auxiliary Conditions

Need for Auxiliary Conditions:

- N undetermined coefficients A_m in homogeneous solution
- Need N auxiliary (boundary) conditions for unique solution

Types of Auxiliary Conditions:

1. **Fixed Values**: Specify $y[-1], y[-2], \ldots, y[-N]$
2. **Initial Rest**: If $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$

Forward and Backward Computation for Specific Class of Difference Equations

Specific Class of Difference Equations:

- Inputs $x[n]$ and Outputs $y[n]$ satisfy an Nth-order linear constant coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Recursive Computation (forward):

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

Recursive Computation (backward):

$$y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_N} x[n-k]$$

Example 3: First-Order Difference Equation

Problem: Solve the difference equation with given input and initial condition.

Difference Equation:

$$y[n] = ay[n-1] + x[n]$$

Input:

$$x[n] = K\delta[n]$$

Auxiliary Condition:

$$y[-1] = c$$

Find: The complete solution $y[n]$ for all n.

Summary: Linear Constant-Coefficient Difference Equations

- **Definition**:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

  Describes the relationship between input $x[n]$ and output $y[n]$ in LTI systems.
- **Key Components**:
  - N and M: Order of the system
  - a_k, b_m: Constant coefficients
  - Recursive and direct computation methods
- **General Solution**:

$$y[n] = y_p[n] + y_h[n]$$

  - $y_p[n]$: Particular solution (due to input $x[n]$)
  - $y_h[n]$: Homogeneous solution (due to initial conditions)
  - Auxiliary conditions: Necessary to determine a unique / particular solutions.
- **Applications**: Digital filters, computational algorithms for LTI systems, and time-domain analysis.