Introduction
LTI Processing
Example: Ideal Lowpass Filter
Example: Ideal Differentiator
Impulse Invariance
Summary

Continuous-Time Processing of Discrete-Time Signals

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University of Nebraska-Lincoln

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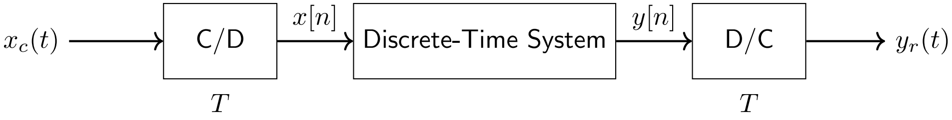
Overview: Discrete-Time Processing System

General System Architecture:



Key Properties:

- Overall system cascade: continuous-time input $\rightarrow$ continuous-time output
- Equivalent to continuous-time LTI system (under conditions)
- Same sampling rate T for C/D and D/C converters
- Properties depend on discrete-time system and sampling rate

Mathematical Summary: C/D Converter (ADC)

Time Domain - Sample Generation:

$$\boxed{x[n] = x_c(nT)}$$

Frequency Domain - Periodic Replication:

$$\boxed{X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)}$$

Key Points:

- DTFT $X(e^{j\omega})$ is periodic with period 2π
- Continuous-time frequency: Ω (rad/s)
- Discrete-time frequency: ω (radians)
- Relationship: $\omega = \Omega T$

Mathematical Summary: D/C Converter (DAC)

Ideal Bandlimited Interpolation:

$$\boxed{y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}}$$

Frequency Domain Relationship:

$$\boxed{Y_r(j\Omega) = \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}}$$

Interpretation:

- Each sample reconstructed as sinc function
- Ideal lowpass filter with cutoff $\Omega_c = \pi/T$
- Extracts baseband from periodic spectrum

Discrete-Time LTI System Processing

LTI System in Frequency Domain:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Combining with D/C Converter:

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$

Using Periodic Replication of $X(e^{j\omega})$:

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

Effective Continuous-Time System

Condition: If $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$ (bandlimited)

Then the ideal reconstruction filter selects only $k = 0$ term:

$$\boxed{Y_r(j\Omega) = H(e^{j\Omega T})X_c(j\Omega), \quad |\Omega| < \pi/T}$$

Effective Frequency Response:

$$\boxed{H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}}$$

Result: Overall system behaves as continuous-time LTI system

Requirements for LTI Behavior

Two Essential Conditions:

1. **Discrete-Time System**:
   - Must be linear and time-invariant
   - Characterized by impulse response $h[n]$ or frequency response $H(e^{j\omega})$
2. **Input Signal and Sampling**:
   - Input $x_c(t)$ must be bandlimited
   - Sampling rate above Nyquist: $\Omega_s = 2\pi/T > 2\Omega_N$
   - Aliased components (if any) must be removed by $H(e^{j\omega})$

Caution: Time-invariance can fail even with identity system if aliasing occurs

Example: Ideal Lowpass Filtering

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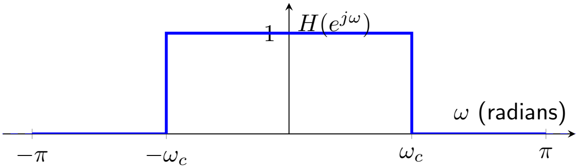
Discrete-Time Filter:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

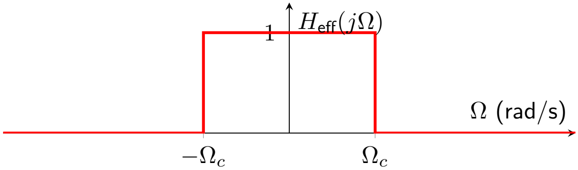


Note: Periodic with period 2π

Effective Continuous-Time Lowpass Filter

For bandlimited inputs at/above Nyquist rate:

$$H_{\text{eff}}(j\Omega) = \begin{cases} 1, & |\Omega| < \omega_c/T \\ 0, & |\Omega| \geq \omega_c/T \end{cases}$$



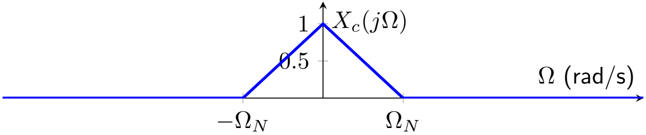
Key Insight: Cutoff frequency $\Omega_c = \omega_c/T$

- Variable T provides tunable continuous-time cutoff
- Fixed discrete-time filter, varying sampling rate

Frequency Domain Illustration (1/3)

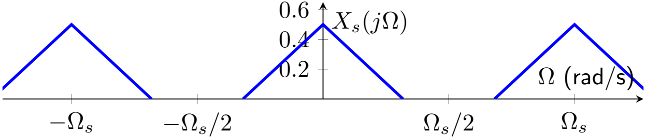
Bandlimited Input Spectrum:

Original Continuous-Time Signal



After Sampling - Periodic Replication:

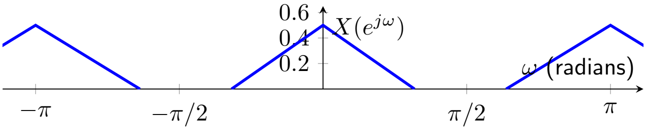
After C/D: Replicated Spectrum



Frequency Domain Illustration (2/3)

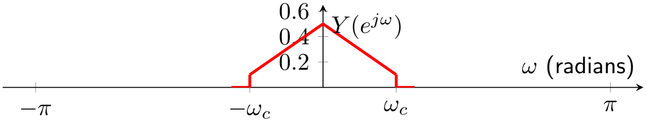
Discrete-Time Domain ($\omega = \Omega T$):

Normalized Frequency - Input to Discrete-Time System



After Discrete-Time Filtering ($\omega_c < \omega_N$):

Output of Discrete-Time Lowpass Filter (Cutoff Below Input Bandwidth)



Relaxed Nyquist Condition

Traditional Nyquist Requirement:

$$\Omega_s \geq 2\Omega_N \quad \Rightarrow \quad (2\pi - \Omega_N T) \geq \Omega_N T$$

With Discrete-Time Filtering:

$$(2\pi - \Omega_N T) \geq \omega_c$$

Interpretation:

- Can tolerate some aliasing if $H(e^{j\omega})$ removes aliased components
- Filter must eliminate frequencies where aliasing occurs
- More flexible than strict bandlimiting requirement
- Allows lower sampling rates in some applications

Example: Ideal Bandlimited Differentiator

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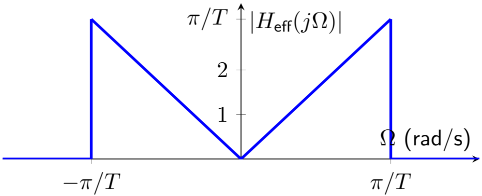
Continuous-Time Differentiator:

$$y_c(t) = \frac{d}{dt}[x_c(t)] \quad \Leftrightarrow \quad H_c(j\Omega) = j\Omega$$

Desired Effective Frequency Response:

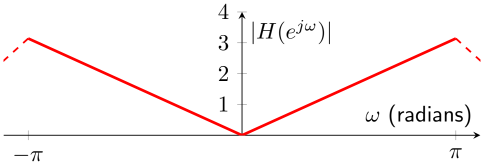
$$H_{\text{eff}}(j\Omega) = \begin{cases} j\Omega, & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$



Discrete-Time Differentiator Implementation

Required Discrete-Time Frequency Response:

$$H(e^{j\omega}) = \frac{j\omega}{T}, \quad |\omega| < \pi$$



Relationship: $H(e^{j\omega}) = H_c(j\omega/T)$ (periodic with period 2π)

Impulse Response of Discrete-Time Differentiator

Inverse DTFT:

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{j\omega}{T} e^{j\omega n} d\omega$$

Result:

$$\boxed{h[n] = \begin{cases} 0, & n = 0 \\ \frac{\cos(\pi n)}{nT}, & n \neq 0 \end{cases}}$$

Properties:

- Non-causal and infinite duration (ideal filter)
- Alternating signs: $h[n] = \frac{(-1)^n}{nT}$ for $n \neq 0$
- Practical implementation requires windowing and truncation

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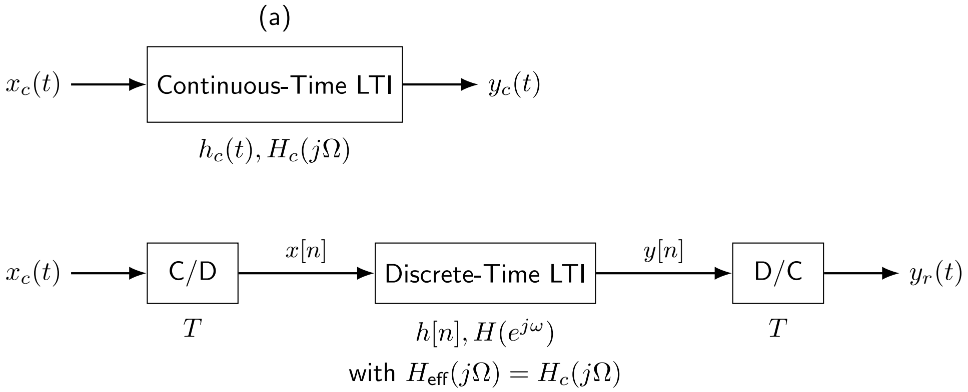
Impulse Invariance: Time Domain Relationship

Goal: Given continuous-time system $H_c(j\Omega)$, design discrete-time $H(e^{j\omega})$



Impulse Invariance: Mathematical Derivation

Step 1 - Apply Sampling Theory:
Start with sampled impulse response:

$$h[n] = h_c(nT)$$

Step 2 - Use C/D Frequency Relationship:
From sampling theory (previous lecture):

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega - 2\pi k}{T}\right)$$

We are left with periodic replication due to discrete sampling

Impulse Invariance: Deriving the Scale Factor

Step 3 - Apply Bandlimited Condition:
If $H_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, only $k = 0$ term survives (Perfect LPF'ing):

$$H(e^{j\omega}) = \frac{1}{T} H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Step 4 - Match Desired Response:
We want: $H(e^{j\omega}) = H_c(j\omega/T)$
But we have: $H(e^{j\omega}) = \frac{1}{T} H_c(j\omega/T)$

Solution: Scale the impulse response by T

$$\boxed{h[n] = T h_c(nT)}$$

This compensates for the $\frac{1}{T}$ scaling factor in the frequency domain

Impulse Invariance: Final Result

Time Domain Relationship:

$$\boxed{h[n] = T h_c(nT)}$$

Frequency Domain Relationship:

$$\boxed{H(e^{j\omega}) = H_c(j\omega/T), \quad |\omega| < \pi}$$

Summary of Derivation:

1. Start with sampled impulse response: $h[n] = h_c(nT)$
2. Apply C/D frequency relationship (periodic replication)
3. Eliminate aliased terms
4. Scale by T to match desired frequency response

Impulse Invariance: Ideal Lowpass Example

Continuous-Time Ideal Lowpass:

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & |\Omega| \geq \Omega_c \end{cases}, \quad h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

Impulse Invariant Discrete-Time Filter:

$$h[n] = T h_c(nT) = T \frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n}$$

where $\omega_c = \Omega_c T$

Resulting Frequency Response:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

Impulse Invariance: Suddenly Applied Exponential

Continuous-Time Exponential System:

$$h_c(t) = Ae^{s_0 t}u(t) \quad \Leftrightarrow \quad H_c(s) = \frac{A}{s - s_0}, \quad \text{Re}(s) > \text{Re}(s_0)$$

Apply Impulse Invariance:

$$h[n] = Th_c(nT) = ATe^{s_0 T n}u[n]$$

Discrete-Time System Function:

$$H(z) = \frac{AT}{1 - e^{s_0 T}z^{-1}}, \quad |z| > |e^{s_0 T}|$$

Frequency Response (if $\text{Re}(s_0) < 0 \rightarrow$ stable):

$$H(e^{j\omega}) = \frac{AT}{1 - e^{s_0 T}e^{-j\omega}}$$

Impulse Invariance: Aliasing Considerations

Exact Relationship Requires:

$$H_c(j\Omega) = 0 \text{ for } |\Omega| \geq \pi/T$$

In Practice:

- Most systems are not strictly bandlimited
- Aliasing will occur in $H(e^{j\omega})$
- Effect may be small if $H_c(j\Omega)$ decays rapidly

Design Strategy:

- Choose T small enough that $H_c(j\Omega) \approx 0$ for $|\Omega| > \pi/T$
- Higher sampling rate reduces aliasing error
- Trade-off between computational complexity and accuracy
- Impulse invariance is widely used for IIR filter design

Summary: Key Equations

C/D Conversion:

$$x[n] = x_c(nT), \quad X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)$$

D/C Conversion:

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

Effective System:

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

Impulse Invariance:

$$h[n] = Th_c(nT), \quad H(e^{j\omega}) = H_c(j\omega/T)$$

Summary: Design Implications

Advantages of Discrete-Time Processing:

- Programmable filter characteristics
- Easily tunable cutoff frequencies (vary T, fixed $H(e^{j\omega})$)
- Precise, stable implementations (analog systems suffer from variances)
- Complex operations (differentiation, filtering) easily realized

Design Considerations:

- Input must be bandlimited or prefiltered (anti-aliasing filter)
- Sampling rate must meet Nyquist (or relaxed) criterion
- Aliasing can be tolerated if filtered out by $H(e^{j\omega})$
- Impulse invariance is widely used for IIR filter design

Applications:

- Highly popular in Audio/video processing, communications
- Control systems
- Any continuous-time signal processing via digital systems