Discrete-Time Processing of Continuous-Time Signals

Maxx Seminario

University of Nebraska-Lincoln

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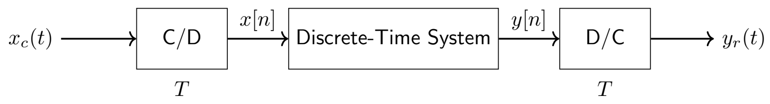
Overview: Discrete-Time Processing System

General System Architecture:



Key Properties:

- Overall system cascade: continuous-time input $\rightarrow$ continuous-time output
- Equivalent to continuous-time LTI system (under conditions)
- Same sampling rate T for C/D and D/C converters
- Properties depend on discrete-time system and sampling rate

Mathematical Summary: C/D Converter (ADC)

Time Domain - Sample Generation:

$$\boxed{x[n] = x_c(nT)}$$

Frequency Domain - Periodic Replication:

$$\boxed{X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)}$$

Key Points:

- DTFT $X(e^{j\omega})$ is periodic with period 2π
- Continuous-time frequency: Ω (rad/s)
- Discrete-time frequency: ω (radians)
- Relationship: $\omega = \Omega T$

Mathematical Summary: D/C Converter (DAC)

Ideal Bandlimited Interpolation:

$$\boxed{y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}}$$

Frequency Domain Relationship:

$$\boxed{Y_r(j\Omega) = \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}}$$

Interpretation:

- Each sample reconstructed as sinc function
- Ideal lowpass filter with cutoff $\Omega_c = \pi/T$
- Extracts baseband from periodic spectrum

Discrete-Time LTI System Processing

LTI System in Frequency Domain:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Combining with D/C Converter:

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$

Using Periodic Replication of $X(e^{j\omega})$:

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

Effective Continuous-Time System

Condition: If $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$ (bandlimited)

Then the ideal reconstruction filter selects only $k = 0$ term:

$$\boxed{Y_r(j\Omega) = H(e^{j\Omega T})X_c(j\Omega), \quad |\Omega| < \pi/T}$$

Effective Frequency Response:

$$\boxed{H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}}$$

Result: Overall system behaves as continuous-time LTI system

Requirements for LTI Behavior

Two Essential Conditions:

1. Discrete-Time System:

- Must be linear and time-invariant
- Characterized by impulse response $h[n]$ or frequency response $H(e^{j\omega})$

2. Input Signal and Sampling:

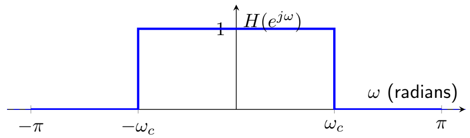
- Input $x_c(t)$ must be bandlimited
- Sampling rate above Nyquist: $\Omega_s = 2\pi/T > 2\Omega_N$
- Aliased components (if any) must be removed by $H(e^{j\omega})$

Caution: Time-invariance can fail even with identity system if aliasing occurs

Example: Ideal Lowpass Filtering

Discrete-Time Filter:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

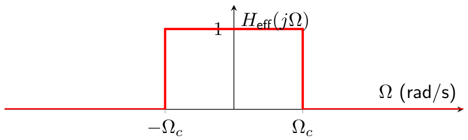


Note: Periodic with period 2π

Effective Continuous-Time Lowpass Filter

For bandlimited inputs at/above Nyquist rate:

$$H_{\text{eff}}(j\Omega) = \begin{cases} 1, & |\Omega| < \omega_c/T \\ 0, & |\Omega| \geq \omega_c/T \end{cases}$$



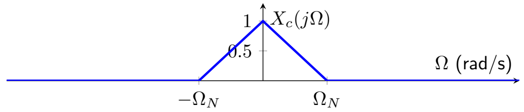
Key Insight: Cutoff frequency $\Omega_c = \omega_c/T$

- Variable T provides tunable continuous-time cutoff
- Fixed discrete-time filter, varying sampling rate

Frequency Domain Illustration (1/3)

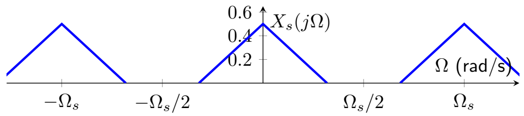
Bandlimited Input Spectrum:

Original Continuous-Time Signal



After Sampling - Periodic Replication:

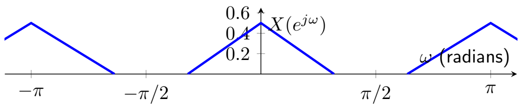
After C/D: Replicated Spectrum



Frequency Domain Illustration (2/3)

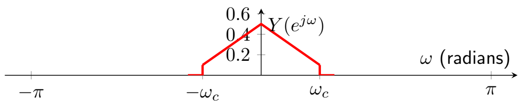
Discrete-Time Domain ($\omega = \Omega T$):

Normalized Frequency - Input to Discrete-Time System



After Discrete-Time Filtering ($\omega_c < \omega_N$):

Output of Discrete-Time Lowpass Filter (Cutoff Below Input Bandwidth)



Relaxed Nyquist Condition

Traditional Nyquist Requirement:

$$\Omega_s \geq 2\Omega_N \quad \Rightarrow \quad (2\pi - \Omega_N T) \geq \Omega_N T$$

With Discrete-Time Filtering:

$$(2\pi - \Omega_N T) \geq \omega_c$$

Interpretation:

- Can tolerate some aliasing if $H(e^{j\omega})$ removes aliased components
- Filter must eliminate frequencies where aliasing occurs
- More flexible than strict bandlimiting requirement
- Allows lower sampling rates in some applications

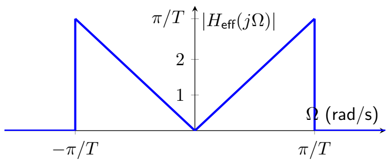
Example: Ideal Bandlimited Differentiator

Continuous-Time Differentiator:

$$y_c(t) = \frac{d}{dt}[x_c(t)] \quad \Leftrightarrow \quad H_c(j\Omega) = j\Omega$$

Desired Effective Frequency Response:

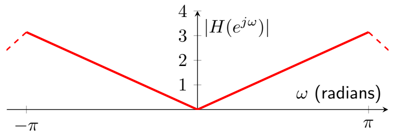
$$H_{\text{eff}}(j\Omega) = \begin{cases} j\Omega, & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$



Discrete-Time Differentiator Implementation

Required Discrete-Time Frequency Response:

$$H(e^{j\omega}) = \frac{j\omega}{T}, \quad |\omega| < \pi$$



Relationship: $H(e^{j\omega}) = H_c(j\omega/T)$ (periodic with period 2π)

Impulse Response of Discrete-Time Differentiator

Inverse DTFT:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{j\omega}{T} e^{j\omega n} d\omega$$

Result:

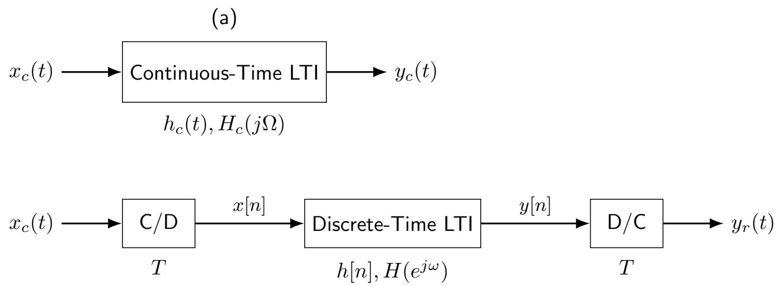
$$\boxed{h[n] = \begin{cases} 0, & n = 0 \\ \frac{\cos(\pi n)}{nT}, & n \neq 0 \end{cases}}$$

Properties:

- Non-causal and infinite duration (ideal filter)
- Alternating signs: $h[n] = \frac{(-1)^n}{nT}$ for $n \neq 0$
- Practical implementation requires windowing and truncation

Impulse Invariance: Time Domain Relationship

Goal: Given continuous-time system $H_c(j\Omega)$, design discrete-time $H(e^{j\omega})$



with $H_{\text{eff}}(j\Omega) = H_c(j\Omega)$

Impulse Invariance: Mathematical Derivation

Step 1 - Apply Sampling Theory:
Start with sampled impulse response:

$$h[n] = h_c(nT)$$

Step 2 - Use C/D Frequency Relationship:
From sampling theory (previous lecture):

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega - 2\pi k}{T}\right)$$

We are left with periodic replication due to discrete sampling

Impulse Invariance: Deriving the Scale Factor

Step 3 - Apply Bandlimited Condition:
If $H_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, only $k = 0$ term survives (Perfect LPF'ing):

$$H(e^{j\omega}) = \frac{1}{T}H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Step 4 - Match Desired Response:
We want: $H(e^{j\omega}) = H_c(j\omega/T)$
But we have: $H(e^{j\omega}) = \frac{1}{T}H_c(j\omega/T)$

Solution: Scale the impulse response by T

$$\boxed{h[n] = Th_c(nT)}$$

This compensates for the $\frac{1}{T}$ scaling factor in the frequency domain

Impulse Invariance: Final Result

Time Domain Relationship:

$$\boxed{h[n] = Th_c(nT)}$$

Frequency Domain Relationship:

$$\boxed{H(e^{j\omega}) = H_c(j\omega/T), \quad |\omega| < \pi}$$

Summary of Derivation:

1. Start with sampled impulse response: $h[n] = h_c(nT)$
2. Apply C/D frequency relationship (periodic replication)
3. Eliminate aliased terms
4. Scale by T to match desired frequency response

Impulse Invariance: Ideal Lowpass Example

Continuous-Time Ideal Lowpass:

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & |\Omega| \geq \Omega_c \end{cases}, \quad h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

Impulse Invariant Discrete-Time Filter:

$$h[n] = Th_c(nT) = T\frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n}$$

where $\omega_c = \Omega_c T$

Resulting Frequency Response:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

Impulse Invariance: Suddenly Applied Exponential

Continuous-Time Exponential System:

$$h_c(t) = Ae^{s_0 t}u(t) \quad \Leftrightarrow \quad H_c(s) = \frac{A}{s - s_0}, \quad \text{Re}(s) > \text{Re}(s_0)$$

Apply Impulse Invariance:

$$h[n] = Th_c(nT) = ATe^{s_0 Tn}u[n]$$

Discrete-Time System Function:

$$H(z) = \frac{AT}{1 - e^{s_0 T}z^{-1}}, \quad |z| > |e^{s_0 T}|$$

Frequency Response (if $\text{Re}(s_0) < 0 \to$ stable):

$$H(e^{j\omega}) = \frac{AT}{1 - e^{s_0 T}e^{-j\omega}}$$

Impulse Invariance: Aliasing Considerations

Exact Relationship Requires:

$$H_c(j\Omega) = 0 \text{ for } |\Omega| \geq \pi/T$$

In Practice:

- Most systems are not strictly bandlimited
- Aliasing will occur in $H(e^{j\omega})$
- Effect may be small if $H_c(j\Omega)$ decays rapidly

Design Strategy:

- Choose T small enough that $H_c(j\Omega) \approx 0$ for $|\Omega| > \pi/T$
- Higher sampling rate reduces aliasing error
- Trade-off between computational complexity and accuracy
- Impulse invariance is widely used for IIR filter design

Summary: Key Equations

C/D Conversion:

$$x[n] = x_c(nT), \quad X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)$$

D/C Conversion:

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

Effective System:

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

Impulse Invariance:

$$h[n] = Th_c(nT), \quad H(e^{j\omega}) = H_c(j\omega/T)$$

Summary: Design Implications

Advantages of Discrete-Time Processing:

- Programmable filter characteristics
- Easily tunable cutoff frequencies (vary T, fixed $H(e^{j\omega})$)
- Precise, stable implementations (analog systems suffer from variances)
- Complex operations (differentiation, filtering) easily realized

Design Considerations:

- Input must be bandlimited or prefiltered (anti-aliasing filter)
- Sampling rate must meet Nyquist (or relaxed) criterion
- Aliasing can be tolerated if filtered out by $H(e^{j\omega})$
- Impulse invariance is widely used for IIR filter design

Applications:

- Highly popular in Audio/video processing, communications
- Control systems
- Any continuous-time signal processing via digital systems

Continuous-Time Processing of Discrete-Time Signals

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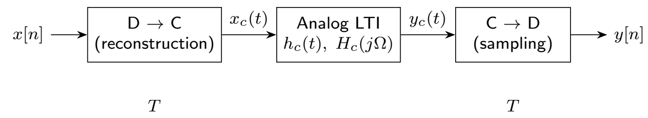
Overview: System Architecture

Complementary to Previous Topic:

- Previously: Discrete-time processing of continuous-time signals
- Now: Continuous-time processing of discrete-time signals

General System Configuration:



Key Characteristics:

- Input and output: discrete-time sequences
- Intermediate processing: continuous-time domain
- Provides useful interpretation of certain discrete-time systems
- Not typically implemented

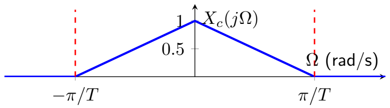
Bandlimited Signal Constraint

Fundamental Property:
The ideal D/C converter produces a bandlimited signal:

$$X_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T$$

Consequence: No aliasing in C/D conversion

- $y_c(t)$ is also bandlimited: $Y_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$
- Sampling $y_c(t)$ at rate $1/T$ satisfies Nyquist criterion
- Perfect reconstruction of $y[n]$ from $y_c(nT)$



Time-Domain Representations

Input Signal - Bandlimited Interpolation:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

Output Signal - After Continuous-Time Processing:

$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

where $y[n] = y_c(nT)$

Key Relationships:

- $x[n] = x_c(nT)$ - samples of reconstructed signal
- $y[n] = y_c(nT)$ - samples of processed signal
- Both sequences connected through continuous-time system

Frequency Domain Representations

Three Key Equations:

1. D/C Conversion:

$$X_c(j\Omega) = TX(e^{j\Omega T}), \quad |\Omega| < \pi/T$$

2. Continuous-Time Processing:

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

3. C/D Conversion:

$$Y(e^{j\omega}) = \frac{1}{T}Y_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Overall Discrete-Time System Response

Combining All Relationships:
Substitute Eq. (1) and (2) into Eq. (3):

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{T}Y_c\left(j\frac{\omega}{T}\right) \\ &= \frac{1}{T}H_c\left(j\frac{\omega}{T}\right)X_c\left(j\frac{\omega}{T}\right) \\ &= \frac{1}{T}H_c\left(j\frac{\omega}{T}\right)\cdot TX(e^{j\omega}) \end{aligned}$$

Result - Effective Discrete-Time Frequency Response:

$$\boxed{H(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi}$$

Therefore the overall system behaves as a discrete-time system with frequency response $H(e^{j\omega})$

Design Relationship

Forward Design: Given desired $H(e^{j\omega})$, find $H_c(j\Omega)$

Solution:

$$\boxed{H_c(j\Omega) = H(e^{j\Omega T}), \quad |\Omega| < \pi/T}$$

Arbitrary Extension:

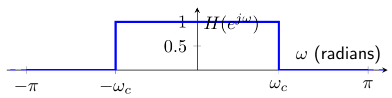
- Since $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, we can choose $H_c(j\Omega)$ arbitrarily above π/T
- Typically (out of convenience): $H_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$
- Makes $H_c(j\Omega)$ bandlimited

Key Notes:

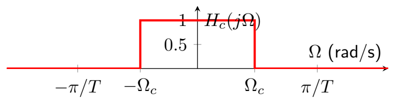
- This is the **inverse** of impulse invariance
- Impulse invariance: $H(e^{j\omega}) = H_c(j\omega/T)$
- This method: $H_c(j\Omega) = H(e^{j\Omega T})$

Frequency Domain Illustration

Discrete-Time Frequency Response $H(e^{j\omega})$:



Continuous-Time Frequency Response $H_c(j\Omega)$:



Relationship: $H_c(j\Omega) = H(e^{j\Omega T})$ for $|\Omega| < \pi/T$

Example: Noninteger Delay System

Discrete-Time Frequency Response:

$$H(e^{j\omega}) = e^{-j\omega\Delta}, \quad |\omega| < \pi$$

Case 1 - Integer Delay ($\Delta = n_0$, integer):

$$y[n] = x[n - n_0]$$

Straightforward interpretation: shift sequence by n_0 samples

Case 2 - Noninteger Delay (Δ not integer):

- Expression $y[n] = x[n - \Delta]$ has no direct meaning
- Cannot shift discrete sequence by fractional samples
- Need continuous-time interpretation

Continuous-Time Interpretation

Apply Design Relationship:

$$H_c(j\Omega) = H(e^{j\Omega T}) = e^{-j\Omega\Delta T}$$

We recognize this is an ideal time delay

$$y_c(t) = x_c(t - \Delta T)$$

Physical Interpretation:

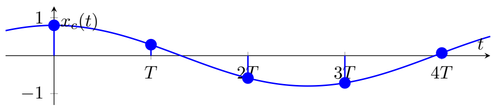
1. Start with discrete sequence $x[n]$
2. Reconstruct bandlimited $x_c(t)$ via D/C converter
3. Delay $x_c(t)$ by ΔT seconds
4. Sample delayed signal to get $y[n] = y_c(nT)$

Therefore the noninteger delay operates on the **interpolated** continuous signal

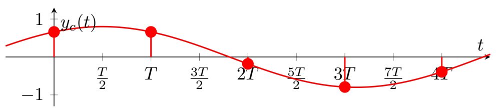
Noninteger Delay: Time Domain Visualization

Example: $\Delta = 0.5$ (half-sample delay)

Input: Interpolated Signal with Samples $x[n]$



Output: Delayed by $T/2$, Sampled at $y[n]$



Example: Moving-Average System

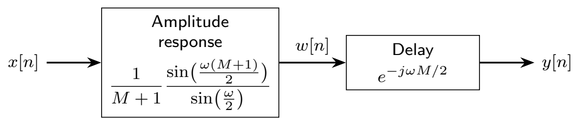
General $(M+1)$**-Point Moving Average**:

$$y[n] = \frac{1}{M+1} \sum_{k=0}^{M} x[n-k]$$

Frequency Response (from DTFT Lecture):

$$H(e^{j\omega}) = \frac{1}{M+1} \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$$

Decomposition:



Moving Average: Integer vs. Noninteger Delay

Case 1 - Odd Number of Points (M even):
Example: $M = 4$ (5-point average)

$$\text{Delay} = \frac{M}{2} = 2 \text{ samples (integer)}$$

$$y[n] = w[n-2]$$

Simple interpretation: 2-sample shift

Case 2 - Even Number of Points (M odd):
Example: $M = 5$ (6-point average)

$$\text{Delay} = \frac{M}{2} = 2.5 \text{ samples (noninteger)}$$

Must use continuous-time interpretation:

- Bandlimited interpolation of $w[n]$
- Continuous delay of $MT/2 = 2.5T$ seconds
- Resampling to get $y[n]$

Moving Average: Numerical Example

Input: $x[n] = \cos(0.25\pi n)$
System: 6-point moving average ($M = 5$)

Frequency Response at Input Frequency:

$$H(e^{j0.25\pi}) = \frac{1}{6}\frac{\sin[3(0.25\pi)]}{\sin(0.125\pi)}e^{-j(0.25\pi)(2.5)}$$

Calculate magnitude:

$$|H(e^{j0.25\pi})| = \frac{1}{6}\frac{\sin(0.75\pi)}{\sin(0.125\pi)} \approx 0.308$$

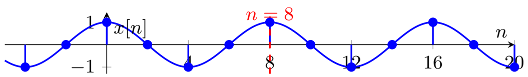
Calculate phase:

$$\angle H(e^{j0.25\pi}) = -0.25\pi \times 2.5 = -0.625\pi$$

Moving Average: Visualization

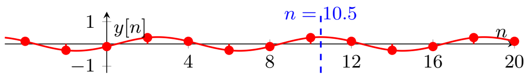
Input Signal: $x[n] = \cos(0.25\pi n)$

Input: Sampled Cosine



Output Signal: $y[n] = 0.308\cos[0.25\pi(n-2.5)]$

Output: Amplitude 0.308, Delay 2.5 samples



Summary: Concepts

System Architecture:

$$x[n] \xrightarrow{\text{D/C}} x_c(t) \xrightarrow{H_c(j\Omega)} y_c(t) \xrightarrow{\text{C/D}} y[n]$$

Fundamental Relationships:

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right) \quad \Leftrightarrow \quad H_c(j\Omega) = H(e^{j\Omega T})$$

Limitations:

- Not typically used for actual implementation
- Requires bandlimited signals for exact analysis
- Mainly a conceptual/analytical tool

Summary: Mathematical Framework

Time Domain:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

$$h[n] = \frac{\sin[\pi(n - \Delta)]}{\pi(n - \Delta)} \text{ for delay } \Delta$$

Frequency Domain:

$$X_c(j\Omega) = TX(e^{j\Omega T}), \quad Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T} Y_c\left(j\frac{\omega}{T}\right) = H_c\left(j\frac{\omega}{T}\right) X(e^{j\omega})$$

Design Equation:

$$\boxed{H_c(j\Omega) = H(e^{j\Omega T}), \quad |\Omega| < \pi/T}$$