Discrete Time Systems and Properties

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What is a Discrete-Time System?

- A **discrete-time system** is mathematically defined as a transformation or operator that maps an input sequence $x[n]$ into an output sequence $y[n]$.
- This can be denoted as:

$$y[n] = T\{x[n]\}$$

- $T\{\cdot\}$ represents a rule or formula for computing output values from input values.
- The output $y[n]$ at each index n may depend on all or part of the entire input sequence $x[n]$.

Pictorial Representation

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[n]$$

- The system transforms the input sequence $x[n]$ into a unique output sequence $y[n]$.

Example 1: The Ideal Delay System

- The **ideal delay system** is defined by:

$$y[n] = x[n - n_d], \quad -\infty < n < \infty$$

- n_d is a fixed positive integer representing the delay.
- The system shifts the input sequence to the right by n_d samples.
- If n_d is negative, the system shifts the input to the left by $|n_d|$ samples (time advance).

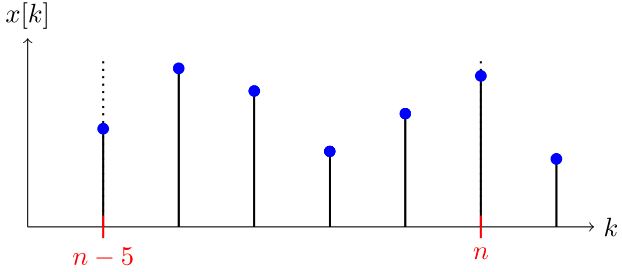
Example 2: Moving Average System

- The **moving-average system** is defined by:

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- M_1 and M_2 are non-negative integers.
- This system computes $y[n]$ as the average of $(M_1 + M_2 + 1)$ samples of $x[n]$ around index n.

Moving Average: Visualization



- For $n = 7$, $M_1 = 0$, $M_2 = 5$: $y[7]$ is the average of the six samples from $n-5$ to n.
- To compute $y[8]$, the region shifts right by one sample.

Example 3: Accumulator System

- **Accumulator system:**

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- **Analog Circuit Equivalent:**
  - The voltage across a capacitor is proportional to the total charge stored:

$$v_C(t) = \frac{1}{C} q(t)$$

$$q(t) = \int_{-\infty}^{t} i(\tau)\, d\tau$$

$$v_C(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau)\, d\tau$$

- The accumulator sums, just as a capacitor integrates current to produce voltage.

Classes of Discrete-Time Systems

- Different classes of systems are defined by placing constraints on the properties of the transformation $T\{\cdot\}$.
- These constraints often lead to general mathematical representations.
- Of particular importance are system properties such as linearity, time-invariance, causality, stability, and memory.

Memoryless Systems

- A system is **memoryless** if the output $y[n]$ at each n depends only on the input $x[n]$ at the same n.
- $y[n] = (x[n])^2$ is memoryless.
- The ideal delay system and the moving average system are **not** memoryless unless $n_d = 0$ or $M_1 = M_2 = 0$, respectively.

$$\text{Ideal delay:} \quad y[n] = x[n - n_d]$$

$$\text{Moving average:} \quad y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

Linear Systems

- A system is **linear** if it satisfies the *principle of superposition*:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$$
$$T\{ax[n]\} = aT\{x[n]\}$$

- In general, for any sequences $x_k[n]$ and scalars a_k:

$$T\left\{\sum_k a_k x_k[n]\right\} = \sum_k a_k T\{x_k[n]\}$$

Is the Ideal Delay System Linear?

- Consider the system $y[n] = x[n - n_d]$.
- Question: Is this system linear?
- What happens if we input a linear combination $ax_1[n] + bx_2[n]$?
- Does the output satisfy $T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$?

Linearity: Ideal Delay System

- **Definition:** $y[n] = x[n - n_d]$
- **Test for Linearity:**
  - For inputs $x_1[n]$ and $x_2[n]$, and scalars a, b:

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

  - Compute:

$$T\{ax_1[n] + bx_2[n]\} = ax_1[n - n_d] + bx_2[n - n_d] = aT\{x_1[n]\} + bT\{x_2[n]\}$$

- **Conclusion:** The ideal delay system is linear.

Is the Moving Average System Linear?

- Consider the system:

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- Question: Is this system linear?
- What happens for the input $ax_1[n] + bx_2[n]$?
- Does the system satisfy the superposition property?

Linearity: Moving Average System

- **Definition:**

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- **Test for Linearity:**
  - For $x_1[n]$, $x_2[n]$, a, b:

$$T\{ax_1[n] + bx_2[n]\} = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} [ax_1[n-k] + bx_2[n-k]]$$

  - This expands to:

$$aT\{x_1[n]\} + bT\{x_2[n]\}$$

- **Conclusion:** The moving average system is linear.

Is $y[n] = (x[n])^2$ Linear?

- Consider the system $y[n] = (x[n])^2$.
- Question: Is this system linear?
- What happens if we input $x_1[n] + x_2[n]$?
- Does the output satisfy the superposition property?

Why is $y[n] = (x[n])^2$ Not Linear?

- **Definition:** $y[n] = (x[n])^2$
- **Test for Linearity:**
  - For $x_1[n]$ and $x_2[n]$:

$$T\{x_1[n] + x_2[n]\} = (x_1[n] + x_2[n])^2 = (x_1[n])^2 + 2x_1[n]x_2[n] + (x_2[n])^2$$

  - $T\{x_1[n]\} + T\{x_2[n]\} = (x_1[n])^2 + (x_2[n])^2$
- **Conclusion:** Extra cross-term $2x_1[n]x_2[n]$ means the system is **not** linear.
- In other words, this cross-term is a second-order nonlinearity, where the two inputs, $x_1[n]$ and $x_2[n]$, mix to produce an output component that depends on both signals simultaneously.

Is the Accumulator System Linear?

- Consider the accumulator system:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- Question: Is this system linear?
- If we input a linear combination, does the output satisfy the superposition property?

Linearity: The Accumulator System

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- The output at time n is the sum of all present and past input samples.
- This system is linear:

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k]$$

$$y_2[n] = \sum_{k=-\infty}^{n} x_2[k]$$

$$y_3[n] = \sum_{k=-\infty}^{n} (ax_1[k] + bx_2[k]) = ay_1[n] + by_2[n]$$

Example: A Nonlinear System

- Consider $w[n] = \log_{10}(|x[n]|)$.
- This system is **not** linear.
- Counterexample: $x_1[n] = 1$, $x_2[n] = 10$.
  - $w_1[n] = 0$, $w_2[n] = 1$
  - $w[n]$ for $x_1[n] + x_2[n] = 11$: $\log_{10}(11) \neq \log_{10}(1) + \log_{10}(10)$
  - Scaling property also fails: $w_2[n] \neq 10w_1[n]$

Time-Invariant Systems

- A system is **time-invariant** if a shift in the input causes an identical shift in the output.
- If $x_1[n] = x[n - n_0]$, then $y_1[n] = y[n - n_0]$ for all n_0.
- All previously studied systems are time invariant.

Time-Invariance: The Accumulator

- For accumulator: $y[n] = \sum_{k=-\infty}^{n} x[k]$
- For shifted input $x_1[n] = x[n - n_0]$:

$$y_1[n] = \sum_{k=-\infty}^{n} x[k - n_0]$$

- Change variable $k_1 = k - n_0$:

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n - n_0]$$

- Thus, the accumulator is time-invariant.

Time-Invariance: The Compressor

$$y[n] = x[Mn], \quad M > 0$$

- M compresses the input by keeping every Mth sample and discarding the rest
- Shift input: $x_1[n] = x[n - n_0]$

$$y_1[n] = x_1[Mn] = x[Mn - n_0]$$

- Delay output:

$$y[n - n_0] = x[M(n - n_0)]$$

- $x[Mn - n_0] \neq x[M(n - n_0)]$ in general
- Not time-invariant

Causality Definition

- A system is **causal** if the output at time n_0 depends only on input values for $n \leq n_0$.
- If $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$.
- Causal systems are non-anticipative.

Causality of Previously Studied Systems

- **Ideal delay:** $y[n] = x[n - n_d]$
- **Moving average:** $y[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x[n-k]$
- **Squaring system:** $y[n] = (x[n])^2$
- **Accumulator:** $y[n] = \sum_{k=-\infty}^{n} x[k]$
- **Compressor:** $y[n] = x[Mn]$

Causality of Previously Studied Systems

- **Ideal delay:** $y[n] = x[n - n_d]$
  Causal if $n_d \geq 0$, noncausal if $n_d < 0$.
- **Moving average:** $y[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x[n-k]$
  Causal if $-M_1 \geq 0$ and $M_2 \geq 0$.
- **Squaring system:** $y[n] = (x[n])^2$
  Always causal.
- **Accumulator:** $y[n] = \sum_{k=-\infty}^{n} x[k]$
  Always causal.
- **Compressor:** $y[n] = x[Mn]$
  Noncausal if $M > 1$.

Causality: Forward and Backward Difference

- Forward difference: $y[n] = x[n+1] - x[n]$
  Not causal, depends on future input.
- Backward difference: $y[n] = x[n] - x[n-1]$
  Causal, depends only on present and past input.

Causality: Forward Difference Counterexample

- Inputs: $x_1[n] = \delta[n-1]$, $x_2[n] = 0$
- Outputs: $y_1[n] = \delta[n] - \delta[n-1]$, $y_2[n] = 0$
- $x_1[n] = x_2[n]$ for $n \leq 0$, but $y_1[0] \neq y_2[0]$
- Thus, forward difference is not causal.

Stability: Definition (BIBO)

- A system is **bounded-input bounded-output (BIBO) stable** if every bounded input produces a bounded output.
- If $|x[n]| \leq B_x < \infty$ for all n,
  then $|y[n]| \leq B_y < \infty$ for all n.
- If any bounded input leads to unbounded output, the system is not stable.

Stability: Previously Discussed Systems

- **Squaring system:** $y[n] = (x[n])^2$
- **Logarithmic system:** $y[n] = \log_{10} |x[n]|$
- **Accumulator:** $y[n] = \sum_{k=-\infty}^{n} x[k]$
- **Moving average:** $y[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x[n-k]$
- **Delay:** $y[n] = x[n-n_d]$
- **Compressor:** $y[n] = x[Mn]$
- **Forward difference:** $y[n] = x[n+1] - x[n]$
- **Backward difference:** $y[n] = x[n] - x[n-1]$

Stability: Previously Discussed Systems Results

- **Squaring system:** Stable ($|y[n]| \leq B_x^2$ for bounded $x[n]$)
- **Logarithmic system:** Unstable ($x[n] = 0$ gives $y[n] = -\infty$)
- **Accumulator:** Unstable (e.g., for $x[n] = u[n]$, $y[n] = n + 1$ unbounded)
- **Moving average:** Stable (output is an average of finitely many bounded values).
- **Delay:** Stable (output is a shifted version of bounded input).
- **Compressor:** Stable (output samples are from bounded input).
- **Forward/Backward diff:** Stable (difference of two bounded values).

Summary of System Properties

- **Memory:**
  - A system is memoryless if the output $y[n]$ at each n depends only on the input $x[n]$ at the same n.
- **Linearity:**
  - A system is linear if it satisfies the principle of superposition:

$$T\{a_1 x_1[n] + a_2 x_2[n]\} = a_1 T\{x_1[n]\} + a_2 T\{x_2[n]\}$$

- **Time-Invariance:**
  - A system is time-invariant if a shift in the input signal causes the same shift in the output:

$$x_1[n] = x[n - n_0] \implies y_1[n] = y[n - n_0]$$

- **Causality:**
  - A system is causal if the output at time n_0 depends only on the input values for $n \leq n_0$.
- **Stability (BIBO):**
  - A system is bounded-input bounded-output (BIBO) stable if every bounded input produces a bounded output:

$$|x[n]| \leq B_x < \infty \implies |y[n]| \leq B_y < \infty$$