The z-Transform

Maxx Seminario

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Overview: The z-Transform

- **Motivation**:
  - Fourier transform doesn't converge for all sequences
  - Need a more general transform that encompasses broader class of signals
  - z-transform notation often more convenient for analysis
- **Key Relationships**:
  - z-transform for discrete-time ↔ Laplace transform for continuous-time
  - Similar relationship to corresponding Fourier transforms
  - Fourier transform is special case: $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$
- **Today's Topics**:
  - Definition and convergence of z-transform
  - Region of Convergence (ROC) properties
  - Examples of common z-transform pairs
  - Properties of rational z-transforms

The z-Transform

Definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \text{(bilateral z-transform)} \tag{1}$$

where z is a complex variable.

z-Transform Operator:

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

Notation: $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$

One-sided z-Transform:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \quad \text{(unilateral z-transform)}$$

Relationship to Fourier Transform

Complex Variable in Polar Form:

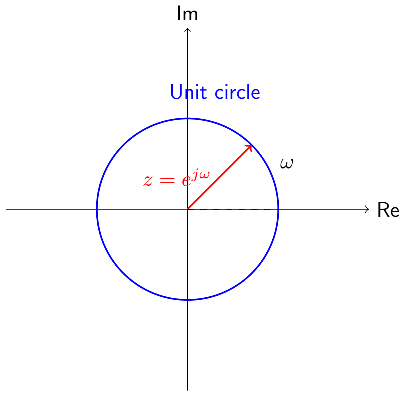
$$z = re^{j\omega} \tag{2}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} \tag{3}$$

$$= \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n} \tag{4}$$

Interpretation:

- z-transform = Fourier transform of $x[n]r^{-n}$
- For $r = 1$ (unit circle): $X(e^{j\omega}) =$ Fourier transform
- $|z| = 1$ defines the unit circle in z-plane



Region of Convergence (ROC)

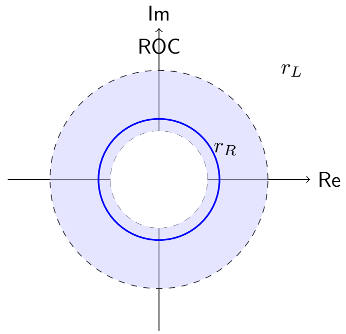
Definition: Set of values of z for which the z-transform converges

Convergence Condition:

$$|X(re^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

Key Properties:

- Convergence depends only on $|z| = r$
- ROC consists of a ring in z-plane: $r_R < |z| < r_L$
- If ROC includes unit circle $\Rightarrow$ Fourier transform exists



Example: Right-Sided Exponential

Signal: $x[n] = a^n u[n]$
z-Transform:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} \quad (5)$$

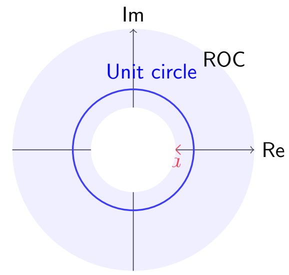
$$= \sum_{n=0}^{\infty} (az^{-1})^n \quad (6)$$

Convergence:

- Requires $|az^{-1}| < 1$
- Therefore: $|z| > |a|$

Closed Form:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad \text{for } |z| > |a|$$



Example: Left-Sided Exponential

Signal: $x[n] = -a^n u[-n-1]$
z-Transform:

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} \quad (7)$$

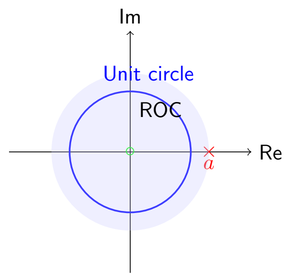
$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n \quad (8)$$

Convergence:

- Requires $|a^{-1}z| < 1$
- Therefore: $|z| < |a|$

Closed Form:

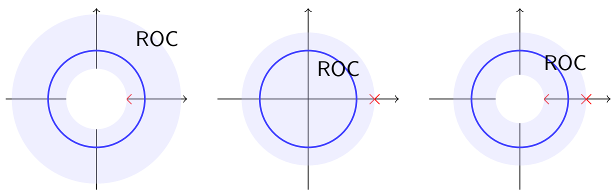
$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad \text{for } |z| < |a|$$



Properties of the ROC

1. **General Form**: $0 \leq r_R < |z| < r_L \leq \infty$ (annulus)
2. **Fourier Transform**: Exists iff ROC includes unit circle
3. **Poles**: ROC cannot contain any poles
4. **Finite-Duration**: ROC is entire z-plane except possibly $z = 0$ or $z = \infty$
5. **Right-Sided**: ROC extends outward from outermost pole
6. **Left-Sided**: ROC extends inward from innermost pole
7. **Two-Sided**: ROC is a ring bounded by poles
8. **Connected Region**: ROC must be connected

ROC for Different Sequence Types



Right-sided | Left-sided | Two-sided

Example: Sum of Two Exponentials

Signal:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

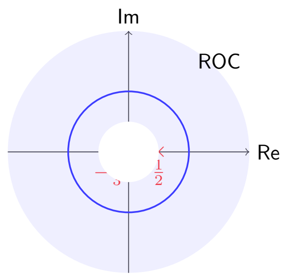
z-Transform (by linearity):

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} \quad (9)$$

$$= \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})} \quad (10)$$

ROC: Intersection of individual ROCs

- First term: $|z| > \frac{1}{2}$
- Second term: $|z| > \frac{1}{3}$
- Combined: $|z| > \frac{1}{2}$



Example: Two-Sided Exponential

Signal:

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

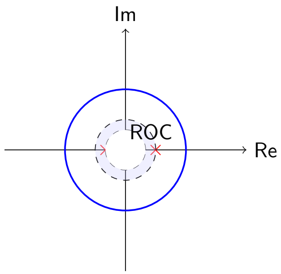
z-Transform:

$$X(z) = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

ROC:

- First term (right-sided): $|z| > \frac{1}{3}$
- Second term (left-sided): $|z| < \frac{1}{2}$
- Combined: $\frac{1}{3} < |z| < \frac{1}{2}$

Note: ROC doesn't include unit circle $\Rightarrow$ no Fourier transform



Finite-Length Sequences

Example: $x[n] = a^n, \quad 0 \le n \le N-1$
z-Transform:

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} \quad (11)$$

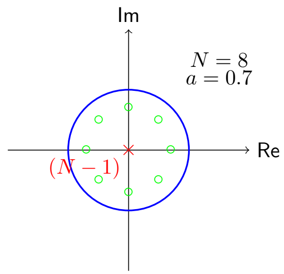
$$= \frac{1 - (az^{-1})^N}{1 - az^{-1}} \quad (12)$$

$$= \frac{z^N - a^N}{z^{N-1}(z-a)} \quad (13)$$

Zeros:

$$z_k = ae^{j2\pi k/N}, \quad k = 0, 1, ..., N-1$$

ROC: Entire z-plane except $z = 0$ (assuming $|a| < \infty$)



Common z-Transform Pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Rational z-Transforms

General Form:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

Key Points:

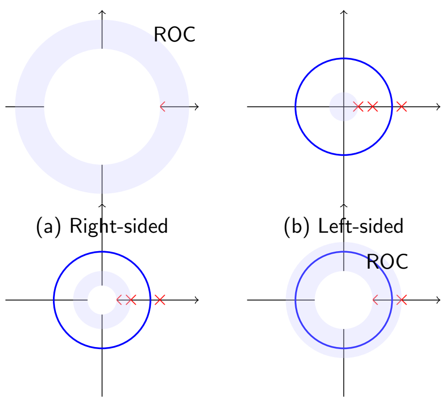
- Zeros: roots of $P(z) = 0$
- Poles: roots of $Q(z) = 0$
- ROC determined by pole locations
- Any sum of exponentials $\Rightarrow$ rational z-transform

Pole-Zero Plot:

- Poles: marked with $\times$
- Zeros: marked with $\circ$
- Must specify ROC to uniquely determine sequence

Example: Multiple ROCs for Same Pole-Zero Pattern

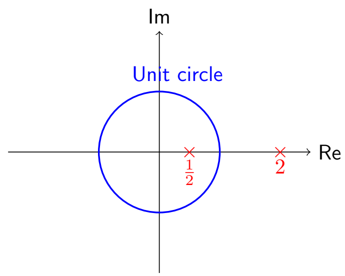
Given: Poles at $z = a, b, c$ with $|a| < |b| < |c|$



(a) Right-sided
(b) Left-sided

Stability and Causality

Example: System with poles at $z = \frac{1}{2}$ and $z = 2$



Three possible ROCs:

1. $|z| < \frac{1}{2}$: Left-sided, not stable
2. $\frac{1}{2} < |z| < 2$: Two-sided, stable, not causal
3. $|z| > 2$: Right-sided, causal, not stable

Summary

- **z-Transform Definition**:
  - Generalization of Fourier transform
  - Power series in complex variable z
  - Reduces to Fourier transform on unit circle
- **Region of Convergence**:
  - Critical for uniquely specifying sequence
  - Depends on sequence type (right/left/two-sided)
  - Cannot contain poles
  - Must be connected annular region
- **Rational z-Transforms**:
  - Result from sums of exponentials
  - Characterized by poles and zeros
  - ROC determines sequence properties
- **Next Time**: z-Transform properties and inverse z-transform

The Inverse z-Transform

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Overview: The Inverse z-Transform

- **Motivation**:
  - Need to move between time-domain and z-domain representations
  - Analysis often involves finding z-transform, manipulating, then inverting
  - Essential for discrete-time signal and system analysis
- **Formal Definition**:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

  where C is a closed contour within the ROC
- **Practical Methods**:
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

Inspection Method

Concept: Recognize common transform pairs "by inspection"

Some Transform Pairs:

- $a^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|$
- $-a^n u[-n-1] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| < |a|$

Example: $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

- If $|z| > \frac{1}{2}$: $x[n] = \left(\frac{1}{2}\right)^n u[n]$
- If $|z| < \frac{1}{2}$: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

Key Point: ROC determines which sequence! Same $X(z)$ can represent different sequences.

Partial Fraction Expansion: Overview

For Rational z-Transforms:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

General Procedure:

1. Factor denominator to find poles d_k
2. Determine expansion form:
   - $M < N$: $X(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$
   - $M \geq N$: $X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$
3. Calculate coefficients: $A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k}$
4. Use ROC to determine sequence type:
   - Poles inside inner ROC boundary → right-sided
   - Poles outside outer ROC boundary → left-sided

Partial Fractions: Simple Poles Example

Example:
$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z| > \frac{1}{2}$

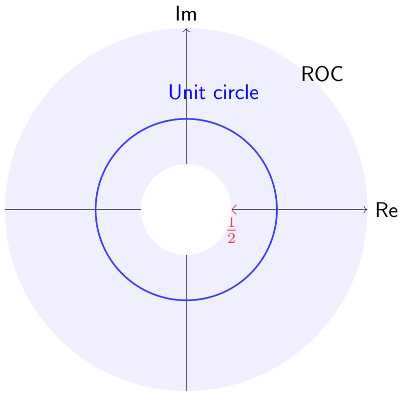
Partial fraction expansion:

$$X(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

Since ROC is $|z| > \frac{1}{2}$: Both poles inside ROC $\rightarrow$ right-sided sequences

Result:

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$



Partial Fractions: Case $M \geq N$

Example: $X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}, \quad |z| > 1$

Long division yields: $B_0 = 2$, remainder $= 5z^{-1} - 1$

Expansion:

$$X(z) = 2 + \frac{-9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$$

Since ROC is $|z| > 1$: All sequences are right-sided

Result:

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Power Series Expansion

Direct from Definition: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

Example 1:

- $X(z) = z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1}) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$
- $x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$

Example: Complete Inverse Transform

Given:

$$X(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$$

ROC: $\frac{1}{4} < |z| < \frac{1}{2}$
Partial fractions:

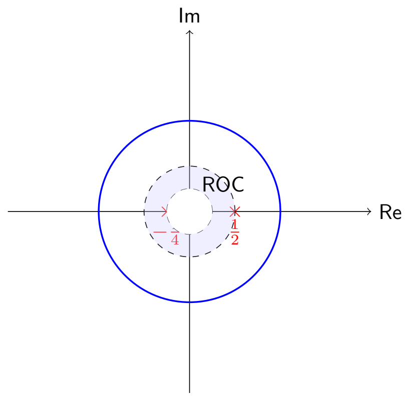
$$X(z) = \frac{3}{1-\frac{1}{2}z^{-1}} + \frac{-2}{1+\frac{1}{4}z^{-1}}$$

ROC analysis:

- Pole at $\frac{1}{2}$ outside ROC $\rightarrow$ left-sided
- Pole at $-\frac{1}{4}$ inside ROC $\rightarrow$ right-sided

Result:

$$x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] - 2\left(-\frac{1}{4}\right)^n u[n]$$



Summary of Inverse z-Transform Methods

1. **Inspection Method**:
   - Best for simple, recognizable forms
2. **Partial Fraction Expansion**:
   - Most useful for rational functions
   - Systematic procedure for simple and multiple poles
   - ROC critical for determining sequence type
3. **Power Series Expansion**:
   - Direct from definition

Key Principle: ROC determines sequence type

- Poles inside inner boundary $\rightarrow$ right-sided sequences
- Poles outside outer boundary $\rightarrow$ left-sided sequences

z-Transform Properties and LTI Systems

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Overview: z-Transform Properties

- **Motivation**:
  - Properties simplify analysis of discrete-time signals and systems
  - Used with inverse z-transform techniques for complex expressions
  - Foundation for solving difference equations algebraically
- **Applications**:
  - Transform difference equations to algebraic equations
  - Analyze LTI systems via system functions
  - Compute convolutions efficiently

Linearity

Property:

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{Z}} aX_1(z) + bX_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Key Points:

- ROC is at least the intersection of individual ROCs
- May be larger if pole-zero cancellation occurs
- Essential for partial fraction decomposition

Example: $x[n] = a^n(u[n] - u[n-N]) = a^n u[n] - a^n u[n-N]$

- Both terms have pole at $z = a$ with ROC $|z| > |a|$
- Linear combination cancels pole $\Rightarrow$ ROC becomes entire z-plane (except $z = 0$)
- Infinite-duration components combine to finite-duration sequence

Time Shifting

Property:

$$x[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}X(z), \quad \text{ROC} = R_x \text{ (except possible changes at } z = 0 \text{ or } z = \infty)$$

Key Points:

- $n_0 > 0$: right shift (delay)
- $n_0 < 0$: left shift (advance)
- Factor z^{-n_0} may add/remove poles at origin or infinity

Example: $X(z) = \frac{1}{z - \frac{1}{4}}, \quad |z| > \frac{1}{4}$

- Rewrite: $X(z) = z^{-1} \cdot \frac{1}{1 - \frac{1}{4}z^{-1}}$
- From $\left(\frac{1}{4}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - \frac{1}{4}z^{-1}}$
- Time-shift property gives: $x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$

Multiplication by Exponential Sequence

Property:

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \quad \text{ROC} = |z_0| R_x$$

Interpretation:

- All poles/zeros scaled by factor z_0
- If $z_0 > 0$ (real): radial scaling in z-plane
- If $|z_0| = 1$, $z_0 = e^{j\omega_0}$: rotation by ω_0
- For Fourier transform: $e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$

Example: Find z-transform of $x[n] = r^n \cos(\omega_0 n) u[n]$

- Express as: $x[n] = \frac{1}{2}(re^{j\omega_0})^n u[n] + \frac{1}{2}(re^{-j\omega_0})^n u[n]$
- Apply property to each term
- Result: $X(z) = \frac{1 - r\cos(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad |z| > r$

Differentiation Property

Property:

$$n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z \frac{dX(z)}{dz}, \quad \text{ROC} = R_x$$

Applications:

- Finding inverse transforms of non-rational functions
- Deriving transforms involving n as a factor
- Computing moments of sequences

Example 1: $X(z) = \log(1 + az^{-1}), \quad |z| > |a|$

- Differentiate: $\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}$
- Apply property: $n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{1 + az^{-1}}$
- Result: $x[n] = \frac{(-1)^{n+1} a^n}{n} u[n-1]$

Example 2: $n a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1 - az^{-1})^2}, \quad |z| > |a|$

Conjugation and Time Reversal

Conjugation Property:

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \quad \text{ROC} = R_x$$

Time Reversal Property:

$$x^*[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(1/z^*), \quad \text{ROC} = 1/R_x$$

For real sequences: $x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z), \quad \text{ROC} = 1/R_x$

Key Points:

- ROC inverted: if $r_R < |z| < r_L$, then new ROC is $1/r_L < |z| < 1/r_R$
- Pole at z_0 becomes pole at $1/z_0$
- Angle negated: $\angle(1/z_0) = -\angle z_0$

Example: $x[n] = a^{-n} u[-n]$ (time-reversed exponential)

- From $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, |z| > |a|$
- Apply time reversal: $X(z) = \frac{1}{1 - az} = \frac{-a^{-1} z^{-1}}{1 - a^{-1} z^{-1}}, |z| < |a^{-1}|$

Convolution Property

Property:

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) X_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Significance:

- Transforms convolution to multiplication
- Fundamental for LTI system analysis
- Basis for efficient filtering algorithms

Derivation: For $y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k]$

- Take z-transform: $Y(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] z^{-n}$
- Change order of summation and substitute $m = n - k$
- Result: $Y(z) = X_1(z) X_2(z)$ for z in both ROCs

ROC Note: May be larger than intersection if pole-zero cancellation occurs

Convolution Property: Example

Example: Convolution of finite sequences

- $x_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$
- $x_2[n] = \delta[n] - \delta[n-1]$

z-Transforms:

- $X_1(z) = 1 + 2z^{-1} + z^{-2}$
- $X_2(z) = 1 - z^{-1}$

Convolution via z-Transform:

$$\begin{aligned} Y(z) = X_1(z)X_2(z) &= (1 + 2z^{-1} + z^{-2})(1 - z^{-1}) \\ &= 1 + z^{-1} - z^{-2} - z^{-3} \end{aligned}$$

Result: $y[n] = \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]$

Key Notes:

- Convolution of sequences $\leftrightarrow$ Polynomial multiplication
- Coefficients of product polynomial = discrete convolution values
- Both sequences finite $\Rightarrow$ ROC is $|z| > 0$

Summary of z-Transform Properties

Property	Time Domain	z-Domain
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$
Exponential multiplication	$z_0^n x[n]$	$X(z/z_0)$
Differentiation	$nx[n]$	$-z\frac{dX(z)}{dz}$
Conjugation	$x^*[n]$	$X^*(z^*)$
Time reversal	$x[-n]$	$X(1/z)$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$
Real part	Re$\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$
Imaginary part	Im$\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$

ROC Considerations:

- Most properties preserve ROC or modify it predictably
- Linearity and convolution: ROC contains intersection
- Time shifting: may add/remove $z = 0$ or $z = \infty$
- Exponential multiplication: scales ROC by $|z_0|$

LTI Systems and the z-Transform

Fundamental Relationship:

- LTI system: $y[n] = x[n] * h[n]$
- z-Transform: $Y(z) = H(z)X(z)$
- $H(z)$ = system function (z-transform of impulse response)

System Function:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Key Properties:

- Poles of $H(z)$ determine system behavior
- ROC determines causality and stability:
  - Causal: ROC is $|z| > r_R$ (outside outermost pole)
  - Stable: ROC includes unit circle
  - Causal + Stable: All poles inside unit circle
- Frequency response: $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$ (if stable)

Example: Convolution via z-Transform - Setup

Problem: Find $y[n] = h[n] * x[n]$ where:

- $h[n] = a^n u[n]$, $|a| < 1$ (exponentially decaying impulse response)
- $x[n] = Au[n]$ (step input)

Step 1 - Find z-Transforms:

- $H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \quad |z| > |a|$
- $X(z) = \sum_{n=0}^{\infty} Az^{-n} = \frac{A}{1-z^{-1}}, \quad |z| > 1$

Step 2 - Multiply Transforms:

$$Y(z) = H(z)X(z) = \frac{A}{(1-az^{-1})(1-z^{-1})}, \quad \text{ROC: } |z| > 1$$

Note: ROC is intersection of individual ROCs. Since $|a| < 1$, we have $|z| > 1$

Example: Convolution via z-Transform - Solution

Step 3 - Partial Fractions:

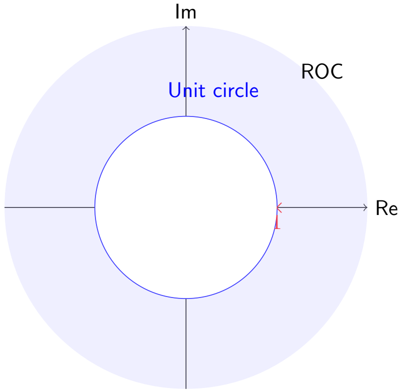
$$Y(z) = \frac{A}{1-a}\left[\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right]$$

Step 4 - Inverse Transform:

- $\frac{1}{1-z^{-1}} \xrightarrow{Z^{-1}} u[n]$
- $\frac{a}{1-az^{-1}} \xrightarrow{Z^{-1}} a \cdot a^n u[n] = a^{n+1}u[n]$

Final Result:

$$y[n] = \frac{A}{1-a}(1-a^{n+1})u[n]$$



Difference Equations and System Functions

General Difference Equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Applying z-Transform:

- Use linearity and time-shifting properties
- Result: $\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$

System Function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

Key Points:

- Numerator $\leftrightarrow$ input coefficients and delays
- Denominator $\leftrightarrow$ output coefficients and delays
- For causal system: ROC is $|z| >$ max pole magnitude
- Stable if all poles inside unit circle

Example: First-Order System

Difference Equation: $y[n] = ay[n-1] + x[n]$

System Function (by inspection):

$$H(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

Impulse Response:

$$h[n] = a^n u[n]$$

- Causal (ROC extends to ∞)
- Stable if $|a| < 1$ (pole inside unit circle)
- Frequency response (if stable): $H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$

Three Methods to Find Output:

1. Iterate difference equation
2. Convolve $x[n]$ with $h[n]$
3. Use z-transforms and partial fractions

Summary

z-Transform Properties:

- Provide powerful tools for signal and system analysis
- Transform complex operations (convolution) to simple ones (multiplication)
- Enable algebraic solution of difference equations

LTI System Analysis:

- System function $H(z)$ completely characterizes LTI system
- Poles determine stability and transient behavior
- Zeros affect frequency response shape
- ROC determines causality and stability

Key Relationships:

- Difference equation $\leftrightarrow$ Rational system function
- Impulse response $\leftrightarrow$ System function
- Convolution $\leftrightarrow$ Multiplication in z-domain
- Stability $\leftrightarrow$ Poles inside unit circle