z-Transform Properties and LTI Systems

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Overview: z-Transform Properties

- **Motivation**:
  - Properties simplify analysis of discrete-time signals and systems
  - Used with inverse z-transform techniques for complex expressions
  - Foundation for solving difference equations algebraically
- **Applications**:
  - Transform difference equations to algebraic equations
  - Analyze LTI systems via system functions
  - Compute convolutions efficiently

Linearity

Property:

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{Z}} aX_1(z) + bX_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Key Points:

- ROC is at least the intersection of individual ROCs
- May be larger if pole-zero cancellation occurs
- Essential for partial fraction decomposition

Example: $x[n] = a^n(u[n] - u[n-N]) = a^n u[n] - a^n u[n-N]$

- Both terms have pole at $z = a$ with ROC $|z| > |a|$
- Linear combination cancels pole $\Rightarrow$ ROC becomes entire z-plane (except $z = 0$)
- Infinite-duration components combine to finite-duration sequence

Time Shifting

Property:

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z), \quad \text{ROC} = R_x \text{ (except possible changes at } z=0 \text{ or } z=\infty)$$

Key Points:

- $n_0 > 0$: right shift (delay)
- $n_0 < 0$: left shift (advance)
- Factor z^{-n_0} may add/remove poles at origin or infinity

Example: $X(z) = \frac{1}{z-\frac{1}{4}}, \quad |z| > \frac{1}{4}$

- Rewrite: $X(z) = z^{-1} \cdot \frac{1}{1-\frac{1}{4}z^{-1}}$
- From $\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}z^{-1}}$
- Time-shift property gives: $x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$

Multiplication by Exponential Sequence

Property:

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \quad \text{ROC} = |z_0| R_x$$

Interpretation:

- All poles/zeros scaled by factor z_0
- If $z_0 > 0$ (real): radial scaling in z-plane
- If $|z_0| = 1$, $z_0 = e^{j\omega_0}$: rotation by ω_0
- For Fourier transform: $e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega - \omega_0)})$

Example: Find z-transform of $x[n] = r^n \cos(\omega_0 n) u[n]$

- Express as: $x[n] = \frac{1}{2}(re^{j\omega_0})^n u[n] + \frac{1}{2}(re^{-j\omega_0})^n u[n]$
- Apply property to each term
- Result: $X(z) = \frac{1 - r\cos(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad |z| > r$

Differentiation Property

Property:

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \quad \text{ROC} = R_x$$

Applications:

- Finding inverse transforms of non-rational functions
- Deriving transforms involving n as a factor
- Computing moments of sequences

Example 1: $X(z) = \log(1 + az^{-1}), \quad |z| > |a|$

- Differentiate: $\frac{dX(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}}$
- Apply property: $nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{1+az^{-1}}$
- Result: $x[n] = \frac{(-1)^{n+1}a^n}{n}u[n-1]$

Example 2: $na^nu[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$

Conjugation and Time Reversal

Conjugation Property:

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \quad \text{ROC} = R_x$$

Time Reversal Property:

$$x^*[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(1/z^*), \quad \text{ROC} = 1/R_x$$

For real sequences: $x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z), \quad \text{ROC} = 1/R_x$

Key Points:

- ROC inverted: if $r_R < |z| < r_L$, then new ROC is $1/r_L < |z| < 1/r_R$
- Pole at z_0 becomes pole at $1/z_0$
- Angle negated: $\angle(1/z_0) = -\angle z_0$

Example: $x[n] = a^{-n}u[-n]$ (time-reversed exponential)

- From $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, |z| > |a|$
- Apply time reversal: $X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, |z| < |a^{-1}|$

Convolution Property

Property:

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{Z}} X_1(z)X_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

Significance:

- Transforms convolution to multiplication
- Fundamental for LTI system analysis
- Basis for efficient filtering algorithms

Derivation: For $y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$

- Take z-transform: $Y(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]z^{-n}$
- Change order of summation and substitute $m = n - k$
- Result: $Y(z) = X_1(z)X_2(z)$ for z in both ROCs

ROC Note: May be larger than intersection if pole-zero cancellation occurs

Convolution Property: Example

Example: Convolution of finite sequences

- $x_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$
- $x_2[n] = \delta[n] - \delta[n-1]$

z-Transforms:

- $X_1(z) = 1 + 2z^{-1} + z^{-2}$
- $X_2(z) = 1 - z^{-1}$

Convolution via z-Transform:

$$\begin{aligned} Y(z) = X_1(z)X_2(z) &= (1 + 2z^{-1} + z^{-2})(1 - z^{-1}) \\ &= 1 + z^{-1} - z^{-2} - z^{-3} \end{aligned}$$

Result: $y[n] = \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]$

Key Notes:

- Convolution of sequences $\leftrightarrow$ Polynomial multiplication
- Coefficients of product polynomial $=$ discrete convolution values
- Both sequences finite $\Rightarrow$ ROC is $|z| > 0$

Summary of z-Transform Properties

Property	Time Domain	z-Domain
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$
Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$
Exponential multiplication	$z_0^n x[n]$	$X(z/z_0)$
Differentiation	$nx[n]$	$-z\frac{dX(z)}{dz}$
Conjugation	$x^*[n]$	$X^*(z^*)$
Time reversal	$x[-n]$	$X(1/z)$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$
Real part	$\text{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$
Imaginary part	$\text{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$

ROC Considerations:

- Most properties preserve ROC or modify it predictably
- Linearity and convolution: ROC contains intersection
- Time shifting: may add/remove $z = 0$ or $z = \infty$
- Exponential multiplication: scales ROC by $|z_0|$

LTI Systems and the z-Transform

Fundamental Relationship:

- LTI system: $y[n] = x[n] * h[n]$
- z-Transform: $Y(z) = H(z)X(z)$
- $H(z) =$ system function (z-transform of impulse response)

System Function:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Key Properties:

- Poles of $H(z)$ determine system behavior
- ROC determines causality and stability:
  - Causal: ROC is $|z| > r_R$ (outside outermost pole)
  - Stable: ROC includes unit circle
  - Causal + Stable: All poles inside unit circle
- Frequency response: $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$ (if stable)

Example: Convolution via z-Transform - Setup

Problem: Find $y[n] = h[n] * x[n]$ where:

- $h[n] = a^n u[n]$, $|a| < 1$ (exponentially decaying impulse response)
- $x[n] = Au[n]$ (step input)

Step 1 - Find z-Transforms:

- $H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \quad |z| > |a|$
- $X(z) = \sum_{n=0}^{\infty} Az^{-n} = \frac{A}{1-z^{-1}}, \quad |z| > 1$

Step 2 - Multiply Transforms:

$$Y(z) = H(z)X(z) = \frac{A}{(1-az^{-1})(1-z^{-1})}, \quad \text{ROC: } |z| > 1$$

Note: ROC is intersection of individual ROCs. Since $|a| < 1$, we have $|z| > 1$

Example: Convolution via z-Transform - Solution

Step 3 - Partial Fractions:

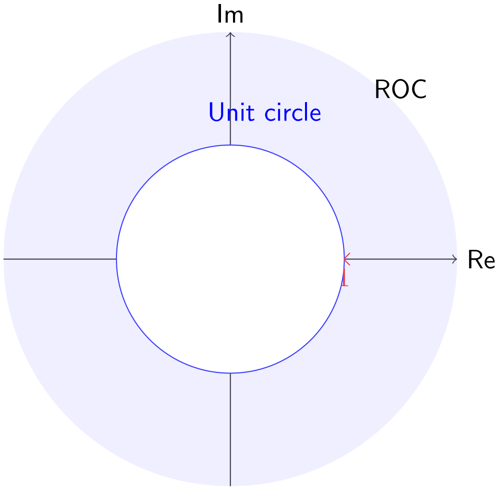
$$Y(z) = \frac{A}{1-a}\left[\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right]$$

Step 4 - Inverse Transform:

- $\frac{1}{1-z^{-1}} \xrightarrow{\mathcal{Z}^{-1}} u[n]$
- $\frac{a}{1-az^{-1}} \xrightarrow{\mathcal{Z}^{-1}} a \cdot a^n u[n] = a^{n+1}u[n]$

Final Result:

$$y[n] = \frac{A}{1-a}(1 - a^{n+1})u[n]$$



Difference Equations and System Functions

General Difference Equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Applying z-Transform:

- Use linearity and time-shifting properties
- Result: $\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$

System Function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

Key Points:

- Numerator $\leftrightarrow$ input coefficients and delays
- Denominator $\leftrightarrow$ output coefficients and delays
- For causal system: ROC is $|z| >$ max pole magnitude
- Stable if all poles inside unit circle

Example: First-Order System

Difference Equation: $y[n] = ay[n-1] + x[n]$

System Function (by inspection):

$$H(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$

Impulse Response:

$$h[n] = a^n u[n]$$

- Causal (ROC extends to ∞)
- Stable if $|a| < 1$ (pole inside unit circle)
- Frequency response (if stable): $H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$

Three Methods to Find Output:

1. Iterate difference equation
2. Convolve $x[n]$ with $h[n]$
3. Use z-transforms and partial fractions

Summary

z-Transform Properties:

- Provide powerful tools for signal and system analysis
- Transform complex operations (convolution) to simple ones (multiplication)
- Enable algebraic solution of difference equations

LTI System Analysis:

- System function $H(z)$ completely characterizes LTI system
- Poles determine stability and transient behavior
- Zeros affect frequency response shape
- ROC determines causality and stability

Key Relationships:

- Difference equation $\leftrightarrow$ Rational system function
- Impulse response $\leftrightarrow$ System function
- Convolution $\leftrightarrow$ Multiplication in z-domain
- Stability $\leftrightarrow$ Poles inside unit circle